Dependence on Parameters

Lecture 6
Math 634
9/13/99

Parameters vs. Initial Conditions

Consider the IVP

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = a, \end{cases} \tag{1}$$

and the paramterized IVP

$$\begin{cases} \dot{x} = f(t, x, \mu) \\ x(t_0) = a, \end{cases} \tag{2}$$

where $\mu \in \mathbb{R}^k$. We are interested in studying how the solution of (1) depends on the initial condition a and how the solution of (2) depends on the parameter μ. In a sense, these two questions are equivalent. For example, if x solves (1) and we let $\tilde{x} := x - a$ and $\tilde{f}(t, \tilde{x}, a) := f(t, \tilde{x} + a)$, then $\tilde{x}$ solves

$$\begin{cases} \dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, a) \\ \tilde{x}(t_0) = 0, \end{cases}$$

so a appears as a parameter rather than an initial condition. If, on the other hand, x solves (2), and we let $\tilde{x} := (x, \mu)$ and $\tilde{f}(t, \tilde{x}) := (f(t, x, \mu), 0)$, then $\tilde{x}$ solves

$$\begin{cases} \dot{\tilde{x}} = \tilde{f}(t, \tilde{x}) \\ \tilde{x}(t_0) = (a, \mu), \end{cases}$$

so μ appears as part of the initial condition, rather than as a parameter in the ODE.

We will concentrate on (2).

Continuous Dependence

The following result can be proved by an approach like that outlined in Exercise 3.

Theorem (Grownwall Inequality) *Suppose that $X(t)$ is a nonnegative, continuous, real-valued function on $[t_0, T]$ and that there are constants $C, K > 0$ such that*

$$X(t) \leq C + K \int_{t_0}^{t} X(s)\, ds$$

for every $t \in [t_0, T]$. Then

$$X(t) \leq C e^{K(t-t_0)}$$

for every $t \in [t_0, T]$.

Using the Grownwall inequality, we can prove that the solution of (2) depends continuously on μ.

Theorem (Continuous Dependence) *Suppose*

$$f : [t_0 - \alpha, t_0 + \alpha] \times \Omega_1 \times \Omega_2 \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$$

is continuous. Suppose, furthermore, that $f(t, \cdot, \mu)$ is Lipschitz continuous with Lipschitz constant $L_1 > 0$ for every $(t, \mu) \in [t_0 - \alpha, t_0 + \alpha] \times \Omega_2$ and $f(t, x, \cdot)$ is Lipschitz continuous with Lipschitz constant $L_2 > 0$ for every $(t, x) \in [t_0 - \alpha, t_0 + \alpha] \times \Omega_1$. If $x_i : [t_0 - \alpha, t_0 + \alpha] \to \mathbb{R}^n$ ($i = 1, 2$) satisfy

$$\begin{cases} \dot{x}_i = f(t, x_i, \mu_i) \\ x_i(t_0) = a, \end{cases}$$

then

$$|x_1(t) - x_2(t)| \leq \frac{L_2}{L_1} |\mu_1 - \mu_2| (e^{L_1 |t - t_0|} - 1) \tag{3}$$

for $t \in [t_0 - \alpha, t_0 + \alpha]$.

This theorem shows continuous dependence on parameters if, in addition to the hypotheses of the Picard-Lindelöf Theorem, the right-hand side of the equation is assumed to be Lipschitz continuous with respect to the parameter (on finite time intervals). The connection between (1) and (2) shows that the hypotheses of the Picard-Lindelöf Theorem are sufficient to guarantee continuous dependence on initial conditions. Note the exponential dependence of the modulus of continuity on $|t - t_0|$.

Proof. For simplicity, we only consider $t \geq t_0$. Note that

$$
\begin{aligned}
|x_1(t) - x_2(t)| &= \left| \int_{t_0}^{t} [f(s, x_1(s), \mu_1) - f(s, x_2(s), \mu_2]\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s, x_1(s), \mu_1) - f(s, x_2(s), \mu_2)|\, ds \\
\leq \int_{t_0}^{t} [|f(s, x_1(s), \mu_1) &- f(s, x_1(s), \mu_2)| + |f(s, x_1(s), \mu_2) - f(s, x_2(s), \mu_2)|]\, ds \\
&\leq \int_{t_0}^{t} [L_2|\mu_1 - \mu_2| + L_1|x_1(s) - x_2(s)|]\, ds
\end{aligned}
$$

Let $X(t) = L_2|\mu_1 - \mu_2| + L_1|x_1(t) - x_2(t)|$. Then

$$
X(t) \leq L_2|\mu_1 - \mu_2| + L_1 \int_{t_0}^{t} X(s)\, ds,
$$

so by the Gronwall Inequality $X(t) \leq L_2|\mu_1 - \mu_2|e^{L_1(t-t_0)}$. This means that (3) holds. □

<u>Exercise 5</u> Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and are each Lipschitz continuous with respect to their second variable. Suppose, also, that x is a global solution to

$$
\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = a, \end{cases}
$$

and y is a global solution to

$$
\begin{cases} \dot{y} = g(t, y) \\ y(t_0) = b. \end{cases}
$$

(a) If $f(t, p) < g(t, p)$ for every $(t, p) \in \mathbb{R} \times \mathbb{R}$ and $a < b$, show that $x(t) < y(t)$ for every $t \geq t_0$.

