# Intervals of Existence

Lecture 5
Math 634
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## Maximal Interval of Existence

We begin our discussion with some definitions and an important theorem of real analysis.

Definition Given $f : \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, we say that $f(t, x)$ is *locally Lipschitz continuous w.r.t. $x$ on $\mathcal{D}$* if for each $(t_0, a) \in \mathcal{D}$ there is a number $L$ and a product set $\mathcal{I} \times \mathcal{U} \subseteq \mathcal{D}$ containing $(t_0, a)$ in its interior such that the restriction of $f(t, \cdot)$ to $\mathcal{U}$ is Lipschitz continuous with Lipschitz constant $L$ for every $t \in \mathcal{I}$.

Definition A subset $\mathcal{K}$ of a topological space is *compact* if whenever $\mathcal{K}$ is contained in the union of a collection of open sets, there is a finite subcollection of that collection whose union also contains $\mathcal{K}$. The original collection is called a *cover* of $\mathcal{K}$, and the finite subcollection is called a *finite subcover* of the original cover.

Theorem (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Now, suppose that $\mathcal{D}$ is an open subset of $\mathbb{R} \times \mathbb{R}^n$, $(t_0, a) \in \mathcal{D}$, and $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz continuous w.r.t. $x$ on $\mathcal{D}$. Then the Picard-Lindelöf Theorem indicates that the IVP

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = a \end{cases} \tag{1}$$

has a solution existing on some time interval containing $t_0$ in its interior and that the solution is unique on that interval. Let's say that an *interval of existence* is an interval containing $t_0$ on which a solution of (1) exists. The following theorem indicates how large an interval of existence may be.

Theorem (Maximal Interval of Existence) *The IVP* (1) *has a maximal interval of existence, and it is of the form $(\omega_-, \omega_+)$, with $\omega_- \in [-\infty, \infty)$ and $\omega_+ \in$*

$(-\infty, \infty]$. *There is a unique solution* $x(t)$ *of* (1) *on* $(\omega_-, \omega_+)$, *and* $(t, x(t))$ *leaves every compact subset* $\mathcal{K}$ *of* $\mathcal{D}$ *as* $t \downarrow \omega_-$ *and as* $t \uparrow \omega_+$.

*Proof.*

Step 1: If $\mathcal{I}_1$ and $\mathcal{I}_2$ are open intervals of existence with corresponding solutions $x_1$ and $x_2$, then $x_1$ and $x_2$ agree on $\mathcal{I}_1 \cap \mathcal{I}_2$.
Let $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$, and let $\mathcal{I}^*$ be the largest interval containing $t_0$ and contained in $I$ on which $x_1$ and $x_2$ agree. By the Picard-Lindelöf Theorem, $\mathcal{I}^*$ is nonempty. If $\mathcal{I}^* \neq \mathcal{I}$, then $\mathcal{I}^*$ has an endpoint $t_1$ in $\mathcal{I}$. By continuity, $x_1(t_1) = x_2(t_1) =: a_1$. The Picard-Lindelöf Theorem implies that

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_1) = a_1 \end{cases} \tag{2}$$

has a local solution that is unique. But restrictions of $x_1$ and $x_2$ near $t_1$ each provide a solution to (2), so $x_1$ and $x_2$ must agree in a neighborhood of $t_1$. This contradiction tells us that $\mathcal{I}^* = \mathcal{I}$.

Now, let $(\omega_-, \omega_+)$ be the union of all open intervals of existence.

Step 2: $(\omega_-, \omega_+)$ is an interval of existence.
Given $t \in (\omega_-, \omega_+)$, pick an open interval of existence $\tilde{\mathcal{I}}$ that contains $t$, and let $x(t) = \tilde{x}(t)$, where $\tilde{x}$ is a solution to (1) on $\tilde{\mathcal{I}}$. Because of step 1, this determines a well-defined function $x : (\omega_-, \omega_+) \to \mathbb{R}^n$; clearly, it solves (1).

Step 3: $(\omega_-, \omega_+)$ is the maximal interval of existence.
An extension argument similar to the one in Step 1 shows that every interval of existence is contained in an open interval of existence. Every open interval of existence is, in turn, a subset of $(\omega_-, \omega_+)$.

Step 4: $x$ is the only solution of (1) on $(\omega_-, \omega_+)$.
This is a special case of Step 1.

Step 5: $(t, x(t))$ leaves every compact subset $\mathcal{K} \subset \mathcal{D}$ as $t \downarrow \omega_-$ and as $t \uparrow \omega_+$.
We only treat what happens as $t \uparrow \omega_+$; the other case is similar.

