COMPACTIFICATION OF THE UNIVERSAL PICARD OVER THE MODULI OF STABLE CURVES

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ABSTRACT. This article provides two different, but closely related, moduli problems, which in characteristic zero provide a type of compactification of the universal Picard over the moduli of stable curves. Although neither is of finite type, both are limits of a sequence of stacks, each of which is a separated algebraic stack of finite type. We discuss relations to previous compactifications and partial compactifications, give a number of examples related to this compactification, and work out the structure of its fibres over certain fixed curves. Some applications are also discussed.

1. INTRODUCTION

The goal of this article is to construct a stack that compactifies the universal Picard over the moduli of stable curves. The *universal Picard* $\mathrm{PIC}^d_{\mathrm{M}_g}$ differs from the relative Picard in that points of the relative Picard that would be identified by automorphisms of the underlying curve are considered isomorphic in the universal Picard (see Definition 1.4.6). Let $\mathfrak{M}_g$ be the stack of smooth curves of genus g, and let $\overline{\mathfrak{M}}_g$ be the Deligne-Mumford compactification of $\mathfrak{M}_g$, namely the stack of stable curves. We seek a stack $\mathfrak{P}_d$ of $\overline{\mathfrak{M}}_g$-schemes such that $\mathfrak{P}_d$ satisfies the valuative criterion for separatedness and properness, and such that $\mathrm{PIC}^d_{\mathrm{M}_g}$, the universal Picard for smooth curves is an open, dense, sub-functor of the moduli of $\mathfrak{P}_d$.

1.1. **Previous Work.** There are several constructions of schemes that are related to this problem. Most of these approaches have the benefit of using geometrically meaningful functors, but they are limited to special families of curves. Among the examples of work in this direction are the results of C. D'Souza [10], Oda and Seshadri [21], Ishida [16], and Altman and Kleiman [4, 5].

D'Souza and Altman-Kleiman provide a compactification of the Jacobian of an integral curve, or the relative Picard of a flat family of integral curves, provided the family is locally projective. But these constructions are not separated in the case of a general (reducible) stable curve. In the special case of an integral curve over a field, the constructions of this paper yield results similar to D'Souza's compactification.

Oda, Seshadri, and Ishida, on the other hand, proved that even when the curve is not integral, it is possible to construct many compactifications of the relative Jacobian or Picard scheme, again using torsion-free sheaves of certain multi-degrees, provided the base of the curve is a field or a pseudo-geometric local ring with infinite residue field. This construction is not well-behaved over a general base.

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Also, C. Simpson[24] has constructed a functor which compactifies the relative Picard functor of a general family of semi-stable curves, and which has a projective coarse moduli. Unfortunately, Simpson's functor is not separated.

An alternative approach is found in the work of L. Caporaso [8] and R. Pandharipande [22]. Rather than restrict to special families, they compactify the universal Picard scheme over the moduli space $\overline{\mathscr{M}}_g$. But just as Simpson's functor, the functors these compactifications coarsely represent are not separated, and these compactifications do not appear to represent any separated, geometrically-meaningful functor. In particular, there is no universal object that lies over these schemes. One aim of this paper is to remedy these problems.

E. Esteves [11] has produced a compactification that is similar in spirit to the constructions of this paper. Esteves provides an algebraic space, which although not of finite type, has the advantage of corresponding to meaningful geometric objects, and it behaves like a compactification of the universal Picard scheme in many ways. This compactification has a universal object, but unfortunately it also is not separated.

This paper provides two compactifications, both separated over $\overline{\mathfrak{M}}_{gn}$. And although the stacks are not of finite type, they meet a form of the valuative criterion of properness; moreover, they have natural substacks which are of finite type, and, in many cases, proper. Thus, in these cases, the substacks provide a natural compactification of the relative Picard. And these substacks suffice for many applications.

Finally, it is interesting to note that Abramovich and Vistoli [2] have recently described a technique for compactifying the stack of maps from $\overline{\mathfrak{M}}_g$ to a Deligne-Mumford stack, provided the target stack has a projective coarse moduli space. If the classifying stack $\mathfrak{B}\mathbb{G}_m$ of the group $\mathbb{G}_m$ were a Deligne-Mumford stack, their construction, applied to maps from $\overline{\mathfrak{M}}_g$ to $\mathfrak{B}\mathbb{G}_m$, should give the "right" compactification of the Picard. However, $\mathfrak{B}\mathbb{G}_m$ is an Artin stack, and their construction will not necessarily generalize to Artin stacks.

The stacks constructed in this paper are closely related to the constructions of Abramovich and Vistoli. Their compactifications correspond to replacing stable curves with *stacks* whose coarse moduli are the usual stable curves. These stacky curves are called *twisted stable curves*. Line bundles on twisted stable curves correspond to torsion-free sheaves on the coarse moduli curve. And a line bundle $\mathscr{L}$ with a specific isomorphism $\mathscr{L}^{\otimes r} \tilde{\rightarrow} \mathscr{M}$ to the r^{th} tensor power $\mathscr{M}$ of the bundle, corresponds to the main geometric object of this paper, namely, an r^{th} root of $\mathscr{M}$. Thus the geometric objects of this paper are essentially coarse moduli for the twisted objects of Abramovich and Vistoli.

1.2. **Main Ideas.** In general, there are two obstructions to constructing a meaningful compactification. First, not every line bundle on the smooth generic fibre of a stable curve can be extended to a line bundle on the whole curve. And second, line bundles that do extend from the generic fibre to the whole curve do not extend uniquely.

A numerical criterion for the admissible degrees of line bundles (i.e. the degree on each irreducible component of the fibres of the curve) will solve the separatedness problem as long as torsion-free sheaves are excluded, but unfortunately, no numerical criterion on the degree of a torsion-free sheaf is sufficiently strong to ensure unique extension of the sheaf to reducible special fibres. An example demonstrating this fact is given in Section 4.3.1. Therefore, another approach is necessary.

Although a line bundle $\mathscr{L}_\eta$ on the generic fibre of a stable curve over a discrete valuation ring will not always extend to the entire curve, some tensor power of the bundle will extend. Moreover, the choice of a suitable N, such that $\mathscr{L}_\eta^N$ does extend to the entire curve, can be made independent of the bundle $\mathscr{L}_\eta$. Namely, it only depends on the underlying curve (see Corollary 3.4.3).

The approach we take, therefore, is to develop a notion of an n^{th} root of the extension of the n^{th} tensor power $\mathscr{L}_\eta^{\otimes n}$. To do this we use techniques developed in [17]. In particular, the idea is to put a numerical criterion on the multi-degree of the extended n^{th} power bundle $\mathscr{M}$, and then define n^{th} roots of this bundle to be rank-one torsion-free sheaves $\mathscr{E}$, together with a map $\mathscr{E}^{\otimes n} \to \mathscr{M}$ having sufficient restrictions to guarantee uniqueness of extension. The numerical criterion is very similar to the standard (Seshadri) definition of semi-stable. However, it is stricter—requiring equality where semi-stability requires only an inequality. One of the surprising results of this paper is the fact that this stricter requirement still permits enough objects to make a proper stack. And indeed, the weaker requirement of semi-stability permits too many objects for the associated stack to be separated.

There are actually two different stacks having similar properties that accomplish this goal. These are the stacks of *quasi-roots* and *roots* of line bundles. These stacks differ only in their definition of families: roots of line bundles are slightly more restrictive than quasi-roots. We define two inductive systems of stacks $\{\mathfrak{QP}_{d,n}\}_n$ and $\{\mathfrak{P}_{d,n}\}_n$ such that each of the terms is a separated algebraic stack, and such that the ind-objects

$$\mathfrak{QP}_d = \varinjlim_{n} \mathfrak{QP}_{d,n!}$$

and

$$\mathfrak{P}_d = \varinjlim_{n} \mathfrak{P}_{d,n!}$$

satisfy a form of valuative criterion of properness (in characteristic 0). And for every n, the universal Picard for smooth curves $\mathrm{Pic}^d_{\mathrm{M}_g}$ is an open, dense, subfunctor of the moduli space of $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$

1.3. **Outline of Paper.** In Section 2 we define roots and quasi-roots of line bundles, and give some necessary background from [17] and [23] regarding torsion-free sheaves. We then give the numerical criterion that guarantees separatedness of the stacks, and define the stacks $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$. In Section 3 the main properties of the stacks $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$ are given, including their algebraic nature and properness. In Section 4 we discuss some of the geometric properties of the fibres of $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$, and give examples of those fibres, as well as examples that the stacks $\mathfrak{QP}_d$ and $\mathfrak{P}_d$ are essentially the best possible. And to conclude, in Section 5 we give an application to torsion points of the Jacobian and modular curves.

1.4. **Conventions and Notation.** Throughout this article we will be working over a fixed base scheme S of finite type over a field or over an excellent Dedekind domain. This is necessary for most of the standard theorems on algebraic stacks and is also necessary for many of the results of [17] to hold.

We will work primarily with *semi-stable curves* of a fixed genus g, by which we mean a flat, proper morphism $\mathscr{X} \to T$ of S-schemes whose geometric fibres $\mathscr{X}_t$ are reduced, connected, one-dimensional schemes, with only ordinary double points, and with $\dim H^1(\mathscr{X}_t, \mathscr{O}_{\mathscr{X}_t}) = g$. A *stable curve* is a semi-stable curve of genus greater than one, with the additional property that any rational irreducible component of a geometric fibre meets the rest of the fibre in at least three points. Irreducible components of a semi-stable curve which are isomorphic to $\mathbb{P}^1$ but meet the curve in only two points will be called *exceptional* curves.

By *line bundle* we mean an invertible (locally free of rank one) coherent sheaf. And in order to define the main objects of interest, we will need torsion-free sheaves.

Definition 1.4.1. A *relatively torsion-free sheaf* (or just torsion-free sheaf) on a semi-stable curve $f : \mathscr{X} \to T$, is a coherent $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{E}$ that is flat over T, such that on each fibre $\mathscr{X}_t = \mathscr{X} \times_T \mathrm{Spec}\,(k(t))$ the induced $\mathscr{E}_t$ has no associated primes of height one.

