

Math 112 – Winter 2004

Departmental Final Exam

PART I: SHORT ANSWER

Problems 1(a) through 1(j) are short answer. Each answer is worth 1 point. Fill in the blanks with the correct answers.

1. a. $\frac{d}{dx}(x^2 + 5) = \underline{\hspace{2cm}}$

b. $\frac{d}{dx} \cos x = \underline{\hspace{2cm}}$

c. $\frac{d}{dx} \left(\frac{1}{x} \right) = \underline{\hspace{2cm}}$

d. $\frac{d}{dx} \tan^{-1} x = \underline{\hspace{2cm}}$

e. $\frac{d}{dx} \ln x = \underline{\hspace{2cm}}$

f. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \underline{\hspace{2cm}}$

g. $\int e^x dx = \underline{\hspace{2cm}}$

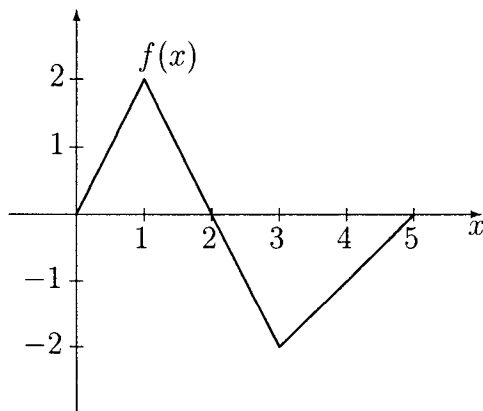
h. $\int \sin x dx = \underline{\hspace{2cm}}$

i. $\int \sec x \tan x dx = \underline{\hspace{2cm}}$

j. $\int \cosh x dx = \underline{\hspace{2cm}}$

1

3. If the graph of f looks like



then $\int_0^5 f(x) dx =$

- | | | |
|----------|---------|---------|
| (a) -3 | (e) 1 | (i) 5 |
| (b) -2 | (f) 2 | (j) 6 |
| (c) -1 | (g) 3 | |
| (d) 0 | (h) 4 | |

4. If Newton's method is used to find a zero of the function

$$f(x) = x^3 + 3x + 3$$

and if $x_1 = 1$, then x_2 is

- | | | |
|------------|------------|-----------------------|
| (a) $13/7$ | (e) $1/6$ | (i) $13/6$ |
| (b) $-6/7$ | (f) $-1/6$ | (j) None of the above |
| (c) $7/6$ | (g) $1/7$ | |
| (d) $-7/6$ | (h) $-1/7$ | |

5. A car accelerates from 0 to 60 miles per hour (88 feet per second) in 30 seconds. If acceleration is constant, how far does the car travel in that time?

- | | |
|---------------|--|
| (a) 2640 feet | (f) 7920 feet |
| (b) 5280 feet | (g) it depends on the kind of stereo system in the car |
| (c) 3960 feet | (h) 1800 feet |
| (d) 1320 feet | (i) 900 feet |
| (e) 660 feet | (j) none of the above |

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6. If $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow 1} g(x)$ both exist, which of the following can you be sure are true?

- I.** $\lim_{x \rightarrow 1} f(x) + g(x) = \lim_{x \rightarrow 1} f(x) + \lim_{x \rightarrow 1} g(x)$
- II.** $\lim_{x \rightarrow 1} f(x)g(x) = f(1) \lim_{x \rightarrow 1} g(x) + g(1) \lim_{x \rightarrow 1} f(x)$
- III.** $\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x)}$
- (a) None of the statements (e) **I** and **II** only
 (b) **I** only (f) **I** and **III** only
 (c) **II** only (g) **II** and **III** only
 (d) **III** only (h) **I**, **II**, and **III**

7. Suppose that $f(3) = 2$, $f'(3) = 3$, $f(9) = 4$, and $f'(9) = 5$. Let

$$g(x) = \frac{f(x^2)}{f(x)}.$$

Find $g'(3)$.

- (a) $\frac{5}{3}$ (f) 10
 (b) -10 (g) 12
 (c) -12 (h) $-\frac{1}{2}$
 (d) $\frac{1}{2}$ (i) 18
 (e) $-\frac{5}{3}$ (j) none of the above

4

The answers to the multiple choice MUST be entered on the grid on page 2. Otherwise, you will not receive credit.

PART III: WRITTEN SOLUTIONS

For problems 8 - 18, write your answers in the space provided. Neatly show your work for full credit.

