Poincaré-Bendixson Theorem
Lecture 39
Math 634
12/3/99

Definition A *periodic orbit* of a continuous dynamical system φ is a set of the form

$$\{\varphi(t,p) \mid t \in [0,T]\}$$

for some time T and point p satisfying $\varphi(T,p) = p$. If this set is a singleton, we say that the periodic orbit is *degenerate*.

Theorem (Poincaré-Bendixson) *Every nonempty, compact ω-limit set of a C^1 planar flow that does not contain an equilibrium point is a (nondegenerate) periodic orbit.*

We will prove this theorem by means of 4 lemmas. Throughout our discussion, we will be referring to a C^1 planar flow φ and the corresponding vector field f.

Definition If $\mathcal{S}$ is a line segment in $\mathbb{R}^2$ and $p_1, p_2, \ldots$ is a (possibly finite) sequence of points lying on $\mathcal{S}$, then we say that this sequence is *monotone on* $\mathcal{S}$ if $(p_j - p_{j-1}) \cdot (p_2 - p_1) \geq 0$ for every $j \geq 2$.

Definition A (possibly finite) sequence $p_1, p_2, \ldots$ of points on a trajectory $\mathcal{T}$ of φ is said to be *monotone on* $\mathcal{T}$ if we can choose a point p and times $t_1 \leq t_2 \leq \cdots$ such that $\varphi(t_j, p) = p_j$ for each j.

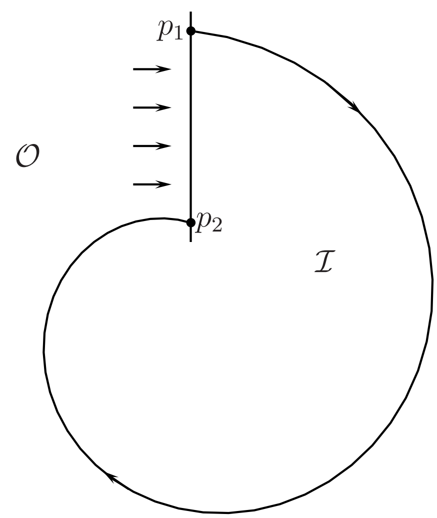
Definition A *transversal* of φ is a line segment $\mathcal{S}$ such that f is not tangent to $\mathcal{S}$ at any point of $\mathcal{S}$.

Lemma *If a (possibly finite) sequence of points $p_1, p_2, \ldots$ lies on the intersection of a transversal $\mathcal{S}$ and a trajectory $\mathcal{T}$, and the sequence is monotone on $\mathcal{T}$, then it is monotone on $\mathcal{S}$.*

Proof. Let p be a point on $\mathcal{T}$. Since $\mathcal{S}$ is closed and f is nowhere tangent to $\mathcal{S}$, the times t at which $\varphi(t,p) \in \mathcal{S}$ form an increasing sequence (possibly biinfinite). Consequently, if the lemma fails then there are times $t_1 < t_2 < t_3$ and distinct points $p_i = \varphi(t_i, p) \in \mathcal{S}$, $i \in \{1, 2, 3\}$, such that

$$\{p_1, p_2, p_3\} = \varphi([t_1, t_3], p) \cap \mathcal{S}$$

and p_3 is between p_1 and p_2. Note that the union of the line segment $\overline{p_1 p_2}$ from p_1 to p_2 with the curve $\varphi([t_1, t_2], p)$ is a simple closed curve in the plane, so by the Jordan Curve Theorem it has an "inside" $\mathcal{I}$ and an "outside" $\mathcal{O}$. Assuming, without loss of generality, that f points *into* $\mathcal{I}$ all along the "interior" of $\overline{p_1 p_2}$, we get a picture something like:



Note that

$$\mathcal{I} \cup \overline{p_1 p_2} \cup \varphi([t_1, t_2], p)$$

is a positively invariant set, so, in particular, it contains $\varphi([t_2, t_3], p)$. But the fact that p_3 is between p_1 and p_2 implies that $f(p_3)$ points into $\mathcal{I}$, so $\varphi(t_3 - \varepsilon, p) \in \mathcal{O}$ for ε small and positive. This contradiction implies that the lemma holds. □

The proof of the next lemma uses something called a *flow box*. A flow box is a (topological) box such that f points into the box along one side, points out of the box along the opposite side, and is tangent to the other two sides, and the restriction of φ to the box is conjugate to unidirectional, constant-velocity flow. The existence of a flow box around any regular point of φ is a consequence of the C^r-rectification Theorem.

Lemma *No ω-limit set intersects a transversal in more than one point.*

Proof. Suppose that for some point x and some transversal $\mathcal{S}$, $\omega(x)$ intersects $\mathcal{S}$ at two distinct points p_1 and p_2. Since p_1 and p_2 are on a transversal, they are regular points, so we can choose disjoint subintervals $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{S}$ containing, respectively, p_1 and p_2, and, for some $\varepsilon > 0$, define flow boxes $\mathcal{B}_1$ and $\mathcal{B}_2$ by

$$\mathcal{B}_i := \{\varphi(t,x) \mid t \in [-\varepsilon, \varepsilon], x \in \mathcal{S}_i\}.$$

Now, the fact that $p_1, p_2 \in \omega(x)$ means that we can pick an increasing sequence of times $t_1, t_2, \ldots$ such that $\varphi(t_j, x) \in \mathcal{B}_1$ if j is odd and $\varphi(t_j, x) \in \mathcal{B}_2$ if j is even. In fact, because of the nature of the flow in $\mathcal{B}_1$ and $\mathcal{B}_2$, we can assume that $\varphi(t_j, x) \in \mathcal{S}$ for each j. Although the sequence $\varphi(t_1, x), \varphi(t_2, x), \ldots$ is monotone on the trajectory $\mathcal{T} := \gamma(x)$, it is not monotone on $\mathcal{S}$, contradicting the previous lemma. $\square$

Definition An *ω-limit point* of a point p is an element of $\omega(p)$.