This paper is only concerned with rank-one torsion-free sheaves. Such sheaves are called *admissible* by Alexeev [3], *sheaves of depth one* by Seshadri [23], and *sheaves of pure dimension one* by Simpson [24]. Of course, on the open set where f is smooth, a torsion-free sheaf is locally free.

Definition 1.4.2. The *degree* of a torsion-free sheaf $\mathscr{E}$ is the integer $\chi(\mathscr{E}) - \chi(\mathscr{O}) = \chi(\mathscr{E}) - g + 1$.

Since the terminology of algebraic stacks is not completely standardized, we give here the definitions used in this paper.

Definition 1.4.3. An *algebraic (Artin) stack* over S is a stack $\mathfrak{X}$ with a smooth atlas $U \to \mathfrak{X}$ (i.e., U is an algebraic space, and the morphism $U \to \mathfrak{X}$ is smooth and surjective), and whose diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable, of finite type, and separated.

Definition 1.4.4. A *Deligne-Mumford Stack* is an algebraic stack $\mathfrak{X}$ which has an étale atlas, or equivalently, [9, Theorem 4.21] has an unramified diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$.

Definition 1.4.5. A *coarse moduli space* for a stack $\mathfrak{X}$ is an algebraic space X and a morphism $p : \mathfrak{X} \to \mathrm{X}$, such that p induces a bijection on (isomorphism classes of) geometric points, and every morphism from $\mathfrak{X}$ to an algebraic space factors through p.

The coarse moduli space X of $\mathfrak{X}$, is the algebraic space representing the étale sheafification of the functor taking any S-scheme T to the set of isomorphism classes of T-valued points of $\mathfrak{X}$.

It is well-known (c.f. [19, Corollary 1.3]) that every Deligne-Mumford stack $\mathfrak{X}$ has a coarse moduli space X. And we will follow the usage of Laumon [20] and use the term *coarse moduli* to describe the associated étale sheaf, even when working with Artin stacks, where representability by an algebraic space is not guaranteed.

Useful general references for Artin stacks are [6] and [20], and for Deligne-Mumford stacks are [9] and [25].

Definition 1.4.6. By *universal Picard for smooth curves of genus g*, we mean the coarse moduli space $\mathrm{Pic}^d_{\mathrm{M}_g}$ of the stack $\mathfrak{Pic}^d_{\mathfrak{M}_g}$ of pairs $(\mathscr{X}/T, \mathscr{L})$, where $\mathscr{X}/T$ is a smooth curve of genus g, and $\mathscr{L}$ is a line bundle on $\mathscr{X}$ of degree d.

This universal Picard is the étale sheafification of the functor taking an S-scheme T to the set of isomorphism classes of smooth curves $\mathscr{X}/T$ and line bundles $\mathscr{L}$ on $\mathscr{X}$. And an isomorphism between $(\mathscr{X}/T, \mathscr{L})$ and $(\mathscr{X}'/T, \mathscr{L}')$ is a pair (σ, τ) with $\sigma : \mathscr{X} \xrightarrow{\sim} \mathscr{X}'$ a T-isomorphism and $\tau : \mathscr{L} \xrightarrow{\sim} \sigma^*\mathscr{L}'$ an isomorphism of line bundles on $\mathscr{X}$.

In particular, for any $\mathfrak{M}_g$-scheme $T \xrightarrow{\mathscr{X}/T} \mathfrak{M}_g$, the fibre product $T \times_{\mathbf{M}_g} \mathrm{Pic}^d_{\mathbf{M}_g}$ is the relative Picard $\mathrm{Pic}_{\mathscr{X}/T}$ *modulo automorphisms of the underlying curve.*

2. Roots and Quasi-Roots of Line Bundles

2.1. **Definition of Quasi-roots.**

The objects of interest in this paper are roots or quasi-roots of line bundles on stable curves.

Definition 2.1.1. For a fixed integer $n \geq 2$, an n^{th} *quasi-root* or *quasi-root of order n* of a line bundle $\mathscr{L}$ on a semi-stable curve $f : \mathscr{X} \to T$ is a pair $(\mathscr{E}, b)$ of a rank-one torsion-free sheaf $\mathscr{E}$ and an $\mathscr{O}_{\mathscr{X}}$-module homomorphism $b : \mathscr{E}^{\otimes n} \to \mathscr{L}$ with the additional conditions that

(1) $n \cdot \deg(\mathscr{E}) = \deg(\mathscr{L})$.
(2) b is an isomorphism on the open set where $\mathscr{E}$ is locally free.
(3) For each closed point t of the base T, and for each singular point $\mathfrak{p}$ of the fibre $\mathscr{X}_t$ where $\mathscr{E}$ is not free, the length of the cokernel of b at $\mathfrak{p}$ is less than n.

The third condition, although it may seem somewhat technical, is necessary to ensure that the homomorphism b is well-behaved. In particular, without it, all the stacks involving quasi-roots would fail to be separated (for more details of the implications of this condition, see [17]).

Definition 2.1.2. Any homomorphism $b : \mathscr{E}^{\otimes n} \to \mathscr{L}$ that meets the third condition of Definition 2.1.1 will be said to have *good cokernel*.

It is worth noting that the condition on the cokernel is not always necessary, and in connection with condition (2) of Definition 2.1.1, it actually implies a much stronger condition. The proof of this fact is interesting not only for its own sake, but also because it illustrates the handy trick of restricting to the normalized curve.

Proposition 2.1.3. *In the case of $n = 2$, the condition on the cokernel is redundant; namely, any pair $(\mathscr{E}, b)$ meeting conditions (1) and (2) must necessarily have good cokernel.*

And even when n is greater than 2, the condition on the degree of $\mathscr{E}$ guarantees that if the length of the cokernel is less than n, then it is exactly $n - 1$ at each singularity.

This follows from the description of a quasi-root $(\mathscr{E}, b)$ of $\mathscr{L}$ on a curve X/k over a field in terms of a line bundle on the normalization $\pi : \tilde{X} \to X$ of X at each of the singularities of $\mathscr{E}$, that is, at the singularities of X where $\mathscr{E}$ fails to be locally free.

Lemma 2.1.4. *Given a curve X/k over a field and a line bundle $\mathscr{L}$ on X, there is a one-to-one correspondence between (isomorphism classes of) n^{th} quasi-roots of $\mathscr{L}$ and line bundles $\mathscr{M}$ on partial normalizations $\pi : \tilde{X} \to X$, such that*

$$\mathscr{M}^{\otimes n} \cong \mathscr{L}(-\sum_{q_i} u_i {p_i}^+ + v_i {p_i}^-).$$

Here the sum is taken over all singularities q_i normalized by π, the points ${p_i}^+$ and ${p_i}^-$ are the preimages of q_i under π, and for each i, the integers u_i and v_i are positive.

Proof (of Lemma). If $\pi^\natural\mathscr{E}$ indicates the line bundle $\pi^*\mathscr{E}/\text{torsion}$, then the following facts are well-known.

(1) Locally on X, at a singularity of $\mathscr{E}$, the completion of the local ring $\hat{\mathscr{O}}_{X,x}$ is of the form $k[[x,y]]/xy$, and the sheaf $\mathscr{E}$ corresponds to a $k[[x,y]]/xy$-module $E \cong xk[[x]] \oplus yk[[y]]$. [23, Chapter 11, Proposition 3].
(2) $\pi_*\pi^\natural\mathscr{E} = \mathscr{E}$ [23, Chapter 11, Proposition 10].
(3) $\deg \pi^\natural\mathscr{E} = \deg \mathscr{E} - \delta$, where δ is the number of singular points normalized by $\pi : \tilde{X} \to X$. [23, Chapter 11, Proposition 10].

From these facts it is easy to see that the n^{th} tensor power $E^{\otimes n}$ is isomorphic to $x^n k[[x]] \oplus y^n k[[y]] \oplus T$, where $\operatorname{Ann} T = (x,y)$. So the map b, being an isomorphism off the singularity, takes x^n in $E^{\otimes n}$ to some element of $k[[x,y]]/xy$ annihilated by y, that is, of the form αx^u, and b takes y^n to some element annihilated by x, that is an element of the form βy^v, where u and v are positive integers, and α and β are units in $k[[x,y]]/xy$. The length of the cokernel of b at the singularity is clearly $u + v - 1$.

From this it is clear that the homomorphism b induces an isomorphism

$$\pi^\natural(b) : (\pi^\natural\mathscr{E})^{\otimes n} \xrightarrow{\sim} \pi^*\mathscr{L}(-\sum(u_i p_i^+ + v_i p_i^-)).$$

And, conversely, b is induced from the composition $(\pi^\natural\mathscr{E})^n \xrightarrow{\pi^\natural(b)} \pi^*\mathscr{L}(-\sum(u_i p_i^+ + v_i p_i^-)) \hookrightarrow \pi^*\mathscr{L}$ by adjointness. Here the sum is taken over all the singularities q_i of $\mathscr{E}$, and p_i^+ and p_i^- are the inverse images of the singularity q_i with respect to π. □

Proof of Proposition 2.1.3. The lemma shows that the degree of $(\pi^\natural\mathscr{E})^{\otimes n}$ is $nd - \sum(u_i + v_i)$. But on the other hand, by Property 3 above, the degree of $\pi^\natural\mathscr{E}$ is $d - \delta$, where δ is the number of singularities of $\mathscr{E}$. Thus we have $\sum(u_i + v_i) = n\delta$. In particular, if $n = 2$, since every term u_i and v_i is at least one and $\sum_{i=1}^{\delta}(u_i + v_i) = 2\delta$, every u_i and v_i is exactly one. Similarly, for a general n, since the individual sums of pairs $u_i + v_i - 1$ are bounded above by $n - 1$ (Condition 3 of Definition 2.1.1), the equality $\sum_{i=1}^{\delta}(u_i + v_i) = n\delta$ implies that each pair sums to n, and the length of the cokernel at each singularity is exactly $n - 1$. □