8. A rectangle has its base on the x -axis and its upper vertices on the curve $y = 12 - x^2$. What are the dimensions of the rectangle having maximum area?
9. A parachutist descends vertically at the rate of 3 meters per second, and a jeep is driving toward the landing spot at the rate of 9 meters per second. When the parachutist is 60 meters from the ground, the jeep is 80 meters from the landing spot. At what rate is the distance between them changing at that instant?

5

10. Evaluate the limit $\lim_{x \rightarrow 0} \frac{x^3}{x - \sin x}$.

11. Find the limit. If the limit does not exist, write “Limit = ∞ ,” or “Limit = $-\infty$,” or *if neither of these is true*, write “Limit doesn’t exist.”

(a) $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$

(b) $\lim_{x \rightarrow 2^-} \frac{x-4}{2-x}$

(c) $\lim_{x \rightarrow \infty} \frac{1-x^2}{1-4x^2}$

6

12. Let H denote the height of a certain tree. Assume the tree grows in height at a rate inversely proportional to its height. If the tree grows 3 feet in the first three years, how tall will it be at the end of 4 years?

13. Let $f(x)$ be a differentiable function, and let

$$g(x) = [f(x)]^2. \tag{1}$$

If

$$\frac{g'(x)}{f'(x)} = e^x$$

for all x , what is the function $f(x)$? (Hint: Differentiate the equation (1).)

7

17. Assume that the equation

$$x + y \ln y = y + x \ln x + 1$$

implicitly defines y as a function of x . Find

(a) y'

(b) y''

18. Let $f(x)$ be a function such that $\lim_{x \rightarrow 5} f(x)$ exists. If

$$\lim_{x \rightarrow 5} (2f(x)) = \lim_{x \rightarrow 5} (3f(x)),$$

use limit theorems to find what $\lim_{x \rightarrow 5} f(x)$ equals.

9

3. The function

$$f(x) = \frac{(x-2)^4(x+2)^3(x-3)^2(x+3)}{(x^2-4)^3(x^2-9)^2}$$

has some points of discontinuity. How many of these are removable discontinuities?

- | | |
|-------|-------|
| (a) 0 | (e) 4 |
| (b) 1 | (f) 5 |
| (c) 2 | (g) 6 |
| (d) 3 | (h) 7 |

4. One of the following functions has a point where the limit *does not exist*, but where at least one of the one-sided limits *does exist*. Which one is it?

- | | |
|---------------------------|------------------------------|
| (a) $\frac{\sin x}{x}$ | (e) $\frac{x^2 - 2x}{x - 2}$ |
| (b) $\tan x$ | (f) $e^{ x }$ |
| (c) $\frac{1}{x}$ | |
| (d) $\frac{x^2 + x }{x}$ | |

5. Let $f(x) = x^2 - 2$. Suppose Newton's Method is used to approximate a root of the equation $f(x) = x^2 - 2 = 0$. If $x_1 = 3$, then x_2 is

- | | | |
|----------------|--------------|------------------------|
| (a) $15/7$ | (e) $27/7$ | (i) $25/26$ |
| (b) $\sqrt{2}$ | (f) 1.4 | (j) None of the above. |
| (c) $-11/6$ | (g) $-25/26$ | |
| (d) $-15/7$ | (h) $11/6$ | |

6. The limit $\lim_{x \rightarrow 0^+} (\sin(3x))^{2x}$ is

- | | | |
|---------------|---------------|------------------------|
| (a) 0 | (e) 1 | (i) $e^{2/3}$ |
| (b) $e^{3/2}$ | (f) $e^{1/3}$ | (j) None of the above. |
| (c) $e^{1/2}$ | (g) e^6 | |
| (d) $e^{1/6}$ | (h) e^{-6} | |

||

PART II: WRITTEN SOLUTIONS

For problems 8 - 19, write your answers in the space provided. Give the best answer and justify it.

8. Find the second derivative of $f(x) = x^2e^{-3x}$.

9. Find the following limits, if they exist.

(a) $\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2}$

(b) $\lim_{x \rightarrow \infty} \frac{2x + \sin x}{x}$

10. Consider the function

$$f(x) = x^2 - 1 - \frac{3}{x}.$$

(a) Starting with the domain $[1, 3]$, apply the *bisection method* until you find an interval of length $\frac{1}{2}$ containing a point x where $f(x) = 0$.

(b) Name the theorem about continuous functions that the bisection method relies upon.

