Math 316 Hwk 1

Problem 1. *Assume that* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *spans the vector space* V*, and let* $\mathbf{v}$ *be any other vector in* V*. Show that* $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is linearly dependent.*

Problem 2. *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *be linearly independent vectors in* $\mathbb{R}^n$*, and let* A *be a nonsingular* $n \times n$ *matrix. Define* $\mathbf{y}_i = A\mathbf{x}_i$ *for* $i = 1, \ldots, k$*. Show that* $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ *are linearly independent*

Problem 3. *Let* X *be a subspace of* W *and* $L : V \longrightarrow W$ *be a linear transformation. The preimage of* X*, denoted* $L^{-1}(X)$*, is defined by*

$$L^{-1}(X) = \{\mathbf{v} \in V \mid L(\mathbf{v}) \in X\}.$$

Prove that $L^{-1}(X)$ *is a subspace of* V*.*

Problem 4. *Prove that the* ℓ^p *norms satisfy the following inequalities:*

(a). $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2$.

(b). $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$.

Hint: Use the Cauchy-Schwarz inequality.

Problem 5. *Let* $d(\mathbf{x}, \mathbf{y})$ *be a metric on a vector space* V*. Show that*

$$\rho(\mathbf{x}, \mathbf{y}) = \frac{d(\mathbf{x}, \mathbf{y})}{1 + d(\mathbf{x}, \mathbf{y})}$$

is also a metric.

Problem 6. *Let* V, W, X *be vector spaces. Assume that* $L : V \longrightarrow W$ *and* $M : W \longrightarrow X$ *are linear transformations. Prove that* $M \circ L : V \longrightarrow X$ *is a linear transformation.*

Problem 7. *A set* $C \subset \mathbb{R}^n$ *is convex if for each* $\mathbf{x}, \mathbf{y} \in C$*, we have that* $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in C$*, whenever* $0 \leq \lambda \leq 1$*.*

(a). *Give the geometric interpretation of a convex set.*

(b). *Provide an example of a set that is convex and one that isn't.*

Problem 8. *The* convex hull *of* $S \subset \mathbb{R}^n$, *denoted* $co(S)$ *is the set of all convex combinations of elements of* S, *that is, the set of all linear combinations*

$$a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$$

such that $a_1 + \cdots + a_n = 1$, *each* $a_j \geq 0$, *and each* $\mathbf{x}_j \in S$, $j = 1, \ldots, n$, $n \in \mathbb{N}$. *Prove that a convex set* C *contains every convex combination of its elements, or in other words* $co(C) \subset C$.

Problem 9. *Let* $\{C_\alpha\}_{\alpha \in J}$ *be a collection of convex sets for some indexing set* J. *Prove that* $\cap_{\alpha \in J} C_\alpha$ *is convex.*

Problem 10. *Let* $S \subset \mathbb{R}^n$. *Prove that* $co(S)$ *is equal to the intersection of all convex sets containing* S.