Definition 2.1.5. If $\mathfrak{q} = (\mathscr{E}, b)$ and $\mathfrak{q}' = (\mathscr{E}', b')$ are n^{th} quasi-roots of line bundles $\mathscr{L}$ and $\mathscr{L}'$, respectively, then an *isomorphism of quasi-roots* from $\mathfrak{q}$ to $\mathfrak{q}'$ is a pair (τ, ϵ) where $\tau : \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ is an isomorphism of line bundles, and $\epsilon : \mathscr{E} \xrightarrow{\sim} \mathscr{E}'$ is an isomorphism of $\mathscr{O}_{\mathscr{X}}$-modules that is compatible with b, b' and τ. In particular, the following diagram commutes.

$$\begin{array}{ccc} \mathscr{E}^{\otimes n} & \xrightarrow{\;b\;} & \mathscr{L} \\ {\scriptstyle \epsilon^{\otimes n}}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathscr{E}'^{\otimes n} & \xrightarrow{\;b'\;} & \mathscr{L}' \end{array}$$

2.1.1. *Example of n^{th} Roots: Two Irreducible Components and One Node.* Consider a stable curve X, over an algebraically closed field, with two smooth, irreducible components C and D, of genus k and $g - k$ respectively, meeting in one double point p. In this case, for any choice of line bundle $\mathscr{L}$, there exists a unique choice of u and v, with $0 \le u < n$, $0 \le v < n$, and $u + v \equiv 0 \pmod{n}$ that makes the degree of $\pi^*\mathscr{L}(-up^+ - vp^-)$ divisible by n on both components. The corresponding roots are locally free if and only if $\mathscr{L}$ has a usual n^{th} root, which is to say, if and only if u and v can be chosen to be 0, or $\deg_C \mathscr{L} \equiv 0 \pmod{n}$. If $\deg_C \mathscr{L} \not\equiv 0 \pmod{n}$ then the resulting root corresponds to an n^{th} root of $\mathscr{L}|_C(-up^+)$ on C and an n^{th} root of $\mathscr{L}|_D(-vp^-)$ on D.

2.1.2. *Example of n^{th} Roots: One Irreducible Component and One Node.* Consider an irreducible stable curve X with one node. In this case there are n different choices of u and v that permit roots of $\mathscr{L}$: either $u = v = 0$, in which case the resulting root is locally free; or $u \in \{1, \ldots, n-1\}$ and $v = n - u$, in which case the resulting root is not locally free. In this second case, the n^{th}-root on X corresponds to an n^{th} root of the bundle $\pi^*\mathscr{L}(-up^+ - vp^-)$.

2.2. **Additional Definitions: The Numerical Criterion.** The stack of all quasi-roots of all line bundles is not separated, since the extension of a line bundle from a smooth generic fibre to a stable special fibre is not unique. But adding the following numerical criterion for the degrees of the line bundles will produce a separated stack.

Definition 2.2.1 (Numerical Criterion)**.** Given a non-zero integer n and a line bundle $\mathscr{A}$ on a stable curve $\mathscr{X}/T$, any line bundle $\mathscr{L}$ on $\mathscr{X}/T$ of degree nd will be said to *meet the numerical criterion*, or to be *$\mathscr{A}$-admissible*, if for every irreducible component C of every geometric fibre $\mathscr{X}_t$ of the curve $\mathscr{X}/T$,

$$\deg_C \mathscr{L} = nd \cdot \frac{\deg_C \mathscr{A}|_C}{\deg \mathscr{A}|_{\mathscr{X}_t}}$$

where $\deg_C \mathscr{A}|_C$ is the degree of $\mathscr{A}$ restricted to C, and $\deg \mathscr{A}|_{\mathscr{X}_t}$ is the total degree of the restriction of the bundle $\mathscr{A}$ to the fibre $\mathscr{X}_t$. Of course, there are no $\mathscr{A}$-admissible line bundles unless $\deg \mathscr{A}|_{\mathscr{X}_t}$ divides $nd \cdot \deg_C \mathscr{A}|_C$ for every component C. Throughout the rest of this paper, given a fixed $\mathscr{A}$, we will assume that $\deg \mathscr{A}$ divides d. This will ensure the existence of $\mathscr{A}$-admissible line bundles on every semi-stable curve, and it is also necessary for the proof of Theorem 3.4.2.

Taking $\mathscr{A}$ to be the canonical (relative dualizing) sheaf ω of $\mathscr{X}/T$ gives a criterion that is similar to the standard (Seshadri) definition of semi-stable. However, our definition of ω-admissibility requires equality, where semi-stability permits an inequality. One surprising result of this paper is that the stricter condition of equality still permits enough limit objects to compactify the Picard stack. And in fact the semi-stability conditions permit too many objects—hence constructions involving such inequalities (e.g. [8, 24]) do not correspond to separated stacks.

Note 2.2.2. One need not use a line bundle to define the numerical criterion. In fact, all that is needed is a rule assigning a degree to every irreducible component of every stable curve, provided that the sum of the degrees for any given stable curve is constant and divides d.

Definition 2.2.3. Given a choice of integers n, and d, and given a choice of a line bundle $\mathscr{A}$ on the universal curve over $\overline{\mathfrak{M}}_g$, such that the degree of $\mathscr{A}$ divides d, the first stack of interest is $\mathfrak{QP}^{\mathscr{A}}_{d,n}$ (or just $\mathfrak{QP}_{d,n}$). It is the stack of triples $(\mathscr{X}/T, \mathfrak{q}, \mathscr{L})$, where $\mathscr{X}/T$ is a stable curve over an S-scheme T, $\mathscr{L}$ is a line bundle of degree nd on $\mathscr{X}$, meeting the numerical criterion, and $\mathfrak{q}$ is an n^{th} quasi-root of $\mathscr{L}$.

Note that if n divides N there is a natural inclusion of $\mathfrak{QP}_{d,n}$ in $\mathfrak{QP}_{d,N}$ defined by taking $(\mathscr{X}/T, \mathscr{E}, b, \mathscr{L}) \mapsto (\mathscr{X}/T, \mathscr{E}, b^{\otimes N/n}, \mathscr{L}^{\otimes N/n})$.

2.3. **Roots.** The other main stack of interest in this paper is a substack $\mathfrak{P}_{d,n}$ of $\mathfrak{QP}_{d,n}$. It consists of quasi-roots that have one additional condition, which we will explain after recalling some background from [17].

2.3.1. *Background on Torsion-free Sheaves and Induced Maps.* All the results in this section are proved in [17].

Proposition 2.3.1. [17, Proposition 3.1 and Section 5.2] *Any rank-one torsion-free sheaf $\mathscr{E}$ on a stable curve $\mathscr{X}/T$ defines a canonical semi-stable curve $\mathscr{X}_{\mathscr{E}}/T$ which has $\mathscr{X}/T$ as its stable model, and which has exactly one exceptional curve in the fibre over each singularity of the sheaf $\mathscr{E}$. Moreover, there is a canonical way to construct a line bundle $\mathscr{O}_{\mathscr{X}_{\mathscr{E}}}(1)$, such that pushing $\mathscr{O}_{\mathscr{X}_{\mathscr{E}}}(1)$ forward along the contraction map $\pi : \mathscr{X}_{\mathscr{E}} \to \mathscr{X}$ (the obvious morphism, contracting all exceptional curves in $\mathscr{X}_{\mathscr{E}}$) gives back $\mathscr{E}$. Namely,*

$$\pi_* \mathscr{O}_{\mathscr{X}_{\mathscr{E}}}(1) \cong \mathscr{E}.$$

In the special case that the sheaf $\mathscr{E}$ has a quasi-root map b, so that $(\mathscr{E}, b)$ is an n^{th} quasi-root of a line bundle $\mathscr{L}$, there is additional canonical construction:

Proposition 2.3.2. [17, Proposition 3.1.5] *If $(\mathscr{E}, b)$ is a quasi-root of the line bundle $\mathscr{L}$, then there is a canonical injective map from $\mathscr{O}_{\mathscr{X}_{\mathscr{E}}}(n) := \mathscr{O}_{\mathscr{X}_{\mathscr{E}}}(1)^{\otimes n}$ to $\pi^*\mathscr{L}$. This map induces another map i by adjointness and push-forward along π; namely, i is the composite map $\mathscr{E}^{\otimes n} \to \pi_*\mathscr{O}_{\mathscr{X}_{\mathscr{E}}}(n) \to \mathscr{L}$. Moreover, i actually makes the pair $(\mathscr{E}, i)$ into a quasi-root of $\mathscr{L}$.*

This induced quasi-root $(\mathscr{E}, i)$ may fail to be isomorphic to $(\mathscr{E}, b)$. In many cases, however, these two quasi-roots are isomorphic. In particular, the difference between $(\mathscr{E}, b)$ and $(\mathscr{E}, i)$ is nilpotent on the base.

Proposition 2.3.3. [17, Corollary 5.4.9] *If T is reduced, then any quasi-root $(\mathscr{E}, b)$ on any stable $\mathscr{X}$ over T is isomorphic to the induced quasi-root $(\mathscr{E}, i)$, as described in the previous proposition.*

2.3.2. *Definition of Roots.*

Definition 2.3.4. An *n^{th} root* of a line bundle $\mathscr{L}$ over a stable curve $\mathscr{X}/T$ is an n^{th} quasi-root $(\mathscr{E}, b)$, with the additional condition that the natural, induced quasi-root $(\mathscr{E}, i)$ is isomorphic to $(\mathscr{E}, b)$.

It is important to note that the difference between quasi-roots and roots is simply a difference in which families are permitted; in particular, the obstruction for a quasi-root to be a root is nilpotent (c.f. Proposition 2.3.3), and thus any quasi-root over a reduced base is a root.

Definition 2.3.5. Given a choice of integers $n \geq 1$ and d, and given a line bundle $\mathscr{A}$ on the universal curve over $\overline{\mathfrak{M}}_g$, such that the degree of $\mathscr{A}$ divides d, the substack $\mathfrak{P}_{d,n}$ of $\mathfrak{QP}_{d,n}$ is the stack of triples $(\mathscr{X}/T, \mathfrak{q}, \mathscr{L})$, where $\mathscr{X}/T$ is a stable curve over an S-scheme T, $\mathscr{L}$ is an $\mathscr{A}$-admissible line bundle of degree nd on $\mathscr{X}$, and $\mathfrak{q}$ is an n^{th} root of $\mathscr{L}$.

