Lienard's Theorem

Lecture 41
Math 634
12/7/99

Recall, that we're going to estimate the change of $R(x,y) := (x^2+y^2)/2$ along the orbit segment connecting $(0,y_0)$ to $(0,\tilde{y}_0)$. Notice that if the point (a,b) and the point (c,d) lie on the same trajectory then

$$R(c,d) - R(a,b) = \int_{(a,b)}^{(c,d)} dR.$$

(The integral is a line integral.) Since $\dot{R} = -xF(x)$, if y is a function of x along the orbit segment connecting (a,b) to (c,d), then

$$R(c,d) - R(a,b) = \int_a^c \frac{\dot{R}}{\dot{x}}\,dx = \int_a^c \frac{-xF(x)}{y(x)-F(x)}\,dx. \tag{1}$$

If, on the other hand, x is a function of y along the orbit segment connecting (a,b) to (c,d), then

$$R(c,d) - R(a,b) = \int_b^d \frac{\dot{R}}{\dot{y}}\,dy = \int_b^d \frac{-x(y)F(x(y))}{-x(y)}\,dy = \int_b^d F(x(y))\,dy. \tag{2}$$

We will chop the orbit segment connecting $(0,y_0)$ to $(0,\tilde{y}_0)$ up into pieces and use (1) and (2) to estimate the change ΔR in R along each piece and, therefore, along the whole orbit segment.

Let $\sigma = \beta + 1$, and let

$$B = \sup_{0 \le x \le \sigma} |F(x)|.$$

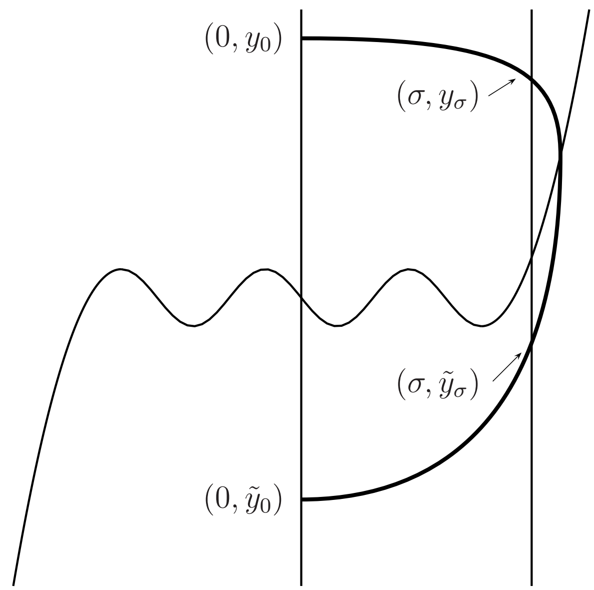
Consider the region

$$\mathcal{R} := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in [0,\sigma], y \in [B+\sigma, \infty) \right\}.$$

In $\mathcal{R}$,

$$\left|\frac{dy}{dx}\right| = \frac{x}{y-F(x)} \le \frac{\sigma}{\sigma} = 1;$$

hence, if $y_0 > B+2\sigma$, then the corresponding trajectory must exit $\mathcal{R}$ through its right boundary, say, at the point (σ, y_σ). Similarly, if $\tilde{y}_0 < -B-2\sigma$, then the trajectory it lies on must have first hit the line $x = \sigma$ at a point $(\sigma, \tilde{y}_\sigma)$. Now, assume that as $y_0 \to \infty$, $\tilde{y}_0 \to -\infty$. (If not, then the claim clearly holds.) Based on this assumption we know that we can pick a value for y_0 and a corresponding value for $\tilde{y}_0$ that are both larger than $B + 2\sigma$ in absolute value, and conclude that the orbit segment connecting them looks qualitatively like:



We will estimate ΔR on the entire orbit segment from $(0, y_0)$ to $(0, \tilde{y}_0)$ by considering separately, the orbit segment from $(0, y_0)$ to (σ, y_σ), the segment from (σ, y_σ) to $(\sigma, \tilde{y}_\sigma)$, and the segment from $(\sigma, \tilde{y}_\sigma)$ to $(0, \tilde{y}_0)$.

