1.5 A Review of Trigonometry

Trigonometry begins with a right triangle. The size of the triangle is not as important as the angles. We focus on one particular angle—call it θ—and on the *ratios* between the three sides x, y, r. The ratios don't change if the triangle is scaled to another size. Three sides give six ratios, which are the basic functions of trigonometry:

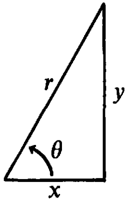

Fig. 1.19

$$\cos\theta = \frac{x}{r} = \frac{\text{near side}}{\text{hypotenuse}} \qquad \sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\sin\theta = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{opposite side}}{\text{near side}} \qquad \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}$$

Of course those six ratios are not independent. The three on the right come directly from the three on the left. And the tangent is the sine divided by the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x}.$$

Note that "tangent of an angle" and "tangent to a circle" and "tangent line to a graph" are different uses of the same word. As the cosine of θ goes to zero, the tangent of θ goes to infinity. The side x becomes zero, θ approaches 90°, and the triangle is infinitely steep. The sine of 90° is $y/r = 1$.

Triangles have a serious limitation. They are excellent for angles up to 90°, and they are OK up to 180°, but after that they fail. We cannot put a 240° angle into a triangle. Therefore we change now to a circle.

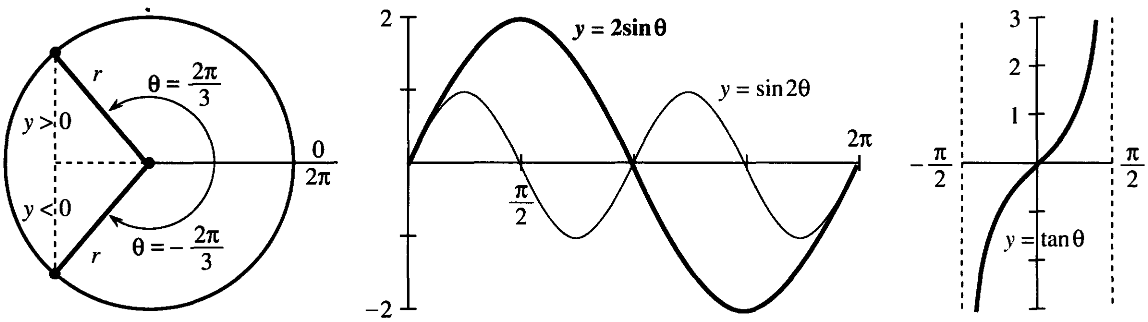

Fig. 1.20 Trigonometry on a circle. Compare $2\sin\theta$ with $\sin 2\theta$ and $\tan\theta$ (periods 2π, π, π).

Angles are measured from the positive x axis (counterclockwise). Thus 90° is straight up, 180° is to the left, and 360° is in the same direction as 0°. (Then 450° is the same as 90°.) Each angle yields a point on the circle of radius r. The coordinates x and y of that point can be negative (*but never* r). As the point goes around the circle, the six ratios $\cos\theta$, $\sin\theta$, $\tan\theta$, ... trace out six graphs. The cosine waveform is the same as the sine waveform—just shifted by 90°.

One more change comes with the move to a circle. Degrees are out. Radians are in. The distance around the whole circle is $2\pi r$. The distance around to other points is θr. ***We measure the angle by that multiple*** θ. For a half-circle the distance is πr, so the angle is π radians—which is 180°. A quarter-circle is $\pi/2$ radians or 90°. ***The distance around to angle*** θ ***is*** r ***times*** θ.

When $r = 1$ this is the ultimate in simplicity: *The distance is* θ. A 45° angle is $\frac{1}{8}$ of a circle and $2\pi/8$ radians—and the length of the circular arc is $2\pi/8$. Similarly for 1°:

$$360° = 2\pi \text{ radians} \qquad 1° = 2\pi/360 \text{ radians} \qquad 1 \text{ radian} = 360/2\pi \text{ degrees.}$$

An angle going clockwise is *negative*. The angle $-\pi/3$ is $-60°$ and takes us $\frac{1}{6}$ of the *wrong* way around the circle. What is the effect on the six functions?