As in the case of $\mathfrak{QP}_{d,n}$, if n divides N, there is a natural inclusion of $\mathfrak{P}_{d,n}$ in $\mathfrak{P}_{d,N}$, defined by taking $(\mathscr{E}, b, \mathscr{L}) \mapsto (\mathscr{E}, b^{\otimes N/n}, \mathscr{L}^{\otimes N/n})$.

3. Properties of Roots and Quasi-roots

The main facts about $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$ are the following:

(1) They both form separated algebraic stacks of finite type over the stack of stable curves $\overline{\mathfrak{M}}_g \times_{\mathbb{Z}} \mathbb{Z}[1/n]$. (See Theorems 3.1.1 and 3.2.1.)

(2) The coarse moduli space of the universal Picard stack is an open dense subspace of the moduli spaces of both $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$. (See Proposition 3.3.3.)

(3) For a fixed family of stable curves $\mathscr{X}/B$ over a smooth base curve B and with smooth generic fibre, there is an integer N such that for any multiple n of N, any line bundle and n^{th} quasi-root on the generic fibre will extend to a line bundle and n^{th} root, meeting the numerical criterion, on the pullback of the curve to a finite cover of B. (See Corollary 3.4.3.)

(4) The limiting ind-objects
$$\mathfrak{QP}_d := \varinjlim_{n}(\mathfrak{QP}_{d,n!} \times_{\mathbb{Z}} \mathbb{Q}),$$
and
$$\mathfrak{P}_d := \varinjlim_{n}(\mathfrak{P}_{d,n!} \times_{\mathbb{Z}} \mathbb{Q}),$$
satisfy the valuative criterion of properness in the sense that if the generic point of a complete discrete valuation ring R maps to a point of $\mathfrak{QP}_d$ or $\mathfrak{P}_d$ with a smooth underlying curve, then this point extends to an R-valued point. (See Corollary 3.4.4.)

In many applications, such as limit linear series, some monodromy problems, and some aspects of the study of the fundamental group of a curve, one is only concerned with the degeneration of certain line bundles from the generic fibre of a family of curves over a smooth curve or even over a discrete valuation ring. The third property above shows that $\mathfrak{P}_{d,n}$ and $\mathfrak{QP}_{d,n}$ are sufficient for applications of this sort.

The proof of algebraicity is given in Subsection 3.1. The proof that the stacks are separated is in Subsection 3.2. The proof that the universal Picard is dense is in Subsection 3.3, and properness is proved in Subsection 3.4.

3.1. **Algebraicity of Roots and Quasi-roots.**

Theorem 3.1.1. *If n is invertible in the base S, the stacks $\mathfrak{P}_{d,n}$ and $\mathfrak{QP}_{d,n}$ each form an algebraic (Artin) stack of finite type over the stack of stable curves $\overline{\mathfrak{M}}_g$.*

The proof of Theorem 3.1 is a direct consequence of the following theory of relatively algebraic stacks.

Definition 3.1.2. A morphism of stacks $f : F \to G$ is called *Artin* if for every morphism $X \to G$ from a representable stack X to G the fibre product $F \times_G X$ is an Artin stack. Alternately, one may say F is *relatively Artin* over G.

We define *relatively Deligne-Mumford* in an identical manner.

Clearly, if G is representable and F is relatively Artin over G, then F is Artin. And if G is representable and F is relatively Deligne-Mumford over G, then F is Deligne-Mumford.

Proposition 3.1.3. *If G is Artin and F is relatively Artin over G, then F is Artin. If G is Deligne-Mumford and F is relatively Deligne-Mumford over G, then F is Deligne-Mumford.*

Proof. To prove that F is Artin (respectively Deligne-Mumford) we need to provide a smooth (respectively, étale) atlas and prove that the diagonal $F \to F \times_S F$ is representable, of finite type, and separated [25, 7.14].

If $U \to G$ is a smooth (étale) atlas for G, then the Artin (Deligne-Mumford) stack $U \times_G F$ has an atlas V, which is also an atlas for F. In particular, since $U \to G$ is smooth (étale) and surjective, and $V \to U \times_G F$ is smooth (étale) and surjective, then so is the composition $V \to U \times_G F \to F$.

Moreover, a straightforward generalization of [25, prop 7.13] shows that the diagonal $F \to F \times_S F$ is representable, of finite type, and separable, if and only if for every pair of morphisms $X \to F$ and $Y \to F$ from representable stacks, the canonical morphism $X \times_F Y \to X \times_S Y$ is representable, is of finite type, and is separable.

Given $X \to F$ and $Y \to F$ from representable stacks, $X \to F$ factors through $F \times_G X$; and we have the following Cartesian diagram:

$$\begin{array}{ccccc} Y & \longleftarrow & Y \times_G X & \longleftarrow & Y \times_F X \\ \downarrow & & \downarrow & & \downarrow \\ F & \longleftarrow & F \times_G X & \longleftarrow & X \\ \downarrow & & \downarrow & & \\ G & \longleftarrow & X & & \end{array}$$

Here $F \times_G X$ is algebraic, and $Y \times_G X$ is representable because G is algebraic and X and Y are representable. Consequently, $Y \times_F X$ is also representable.

Furthermore, $Y \times_G X \to Y \times_S X$ is of finite type and separable, so all that remains is to show that the morphism $Y \times_F X \to Y \times_G X$ is of finite type and separable. However, $F \times_G X$ is algebraic, hence the canonical morphism $\Phi : Y \times_F X \xrightarrow{\sim} (Y \times_G X) \times_{(F \times_G X)} X \to (Y \times_G X) \times_S X$ is of finite type and separable. Since the projection $Y \times_F X \xrightarrow{\pi_2} X$ factors through the projection $Y \times_G X \xrightarrow{p_2} X$, this morphism Φ factors as $Y \times_F X \to Y \times_G X \xrightarrow{(1,p_2)} (Y \times_G X) \times_S X$, so we have the following commutative diagram, with the horizontal morphisms both separated and of finite type.

$$\begin{array}{ccc} Y \times_F X & \longrightarrow & Y \times_G X \times_S X \\ \downarrow & \nearrow^{(1,p_2)} & \downarrow (p_1 \times 1) \\ Y \times_G X & \longrightarrow & Y \times_S X \end{array}$$

The diagonal morphism is separated because the bottom (horizontal) morphism is separated. But now the left (vertical) morphism must be both separated and of finite type because the top morphism is separated and of finite type and the diagonal is separated [15, 5.3.19(v), and 6.3.4(v)]. ◻

Proof of Theorem 3.1.1. The fact that $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$ are algebraic stacks now follows easily from the fact that they are relatively algebraic over $\overline{\mathfrak{M}}_g$. In particular, the stack of triples $(\mathscr{X}/T, \mathscr{E}, \mathscr{L})$ of a stable curve $\mathscr{X}/T$, a line bundle $\mathscr{L}$ on $\mathscr{X}$, and a coherent sheaf $\mathscr{E}$ on $\mathscr{X}$, is relatively algebraic over $\overline{\mathfrak{M}}_g$ because for any fixed stable curve $\mathscr{X}/T$, the stack of line bundles on $\mathscr{X}$ and the stack of coherent sheaves on $\mathscr{X}$ are both algebraic [20, 4.14.2.1], as is their fibre product [20, 3.4.i]. Moreover, the substack of triples such that $\mathscr{E}$ is torsion-free and of rank one is an open substack. And the substack such that $\mathscr{L}$ meets the numerical criterion is an open substack, and hence algebraic [20, 3.4.ii].

The stack $\mathfrak{Hom}_{\mathscr{X}/T}(\mathscr{E}^{\otimes n}, \mathscr{L})$ of quadruples $(\mathscr{X}/T, \mathscr{E}, b, \mathscr{L})$ is relatively representable over the open substack of rank one torsion-free $\mathscr{A}$-admissible triples because each $\mathscr{X}/T$ is flat and projective [13, n$^{\text{o}}$ 221 §4]. And finally, the property that a homomorphism $b : \mathscr{E}^{\otimes n} \to \mathscr{L}$ is an isomorphism off the discriminant locus, and the property of having good cokernel are both open conditions [17, Proposition 4.1.5]. So the stack $\mathfrak{QP}_{d,n}$ is an open substack of $\mathfrak{Hom}_{\mathscr{X}/T}(\mathscr{E}^{\otimes n}, \mathscr{L})$, and thus is algebraic.

Proposition 2.3.2 shows that over $\mathfrak{QP}_{d,n}$ there exists not only the universal quasi-root $(\mathscr{X}/T, \mathscr{E}, b, \mathscr{L}) = Q$, but also an induced root $(\mathscr{X}/T, \mathscr{E}, i, \mathscr{L}) = I$. And the stack $\mathfrak{P}_{d,n}$ is exactly the open substack where Q and I agree. That is, for every $Q : T \to \mathfrak{QP}_{d,n}$, there is an induced $I : T \to \mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n} \times T = T \times_{Q,I} T$. Since $\mathfrak{QP}_{d,n}$ is algebraic, this product is representable. Hence $\mathfrak{P}_{d,n}$ is relatively representable over the algebraic stack $\mathfrak{QP}_{d,n}$, and so is an algebraic stack. □

3.2. **Separatedness.**

Theorem 3.2.1. *$\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$ are separated over $\overline{\mathfrak{M}}_g \times_{\mathbb{Z}} \mathbb{Z}[1/n]$.*

Proof. By the valuative criterion [20, 3.19], it suffices to show that given two n^{th} quasi-roots $(\mathscr{E}, b, \mathscr{L})/\mathscr{X}/R$ and $(\mathscr{E}', b', \mathscr{L}')/\mathscr{X}/R$, both over $\mathscr{X}/\operatorname{Spec} R$, where R is a complete discrete valuation ring, and given an isomorphism $\Phi_\eta : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ defined on the generic fibres, then Φ_η will always extend to an isomorphism Φ over all of $\operatorname{Spec} R$.