Let a compact subset $\mathcal{K}$ of $\mathcal{D}$ be given. For each point $(t, a) \in \mathcal{K}$, pick numbers $\alpha(t, a) > 0$ and $\beta(t, a) > 0$ such that

$$[t - 2\alpha(t, a), t + 2\alpha(t, a)] \times \overline{\mathcal{B}(a, 2\beta(t, a))} \subset \mathcal{D}.$$

Note that the collection of sets

$$\left\{ (t - \alpha(t, a), t + \alpha(t, a)) \times \mathcal{B}(a, \beta(t, a)) \;\middle|\; (t, a) \in \mathcal{K} \right\}$$

is a cover of $\mathcal{K}$. Since $\mathcal{K}$ is compact, a finite subcollection, say

$$\big\{(t_i - \alpha(t_i, a_i), t_i + \alpha(t_i, a_i)) \times \mathcal{B}(a_i, \beta(t_i, a_i))\big\}_{i=1}^m,$$

covers $\mathcal{K}$. Let

$$\mathcal{K}' := \bigcup_{i=1}^m \Big([t_i - 2\alpha(t_i, a_i), t_i + \alpha(t_i, a_i)] \times \overline{\mathcal{B}(a_i, 2\beta(t_i, a_i))}\Big),$$

let

$$\tilde{\alpha} := \min\big\{\alpha(t_i, a_i)\big\}_{i=1}^m,$$

and let

$$\tilde{\beta} := \min\big\{\beta(t_i, x_i)\big\}_{i=1}^m.$$

Since $\mathcal{K}'$ is a compact subset of $\mathcal{D}$, there is a constant $M > 0$ such that $f$ is bounded by $M$ on $\mathcal{K}'$. By the triangle inequality,

$$[t_0 - \tilde{\alpha}, t_0 + \tilde{\alpha}] \times \overline{\mathcal{B}(a, \tilde{\beta})} \subseteq \mathcal{K}',$$

for every $(t_0, a) \in \mathcal{K}$, so $f$ is bounded by $M$ on each such product set. According to the Picard-Lindelöf Theorem, this means that for every $(t_0, a) \in \mathcal{K}$ a solution to $\dot{x} = f(t, x)$ starting at $(t_0, a)$ exists for at least $\min\{\tilde{\alpha}, \tilde{\beta}/M\}$ units of time. Hence, $x(t) \notin \mathcal{K}$ for $t > \omega_+ - \min\{\tilde{\alpha}, \tilde{\beta}/M\}$. $\square$

Corollary *If $\mathcal{D}'$ is a bounded set and $\mathcal{D} = (c, d) \times \mathcal{D}'$ (with $c \in [-\infty, \infty)$ and $d \in (-\infty, \infty]$), then either $\omega_+ = d$ or $x(t) \to \partial\mathcal{D}'$ as $t \uparrow \omega_+$, and either $\omega_- = c$ or $x(t) \to \partial\mathcal{D}'$ as $t \downarrow \omega_-$.*

Corollary *If $\mathcal{D} = (c, d) \times \mathbb{R}^n$ (with $c \in [-\infty, \infty)$ and $d \in (-\infty, \infty]$), then either $\omega_+ = d$ or $|x(t)| \uparrow \infty$ as $t \uparrow \omega_+$, and either $\omega_- = c$ or $|x(t)| \uparrow \infty$ as $t \downarrow \omega_-$.*

If we're dealing with an autonomous equation on a bounded set, then the first corollary applies to tell us that the only way a solution could fail to exist for all time is for it to approach the boundary of the spatial domain. (Note that this is not the same as saying that $x(t)$ converges to a particular point on the boundary; can you give a relevant example?) The second corollary says that autonomous equations on all of $\mathbb{R}^n$ have solutions that exist until they become unbounded.

## Global Existence

For the solution set of the autonomous ODE $\dot{x} = f(x)$ to be representable by a dynamical system, it is necessary for solutions to exist for all time. As the discussion above illustrates, this is not always the case. When solutions do die out in finite time by hitting the boundary of the phase space $\Omega$ or by going off to infinity, it may be possible to change the vector field $f$ to a vector field $\tilde{f}$ that points in the same direction as the original but has solutions that exist for all time.

For example, if $\Omega = \mathbb{R}^n$, then we could consider the modified equation

$$\dot{x} = \frac{f(x)}{1 + |f(x)|}.$$

Clearly, $|\dot{x}| < 1$, so it is impossible for $|x|$ to approach infinity in finite time.

If, on the other hand, $\Omega \neq \mathbb{R}^n$, then consider the modified equation

$$\dot{x} = \frac{f(x)}{1 + |f(x)|} \cdot \frac{d(x, \mathbb{R}^n \setminus \Omega)}{1 + d(x, \mathbb{R}^n \setminus \Omega)},$$

where $d(x, \mathbb{R}^n \setminus \Omega)$ is the distance from $x$ to the complement of $\Omega$. It is not hard to show that it is impossible for a solution $x$ of this equation to become unbounded or to approach the complement of $\Omega$ in finite time, so, again, we have global existence.

It may or may not seem obvious that if two vector fields point in the same direction at each point, then the solution curves of the corresponding ODEs in phase space match up. In the following exercise, you are asked to prove that this is true.

<u>Exercise 4</u> Suppose that $\Omega$ is a subset of $\mathbb{R}^n$, that $f : \Omega \to \mathbb{R}^n$ and $g : \Omega \to \mathbb{R}^n$ are (continuous) vector fields, and that there is a continuous function $h : \Omega \to (0, \infty)$ such that $g(u) = h(u)f(u)$ for every $u \in \Omega$. If $x$ is the only solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = a \end{cases}$$

(defined on the maximal interval of existence) and $y$ is the only solution of

$$\begin{cases} \dot{y} = g(y) \\ y(0) = a, \end{cases}$$

(defined on the maximal interval of existence), show that there is an increasing function $j : \mathrm{dom}(y) \to \mathrm{dom}(x)$ such that $y(t) = x(j(t))$ for every $t \in \mathrm{dom}(y)$.