Certainly the radius r is not changed when we go to $-\theta$. Also x is not changed (see Figure 1.20a). But y reverses sign, because $-\theta$ is below the axis when $+\theta$ is above. This change in y affects y/r and y/x but not x/r:

$$\cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta \qquad \tan(-\theta) = -\tan\theta.$$

The cosine is *even* (no change). The sine and tangent are *odd* (change sign).

The same point is $\frac{5}{6}$ of the *right* way around. Therefore $\frac{5}{6}$ of 2π radians (or 300°) gives the same direction as $-\pi/3$ radians or $-60°$. ***A difference of*** 2π ***makes no difference to*** x, y, r. Thus $\sin\theta$ and $\cos\theta$ and the other four functions have period 2π. We can go five times or a hundred times around the circle, adding 10π or 200π to the angle, and the six functions repeat themselves.

EXAMPLE Evaluate the six trigonometric functions at $\theta = 2\pi/3$ (or $\theta = -4\pi/3$).

This angle is shown in Figure 1.20a (where $r = 1$). The ratios are

$$\cos\theta = x/r = -1/2 \qquad \sin\theta = y/r = \sqrt{3}/2 \qquad \tan\theta = y/x = -\sqrt{3}$$

$$\sec\theta = -2 \qquad \csc\theta = 2/\sqrt{3} \qquad \cot\theta = -1/\sqrt{3}$$

Those numbers illustrate basic facts about the sizes of four functions:

$$|\cos\theta| \leqslant 1 \qquad |\sin\theta| \leqslant 1 \qquad |\sec\theta| \geqslant 1 \qquad |\csc\theta| \geqslant 1.$$

The tangent and cotangent can fall anywhere, as long as $\cot\theta = 1/\tan\theta$.

The numbers reveal more. The tangent $-\sqrt{3}$ is the ratio of sine to cosine. The secant -2 is $1/\cos\theta$. Their squares are 3 and 4 (differing by 1). That may not seem remarkable, but it is. There are three relationships in the *squares* of those six numbers, and they are the key identities of trigonometry:

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Everything flows from the Pythagoras formula $x^2 + y^2 = r^2$. Dividing by r^2 gives $(x/r)^2 + (y/r)^2 = 1$. That is $\cos^2\theta + \sin^2\theta = 1$. Dividing by x^2 gives the second identity, which is $1 + (y/x)^2 = (r/x)^2$. Dividing by y^2 gives the third. All three will be needed throughout the book—and the first one has to be unforgettable.

DISTANCES AND ADDITION FORMULAS

To compute the distance between points we stay with Pythagoras. The points are in Figure 1.21a. They are known by their x and y coordinates, and d is the distance between them. The third point completes a right triangle.

For the x distance along the bottom we don't need help. It is $x_2 - x_1$ (or $|x_2 - x_1|$ since distances can't be negative). The distance up the side is $|y_2 - y_1|$. Pythagoras immediately gives the distance d:

$$\textbf{\textit{distance between points}} = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$

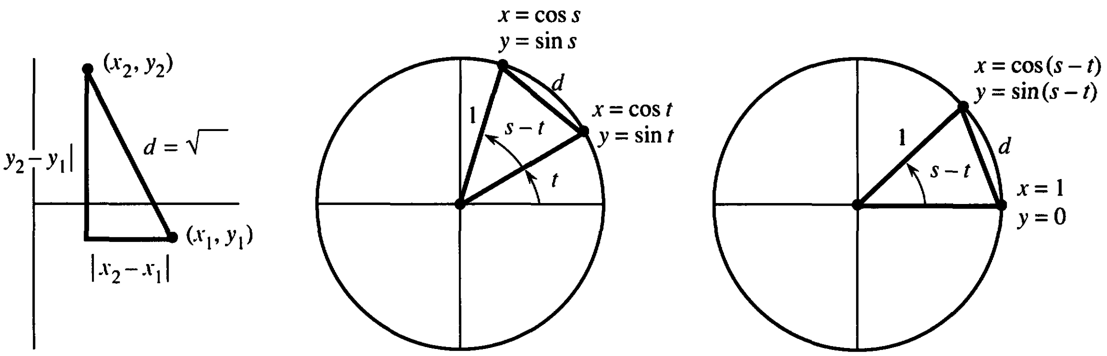

Fig. 1.21 Distance between points and equal distances in two circles.