In the special case that $\mathscr{L} \cong \mathscr{L}' \cong \omega_{\mathscr{X}/R}$, the result is proved in [17, Proposition 4.1.16], assuming that n is invertible on the base S; but the proof there uses only the fact that $\mathscr{L} \cong \mathscr{L}'$. Thus to prove the result in this more general case, it is enough to check that for any two line bundles $\mathscr{L}$ and $\mathscr{L}'$ meeting the numerical criterion on $\mathscr{X}/R$, with an isomorphism $\phi_\eta : \mathscr{L}_\eta \xrightarrow{\sim} \mathscr{L}'_\eta$ defined on the generic fibre, the isomorphism ϕ_η extends to an isomorphism over all of $\mathscr{X}$. To see this, it is enough to assume that the isomorphism ϕ_η is the identity.

It is well known that if $\mathscr{X}$ is regular (and hence the generic fibre of $\mathscr{X}/R$ is smooth), then any two line bundles $\mathscr{L}$ and $\mathscr{L}'$ that are the same on the generic fibre differ only by a line bundle of the form $\mathscr{O}(\sum a_i C_i)$, where the a_i are integers and the C_i are irreducible components of the special fibre of $\mathscr{X}/R$. That is to say, $\mathscr{L}' = \mathscr{L} \otimes \mathscr{O}(\sum a_i C_i) = \mathscr{L}(\sum a_i C_i)$. In this case, the degree of $\mathscr{L}'$ on one irreducible component C differs from the degree of $\mathscr{L}$ on C by the integer $\sum_i a_i(C.C_i)$; and when $C_i \neq C$, then $(C.C_i)$ is the number of points in the intersection of C_i with C (because $\mathscr{X}$ is semi-stable), and when $C = C_i$, then $-C.C_i$ is the total number of intersection points that C has with the rest of the curve.

The numerical criterion guarantees that the degree of both line bundles $\mathscr{L}$ and $\mathscr{L}'$ is the same on every C_i in the special fibre of $\mathscr{X}/R$. And it is easy to see that the only way to choose the a_i so that the degree of $\mathscr{O}(\sum a_i C_i)$ equals zero on all irreducible components is to choose all the a_i equal to each other. But in this case, $\mathscr{O}(\sum a_i C_i) = \mathscr{O}(a \sum C_i) = \mathscr{O}$, and $\mathscr{L} = \mathscr{L}'$.

In the more general case that the generic fibre is smooth but the family $\mathscr{X}$ is not regular, resolve the singularities of $\mathscr{X}$ by blowing up. Let the resulting semi-stable curve be $\nu : \tilde{\mathscr{X}} \to \mathscr{X}$. Then over each singularity of $\mathscr{X}$ there is a chain of exceptional curves in $\tilde{\mathscr{X}}$. The pull-backs $\nu^*\mathscr{L}$ and $\nu^*\mathscr{L}'$ both have degree zero on the exceptional curves of $\tilde{\mathscr{X}}$, and they again differ by $\nu^*(\mathscr{L}^{-1} \otimes \mathscr{L}') = \mathscr{O}_{\tilde{\mathscr{X}}}(\sum a_i \tilde{C}_i)$. The degree of this bundle is zero on all irreducible components of the special fibre. And again the only way to choose a_i to make the bundle $\nu^*(\mathscr{L}^{-1} \otimes \mathscr{L}')$ have degree zero on every irreducible component of the special fibre of $\tilde{\mathscr{X}}$ is to choose all the a_i equal, in which case $\nu^*\mathscr{L} = \nu^*\mathscr{L}'$. And thus $\mathscr{L} = \mathscr{L}'$.

And finally, in the case that the generic fibre is not regular, let $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$ be the normalization of $\mathscr{X}$. Then each connected component of $\tilde{\mathscr{X}}$ has smooth generic fibre. Thus by the previous argument, $\pi^*\mathscr{L} = \pi^*\mathscr{L}'$, and $\mathscr{L} = \mathscr{L}'$. □

3.3. **Relation to the Universal Picard.** Although the stacks over smooth curves $\mathfrak{P}_{d,n} \times_{\overline{\mathfrak{M}}_g} \mathfrak{M}_g$ and $\mathfrak{Pic}^d_{\mathfrak{M}_g}$ are not isomorphic, there is a natural morphism from $\mathfrak{P}_{d,n} \times_{\overline{\mathfrak{M}}_g} \mathfrak{M}_g$ to $\mathfrak{Pic}^d_{\mathfrak{M}_g}$ which induces an isomorphism on the moduli. That is to say, $\mathfrak{P}_{d,n} \times_{\overline{\mathfrak{M}}_g} \mathfrak{M}_g$ is an étale gerbe over $\mathrm{Pic}^d_{\mathrm{M}_g}$.

Proposition 3.3.1. *The étale sheafification $\mathrm{P}^{smooth}_{d,n}$ of roots of line bundles on smooth curves is isomorphic to the universal Picard for smooth curves $\mathrm{Pic}^d_{\mathrm{M}_g}$.*

Proof. Roots of line bundles on smooth curves correspond to isomorphism classes of triples $(\mathscr{X}/T, \mathscr{M}, b)$, where $\mathscr{X}$ is a smooth curve of genus g, $\mathscr{M}$ is a line bundle of degree d, and b is an automorphism $\mathscr{M}^{\otimes n} \xrightarrow{\sim} \mathscr{M}^{\otimes n} = \mathscr{L}$.

So there is a natural morphism of sheaves $f : \mathrm{P}^{smooth}_{d,n} \to \mathrm{Pic}^d_{\mathrm{M}_g}$, which is given by forgetting the automorphism: $(\mathscr{X}/T, \mathscr{M}, b) \mapsto (\mathscr{X}/T, \mathscr{M})$. Moreover, the automorphism b corresponds to an element of $H^0(\mathscr{X}, \mathscr{O}^*_{\mathscr{X}}) = \mathscr{O}^*_T$, and any two choices of automorphism $b, b' \in \mathrm{Aut}\,(\mathscr{M}^{\otimes n})$ induce isomorphic triples $(\mathscr{X}/T, \mathscr{M}, b)$ and $(\mathscr{X}, \mathscr{M}, b')$ if and only if the element $b^{-1} \cdot b'$ has an n^{th} root $\gamma : \mathscr{M} \to \mathscr{M}$ so that the diagram

$$\begin{array}{ccc} \mathscr{M}^{\otimes n} & \xrightarrow{b'} & \mathscr{M}^{\otimes n} \\ \downarrow{\scriptstyle \gamma^{\otimes n}} & & \| \\ \mathscr{M}^{\otimes n} & \xrightarrow{b} & \mathscr{M}^{\otimes n} \end{array}$$

commutes. In particular, over a strictly Henselian local ring R, any two triples $(\mathscr{X}/R, \mathscr{M}, b)$ and $\mathscr{X}/R, \mathscr{M}, b')$ are isomorphic, and the morphism of sheaves $f : \mathrm{P}^{smooth}_{d,n} \to \mathrm{Pic}^d_{\mathrm{M}_g}$ induces an isomorphism of R-valued points. In particular, f induces an isomorphism on the stalks $\mathrm{P}^{smooth}_{d,n}(\mathscr{O}^{sh}_{\bar{s},S}) \xrightarrow{\sim} \mathrm{Pic}^d_{\mathrm{M}_g}(\mathscr{O}^{sh}_{\bar{s},S})$ and thus is an isomorphism of étale sheaves [1, VIII.3.5]. □

Theorem 3.3.2. *The universal Picard $\mathrm{Pic}^d_{\mathrm{M}_g}$ is an open, dense subfunctor of the moduli $\mathrm{P}_{d,n}$ of $\mathfrak{P}_{d,n}$ and of the moduli $\mathrm{QP}_{d,n}$ of $\mathfrak{QP}_{d,n}$.*

The theorem follows immediately from Proposition 3.3.1 and the following deformation, which shows that triples $(X, \mathfrak{q}, \mathscr{L})$ of a smooth curve and an n^{th} quasi-root of a line bundle $\mathscr{L}$ form a dense open substack of $\mathfrak{P}_{d,n}$. And hence their moduli $\mathrm{Pic}^d_{\mathbf{M}_g}$ forms an open dense subfunctor of the moduli of $\mathfrak{QP}_{d,n}$. Recall that any quasi-root over a field is actually a root, so the existence of the deformation actually proves this stack is dense in both $\mathfrak{P}_{d,n}$ and $\mathfrak{QP}_{d,n}$.

Proposition 3.3.3. *Given any n^{th} root $(\overline{\mathscr{E}}, \bar{b}, \overline{\mathscr{L}})$ on a stable curve X over a field k, there is a canonical deformation to an n^{th} root $(\mathscr{E}, b, \mathscr{L})$ on $\mathscr{X}$ over $\mathscr{M}$ where $\mathscr{X}/\mathscr{M}$ is the pullback of the universal deformation*

$$\mathfrak{X} \to \mathit{Spec}\, \mathfrak{o}_k[[t_1, \ldots, t_\delta, t_{\delta+1}, \ldots, t_{3g-3}]] = \mathscr{M}$$

of the curve X/k along the homomorphism

$$\mathfrak{o}_k[[t_1, \ldots, t_\delta, t_{\delta+1}, \ldots, t_{3g-3}]] \to \mathfrak{o}_k[[\tau_1, \ldots, \tau_\delta, t_{\delta+1}, \ldots, t_{3g-3}]]$$

via $t_i \mapsto \tau_i^n$ for each $i \in \{1, \ldots, \delta\}$.

Proof. In [17, 4.2.1] the result is proved for $\overline{\mathscr{L}} \cong \omega_{X/k}$. That proof consists of constructing a semi-stable curve $\tilde{\mathscr{X}}_{\mathscr{E}}$ over $\mathscr{X}$ with a line bundle $\mathscr{O}_{\tilde{\mathscr{X}}_{\mathscr{E}}}(1)$ on it that pushes down to the desired $\mathscr{E}$ on $\mathscr{X}$. And it then shows that any such line bundle has a natural homomorphism from its n^{th} power to ω that induces a homomorphism from $\mathscr{E}^{\otimes n}$ to ω. Moreover, the proof of [17, 4.2.1] depends only on the fact that the bundle $\omega_{X/k}$ has a canonical extension to a line bundle on $\mathscr{X}$. It suffices, therefore, to show that given any $\bar{\mathscr{L}}$ on X/k, there is an extension $\mathscr{L}$ on $\mathscr{X}/\mathscr{M}$ of $\bar{\mathscr{L}}$. But since the curve $\mathfrak{X}$ is regular, $\bar{\mathscr{L}}$ extends to a line bundle on $\mathfrak{X}/\mathscr{M}$ [7, 2.1]; and thus the pull-back of the extension to $\mathscr{X}/\mathscr{M}$ is an extension of $\bar{\mathscr{L}}$, as desired. □

3.4. **Properness.** The following proposition shows that for any fixed line bundle the stack of roots of that bundle is proper.