Lemma *Every ω-limit point of an ω-limit point lies on a periodic orbit.*

Proof. Suppose that $p \in \omega(q)$ and $q \in \omega(r)$. If p is a singular point, then it obviously lies on a (degenerate) periodic orbit, so suppose that p is a regular point. Pick $\mathcal{S}$ to be a transversal containing p in its "interior". By putting a suitable flow box around p, we see that, since $p \in \omega(q)$, the solution beginning at q must repeatedly cross $\mathcal{S}$. But $q \in \omega(r)$ and ω-limit sets are invariant, so the solution beginning at q remains confined within $\omega(r)$. Since $\omega(r) \cap \mathcal{S}$ contains at most one point, the solution beginning at q must repeatedly cross $\mathcal{S}$ at the same point; *i.e.*, q lies on a periodic orbit. Since $p \in \omega(q)$, p must lie on this same periodic orbit. $\square$

Lemma *If an ω-limit set $\omega(x)$ contains a nondegenerate periodic orbit $\mathcal{P}$, then $\omega(x) = \mathcal{P}$.*

Proof. Fix $q \in \mathcal{P}$. Pick $T > 0$ such that $\varphi(T, q) = q$. Let $\varepsilon > 0$ be given. By continuous dependence, we can pick $\delta > 0$ such that $|\varphi(t, y) - \varphi(t, q)| < \varepsilon$ whenever $t \in [0, 3T/2]$ and $|y - q| < \delta$. Pick a transversal $\mathcal{S}$ of length less than δ with q in its "interior", and create a flow box

$$\mathcal{B} := \{\varphi(t, x) \mid x \in \mathcal{S}, t \in [-\rho, \rho]\}$$

for some $\rho \in (0, T/4]$. By continuity of $\varphi(T, \cdot)$, we know that we can pick a subinterval $\mathcal{S}'$ of $\mathcal{S}$ that contains q and that satisfies $\varphi(T, \mathcal{S}') \subset \mathcal{B}$. Let t_j be the jth smallest element of

$$\{t \geq 0 \mid \varphi(t, x) \in \mathcal{S}'\}.$$

Because $\mathcal{S}'$ is a transversal and $q \in \omega(x)$, the t_j are well-defined and increase to infinity as $j \uparrow \infty$. Also, by the lemma on monotonicity, $|\varphi(t_j, x) - q|$ is a decreasing function of j.

Note that for each $j \in \mathbb{N}$, $\varphi(T, \varphi(t_j, x)) \in \mathcal{B}$, so, by construction of $\mathcal{S}$ and $\mathcal{B}$, $\varphi(t, \varphi(T, \varphi(t_j, x))) \in \mathcal{S}$ for some $t \in [-T/2, T/2]$. Pick such a t. The lemma on monotonicity implies that

$$\varphi(t, \varphi(T, \varphi(t_j, x))) \in \mathcal{S}'.$$

This, in turn, implies that $t + T + t_j \in \{t_1, t_2, \ldots\}$, so

$$t_{j+1} - t_j \leq 3T/2. \tag{1}$$

Now, suppose that $t \geq t_1$. Then $t \in [t_j, t_{j+1})$ for some $j \geq 1$. For this j,

$$|\varphi(t, x) - \varphi(t - t_j, p)| = |\varphi(t - t_j, \varphi(t_j, x)) - \varphi(t - t_j, p)| < \varepsilon,$$

since, by (1), $|t - t_j| < |t_{j+1} - t_j| < 3T/2$ and since, because $\varphi(t_j, x) \in \mathcal{S}' \subseteq \mathcal{S}$, $|p - \varphi(t_j, x)| < \delta$.

Since ε was arbitrary, we have shown that

$$\lim_{t \uparrow \infty} d(\varphi(t, x), \mathcal{P}) = 0.$$

Thus, $\mathcal{P} = \omega(x)$, as was claimed. □

Now, we get to the proof of the Poincaré-Bendixson Theorem itself. Suppose $\omega(x)$ is compact and nonempty. Pick $p \in \omega(x)$. Since $\gamma^+(p)$ is contained in the compact set $\omega(x)$, we know $\omega(p)$ is nonempty, so we can pick $q \in \omega(p)$. Note that q is an ω-limit point of an ω-limit point, so, by the third lemma, q lies on a periodic orbit $\mathcal{P}$. Since $\omega(p)$ is invariant, $\mathcal{P} \subseteq \omega(p) \subseteq \omega(x)$. If $\omega(x)$ contains no equilibrium point, then $\mathcal{P}$ is nondegenerate, so, by the fourth lemma, $\omega(x) = \mathcal{P}$.