# Center Manifolds

Lecture 37
Math 634
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## Definition

Recall that for the linear differential equation

$$\dot{x} = Ax \tag{1}$$

the corresponding invariant subspaces $\mathcal{E}^u$, $\mathcal{E}^s$, and $\mathcal{E}^c$ had the characterizations

$$\mathcal{E}^u = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists c > 0 \text{ s.t. } \lim_{t \downarrow -\infty} |e^{-ct} e^{tA} x| = 0 \right\},$$

$$\mathcal{E}^s = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists c > 0 \text{ s.t. } \lim_{t \uparrow \infty} |e^{ct} e^{tA} x| = 0 \right\},$$

and

$$\mathcal{E}^c = \left\{ x \in \mathbb{R}^n \;\middle|\; \forall c > 0, \lim_{t \downarrow -\infty} |e^{ct} e^{tA} x| = 0 \text{ and } \lim_{t \uparrow \infty} |e^{-ct} e^{tA} x| = 0 \right\}.$$

The Stable Manifold Theorem tells us that for the nonlinear differential equation

$$\dot{x} = f(x), \tag{2}$$

with $f(0) = 0$, the stable manifold $W^s(0)$ and the unstable manifold $W^u(0)$ have characterizations similar to $\mathcal{E}^s$ and $\mathcal{E}^u$, respectively:

$$W^s(0) = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists c > 0 \text{ s.t. } \lim_{t \uparrow \infty} |e^{ct} \varphi(t, x)| = 0 \right\},$$

and

$$W^u(0) = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists c > 0 \text{ s.t. } \lim_{t \downarrow -\infty} |e^{-ct} \varphi(t, x)| = 0 \right\},$$

where $\varphi$ is the flow generated by (2). (This was only verified when the equilibrium point at the origin was hyperbolic, but a similar result holds in general.)

Is there a useful way to modify the characterization of $\mathcal{E}^c$ similarly to get a characterization of a *center manifold* $W^c(0)$? Not really. The main problem is that the characterizations of $\mathcal{E}^s$ and $\mathcal{E}^u$ only depend on the *local* behavior of solutions when they are near the origin, but the characterization of $\mathcal{E}^c$ depends on the behavior of solutions that are, possibly, far from 0.

Still, the idea of a center manifold as some sort of nonlinear analogue of $\mathcal{E}^c(0)$ is useful. Here's one widely-used definition:

Definition Let $A = Df(0)$. A *center manifold* $W^c(0)$ of the equilbrium point 0 of (2) is an invariant manifold whose dimension equals the dimension of the invariant subspace $\mathcal{E}^c$ of (1) and which is tangent to $\mathcal{E}^c$ at the origin.

## Nonuniqueness

While the fact that stable and unstable manifolds are really manifolds is a theorem (namely, the Stable Manifold Theorem), a center manifold is a manifold *by definition*. Also, note that we refer to *the* stable manifold and *the* unstable manifold, but we refer to *a* center manifold. This is because center manifolds are not necessarily unique. An extremely simple example of nonuniqueness (commonly credited to Kelley) is the planar system

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y. \end{cases}$$

Clearly, $\mathcal{E}^c$ is the $x$-axis, and solving the system explicitly reveals that for any constant $c \in \mathbb{R}$ the curve

$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x < 0 \text{ and } y = ce^{1/x}\right\} \cup \left\{(x, 0) \in \mathbb{R}^2 \;\middle|\; x \geq 0\right\}$$

is a center manifold.

## Existence

There is a Center Manifold Theorem just like there was a Stable Manifold Theorem. However, the goal of the Center Manifold Theorem is not to characterize a center manifold; that is done by the definition. The Center Manifold Theorem asserts the *existence* of a center manifold.

We will not state this theorem precisely nor prove it, but we can give some indication how the proof of existence of a center manifold might go. Suppose that none of the eigenvalues of $Df(0)$ have real part equal to $\alpha$, where $\alpha$ is a given real number. Then we can split the eigenvalues up into two sets: Those with real part less than $\alpha$ and those with real part greater than $\alpha$. Let $\mathcal{E}^-$ be the vector space spanned by the generalized eigenvectors corresponding to the first set of eigenvalues, and let $\mathcal{E}^+$ be the vector space spanned by the generalized eigenvectors corresponding to the second set of eigenvalues. If we cut off $f$ so that it is stays nearly linear throughout $\mathbb{R}^n$, then an analysis very much like that in the proof of the Stable Manifold Theorem can be done to conclude that there are invariant manifolds called the *pseudo-stable manifold* and the *pseudo-unstable manifold* that are tangent, respectively, to $\mathcal{E}^-$ and $\mathcal{E}^+$ at the origin. Solutions $x(t)$ in the first manifold satisfy $e^{-\alpha t}x(t) \to 0$ as $t \uparrow \infty$, and solutions in the second manifold satisfy $e^{-\alpha t}x(t) \to 0$ as $t \downarrow -\infty$.

Now, suppose that $\alpha$ is chosen to be negative but larger than the real part of the eigenvalues with negative real part. The corresponding pseudo-unstable manifold is called a *center-unstable manifold* and is written $W^{cu}(0)$. If, on the other hand, we choose $\alpha$ to be between zero and all the positive real parts of eigenvalues, then the resulting pseudo-stable manifold is called a *center-stable manifold* and is written $W^{cs}(0)$. It turns out that

$$W^c(0) := W^{cs}(0) \cap W^{cu}(0)$$

is a center manifold.

## Center Manifold as a Graph

Since a center manifold $W^c(0)$ is tangent to $\mathcal{E}^c$ at the origin it can, at least locally, be represented as the graph of a function $h : \mathcal{E}^c \to \mathcal{E}^s \oplus \mathcal{E}^u$. Suppose, for simplicity, that (2) can be rewritten in the form

$$\begin{cases} \dot{x} = Ax + F(x, y) \\ \dot{y} = By + G(x, y), \end{cases} \tag{3}$$

where $x \in \mathcal{E}^c$, $y \in \mathcal{E}^s \oplus \mathcal{E}^u$, the eigenvalues of $A$ all have zero real part, all of the eigenvalues of $B$ have nonzero real part, and $F$ and $G$ are higher order terms. Then, for points $x + y$ lying on $W^c(0)$, $y = h(x)$. Inserting that into (3) and using the chain rule, we get

$$Dh(x)[Ax + F(x, h(x))] = Dh(x)\dot{x} = \dot{y} = Bh(x) + G(x, h(x)).$$

Thus, if we define an operator $\mathcal{M}$ by the formula

$$(\mathcal{M}\phi)(x) := D\phi(x)[Ax + F(x, \phi(x))] - B\phi(x) + G(x, \phi(x)),$$

the function $h$ whose graph is the center manifold is a solution of the equation $\mathcal{M}h = 0$.