Math 311 Hwk 3

Show your work. Always provide both code and output.

Problem 1 (25 points). *If A is an $m \times n$ matrix, show that*

$$\|A\|_\infty = \sup_{1 \le i \le m} \left(\sum_{j=1}^{n} |a_{ij}| \right).$$

<u>*Solution*</u>*:* Note that

$$Ax = \left[\sum_{j=1}^{n} a_{1j} \; \sum_{j=1}^{n} a_{2j} \; \cdots \; \sum_{j=1}^{n} a_{mj} \right]^T.$$

Hence

$$\begin{aligned}
\|A\|_\infty &= \sup_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} \, x_j \right| \\
&\le \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}||x_j| \\
&\le \left(\sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \right) \sup_{1 \le j \le n} |x_j| \\
&= \left(\sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \right) \|x\|_\infty.
\end{aligned}$$

Hence for all $x \neq 0$, we have

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \le \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Taking the supremum of the left-hand size yields

$$\|A\|_\infty \le \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|. \tag{1}$$

Now let k be the row satisfying

$$\sum_{j=1}^{n} |a_{kj}| = \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

If

$$x = \left[\frac{\overline{a_{k1}}}{|a_{k1}|} \; \frac{\overline{a_{k2}}}{|a_{k2}|} \; \cdots \; \frac{\overline{a_{kn}}}{|a_{kn}|} \right]^T,$$

then $\|x\| = 1$ and

$$\|Ax\|_\infty = \sup_{1\le i\le n} \left| \sum_{j=1}^n a_{ij}\, x_j \right| \ge \sum_{j=1}^n |a_{kj}| = \sup_{1\le i\le n} \sum_{j=1}^n |a_{ij}|.$$

Hence

$$\|A\|_\infty \ge \sup_{1\le i\le n} \sum_{j=1}^n |a_{ij}|. \tag{2}$$

Combining (1) and (2) yields equality.

Remark 1. *In the second part of the solution, we defined*

$$x = \left[\frac{\overline{a_{k1}}}{|a_{k1}|} \; \frac{\overline{a_{k2}}}{|a_{k2}|} \; \cdots \; \frac{\overline{a_{kn}}}{|a_{kn}|} \right]^T.$$

We took into account the possibility that A could have complex values. In that situation, $a_{kj}x_j = |a_{kj}|$. If the values of A were real, we would have used

$$x = [sgn(a_{k1})\ sgn(a_{k2})\ \cdots\ sgn(a_{k1})]^T.$$

Students who did it the latter route will still receive full credit, but in the future think generally.

Problem 2 (25 points). *Write a Malab function called* `jacobIt` *that performs Jacobi iteration. Make it so that the program gracefully deals with non-convergent sequences.*

Solution: Make sure that the graceful check is on the eigenvalues all having modulus less than (or in some rare cases equal to) 1. It is possible for Jacobi iteration to converge without the matrix being diagonally dominant, so using that as a test is not correct.

Problem 3 (10 points). *Determine whether the following sets form subspaces of $C[-1,1]$.*

(a). *The set of functions f in $C[-1,1]$ such that $f(-1) = f(1)$.*

(b). *The set of even functions in $C[-1,1]$.*

(c). *The set of continuous nondecreasing functions on $[-1,1]$.*

(d). *The set of functions f in $C[-1,1]$ such that $f(-1) = 0$ or $f(1) = 0$.*

<u>Solution</u>:

(a). Yes.

(b). Yes.

(c). No. Consider -1 times any nondecreasing function.

(d). No. Consider $f(x) = x + 1$ and $g(x) = x - 1$.

Problem 4 (10 points). *Let $A \in M_n(\mathbb{R})$ be fixed. Determine which of the following are subspaces of $M_n(\mathbb{R})$.*

(a). $S_1 = \{B \in M_n(\mathbb{R}) \mid AB = BA\}$

(b). $S_2 = \{B \in M_n(\mathbb{R}) \mid AB \neq BA\}$

(c). $S_3 = \{B \in M_n(\mathbb{R}) \mid BA = 0\}$

<u>Solution</u>:

(a). Yes. Note that

$$\begin{aligned} A(b_1B_1 + b_2B_2) - (b_1B_1 + b_2B_2)A &= b_1(AB_1 - B_1A) + b_2(AB_2 - B_2A) \\ &= b_1 0 + b_2 0 = 0. \end{aligned}$$

(b). No. The zero matrix is not even in S_2.

(c). Yes. Note that $(b_1B_1 + b_2B_2)A = b_1B_1A + b_2B_2A = b_1 0 + b_2 0 = 0$.

Problem 5 (10 points). *Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for the vector space V, and let $\mathbf{v}$ be any other vector in V. Show that $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent.*

Solution: Since $\{\mathbf{v}_1 \ldots, \mathbf{v}_n\}$ spans V, then $\mathbf{v}$ can be written as some linear combination

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n.$$

Hence $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent since

$$(-1)\mathbf{v} + a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n = 0.$$

Problem 6 (10 points). *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent vectors in a vector space V. Show that $\mathbf{v}_2, \ldots, \mathbf{v}_n$ cannot span V.*

Solution: Suppose $\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ spans V. Then $\mathbf{v}_1$ can be written as some linear combination

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n,$$

and hence

$$(-1)\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

This contradicts the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Problem 7 (10 points). *Let $S = \{x, 1\}$ and $T = \{2x - 1, 2x + 1\}$ be bases for $\mathbb{R}_1[x]$.*

(a). *Find the transition matrix from T to S.*

(b). *Use your answer in part (a) to find the transition matrix from S to T.*

Solution:

(a). Note

$$[2x - 1, 2x + 1] = [x, 1]\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.$$

(b). We use the inverse from (a)

$$[x, 1] = [2x - 1, 2x + 1]\begin{pmatrix} 1/4 & -1/2 \\ 1/4 & 1/2 \end{pmatrix}.$$