Constant Coefficient Linear Equations

Lecture 7
Math 634
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Linear Equations

Definition Given

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,$$

we say that the first-order ODE

$$\dot{x} = f(t, x) \tag{1}$$

is *linear* if every linear combination of solutions of (1) is a solution of (1).

Definition Given two vector spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the space of linear maps from $\mathcal{X}$ to $\mathcal{Y}$.

Exercise 6 Show that if (1) is linear (and f is continuous) then there is a function $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $f(t, p) = A(t)p$, for every $(t, p) \in \mathbb{R} \times \mathbb{R}^n$.

ODEs of the form $\dot{x} = A(t)x + g(t)$ are also often called linear, although they don't satisfy the definition given above. These are called *inhomogeneous*; ODEs satisfying the previous definition are called *homogeneous*.

Constant Coefficients and the Matrix Exponential

Here we will study the autonomous IVP

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0, \end{cases} \tag{2}$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, or equivalently A is a (constant) $n \times n$ matrix.

If $n = 1$, then we're dealing with $\dot{x} = ax$. The solution is $x(t) = e^{ta}x_0$. When $n > 1$, we will show that we can similarly define e^{tA} in a natural way, and the solution of (2) will be given by $x(t) = e^{tA}x_0$.

Given $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, we define its matrix exponential

$$e^B := \sum_{k=0}^{\infty} \frac{B^k}{k!}.$$

We will show that this series converges, but first we specify a norm on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

Definition The operator norm $\|B\|$ of an element $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is defined by

$$\|B\| = \sup_{x \neq 0} \frac{|Bx|}{|x|} = \sup_{|x|=1} |Bx|.$$

$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a Banach space under the operator norm. Thus, to show that the series for e^B converges, it suffices to show that

$$\left\| \sum_{k=\ell}^{m} \frac{B^k}{k!} \right\|$$

can be made arbitrarily small by taking $m \geq \ell \geq N$ for N sufficiently large.

Suppose $B_1, B_2 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and B_2 does not map everything to zero. Then

$$\begin{aligned} \|B_1 B_2\| &= \sup_{x \neq 0} \frac{|B_1 B_2 x|}{|x|} = \sup_{B_2 x \neq 0, x \neq 0} \frac{|B_1 B_2 x|}{|B_2 x|} \cdot \frac{|B_2 x|}{|x|} \\ &\leq \left(\sup_{y \neq 0} \frac{|B_1 y|}{|y|} \right) \left(\sup_{x \neq 0} \frac{|B_2 x|}{|x|} \right) = \|B_1\| \cdot \|B_2\|. \end{aligned}$$

If B_2 does map everything to zero, then $\|B_2\| = \|B_1 B_2\| = 0$, so $\|B_1 B_2\| \leq \|B_1\| \cdot \|B_2\|$, obviously. Thus, the operator norm is *submultiplicative*. Using this property, we have

$$\left\| \sum_{k=\ell}^{m} \frac{B^k}{k!} \right\| \leq \sum_{k=\ell}^{m} \left\| \frac{B^k}{k!} \right\| \leq \sum_{k=\ell}^{m} \frac{\|B\|^k}{k!}.$$

Since the regular exponential series (for real arguments) has an infinite radius of convergence, we know that the last quantity in this estimate goes to zero as $\ell, m \uparrow \infty$.

Thus, e^B makes sense, and, in particular, e^{tA} makes sense for each fixed $t \in \mathbb{R}$ and each $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. But does $x(t) := e^{tA}x_0$ solve (2)? To check that, we'll need the following important property.

Lemma *If $B_1, B_2 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $B_1 B_2 = B_2 B_1$, then $e^{B_1+B_2} = e^{B_1}e^{B_2}$.*

Proof. Using commutativity, we have

$$\begin{aligned}
e^{B_1}e^{B_2} &= \left(\sum_{j=0}^{\infty} \frac{B_1^j}{j!}\right)\left(\sum_{k=0}^{\infty} \frac{B_2^k}{k!}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_1^j B_2^k}{j!k!} = \sum_{i=0}^{\infty}\sum_{j+k=i} \frac{B_1^j B_2^k}{j!k!} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{B_1^j B_2^{(i-j)}}{j!(i-j)!} = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \binom{i}{j} \frac{B_1^j B_2^{(i-j)}}{i!} \\
&= \sum_{i=0}^{\infty} \frac{(B_1+B_2)^i}{i!} = e^{(B_1+B_2)}.
\end{aligned}$$

□

Now, if $x : \mathbb{R} \to \mathbb{R}^n$ is defined by $x(t) := e^{tA}x_0$, then

$$\begin{aligned}
\frac{d}{dt}x(t) &= \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h\to 0} \frac{e^{(t+h)A}x_0 - e^{tA}x_0}{h} \\
&= \left(\lim_{h\to 0} \frac{e^{(t+h)A} - e^{tA}}{h}\right) x_0 = \left(\lim_{h\to 0} \frac{e^{hA} - I}{h}\right) e^{tA}x_0 \\
&= \left(\lim_{h\to 0} \sum_{k=1}^{\infty} \frac{h^{k-1}A^k}{k!}\right) e^{tA}x_0 = Ae^{tA}x_0 = Ax(t),
\end{aligned}$$

so $x(t) = e^{tA}x_0$ really does solve (2).