Proposition 3.4.1. *Given a line bundle $\mathscr{L}$ on a curve $\mathscr{X}/V$ where V is of dimension one, n is invertible in $\mathscr{O}_V$, and $\mathscr{X}$ has smooth generic fibre, any n^{th} (quasi-) root of $\mathscr{L}$ defined on $\mathscr{X}$ over an open subset U of V will extend to an n^{th} (quasi-) root of $\mathscr{L}$ on the pullback of $\mathscr{X}$ to a finite (degree-n) cover of V.*

Proof. This is an easy generalization of the proof of the properness theorem of [17, Section 4.2] □

Given a fixed n and a stable curve $\mathscr{X}$ over the spectrum V of a discrete valuation ring R, with $\mathscr{X}$ having smooth generic fibre $\mathscr{X}_\eta$, it is not true that every pair $(\mathfrak{q}, \mathscr{L})_\eta$ on the generic fibre extends to an n^{th} root on the whole curve. In particular, $\mathscr{L}_\eta$ will not necessarily extend to a line bundle on all of $\mathscr{X}$, and even if it does, it is not true that the extension can always be chosen to meet the numerical criterion.

Nevertheless, the following theorem shows that for any fixed curve over a discrete valuation ring, all n^{th} roots extend from the generic to the special fibre if n is large enough, or rather, sufficiently divisible.

Theorem 3.4.2. *For a fixed stable curve with smooth generic fibre over a discrete valuation ring R, with field of fractions K, there is an integer N such that for any multiple n of N, any line bundle $\mathscr{L}_\eta$ and any n^{th} root of $\mathscr{L}_\eta$ on the generic fibre will extend (up to finite* *extension of K) to an n^{th} root of a line bundle $\mathscr{L}$, meeting the numerical criterion on the whole curve.*

Proof. Let R be a discrete valuation ring, and let $V = \operatorname{Spec} R$. Let $\mathscr{X}/V$ be the curve in question; let $\mathscr{X}_\eta$ be the generic fibre, which is smooth; and let $((\mathscr{E}, b), \mathscr{L})_\eta$ be a root structure on the generic fibre $\mathscr{X}_\eta$. By Proposition 3.4.1 it suffices to show that there is an N such that for any positive multiple n of N, the line bundle $\mathscr{L}_\eta = \mathscr{E}_\eta^{\otimes n}$ can be extended to one meeting the numerical criterion.

Resolving the singularities of $\mathscr{X}$ by blowing up produces a semi-stable curve $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$ with a chain of exceptional curves over each singularity of $\mathscr{X}$. $\mathscr{E}_\eta$ is torsion-free on the smooth curve $\mathscr{X}_\eta$, and hence is locally free. Thus $\mathscr{E}_\eta$ will extend (not uniquely) to a line bundle $\tilde{\mathscr{E}}$ on all of $\tilde{\mathscr{X}}$, and $\pi_* \tilde{\mathscr{E}}^{\otimes n}$ is a line bundle on $\mathscr{X}$ if and only if the degree of $\tilde{\mathscr{E}}^{\otimes n}$ on each exceptional curve is zero. As discussed in Section 3.2, any two line bundles $\mathscr{M}$ and $\mathscr{N}$ on $\mathscr{X}/V$ that agree on the generic fibre are related by $\mathscr{M} \cong \mathscr{N} \otimes \mathscr{O}(\sum a_i C_i)$, where the C_i are the irreducible components of the special fibre of $\tilde{\mathscr{X}}$, and a_i are integers. In particular, it suffices to show that for some $\mathbf{a} = (a_1, \ldots, a_k)$, the bundle $\tilde{\mathscr{E}}^{\otimes n} \otimes \mathscr{O}(\sum a_i C_i)$ has degree 0 on every exceptional curve and on the non-exceptional curves has the degree specified by the numerical criterion.

Since the degrees specified by the numerical criterion are all divisible by n, it suffices to show that for any multi-degree $(m_1, m_2, \ldots, m_k) = \mathbf{m} \in \mathbb{Z}^k$, with $\sum m_i = 0$, there is a bundle $\mathscr{O}(\sum a_i C_i)$ such that the multi-degree of $\mathscr{O}(\sum a_i C_i)$ is $n \cdot \mathbf{m}$.

Basic intersection theory shows that the intersection matrix $\Delta = [C_i \cdot C_j]$ of the irreducible components is negative semi-definite of rank $k-1$, and that the multi-degree of $\mathscr{O}(\sum a_i C_i)$ is just $\Delta \mathbf{a}$. So if N is any principal, $(k-1) \times (k-1)$-minor of Δ, then whenever N divides n, the equation $\Delta \mathbf{a} = \mathbf{m}$ has a solution $\mathbf{a}$ in $(\mathbb{Z}[1/n])^k$, provided $\sum m_i = 0$. And thus $\Delta \mathbf{a} = n\mathbf{m}$ will have a solution $\mathbf{a}$ in $\mathbb{Z}^k$, for any multiple n of N. □

Corollary 3.4.3. *For a fixed family of stable curves $\mathscr{X}/B$ over a smooth base curve B and with smooth generic fibre, there is an integer N such that for any multiple n of N, any line bundle and n^{th} quasi-root on the generic fibre will extend to a line bundle and n^{th} root, meeting the numerical criterion, on the pullback of the curve to a finite cover of B.*

Proof. By Proposition 3.4.1 it suffices to show that there is an N such that for every positive multiple n of N every line bundle $\mathscr{L}_\eta$ on the generic fibre extends to a line bundle meeting the numerical criterion on all of $\mathscr{X}/B$.

For each point $b \in B$ of the singular locus $SL := \{b \in B | \mathscr{X}_b \text{ is singular}\}$, Theorem 3.2.1 guarantees the existence of an integer N_b with the property that for any n^{th} root $\mathscr{E}_\eta$, defined on the generic fibre of $\mathscr{S}/B$, and for any multiple n of N_b, the n^{th} power $\mathscr{E}_\eta^{\otimes n}$ extends to a line bundle $\mathscr{L}_{(b)}$ meeting the numerical criterion on all of $\mathscr{X} \times_B \operatorname{Spec} \mathscr{O}_{B,b}$, where $\mathscr{O}_{B,b}$ is the local ring of B at b.

Let N be the product $N := \prod_{b \in SL} N_b$. Since SL is a finite set, this is well-defined, and by descent this gives a line bundle $\mathscr{L}$, extending $\mathscr{L}_\eta$, and meeting the numerical criterion on all of $\mathscr{X}$, (here the cover of B is $(\coprod_{b \in SL} \operatorname{Spec} \mathscr{O}_{B,b}) \amalg (B - SL)$, and the descent data are the obvious ones). □

If n divides m, then there is a canonical inclusion $\mathfrak{QP}_{d,n} \to \mathfrak{QP}_{d,m}$ and $\mathfrak{P}_{d,n} \to \mathfrak{P}_{d,m}$ defined by taking $(\mathscr{E}, b, \mathscr{L}) \mapsto (\mathscr{E}, b^{\otimes m/n}, \mathscr{L}^{\otimes m/n})$. Consider the ind-objects

$$\mathfrak{QP}_d := \varinjlim_{n} \mathfrak{QP}_{d,n!} \times_{\mathbb{Z}} \mathbb{Q}$$

and

$$\mathfrak{P}_d := \varinjlim_{n} \mathfrak{P}_{d,n!} \times_{\mathbb{Z}} \mathbb{Q}.$$

Although these are not of finite type, they satisfy a form of the valuative criterion of properness.

Corollary 3.4.4. *Let K be the field of fractions of a complete, discrete valuation ring R. Given any K-valued point of the smooth locus of $\mathfrak{QP}_d$ (or of $\mathfrak{P}_d$), there is a finite extension field L of K, such that if V is the ring of integers in L, then there is a V-valued point of $\mathfrak{QP}_d$ (respectively $\mathfrak{P}_d$), extending the induced L-valued point.*

Proof. This valuative criterion is an immediate consequence of Theorem 3.4.2. Also, recall that since any quasi-root over a field or a valuation ring is actually a root, Theorem 3.4.2 shows that this weakened valuative criterion holds for both $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$. □

If the stacks $\mathfrak{QP}_d$ and $\mathfrak{P}_d$ were of finite type, Corollary 3.4.4 would suffice to prove properness because the smooth curves with quasi-roots form a dense open substack, and it is enough to check the valuative criterion in the special case that the generic point of R is contained in an open dense substack (c.f. [9, pg. 109] or [14, 7.3.10 ii]).

4. Fibres and Other Examples

4.1. **Structure of the Fibres Over a Fixed Curve.** Let $P_{X,n}$ be the fibre of the moduli space of $\mathfrak{P}_{d,n}$ (and also of $\mathfrak{QP}_{d,n}$) lying over the point of $\overline{\mathfrak{M}}_g$ corresponding to the curve X. That is to say, $P_{X,n}$ is the set of isomorphism classes of roots $(\mathfrak{q}, \mathscr{L})$ on a fixed stable curve X/k, with $\mathfrak{q}$ an n^{th} root of $\mathscr{L}$, and $\mathscr{L}$ meeting the numerical criterion. In this section we develop a combinatorial description of these fibres and give some specific examples.