13

11. Recall that the definition of $\lim_{x \rightarrow \infty} f(x) = \infty$ is:

For every $A > 0$ there exists $B > 0$ such that
if $x > B$ then $f(x) > A$.

Now write the definition of $\lim_{x \rightarrow \infty} f(x) = L$.

12. Is the following statement true or false?

“If $f(x)$ is continuous at a , then $f(x)$ is differentiable at a .”

If it is true explain why, using the definition of continuity and differentiability. If it is false exhibit a function $f(x)$ that is continuous at a and explain why it is not differentiable at a .

13. Air is pumped into a spherical balloon at the rate of 2 cubic feet per minute. Determine the rate of change of the radius r of the balloon at the instant when the circumference is 8 feet.

[Note: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

14. An open rectangular box is to be made from a 3 by 8 foot piece of cardboard by cutting a square out of each corner, folding up the sides, and taping the sides together at the corners. What size square should be cut from each corner so that the resulting box has maximum volume?

15. Find the global maximum and global minimum of the function $f(x) = \frac{x}{x^2 + 4}$ on the domain $0 \leq x \leq 6$.

16. Compute $\int_1^4 \frac{(\ln x)^3}{x} dx$

17. Solve the separable differential equation $yy' = 3(1 + x^2)$ subject to the boundary condition $y(1) = 2$.

18. Let $I = \int_0^3 \sqrt{x^3 + 9} \, dx$, $J = \int_2^3 \sqrt{x^3 + 9} \, dx$, and $K = \int_{-1}^0 \sqrt{x^3 + 9} \, dx$. Express $\int_2^{-1} 3\sqrt{x^3 + 9} \, dx$ in terms of I , J , and K .

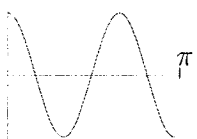
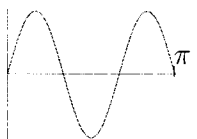
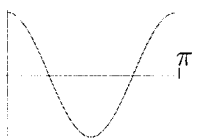
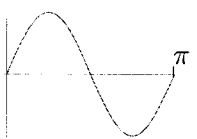
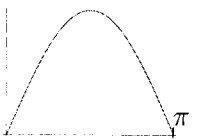
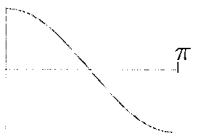
19. Use the identity $\tan x = \frac{\sin x}{\cos x}$ and the quotient rule to prove the formula

$$\frac{d}{dx} \tan x = \sec^2 x.$$

PART I: MULTIPLE CHOICE

- Problems 1-7 are multiple choice. Each multiple choice problem is worth 4 points
- **Evening Sections** - Your answer must appear both on this test paper and on the bubble sheet. **Circle the letter of the best answer for each question below. Also encode the correct answer on the bubble sheet.** Failure to record the answer in both places may result in loss of credit on your exam. Ambiguous responses will receive no credit.
- **Daytime Sections** - Circle the letter of the best answer for each question below. Ambiguous responses will receive no credit.

1. Which is the graph of $y = \sin 2x$ on $[0, \pi]$?

- (a) 
- (b) 
- (c) 
- (d) 
- (e) 
- (f) 
- (g) None of the above

2. Using properties of logarithms, simplify the expression $\log_2(x2^{-2})$.

- (a) $-2 + \log_2 x$
- (b) $2 + \log_{-2} x$
- (c) $2 \log_{-2} x$
- (d) $-2 - 2 \log_2 x$
- (e) $-4 + \log_2 x$
- (f) $-2 \log_2 x$
- (g) None of the above

3. Find the value of $\tan x$ given that $x = \sin^{-1}(\frac{1}{2})$.

- (a) $\frac{1}{2}$
- (b) $\frac{1}{\sqrt{3}}$
- (c) $\frac{2}{\sqrt{3}}$
- (d) $\sqrt{3}$
- (e) $\frac{4}{\sqrt{3}}$
- (f) $\frac{\sqrt{3}}{2}$
- (g) None of the above

4. Find the maximum of the function $y = x^3 - 3x$ for $x \in [-2, 4]$.

- (a) 0
- (b) -2
- (c) 2
- (d) 52
- (e) -3
- (f) 8
- (g) 26
- (h) 18
- (i) None of the above

5. Find the x values of the point or points on the graph of the function $f(x) = x^3 - 3x + 9$ at which the tangent line has slope 9.

