Stable Manifold Theorem: Part 6

Lecture 36
Math 634
11/22/99

Higher Differentiability

Lemma (Higher Differentiability) *If F is C^k, then h is C^k.*

Proof. We've already seen that this holds for $k = 1$. We show that it is true for all k by induction. Let $k \geq 2$, and assume that the lemma works for $k-1$. Define a new map $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ by the formula

$$H\left(\begin{bmatrix} p \\ v \end{bmatrix}\right) := \begin{bmatrix} F(p) \\ DF(p)v \end{bmatrix}.$$

Since F is C^k, H is C^{k-1}. Note that

$$H^2\left(\begin{bmatrix} p \\ v \end{bmatrix}\right) = \begin{bmatrix} F(F(p)) \\ DF(F(p))DF(p)v \end{bmatrix} = \begin{bmatrix} F^2(p) \\ D(F^2)(p)v \end{bmatrix},$$

$$H^3\left(\begin{bmatrix} p \\ v \end{bmatrix}\right) = \begin{bmatrix} F(F^2(p)) \\ DF(F^2(p))D(F^2)(p)v \end{bmatrix} = \begin{bmatrix} F^3(p) \\ D(F^3)(p)v \end{bmatrix},$$

and, in general,

$$H^j\left(\begin{bmatrix} p \\ v \end{bmatrix}\right) = \begin{bmatrix} F^j(p) \\ D(F^j)(p)v \end{bmatrix}.$$

Also,

$$DH\left(\begin{bmatrix} p \\ v \end{bmatrix}\right) = \left[\begin{array}{c|c} DF(p) & 0 \\ \hline D^2F(p)v & DF(p) \end{array}\right],$$

so

$$DH\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \left[\begin{array}{c|c} DF(0) & 0 \\ \hline 0 & DF(0) \end{array}\right],$$

which is hyperbolic and invertible, since $DF(0)$ is. Applying the induction hypothesis, we can conclude that the fixed point of H at the origin (in $\mathbb{R}^n \times \mathbb{R}^n$) has a local stable manifold $\mathcal{W}$ that is C^{k-1}.

Fix $q \in W_r^s$, and note that $F^j(q) \to 0$ as $j \uparrow \infty$ and

$$\mathcal{P}_q = \left\{ v \in \mathbb{R}^n \;\middle|\; \lim_{j\uparrow\infty} D(F^j)(q)v = 0 \right\}.$$

This means that

$$\mathcal{P}_q = \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} q \\ v \end{bmatrix} \in \mathcal{W} \right\}.$$

Since $\mathcal{W}$ has a C^{k-1} dependence on q, so does $\mathcal{P}_q$. Hence, h is C^k. □

Flows

Now we discuss how the Stable Manifold Theorem for maps implies the Stable Manifold Theorem for flows. Given $f : \Omega \to \mathbb{R}^n$ satisfying $f(0) = 0$, let $F = \varphi(1, \cdot)$, where φ is the flow generated by the differential equation

$$\dot{x} = f(x). \tag{1}$$

If f is C^k, so is F. Clearly, F is invertible and $F(0) = 0$. Our earlier discussion on differentiation with respect to initial conditions tells us that

$$\frac{d}{dt} D_x\varphi(t,x) = Df(\varphi(t,x))D_x\varphi(t,x)$$

and $D_x\varphi(0,x) = I$, where D_x represents differentiation with respect to x. Setting

$$g(t) = D_x\varphi(t,x)\big|_{x=0},$$

this implies, in particular, that

$$\frac{d}{dt} g(t) = Df(0)g(t)$$

and $g(0) = I$, so

$$g(t) = e^{tDf(0)}.$$

Setting $t = 1$, we see that

$$e^{Df(0)} = g(1) = D_x\varphi(1,x)|_{x=0} = D_xF(x)|_{x=0} = DF(0).$$

Thus, $DF(0)$ is invertible, and if (1) has a hyperbolic equilibrium at the origin then $DF(0)$ is hyperbolic.

Since F satisfies the hypotheses of the Stable Manifold Theorem for maps, we know that F has a local stable manifold W_r^s on some box $\mathcal{B}(r)$. Assume that $\alpha < 1$ and that r is small enough that the vector field of (1) points *into* $\mathcal{B}(r)$ on $C^s(\alpha) \cap \partial\mathcal{B}(r)$. (See the estimates in Lecture 21.) The requirements for a point to be in W_r^s are no more restrictive then the requirements to be in the local stable manifold $\mathcal{W}_r^s$ of the origin with respect to the flow, so $\mathcal{W}_r^s \subseteq W_r^s$.

We claim that, in fact, these two sets are equal. Suppose they are not. Then there is a point $q \in W_r^s \setminus \mathcal{W}_r^s$. Let $x(t)$ be the solution of (1) satisfying $x(0) = q$. Since $\lim_{j\uparrow\infty} F^j(q) = 0$ and, in a neighborhood of the origin, there is a bound on the factor by which $x(t)$ can grow in 1 unit of time, we know that $x(t) \to 0$ as $t \uparrow \infty$. Among other things, this implies that

(a) $x(t) \notin W_r^s$ for some $t > 0$, and

(b) $x(t) \in W_r^s$ for all t sufficiently large.

Since W_r^s is a closed set and x is continuous, **(a)** and **(b)** say that we can pick t_0 to be the earliest time such that $x(t) \in W_r^s$ for every $t \geq t_0$.

Now, consider the locationof $x(t)$ for t in the time interval $[t_0-1, t_0)$. Since $x(0) \in W_r^s$, we know that $x(j) \in W_r^s$ for every $j \in \mathbb{N}$. In particular, we can choose $t_1 \in [t_0-1, t_0)$ such that $x(t_1) \in W_r^s$. By definition of t_0, we can choose $t_2 \in (t_1, t_0)$ such that $x(t_2) \notin W_r^s$. By the continuity of x and the closedness of W_r^s, we can pick t_3 to the be the last time before t_2 such that $x(t_3) \in W_r^s$. By definition of W_r^s, if $t \in [t_0 - 1, t_0)$ and $x(t) \notin W_r^s$, then $x(t) \notin \mathcal{B}(r)$; hence, $x(t)$ must leave $\mathcal{B}(r)$ at time $t = t_3$. But this contradicts the fact that the vector field points into $\mathcal{B}(r)$ at $x(t_3)$, since $x(t_3) \in C^s(\alpha) \cap \partial\mathcal{B}(r)$. This contradiction implies that no point $q \in W_r^s \setminus \mathcal{W}_r^s$ exists; *i.e.*, $W_r^s = \mathcal{W}_r^s$.

The exponential decay of solutions of the flow on the local stable manifold is a consequence of the similar decay estimate for the map, along with the observation that, near 0, there is a bound to the factor by which a solution can grown in 1 unit of time.