By applying this distance formula in two identical circles, we discover the cosine of $s - t$. (Subtracting angles is important.) In Figure 1.21b, the distance squared is

$$\begin{aligned} d^2 &= (\text{change in } x)^2 + (\text{change in } y)^2 \\ &= (\cos s - \cos t)^2 + (\sin s - \sin t)^2. \end{aligned} \tag{2}$$

Figure 1.21c shows the same circle and triangle (but rotated). The same distance squared is

$$d^2 = (\cos(s - t) - 1)^2 + (\sin(s - t))^2. \tag{3}$$

Now multiply out the squares in equations (2) and (3). Whenever $(\text{cosine})^2 + (\text{sine})^2$ appears, replace it by 1. The distances are the same, so (2) = (3):

$$(2) = 1 + 1 - 2\cos s\cos t - 2\sin s\sin t$$

$$(3) = 1 + 1 - 2\cos(s - t).$$

After canceling $1 + 1$ and then -2, we have the "***addition formula***" for $\cos(s-t)$:

$$\text{The cosine of } s-t \text{ equals } \cos s \cos t + \sin s \sin t. \tag{4}$$

$$\text{The cosine of } s+t \text{ equals } \cos s \cos t - \sin s \sin t. \tag{5}$$

The easiest is $t = 0$. Then $\cos t = 1$ and $\sin t = 0$. The equations reduce to $\cos s = \cos s$.

To go from (4) to (5) in all cases, replace t by $-t$. No change in $\cos t$, but a "minus" appears with the sine. In the special case $s = t$, we have $\cos(t+t) = (\cos t)(\cos t) - (\sin t)(\sin t)$. This is a much-used formula for $\cos 2t$:

$$\textbf{\textit{Double angle}}: \cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t. \tag{6}$$

I am constantly using $\cos^2 t + \sin^2 t = 1$, to switch between sines and cosines.

We also need addition formulas and double-angle formulas for the *sine* of $s - t$ and $s + t$ and $2t$. For that we connect sine to cosine, rather than $(\text{sine})^2$ to $(\text{cosine})^2$. The connection goes back to the ratio y/r in our original triangle. This is the sine of the angle θ and also the cosine of the *complementary angle* $\pi/2 - \theta$:

$$\sin\theta = \cos(\pi/2 - \theta) \qquad \text{and} \qquad \cos\theta = \sin(\pi/2 - \theta). \tag{7}$$

The complementary angle is $\pi/2 - \theta$ because the two angles add to $\pi/2$ (a right angle). By making this connection in Problem 19, formulas (4–5–6) move from cosines to sines:

$$\sin(s-t) = \sin s \cos t - \cos s \sin t \tag{8}$$

$$\sin(s+t) = \sin s \cos t + \cos s \sin t \tag{9}$$

$$\sin 2t = \sin(t+t) = 2\sin t \cos t \tag{10}$$

I want to stop with these ten formulas, even if more are possible. Trigonometry is full of identities that connect its six functions—basically because all those functions come from a single right triangle. The x, y, r ratios and the equation $x^2 + y^2 = r^2$ can be rewritten in many ways. But you have now seen the formulas that are needed by calculus.† They give derivatives in Chapter 2 and integrals in Chapter 5. And it is typical of our subject to add something of its own—a limit in which an angle approaches zero. *The essence of calculus is in that limit.*

Review of the ten formulas Figure 1.22 shows $d^2 = (0 - \frac{1}{2})^2 + (1 - \sqrt{3}/2)^2$.