First, consider the first segment. On this segment, the y-coordinate is a function $y(x)$ of the x-coordinate. Thus,

$$\begin{aligned} |R(\sigma, y_\sigma) - R(0, y_0)| &= \left| \int_0^\sigma \frac{-xF(x)}{y(x) - F(x)} \, dx \right| \leq \int_0^\sigma \left| \frac{-xF(x)}{y(x) - F(x)} \right| dx \\ &\leq \int_0^\sigma \frac{\sigma B}{y_0 - B - \sigma} \, dx = \frac{\sigma^2 B}{y_0 - B - \sigma} \to 0 \end{aligned}$$

as $y_0 \to \infty$. A similar estimate shows that $|R(0, \tilde{y}_0) - R(\sigma, \tilde{y}_\sigma)| \to 0$ as $y_0 \to \infty$.

On the middle segment, the x-coordinate is a function $x(y)$ of the y-coordinate y. Hence,

$$R(\sigma, \tilde{y}_\sigma) - R(\sigma, y_\sigma) = \int_{y_\sigma}^{\tilde{y}_\sigma} F(x(y))\, dy \le -|y_\sigma - \tilde{y}_\sigma| F(\sigma) \to -\infty$$

as $y_0 \to \infty$.

Putting these three estimates together, we see that

$$R(0, \tilde{y}_0) - R(0, y_0) \to -\infty$$

as $y_0 \to \infty$, so $|\tilde{y}_0| < |y_0|$ if y_0 is sufficiently large. This shows that the orbit connecting these two points forms part of the boundary of a compact, positively invariant set that surrounds (but omits) the origin. By the Poincaré-Bendixson Theorem, there must be a limit cycle in this set.

Now for the second half of Lienard's Theorem. We need to show that if $\alpha = \beta$ (*i.e.*, if F has a unique positive zero) then the limit cycle whose existence we've deduced is the only nondegenerate periodic orbit and it attracts all points other than the origin. If we can show the uniqueness of the limit cycle, then the fact that we can make our compact, positively invariant set as large as we want and make the hole cut out of its center as small as we want will imply that it attracts all points other than the origin. Note also, that our observations on the general direction of the flow imply that any nondegenerate periodic orbit must circle the origin in the clockwise direction.

So, suppose that $\alpha = \beta$ and consider, as before, orbit segments that start on the positive y-axis at a point $(0, y_0)$ and end on the negative y-axis at a point $(0, \tilde{y}_0)$. Such orbit segments are "nested" and fill up the (open) right half-plane. We need to show that only one of them satisfies $\tilde{y}_0 = -y_0$. In other words, we claim that there is only one segment that gives

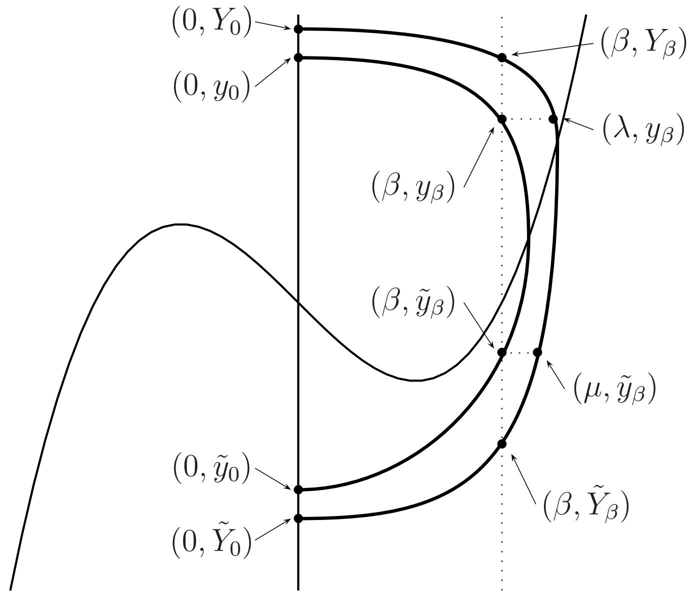
$$R(0, \tilde{y}_0) - R(0, y_0) = 0.$$

Now, if such a segment hits the x-axis on $[0, \beta]$, then $x \le \beta$ all along that segment, and $F(x) \le 0$ with equality only if $(x, y) = (\beta, 0)$. Let $x(y)$ be the x-coordinate as a function of y and observe that