- (a) ± 9
- (b) ± 4
- (c) ± 5
- (d) ± 7
- (e) ± 1
- (f) ± 2
- (g) None of the above

6. Differentiate 7^x .

- (a) $7^x \ln 7$
- (b) $7^x \ln 7 + C$
- (c) $\frac{7^{x+1}}{x+1} + C$
- (d) 7^x
- (e) $x7^{x-1}$
- (f) $\frac{7^x}{\ln 7}$
- (g) $\frac{7^x}{\ln 7} + C$
- (h) $\frac{7^{x+1}}{x+1}$
- (i) None of the above

7. $x \cos y + 3x^2y^2 = 5$. Find $\frac{dy}{dx}$.

- (a) $\frac{\cos y + 3xy^2}{x(\sin y - 2xy)}$
- (b) $\frac{\cos y + 2xy^2}{x(\sin y + 6xy)}$
- (c) $\frac{\sin y + 2xy^2}{x(2\sin y + 6xy)}$
- (d) $\frac{\cos y + 6xy^2}{x(\sin y - 6xy)}$
- (e) $\frac{\cos y - 3xy^2}{x^2(\sin y - 6xy)}$
- (f) $\frac{\cos y - 6x^2y^2}{2x(\sin y - 2xy)}$
- (g) $\frac{\cos y - 2xy^2}{x(\sin y + 2xy)}$
- (h) None of the above

PART II: WRITTEN SOLUTIONS

For problems 8-19 give the best answer and justify this answer with suitable reasons and relevant work.

8. Suppose $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$. Show $\lim_{x \rightarrow 0} f(x) = 0$.
9. A piece of property is to be fenced on the front and two sides. Available money is \$400. Fencing for the sides costs \$1 per foot and fencing for the front costs \$1.60 per foot. What are the dimensions of the rectangular lot having largest area subject to these constraints?
10. A kite is flying at an angle of elevation of $\frac{1}{3}\pi$. The kite string is being taken in at the rate of 1 foot per second. If the angle of elevation does not change, how fast is the kite losing altitude?

Departmental Final Exam Form K
PART I: MULTIPLE CHOICE (3.5 POINTS EACH)

1. Using properties of logarithms, solve $5^{x-1} = 11^{2x+5}$.

(a) $\frac{\ln 5 + 2 \ln 2 + \ln 3}{\ln 5 - 2 \ln 11}$

(b) $\frac{\ln 5 + 5 \ln 11}{\ln 5 - 2 \ln 11}$

(c) $-\frac{15}{17}$

(d) $\frac{\ln 2 + 5 \ln 11}{\ln 5 - 3 \ln 11}$

(e) -6

(f) There is no solution to this equation because $11 \neq 5$.

2. Integrate $\int_0^{\pi/4} \sec x \tan x \, dx$.

(a) 1

(b) $\sqrt{2}/2$

(c) $1 - \frac{1}{4}\pi$

(d) $\sqrt{3}/2$

(e) 2

(f) $\sqrt{2} - 1$

(g) The integral does not exist.

3. Find $\lim_{x \rightarrow 0} \frac{\sin(5x)}{\tan(11x)}$.

(a) $\sin\left(\frac{5}{11}\right)$

(b) $\frac{0}{0}$

(c) $\sin\left(\frac{11}{5}\right)$

(d) $\frac{5}{11}$

(e) $\frac{11}{5}$

(f) The function is not continuous at 0 and so there is no limit for this expression.

(g) It is impossible to take this limit because you can't divide by 0.

4. It is desired to find the positive solution to

$$x^3 - 2 = 0$$

using Newton's method. Using $x_1 = 1$, find x_2 .

(a) $\frac{7}{3}$

(b) $-\frac{5}{3}$

(c) $\frac{4}{3}$

(d) $\frac{16}{3}$

(e) $\frac{10}{3}$

(f) $\frac{1}{3}$

(g) $-\frac{11}{3}$

5. Let $f(x) = 2x^3 - 12x^2 + 18x$. Find the x coordinates of the critical points.

(a) 1, 3
 (b) 1, 4
 (c) 1, 5
 (d) 0, 4
 (e) 0, 3
 (f) -1, 3

6. Find the general antiderivative of $f(x) = 7^x$.

(a) $7^x \ln 7 + C$
 (b) $7^x \ln 7$
 (c) $7^{x+1} + C$
 (d) $\frac{7^{x+1}}{x+1} + C$
 (e) $\frac{7^x}{\ln 7} + C$
 (f) $x7^{x-1} + C$
 (g) $x7^{x-1}$

7. Let $f(x) = \frac{x^2 - 9}{x - 3}$ for $x \neq 3$. How should f be defined at $x = 3$ to make f continuous at 3?

(a) $f(3) = 6$
 (b) $f(3) = 14$
 (c) $f(3) = \frac{0}{0}$
 (d) $f(3) = 0$
 (e) $f(3) = 1$
 (f) It is impossible to divide by 0 so this function cannot be defined at $x = 3$.