$$\cos\frac{\pi}{6} = \cos\frac{\pi}{2}\cos\frac{\pi}{3} + \sin\frac{\pi}{2}\sin\frac{\pi}{3} \qquad (s-t) \qquad \sin\frac{\pi}{6} = \sin\frac{\pi}{2}\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\sin\frac{\pi}{3}$$

$$\cos\frac{5\pi}{6} = \cos\frac{\pi}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{2}\sin\frac{\pi}{3} \qquad (s+t) \qquad \sin\frac{5\pi}{6} = \sin\frac{\pi}{2}\cos\frac{\pi}{3} + \cos\frac{\pi}{2}\sin\frac{\pi}{3}$$

$$\cos 2\frac{\pi}{3} = \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3} \qquad (2t) \qquad \sin 2\frac{\pi}{3} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}$$

$$\cos\frac{\pi}{6} = \sin\frac{\pi}{3} = \sqrt{3}/2 \qquad \left(\frac{\pi}{2} - \theta\right) \qquad \sin\frac{\pi}{6} = \cos\frac{\pi}{3} = 1/2$$

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†Calculus turns (6) around to $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$.

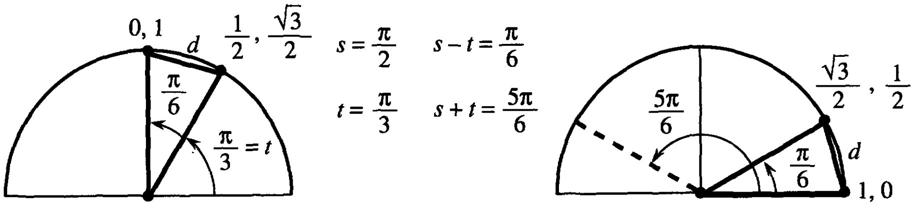

Fig. 1.22

1.5 EXERCISES

Read-through questions

Starting with a ___a___ triangle, the six basic functions are the ___b___ of the sides. Two ratios (the cosine x/r and the ___c___) are below 1. Two ratios (the secant r/x and the ___d___) are above 1. Two ratios (the ___e___ and the ___f___) can take any value. The six functions are defined for all angles θ, by changing from a triangle to a ___g___.

The angle θ is measured in ___h___. A full circle is $\theta =$ ___i___, when the distance around is $2\pi r$. The distance to angle θ is ___j___. All six functions have period ___k___. Going clockwise changes the sign of θ and ___l___ and ___m___. Since $\cos(-\theta) = \cos\theta$, the cosine is ___n___.

Coming from $x^2 + y^2 = r^2$ are the three identities $\sin^2\theta + \cos^2\theta = 1$ and ___o___ and ___p___. (Divide by r^2 and ___q___ and ___r___.) The distance from (2, 5) to (3, 4) is $d =$ ___s___. The distance from (1, 0) to $(\cos(s-t), \sin(s-t))$ leads to the addition formula $\cos(s-t) =$ ___t___. Changing the sign of t gives $\cos(s+t) =$ ___u___. Choosing $s = t$ gives $\cos 2t =$ ___v___ or ___w___. Therefore $\frac{1}{2}(1 + \cos 2t) =$ ___x___, a formula needed in calculus.

1 In a 60–60–60 triangle show why $\sin 30° = \frac{1}{2}$.

2 Convert π, 3π, $-\pi/4$ to degrees and 60°, 90°, 270° to radians. What angles between 0 and 2π correspond to $\theta = 480°$ and $\theta = -1°$?

3 Draw graphs of $\tan\theta$ and $\cot\theta$ from 0 to 2π. What is their (shortest) period?

4 Show that $\cos 2\theta$ and $\cos^2\theta$ have period π and draw them on the same graph.

5 At $\theta = 3\pi/2$ compute the six basic functions and check $\cos^2\theta + \sin^2\theta$, $\sec^2\theta - \tan^2\theta$, $\csc^2\theta - \cot^2\theta$.

6 Prepare a table showing the values of the six basic functions at $\theta = 0, \pi/4, \pi/3, \pi/2, \pi$.

7 The area of a circle is πr^2. What is the area of the sector that has angle θ? It is a fraction ________ of the whole area.