(b) If $f(t, p) \leq g(t, p)$ for every $(t, p) \in \mathbb{R} \times \mathbb{R}$ and $a \leq b$, show that $x(t) \leq y(t)$ for every $t \geq t_0$. (Hint: You may want to use the results of part (a) along with a limiting argument.)

Differentiable Dependence

Theorem (Differentiable Dependence) *Suppose $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and is continuously differentiable with respect to x and μ. Then the solution $x(\cdot, \mu)$ of*

$$\begin{cases} \dot{x} = f(t, x, \mu) \\ x(t_0) = a \end{cases} \tag{4}$$

is differentiable with respect to μ, and $y = x_\mu := \partial x / \partial \mu$ satisfies

$$\begin{cases} \dot{y} = f_x(t, x(t, \mu), \mu) y + f_\mu(t, x(t, \mu), \mu) \\ y(t_0) = 0. \end{cases} \tag{5}$$

That x_μ, if it exists, should satisfy the IVP (5) is not terribly surprising; (5) can be derived (formally) by differentiating (4) with respect to μ. The real difficulty is showing that x_μ exists. The key to the proof is to use the fact that (5) has a solution y and then to use the Gronwall inequality to show that difference quotients for x_μ converge to y.

Proof. Given μ, it is not hard to see that the right-hand side of the ODE in (5) is continuous in t and y and is locally Lipschitz continuous with respect to y, so by the Picard-Lindelöf Theorem we know that (5) has a unique solution $y(\cdot, \mu)$. Let

$$w(t, \mu, \Delta\mu) := \frac{x(t, \mu + \Delta\mu) - x(t, \mu)}{\Delta\mu}.$$

We want to show that $w(t, \mu, \Delta\mu) \to y(t, \mu)$ as $\Delta\mu \to 0$.

Let $z(t, \mu, \Delta\mu) := w(t, \mu, \Delta\mu) - y(t, \mu)$. Then

$$\frac{dz}{dt}(t, \mu, \Delta\mu) = \frac{dw}{dt}(t, \mu, \Delta\mu) - f_x(t, x(t, \mu), \mu) y(t, \mu) - f_\mu(t, x(t, \mu), \mu),$$

and

$$\begin{aligned} \frac{dw}{dt}(t, \mu, \Delta\mu) &= \frac{f(t, x(t, \mu + \Delta\mu), \mu + \Delta\mu) - f(t, x(t, \mu), \mu)}{\Delta\mu} \\ &= \frac{f(t, x(t, \mu + \Delta\mu), \mu + \Delta\mu) - f(t, x(t, \mu), \mu + \Delta\mu)}{\Delta\mu} \\ &\quad + \frac{f(t, x(t, \mu), \mu + \Delta\mu) - f(t, x(t, \mu), \mu)}{\Delta\mu} \\ &= f_x(t, x(t, \mu) + \theta_1 \Delta x, \mu + \Delta\mu) w(t, \mu, \Delta\mu) + f_\mu(t, x(t, \mu), \mu + \theta_2 \Delta\mu), \end{aligned}$$

for some $\theta_1, \theta_2 \in [0,1]$ (by the Mean Value Theorem), where

$$\Delta x := x(t, \mu + \Delta\mu) - x(t, \mu).$$

Hence,

$$\begin{aligned}
|\frac{dz}{dt}(t,\mu,\Delta\mu)| &\leq |f_\mu(t, x(t,\mu), \mu + \theta_2\Delta\mu) - f_\mu(t, x(t,\mu), \mu)| \\
&+ |f_x(t, x(t,\mu) + \theta_1\Delta x, \mu + \Delta\mu)| \cdot |z(t,\mu,\Delta\mu)| \\
&+ |f_x(t, x(t,\mu) + \theta_1\Delta x, \mu + \Delta\mu) - f_x(t, x(t,\mu), \mu + \Delta\mu)| \cdot |y(t,\mu)| \\
&+ |f_x(t, x(t,\mu), \mu + \Delta\mu) - f_x(t, x(t,\mu), \mu)| \cdot |y(t,\mu)| \\
&\leq p(t,\mu,\Delta\mu) + (|f_x(t, x(t,\mu), \mu)| + p(t,\mu,\Delta\mu))|z(t,\mu,\Delta\mu)|,
\end{aligned}$$

where $p(t,\mu,\Delta\mu) \to 0$ as $\Delta\mu \to 0$, uniformly on bounded sets.

Letting $X(t) = \varepsilon + (K + \varepsilon)|z|$, we see that if

$$|f_x(t, x(t,\mu), \mu)| \leq K \tag{6}$$

and

$$|p(t,\mu,\Delta\mu)| < \varepsilon, \tag{7}$$

then

$$|z(t)| \leq \int_{t_0}^{t} \left|\frac{dz}{ds}\right| ds \leq \int_{t_0}^{t} X(s)\, ds$$

so

$$X(t) \leq \varepsilon + (K + \varepsilon)\int_{t_0}^{t} X(s)\, ds,$$

which gives $X(t) \leq \varepsilon e^{(K+\varepsilon)(t-t_0)}$, by Gronwall's inequality. This, in turn, gives

$$|z| \leq \frac{\varepsilon(e^{(K+\varepsilon)(t-t_0)} - 1)}{K + \varepsilon}.$$

Given $t \geq t_0$, pick K so large that (6) holds. As $\Delta\mu \to 0$, we can take ε arbitrarily small and still have (7) hold, to see that

$$\lim_{\Delta\mu \to 0} z(t,\mu,\Delta\mu) = 0.$$

□