Proposition 4.1.1. *For any integers n and d, and any semi-stable curve X/k over a field, there is a one-to-one correspondence between (isomorphism classes of) $\mathscr{A}$-admissible bundles and roots $(\mathfrak{q}, \mathscr{L})$, and the set of triples $(\nu, \mathscr{M}, \mathscr{L})$, where $\nu \in H_1(\Gamma, \mathbb{Z}/n\mathbb{Z})$ is a $\mathbb{Z}/n\mathbb{Z}$ one-cycle for the dual graph Γ of X, and $\mathscr{M}$ is a line bundle on the partial normalization $\pi : X^\nu \to X$ of X at the support of ν, and $\mathscr{L}$ is a line bundle on X, such that $\mathscr{M}^{\otimes n} \cong \pi^*(\mathscr{L})(-\nu) := \pi^*\mathscr{L}(-\sum u_i p_i^+ + v_i p_i^-)$, as explained below.*

Proof. As described in Lemma 2.1.4, each $(\mathfrak{q}, \mathscr{L}) = ((\mathscr{E}, b), \mathscr{L})$ induces a line bundle $\pi^\natural \mathscr{E} := \pi^*\mathscr{E}/(\text{torsion})$ on the partial normalization $\tilde{X} \xrightarrow{\pi} X$ of X at the singular points of $\mathscr{E}$. The map b induces an isomorphism $(\pi^\natural\mathscr{E})^{\otimes n} \xrightarrow{\sim} \pi^*\mathscr{L}(-\sum u_i p_i^+ + v_i p_i^-)$, where each pair $\{p_i^+, p_i^-\}$ is the inverse image of a singular point q_i of $\mathscr{E}$, i.e., $\{p_i^+, p_i^-\} = \pi^{-1}\{q_i\}$. And the conditions on the cokernel guarantee that $u_i + v_i = n$ for all i.

Let Γ be the dual graph of X (every irreducible component of X corresponds to a vertex of Γ, and every singular point of X corresponds to an edge joining the vertices associated to the components containing the singularity), and fix an orientation of Γ. A choice of u_i and v_i corresponds uniquely to a $\mathbb{Z}/n\mathbb{Z}$ one-chain ν on Γ. Denote the line bundle $\mathscr{O}(-\sum u_i p_i^+ + v_i p_i^-)$ by $\mathscr{O}(-\nu)$ and the bundle $\pi^*\mathscr{L}(-\sum u_i p_i^+ + v_i p_i^-)$ by $\pi^*\mathscr{L}(-\nu)$.

The multi-degree function defines a map from line bundles on X to 0-chains in $C_0(\Gamma, \mathbb{Z})$; namely, if $[D]$ denotes the vertex of Γ associated to the irreducible component D of X, and if $\deg_D \mathscr{L}$ denotes the degree of $\mathscr{L}$ on D, then the 0-chain associated to $\mathscr{L}$ is

$$\mathrm{mdeg}\mathscr{L} := \sum_D \deg_D \mathscr{L} \cdot [D].$$

And since $\deg \mathscr{A}$ divides d,

$$\mathbf{a} := \frac{nd}{\deg \mathscr{A}} \cdot \mathrm{mdeg}\mathscr{A}$$

is also a 0-chain in $C_0(\Gamma, \mathbb{Z})$.

It is easy to see that the function $\partial : C_1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \to C_0(\Gamma, \mathbb{Z}/n, \mathbb{Z})$, defined as $\partial(\nu) := \mathrm{mdeg}\mathscr{O}(\nu) \pmod{n}$, is the usual boundary map of graph homology.

And a line bundle $\mathscr{L}$ is $\mathscr{A}$-admissible if and only if $\mathrm{mdeg}\mathscr{L} = \mathbf{a}$ in $C_0(\Gamma, \mathbb{Z})$. By Lemma 2.1.4 $\mathscr{L}$ is part of an n^{th} root structure if and only if there is a $\nu \in C_1(\Gamma, \mathbb{Z}/n\mathbb{Z})$ such that if $\pi : X^\nu \to X$ is the partial normalization of X on the support of ν, then $\pi^*\mathscr{L}(-\nu)$ is an n^{th} power of some bundle $\mathscr{M}$ on X^ν. And since the Picard group of X^ν is n-divisible, this occurs if and only if the multidegree of $\mathscr{L}(-\nu)$ is divisible by n; that is to say, if and only if

$$\mathbf{a} - \partial\nu \equiv 0 \text{ in } C_1(\Gamma, \mathbb{Z}/n\mathbb{Z}).$$

Since $\mathbf{a}$ is congruent to zero in $C_0(\Gamma, \mathbb{Z}/n\mathbb{Z})$, this occurs if and only if $-\partial\nu = 0$; that is, if and only if ν is an element of $H_1(\Gamma, \mathbb{Z}/n\mathbb{Z})$. Thus every quasi-root $(\mathscr{E}, b, \mathscr{L})$ induces a unique choice of $(\nu, \mathscr{M}, \mathscr{L})$ with $\nu \in H_1(\Gamma, \mathbb{Z}/n\mathbb{Z})$, and $\mathscr{M} = \pi^\natural\mathscr{E}$. And conversely, every triple $(\nu, \mathscr{M}, \mathscr{L})$ with $\nu \in H_1(\Gamma, \mathbb{Z}/n\mathbb{Z})$ and $\mathscr{M}^{\otimes n} \cong \mathscr{L}$ induces $\mathscr{E} = \pi_*\mathscr{M}$, and b is induced by adjointness from $\mathscr{M}^{\otimes n} \xrightarrow{\sim} \pi^*\mathscr{L}(-\nu) \hookrightarrow \pi^*\mathscr{L}$, as in Lemma 2.1.4.

□

Let $\mathrm{Pic}^{\mathbf{a}} X$ indicate the line bundles on X with multi-degree equal to $\mathbf{a}$, and $\mathrm{Pic}^{\mathbf{d}_\nu} X^\nu$ indicate the line bundles of multi-degree $\mathbf{d}_\nu := (\mathbf{a} - \partial\nu)/n$ on the normalized curve X^ν.

We see from the previous proposition that n^{th} roots over a fixed curve X are parametrized by an element $\mathscr{L}$ in $\mathrm{Pic}^{\mathbf{a}} X$, a choice of ν in $H_1(\Gamma, \mathbb{Z}/n\mathbb{Z})$, and a line-bundle n^{th} root of the normalization of $\mathscr{L}$. This means that for each ν we have the following diagram:

$$\begin{array}{ccc} & & \mathrm{Pic}^{\mathbf{d}_\nu} X^\nu \\ & & \Big\downarrow [\otimes n] \\ \mathrm{Pic}^{\mathbf{a}} X & \xrightarrow{\mathscr{O}(-\nu) \otimes \pi^*(\cdot)} & \mathrm{Pic}^{n\mathbf{d}_\nu} X^\nu \end{array}$$

Here the horizontal arrow is the map taking $\mathscr{L}$ to $\pi^*\mathscr{L}(-\nu)$, and the vertical arrow is the map taking $\mathscr{M}$ to $\mathscr{M}^{\otimes n}$, and the space of n^{th} roots corresponding to a given

ν is just the fibre product of these two maps. The fibre $P_{X,n}$ is

$$P_{X,n} = \coprod_{\nu \in H_1(\Gamma, \mathbb{Z}/n\mathbb{Z})} \operatorname{Pic}^{\mathbf{a}} X \times_{\operatorname{Pic}^{n\mathbf{d}_\nu} X^\nu} \operatorname{Pic}^{\mathbf{d}_\nu} X^\nu.$$

In particular, since the vertical map is finite, the fibre $P_{X,n}$ has dimension g, just as in the case when X is smooth.

4.2. **Examples of Fibres.**

4.2.1. *Fibres Over a Curve of Compact Type.* If the curve X is of compact type, i.e., if Γ is a tree, then $H_1(\Gamma) = 0$. In this case, the horizontal map in the above diagram is an isomorphism. Thus $P_{X,n}$ is simply the set of all line bundles $\mathscr{M}$ on X of multi-degree $\mathbf{a}/n$, that is $P_{X,n} = \operatorname{Pic}^{\mathbf{a}/n} X$.

4.2.2. *Fibres Over an Integral Curve.* If X is an integral curve with δ double points, then rank $H_1(\Gamma) = \delta$. Moreover, $\Gamma(X)$ is a bouquet of circles with only one vertex, so every one-chain is a one-cycle. Therefore, given any choice of a normalization X^ϱ of X and any line bundle $\mathscr{M}$ of degree $d - |\varrho|$ on that normalization ($|\varrho|$ is the number of points normalized by ϱ) there are $(n-1)^{|\varrho|}$ choices of a one-chain ν supported on ϱ, and all are one-cycles. For each of these choices of ν and $\mathscr{M}$, the possible choices for $\mathscr{L}$ on X are parameterized by $\mathbb{G}_m^{|\varrho|}$. However, this is non-canonical, as it requires a choice of trivialization of $\pi^*\mathscr{L} = \mathscr{M}^{\otimes n}$ in order to produce a line bundle $\mathscr{L}$ on X. Note also that although the dimension of $P_{X,n}$ is g, the number of components that it has depends on n.

If X is integral, there exists a natural forgetful map from $P_{X,n}$ onto D'Souza's compactification (the moduli of rank-one, torsion-free sheaves of degree d on an integral curve); namely, the structure map $(\mathscr{E}, b, \mathscr{L}) \mapsto \mathscr{E}$. The fibres of this map are isomorphic to $\mathbb{G}_m^{|\varrho|}$.

4.2.3. *Fibres Over a Curve With Two Smooth, Irreducible Components, Intersecting in Two Points.* If X is a curve with two irreducible components, intersecting in exactly two points, both components without self-intersections, then rank $H_1(\Gamma) = 1$ and there are n one-cycles. The fibre $P_{X,n}$ consists of two components—one corresponding to locally free root structures (i.e., where $\nu = 0$, and $\mathscr{E}$ is a line bundle), and the other corresponding to root structures that are singular at both intersection points (i.e., $\mathscr{E}$ is not locally free at the intersection points). Forgetting the additional structure of the map b and the target bundle $\mathscr{L}$, the first component consists of line bundles of degree d (corresponding to the cycle $\nu = 0$) and multi-degree $\mathbf{d} = \mathbf{a}/n$. And the second component is made up of rank-one torsion-free sheaves of total degree d and multi-degree $\mathbf{d} - 1 = (d_1 - 1, d_2 - 1)$ that are singular at both of the intersection points.