8 Find the distance from (1, 0) to (0, 1) along (a) a straight line (b) a quarter-circle (c) a semicircle centered at $(\frac{1}{2}, \frac{1}{2})$.

9 Find the distance d from (1, 0) to $(\frac{1}{2}, \sqrt{3}/2)$ and show on a circle why $6d$ is less than 2π.

10 In Figure 1.22 compute d^2 and (with calculator) $12d$. Why is $12d$ close to and below 2π?

11 Decide whether these equations are true or false:

(a) $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$

(b) $\dfrac{\sec\theta + \csc\theta}{\tan\theta + \cot\theta} = \sin\theta + \cos\theta$

(c) $\cos\theta - \sec\theta = \sin\theta\tan\theta$

(d) $\sin(2\pi - \theta) = \sin\theta$

12 Simplify $\sin(\pi-\theta)$, $\cos(\pi-\theta)$, $\sin(\pi/2+\theta)$, $\cos(\pi/2+\theta)$.

13 From the formula for $\cos(2t+t)$ find $\cos 3t$ in terms of $\cos t$.

14 From the formula for $\sin(2t+t)$ find $\sin 3t$ in terms of $\sin t$.

15 By averaging $\cos(s-t)$ and $\cos(s+t)$ in (4–5) find a formula for $\cos s\cos t$. Find a similar formula for $\sin s\sin t$.

16 Show that $(\cos t + i\sin t)^2 = \cos 2t + i\sin 2t$, if $i^2 = -1$.

17 Draw $\cos\theta$ and $\sec\theta$ on the same graph. Find all points where $\cos\theta = \sec\theta$.

18 Find all angles s and t between 0 and 2π where $\sin(s+t) = \sin s + \sin t$.

19 Complementary angles have $\sin\theta = \cos(\pi/2 - \theta)$. Write $\sin(s+t)$ as $\cos(\pi/2 - s - t)$ and apply formula (4) with $\pi/2 - s$ instead of s. In this way derive the addition formula (9).

20 If formula (9) is true, how do you prove (8)?

21 Check the addition formulas (4–5) and (8–9) for $s = t = \pi/4$.

22 Use (5) and (9) to find a formula for $\tan(s+t)$.

In 23–28 find *every* θ that satisfies the equation.

23 $\sin\theta = -1$

24 $\sec\theta = -2$

25 $\sin\theta = \cos\theta$

26 $\sin\theta = \theta$

27 $\sec^2\theta + \csc^2\theta = 1$

28 $\tan\theta = 0$

29 Rewrite $\cos\theta + \sin\theta$ as $\sqrt{2}\sin(\theta+\phi)$ by choosing the correct "phase angle" ϕ. (Make the equation correct at $\theta = 0$. Square both sides to check.)

30 Match $a\sin x + b\cos x$ with $A\sin(x+\phi)$. From equation (9) show that $a = A\cos\phi$ and $b = A\sin\phi$. Square and add to find $A =$ ________. Divide to find $\tan\phi = b/a$.

31 Draw the base of a triangle from the origin $O = (0, 0)$ to $P = (a, 0)$. The third corner is at $Q = (b\cos\theta, b\sin\theta)$. What are the side lengths OP and OQ? From the distance formula (1) show that the side PQ has length

$$d^2 = a^2 + b^2 - 2ab\cos\theta \quad \text{(law of cosines).}$$

32 Extend the same triangle to a parallelogram with its fourth corner at $R = (a + b\cos\theta, b\sin\theta)$. Find the length squared of the other diagonal OR.

Draw graphs for equations 33–36, and mark three points.

33 $y = \sin 2x$

34 $y = 2\sin\pi x$

35 $y = \frac{1}{2}\cos 2\pi x$

36 $y = \sin x + \cos x$

37 Which of the six trigonometric functions are infinite at what angles?

38 Draw rough graphs or computer graphs of $t\sin t$ and $\sin 4t\sin t$ from 0 to 2π.