Math 316 Practice Exam 1

Problem 1. *Prove that* $x_n \to x$ *implies* $x_n^2 \to x^2$

Problem 2. *Let* $E \subset \mathbb{R}^n$ *be nonempty. For* $x \in \mathbb{R}^n$, *define*

$$D(x) = \inf\{|x - e| \mid e \in E\}.$$

Show that D *is a continuous function on* $\mathbb{R}^n$.

Problem 3. *Let* (X, d) *be a metric space such that* $d(x, y) < 1$ *for all* $x, y \in X$ *and let* $f : X \longrightarrow \mathbb{R}$ *be uniformly continuous. Does it follow that* f *must be bounded? Justify your answer with either a proof or a counterexample.*

Problem 4. *Let* $\mathbb{R}^{2\times 2}$ *denote the set of all* 2×2 *matrices with real coefficients. Make it a metric space by identifying* $\mathbb{R}^{2\times 2}$ *with* $\mathbb{R}^4$ *via*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (a, b, c, d).$$

Let X *denote the subset of all invertible* 2×2 *matrices. Is* X *connected? Prove your answer.*

Problem 5.

(a). *If* $B \subset \mathbb{R}^n$ *is a bounded set and* $f : B \longrightarrow \mathbb{R}$ *is uniformly continuous, show that* $f(B)$ *is bounded.*

(b). *Give an example to show that this does not necessarily follow if* f *is merely continuous on* B.

Problem 6. *Does the following limit exist? Justify your answer.*

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4 + y^4}.$$

Problem 7. *Let* $E \subset \mathbb{R}^n$ *and* $f : E \longrightarrow \mathbb{R}^m$. *Give an example of a continuous function* f *and a Cauchy sequence* $\{x_k\}_{k=1}^{\infty} \subset E$ *for which* $\{f(x_k\}_{k=1}^{\infty}$ *is* **not** *a Cauchy sequence in* $\mathbb{R}^m$.

Problem 8. *Define* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *and* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *by*

$$f(x, y) = (\sin(y) - x, e^x - y) \quad \textit{and} \quad g(x, y) = (xy, x^2 + y^2).$$

Compute $(g \circ f)'(0, 0)$.

Problem 9. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined by*

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

Show that f *is not differentiable at* $(0, 0)$.

Problem 10. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined by*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

Then show that the partial derivatives exist at $(0, 0)$ *but are discontinuous there.*