8. Find $\lim_{x \rightarrow \infty} (\sqrt{25x^2 - 5x} - 5x)$.

(a) $-\frac{1}{2}$
 (b) $-\frac{3}{10}$
 (c) $-\frac{7}{10}$
 (d) $\infty - \infty$
 (e) 0
 (f) $-\frac{2}{5}$
 (g) The limit does not exist.

Neatly write solutions to each of the following questions in the space provided. Show your work.

9. Differentiate $\left(-2 \sin (3 x)+2 e^{3 x}\right)^{16}$.

10. $x y=\sin (3 x-7 y)$. Find $\frac{d y}{d x}$.

11. Solve the problem consisting of the differential equation $\frac{d y}{d x}=\frac{2 x}{7 y}$ and the condition $y(0)=2$.

12. Find the point on the curve $y = \sqrt{64 - 6x}$ which is closest to $(0, 0)$.

13. Sand is dumped off a conveyor belt into a pile at the rate of 2 cubic feet per minute. The sand pile is shaped like a cone whose height and base diameter are always equal. At what rate is the height of the pile growing when the pile is 5 feet high? (The volume of a cone is $\frac{1}{3}\pi r^2 h$ where r is the radius and h is the height.)

14. Find $\lim_{x \rightarrow 0} \frac{1 - e^x}{\sin(3x)}$.

18. Prove $\lim_{x \rightarrow 5} (2x + 4) = 14$.

19. Let $I = \int_0^4 \sqrt{1+x^4} dx$, $J = \int_2^4 \sqrt{1+x^4} dx$, and $K = \int_0^5 \sqrt{1+x^4} dx$.

Find $3 \int_0^2 \sqrt{1+x^4} dx + 6 \int_2^5 \sqrt{1+x^4} dx$ in terms of I, J, K .

20. Find $\int x^3 \sqrt{5x^2 + 1} dx$.

Departmental Final Exam Form B
PART I: MULTIPLE CHOICE (3 POINTS EACH)

- $f(x) = x^3 + 9x^2 + 27x + 27$ has a point of inflection at $x =$
 (a) -2 (b) -1 (c) -3 (d) 2
 (e) $-\frac{7}{2}$ (f) -4 (g) -5 (h) None of the above
- If $\lim_{x \rightarrow -2} f(x) = 7$, then $f(-2)$
 (a) Equals 7. (b) Is arbitrarily close to 7.
 (c) Does not exist. (d) Cannot be determined from the given information.
- Find $\lim_{x \rightarrow 0} \left(\frac{\sqrt{3+x} - \sqrt{3}}{x} \right)$
 (a) 1 (b) 6 (c) $\frac{5}{3\sqrt{2}}$ (d) $\frac{0}{0}$
 (e) $\frac{1}{2\sqrt{2}}$ (f) $\frac{1}{2\sqrt{3}}$ (g) $\frac{1}{4}$ (h) None of the above
- Use differentials to approximate $\sqrt{66}$.
 (a) $8\frac{1}{8}$ (b) $9\frac{1}{8}$ (c) $6\frac{1}{8}$ (d) $8\frac{3}{16}$
 (e) $8\frac{1}{9}$ (f) $8\frac{1}{10}$ (g) $8\frac{1}{4}$ (h) None of the above
- Find $\int \frac{\cos x}{1 + \sin x} dx$.
 (a) $\ln|1 + \sin x| + C$ (b) $\ln|2 + \sin x| + C$ (c) $\ln|1 + \sin x| + \sin x + C$
 (d) $\ln|1 + \sin x| + 3\sin 2x + C$ (e) $\ln|1 + 2\sin x| + C$ (f) None of the above
- Find $\frac{d}{dx}(x \sin x + \cos x)$
 (a) $\sin x + x \cos x$ (b) $x \cos x$ (c) $\sin x$
 (d) 1 (e) 0 (f) $1 + x \cos x$
 (g) $2 + x \cos x$ (h) None of the above
- Which of the following equations is false?
 (a) $\int_0^1 x^2 dx = 1/3$ (b) $\int_0^\pi \sin x dx = 2$ (c) $\int_{-4}^4 (x^5 + 4x^3 + x) dx = 0$
 (d) $\int_{-1}^1 \frac{1}{x^2} dx = -2$ (e) $\int_0^{10} e^{-x} dx = 1 - e^{-10}$ (f) $\int_0^{\pi/4} \sec^2 x dx = 1$
 (g) $\int_0^1 \cosh x dx = \sinh 1$ (h) $\int_{-4}^4 \sqrt{16 - x^2} dx = 8\pi$ (i) $\int_0^1 \sqrt{x} dx = \frac{2}{3}$
 (j) None of the above
- Find $\lim_{x \rightarrow 0} \frac{\tan x}{x \cos x}$.
 (a) $\frac{3}{2}$ (b) 0 (c) $\frac{5}{2}$ (d) $\frac{7}{4}$
 (e) $\frac{1}{2}$ (f) $\frac{9}{2}$ (g) 1 (h) None of the above