4.3. $\mathfrak{QP}_d$ **and** $\mathfrak{P}_d$ **are the Best Possible.** A natural question to ask about the construction of $\mathfrak{P}_{d,n}$, $\mathfrak{QP}_{d,n}$, $\mathfrak{P}_d$, and $\mathfrak{QP}_d$, is whether something simpler might suffice for the compactification. The answer is unfortunately no, as the following two examples show.

In particular, the first example shows that any stack that depends alone on a numerical criterion on the degrees of torsion-free sheaves cannot be separated. Thus additional structure, such as that provided by the quasi-root structure, is necessary to produce a separated stack.

The second example shows that, in general, no finite n is sufficient to produce a proper stack. Thus the limit objects $\mathfrak{QP}_d$ and $\mathfrak{P}_d$ must be used instead of $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$. However, in many cases $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$ are sufficient for applications.

4.3.1. *Example: No Numerical Criterion Guarantees Separatedness.* Consider a regular, semi-stable curve $\mathscr{X}$, over a discrete valuation ring Spec R, such that the generic fibre of $\mathscr{X}$ is smooth. Moreover, assume that the special fibre of $\mathscr{X}$ is the union of two irreducible, non-exceptional curves C and D, joined by a single chain of exceptional curves $\{E_i\}$ of length $n \geq 2$. In other words, the dual graph of the special fibre is of type A_{n+1}, with the vertices representing C and D on the ends.

Let $\mathscr{L}_0$ be a line bundle on the special fibre with degree 0 on each irreducible component, except on E_i for some i, with $2 \leq i < n$. On E_i assume that $\mathscr{L}_0$ has degree 1. Such a line bundle extends to $\mathscr{L}$ on all of $\mathscr{X}$ (c.f. [7, 2.1]). And if $\overline{\mathscr{X}}$ is the stable model of $\mathscr{X}$, and $\pi : \mathscr{X} \to \overline{\mathscr{X}}$ is the natural contraction map, then $\pi_*\mathscr{L}$ is a rank-one torsion-free sheaf of degree zero on all irreducible components of the special fibre (namely, on both C and D) [17, Section 3.1.2]. However, the bundle $\mathscr{L}' = \mathscr{L} \otimes \mathscr{O}(C + \sum_{j \leq i} E_j)$ has the same generic fibre as $\mathscr{L}$, and its degree on each irreducible component of the special fibre is zero except on E_{i+1}, where it has degree 1. The push-forward $\pi_*\mathscr{L}'$ is a torsion-free sheaf which also has degree zero on all of the irreducible components of the special fibre. On the generic fibre the two are isomorphic,

$$\pi_*\mathscr{L}_\eta \cong \pi_*\mathscr{L}'_\eta,$$

but $\pi_*\mathscr{L}$ is not isomorphic to $\pi_*\mathscr{L}'$ on all of $\mathscr{X}$. Thus the degree of the torsion-free sheaf alone is not enough to guarantee uniqueness of the extensions from the generic fibre.

In the special case that the underlying curve is geometrically irreducible, D'Souza [10] shows that there are numerical criteria that suffice to guarantee uniqueness of extension, but the above example shows that this cannot hold in the case of a general stable curve.

4.3.2. *Example: $\mathfrak{QP}_{d,n}$ and $\mathfrak{P}_{d,n}$ Are Not Proper for Any Finite n.* As in the previous example, consider $\mathscr{X}$/Spec (R) a regular, semi-stable curve over a discrete valuation ring R, with smooth generic fibre. And assume that the special fibre consists of a single, non-exceptional curve C, joined to itself by a chain of n exceptional curves. The intersection matrix for the irreducible components of the special fibre is of the form

$$\Delta = \begin{bmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & \mathbf{0} & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{bmatrix}.$$

Let $\mathscr{L}$ be the extension to all of $\mathscr{X}$ of a line bundle on the special fibre that has degree e_i on the exceptional curves E_i, and degree $(d - \sum e_i)$ on the non-exceptional curve C. Such an $\mathscr{L}$ exists for every choice of $\{e_i\}$ [7, 2.1]. $\mathscr{L}_\eta$ has an extension to the stable model of $\mathscr{X}$ if and only if there is a choice of integers $\mathbf{a} \in \mathbb{Z}^{n+1}$ such that $\Delta\mathbf{a}^T = [d - \sum e_i, e_1, e_2, \ldots, e_n]$. And this is equivalent to saying that there is a choice of integers $\alpha \in \mathbb{Z}^n$ such that

$$\Delta' \alpha^T = [e_1, \ldots, e_n] ,$$

where $\Delta' = [E_i.E_j]$. However, Δ' has determinant $-(n+1)$, and thus for a general choice of $\{e_i\}$ there is no integer solution. And the only way to ensure that such a line bundle has an extension to the stable model of $\mathscr{X}$ is to take, instead, its $n+1^{\text{st}}$ tensor power, or some integer multiple of that. Finally, because there exist curves as described for arbitrary n, the stacks $\mathfrak{Q}\mathfrak{P}_{d,n}$ and $\mathfrak{P}_{d,n}$ are not complete for any finite n.

5. Application: Degeneration of Torsion Points of the Jacobian

One application of this compactification is that it gives a compactification over $\mathbb{Z}[1/n]$ of the moduli of pairs $(X, \mathscr{L})$, where $\mathscr{L}$ is a line bundle on X corresponding to an n-torsion point of the Jacobian of X; that is, $\mathscr{L}^{\otimes n} \cong \mathscr{O}_X$.

In this case, since $\mathscr{O}_X$ always extends from the generic to the special fibres, Proposition 3.4.1 and Theorem 3.1.1 show that the $\mathfrak{P}_{0,n}$-substack $\overline{\mathfrak{M}}_g(n, \mathscr{O})$ of n^{th} roots $(\mathscr{X}/T, \mathscr{E}, b, \mathscr{O}_{\mathscr{X}/T})$ of the trivial bundle is a proper algebraic stack of finite type over $\mathbb{Z}[1/n]$, and it is finite over $\overline{\mathfrak{M}}_g$.

It is straightforward to check that the diagonal $\overline{\mathfrak{M}}_g(n, \mathscr{O}) \to \overline{\mathfrak{M}}_g(n, \mathscr{O}) \times_{\overline{\mathfrak{M}}_g} \overline{\mathfrak{M}}_g(n, \mathscr{O})$ is unramified (c.f. [17, Proposition 4.1.15]), and hence the stack $\overline{\mathfrak{M}}_g(n, \mathscr{O})$ is a Deligne-Mumford stack [9, Theorem 4.21].

By Proposition 3.3.1, the geometric points of the open substack $\mathfrak{M}_g(n, \mathscr{O})$ of n^{th} roots of $\mathscr{O}$ over smooth curves, correspond (up to isomorphism) to pairs $(X, \mathscr{M})$, with $\mathscr{M}^{\otimes n} \cong \mathscr{O}$. And the coarse moduli space of $\mathfrak{M}_g(n, \mathscr{O})$ is the algebraic space representing the étale sheafification of the functor taking an S-scheme T to the set of isomorphism classes of pairs $(X/T, \mathscr{M})$ with $\mathscr{M}^{\otimes n}$ isomorphic to $\mathscr{O}$.

In the special case that the genus g is 1, the moduli space $\mathrm{M}_{1,1}(n, \mathscr{O})$ of $\mathfrak{M}_{1,1}(n, \mathscr{O})$ is the union of the modular curves $Y_1(d)$, where d runs through all divisors of n. Recall that $Y_1(d)$ is the moduli of elliptic curves with points of exact order d. Consequently, $\overline{\mathrm{M}}_{1,1}(n, \mathscr{O})$ gives a compactification of $\coprod_{d|n} Y_1(n)$ whose normalization agrees with the "normalization near infinity"—the modular curves $\coprod_{d|n} X_1(n)$ (c.f. [18]).

The points at infinity correspond to roots of $\mathscr{O}$ on the (one-pointed) nodal curve C of arithmetic genus 1 and geometric genus 0. There are n locally-free roots of $\mathscr{O}_C$, corresponding to the n roots of unity that could be used to patch $\mathscr{O}_{\mathbb{P}^1}$ into an n^{th} root of $\mathscr{O}_C$. And there are $n-1$ non-trivial cycles ν in $H_1(\Gamma(C), \mathbb{Z}/n\mathbb{Z})$, corresponding to torsion-free but not locally-free roots. By Proposition 4.1.1 these correspond to the unique n^{th} root of $\pi^*\mathscr{O}_C(-\nu) = \mathscr{O}_{\mathbb{P}^1}(-\nu)$. Although the bundle $\mathscr{E} \cong \pi * \mathscr{O}(-1)$ is the same for each choice of (non-zero) ν, the homomorphism b distinguishes the different values of ν.

The curve C has a unique involution fixing the marked point and the node, and this corresponds to interchanging the normalized points p^+ and p^-, and in particular, identifies ν to $-\nu$. Similarly, the involution identifies the locally free root defined by an n^{th} root of unity ζ to the root defined by ζ^{-1}. Thus there are

$$n = 1 + 1/2(n - 1 + n - 1)$$

total points at infinity. In the special case of n prime, this corresponds exactly to the one cusp of $X(1) \cong \mathbb{P}^1$ and the $n-1$ cusps of $X_1(n)$.

6. Conclusion

We have provided two related moduli problems $\mathfrak{QP}_d$ and $\mathfrak{P}_d$, which in characteristic zero provide a type of compactification of the universal Picard scheme over the moduli of stable curves. Although neither stack is of finite type, both are inductive limits of a sequence of stacks, each of which is a separated algebraic stack of finite type. Each of the terms of the sequence contains the relative Picard functor for the universal smooth curve over the stack $\mathfrak{M}_g$ as an open dense substack. Moreover, both $\mathfrak{QP}_d$ and $\mathfrak{P}_d$ meet the valuative criterion for properness, which, if the stacks were of finite type, would suffice to prove properness. Finally, in light of the examples given in Section 4.3, these two stacks are essentially the best possible, since any weakening of the conditions would give a stack that was either not separated or not proper.

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