$$R(0, \tilde{y}_0) - R(0, y_0) = \int_{y_0}^{\tilde{y}_0} F(x(y))\, dy > 0. \tag{3}$$

We claim that for values of y_0 generating orbits intersecting the x-axis in (β, ∞), $R(0, \tilde{y}_0) - R(0, y_0)$ is a strictly decreasing function of y_0. In combination with (3) (and the fact that $R(0, \tilde{y}_0) - R(0, y_0) < 0$ if y_0 is sufficiently large), this will finish the proof.

Consider 2 orbits (whose coordinates we denote (x, y) and (X, Y)) that intersect the x-axis in (β, ∞) and contain selected points as shown in the following diagram.



Note that

$$\begin{aligned} R(0, \tilde{Y}_0) - R(0, Y_0) &= R(0, \tilde{Y}_0) - R(\beta, \tilde{Y}_\beta) \\ &\quad + R(\beta, \tilde{Y}_\beta) - R(\mu, \tilde{y}_\beta) \\ &\quad + R(\mu, \tilde{y}_\beta) - R(\lambda, y_\beta) \\ &\quad + R(\lambda, y_\beta) - R(\beta, Y_\beta) \\ &\quad + R(\beta, Y_\beta) - R(0, Y_0) \\ &=: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5. \end{aligned} \tag{4}$$

Let $X(Y)$ and $x(y)$ give, respectively, the first coordinate of a point on the outer and inner orbit segments as a function of the second coordinate. Similarly, let $Y(X)$ and $y(x)$ give the second coordinates as functions of the first coordinates (on the segments where that's possible). Estimating, we find that

$$\Delta_1 = \int_\beta^0 \frac{-XF(X)}{Y(X)-F(X)}\,dX < \int_\beta^0 \frac{-xF(x)}{y(x)-F(x)}\,dx = R(0,\tilde{y}_0) - R(\beta,\tilde{y}_\beta), \tag{5}$$

$$\Delta_2 = \int_{\tilde{y}_\beta}^{\tilde{Y}_\beta} F(X(Y))\,dY < 0, \tag{6}$$

$$\Delta_3 = \int_{y_\beta}^{\tilde{y}_\beta} F(X(Y))\,dY < \int_{y_\beta}^{\tilde{y}_\beta} F(x(y))\,dy = R(\beta,\tilde{y}_\beta) - R(\beta,y_\beta), \tag{7}$$

$$\Delta_4 = \int_{Y_\beta}^{y_\beta} F(X(Y))\,dY < 0, \tag{8}$$

and

$$\Delta_5 = \int_0^\beta \frac{-XF(X)}{Y(X)-F(X)}\,dX < \int_0^\beta \frac{-xF(x)}{y(x)-F(x)}\,dx = R(\beta,y_\beta) - R(0,y_0). \tag{9}$$

By plugging, (5), (6), (7), (8), and (9) into (4), we see that

$$\begin{aligned} R(0,\tilde{Y}_0) - R(0,Y_0) < {} & [R(0,\tilde{y}_0) - R(\beta,\tilde{y}_\beta)] + 0 + [R(\beta,\tilde{y}_\beta) - R(\beta,y_\beta)] + 0 \\ & + [R(\beta,y_\beta) - R(0,y_0)] = R(0,\tilde{y}_0) - R(0,y_0). \end{aligned}$$

This gives the claimed monotonicity and completes the proof.