Computing and Using Center Manifolds

Lecture 38
Math 634
12/1/99

Approximation

Recall that we projected our equation onto $\mathcal{E}^c$ and onto $\mathcal{E}^s \oplus \mathcal{E}^u$ to get the system

$$\begin{cases} \dot{x} = Ax + F(x,y) \\ \dot{y} = By + G(x,y), \end{cases} \tag{1}$$

and that we were looking for a function $h : \mathcal{E}^c \to \mathcal{E}^s \oplus \mathcal{E}^u$ satisfying $(\mathcal{M}h) \equiv 0$, where

$$(\mathcal{M}\phi)(x) := D\phi(x)[Ax + F(x,\phi(x))] - B\phi(x) + G(x,\phi(x)).$$

Except in the simplest of cases we have no hope of trying to get an explicit formula for h, but because of the following theorem of Carr we can approximate h to arbitrarily high orders.

Theorem (Carr) *Let ϕ be a C^1 mapping of a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}^n$ that satisfies $\phi(0) = 0$ and $D\phi(0) = 0$. Suppose that*

$$(\mathcal{M}\phi)(x) = O(|x|^q)$$

as $x \to 0$ for some constant $q > 1$. Then

$$|h(x) - \phi(x)| = O(|x|^q)$$

as $x \to 0$.

Stability

If we put $y = h(x)$ in the first equation in (1), we get the *reduced equation*

$$\dot{x} = Ax + F(x, h(x)), \tag{2}$$

which describes the evolution of the $\mathcal{E}^c$ coordinate of solutions on the center manifold. Another theorem of Carr's states that if all the eigenvalues of

$Df(0)$ are in the closed left half-plane, then the stability type of the origin as an equilibrium solution of (1) (Lyapunov stable, asymptotically stable, or unstable) matches the stability type of the origin as an equilibrium solution of (2).

These results of Carr are sometimes useful in computing the stability type of the origin. Consider, for example, the following system:

$$\begin{cases} \dot{x} = xy + ax^3 + by^2x \\ \dot{y} = -y + cx^2 + dx^2y, \end{cases}$$

where x and y are real variables and a, b, c, and d are real parameters. We know that there is a center manifold, tangent to the x-axis at the origin, that is (locally) of the form $y = h(x)$. The reduced equation on the center manifold is

$$\dot{x} = xh(x) + ax^3 + b[h(x)]^2x. \tag{3}$$

To determine the stability of the origin in (3) (and, therefore, in the original system) we need to approximate h. Therefore, we consider the operator $\mathcal{M}$ defined by

$$(\mathcal{M}\phi)(x) = \phi'(x)[x\phi(x) + ax^3 + b(\phi(x))^2x] + \phi(x) - cx^2 - dx^2\phi(x),$$

and seek polynomial ϕ (satisfying $\phi(0) = \phi'(0) = 0$) for which $(\mathcal{M}\phi)(x)$ is of high order in x. By inspection, if $\phi(x) = cx^2$ then $(\mathcal{M}\phi)(x) = O(x^4)$, so $h(x) = cx^2 + O(x^4)$, and (3) becomes

$$\dot{x} = (a + c)x^3 + O(x^5).$$

Hence, the origin is asymptotically stable if $a+c < 0$ and is unstable if $a+c > 0$. What about the borderline case when $a+c = 0$? Suppose that $a+c = 0$ and let's go back and try a different ϕ, namely, one of the form $\phi(x) = cx^2 + kx^4$. Plugging this in, we find that $(\mathcal{M}\phi)(x) = (k - cd)x^4 + O(x^6)$, so if we choose $k = cd$ then $(\mathcal{M}\phi)(x) = O(x^6)$; thus, $h(x) = cx^2 + cdx^4 + O(x^6)$. Inserting this in (3), we get

$$\dot{x} = (cd + bc^2)x^5 + O(x^7),$$

so the origin is asymptotically stable if $cd + bc^2 < 0$ (and $a + c = 0$) and is unstable if $cd + bc^2 > 0$ (and $a + c = 0$).

What if $a + c = 0$ *and* $cd + bc^2 = 0$? Suppose that these two conditions hold, and consider ϕ of the form $\phi(x) = cx^2 + cdx^4 + kx^6$ for some $k \in \mathbb{R}$ yet to be determined. Calculating, we discover that $(\mathcal{M}\phi)(x) = (k - b^2c^3)x^6 + O(x^8)$, so by choosing $k = b^2c^3$, we see that $h(x) = cx^2 + cdx^4 + b^2c^3x^6 + O(x^8)$. Inserting this in (3), we see that (if $a + c = 0$ and $cd + bc^2 = 0$)

$$\dot{x} = -b^2c^3x^7 + O(x^9).$$

Hence, if $a + c = cd + bc^2 = 0$ and $b^2c > 0$ then the origin is asymptotically stable, and if $a + c = cd + bc^2 = 0$ and $b^2c < 0$ then the origin is unstable.

It can be checked that in the remaining borderline case $a + c = cd + bc^2 = b^2c = 0$, $h(x) \equiv cx^2$ so the reduced equation is simply $\dot{x} = 0$. Hence, in this case, the origin is Lyapunov stable, but not asymptotically stable.

Bifurcation Theory

Bifurcation theory studies fundamental changes in the structure of the solutions of a differential equation or a dynamical system in response to change in a parameter. Consider the parametrized equation

$$\dot{x} = F(x, \varepsilon), \tag{4}$$

where $x \in \mathbb{R}^n$ is a variable and $\varepsilon \in \mathbb{R}^p$ is a parameter. Suppose that $F(0, \varepsilon) = 0$ for every ε, that the equilibrium solution at $x = 0$ is stable when $\varepsilon = 0$, and that we are interested in the possibility of persistent structures (*e.g.*, equilibria or periodic orbits) bifurcating out of the origin as ε is made nonzero. This means that all the eigenvalues of $D_xF(0, 0)$ have nonpositive real part, so we can project (4) onto complementary subspaces of $\mathbb{R}^n$ and get the equivalent system

$$\begin{cases} \dot{u} = Au + f(u, v, \varepsilon) \\ \dot{v} = Bv + g(u, v, \varepsilon), \end{cases}$$

with the eigenvalues of A lying on the imaginary axis and the eigenvalues of B lying in the open right half-plane. Since the parameter ε does not depend on time, we can append the equation $\dot{\varepsilon} = 0$ to get the expanded system

$$\begin{cases} \dot{u} = Au + f(u, v, \varepsilon) \\ \dot{v} = Bv + g(u, v, \varepsilon) \\ \dot{\varepsilon} = 0. \end{cases} \tag{5}$$

The Center Manifold Theorem asserts the existence of a center manifold for the origin that is locally given by points (u, v, ε) satisfying an equation of the form

$$v = h(u, \varepsilon).$$

Furthermore, according to a theorem of Carr, every solution $(u(t), v(t), \varepsilon)$ of (5) for which $(u(0), v(0), \varepsilon)$ is sufficiently close to zero converges exponentially quickly to a solution on the center manifold as $t \uparrow \infty$. In particular, no persistent structure near the origin lies off the center manifold of this expanded system. Hence, it suffices to consider persistent structures for the lower-dimensional equation

$$\dot{u} = Au + f(u, h(u, \varepsilon), \varepsilon).$$