PART II: WRITTEN SOLUTIONS

Neatly write solutions to each of the following questions in the space provided. Show your work.

9. (4 points each) Evaluate the following derivatives:

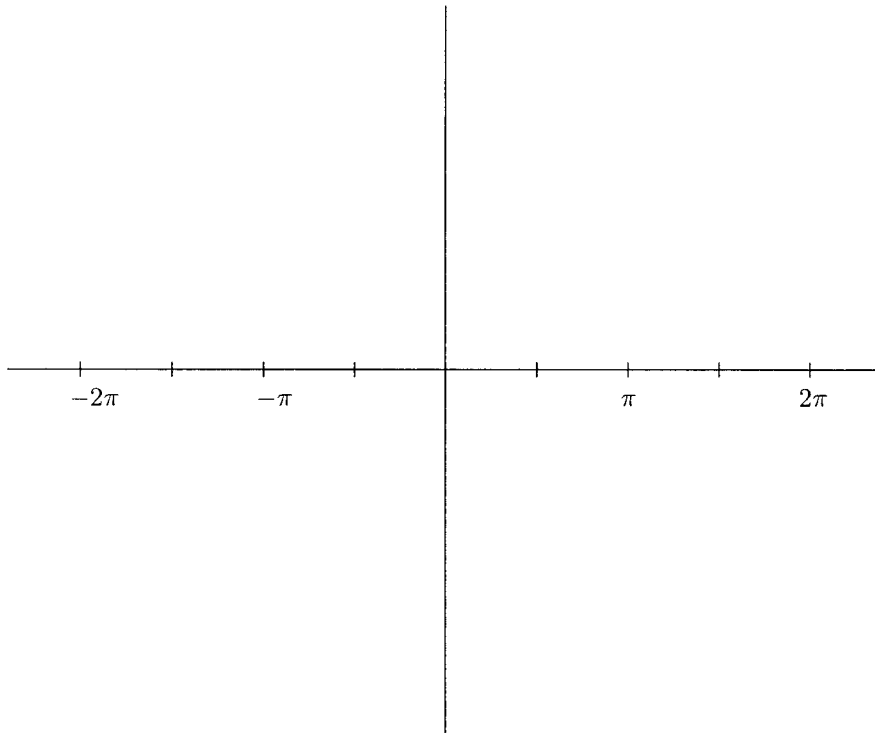
(a) $\frac{d}{dx} \left(\frac{\sin x}{x^2} \right)$

(b) $\frac{d}{dx} \left(x\sqrt{2x^2 + 1} \right)$

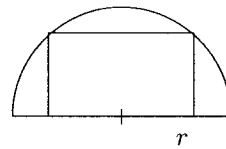
(c) $\frac{d}{dx} \left(\ln \left| \frac{x}{x+1} \right| \right)$

10. (5 points) Find the absolute maximum and absolute minimum of $f(x) = x + \frac{16}{x}$ on the interval $[1, 32]$.

11. (10 points) Graph the function $y = (x/2) - \tan(x/4)$. Label on your graph the asymptotes and relative extreme points.



12. (10 points) Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r where $r = 2$.



17. (5 points) Find $\lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} =$

13. (8 points) A kite 100 feet above the ground is being blown away from the person holding its string⁵ in a direction parallel to the ground at the rate of 10 feet per second. At what rate must the string be let out when the length of string already let out is 200 feet?

