Math 316 Hwk 5

Problem 1. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *satisfy* $f(x,y) = xye^{x+y}$.

(a). *Find the directional derivative of* f *in the direction* $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$ *at the point* $(1,-1)$.

(b). *Find the direction in which* f *is increasing the fastest at* $(1,-1)$.

Problem 2. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f *is not differentiable at* $(0,0)$*, but that both partial derivatives exist there.*

Problem 3. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f *has a directional derivative in every direction at* $(0,0)$*, but is not differentiable there.*

Problem 4. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f *is differentiable at* $(0,0)$*. Then show that the partial derivatives are bounded near* $(0,0)$*, but are discontinuous there.*

Problem 5. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f *is differentiable at* $(0,0)$*, but has unbounded partial derivatives near* $(0,0)$.

Problem 6. *Let* $A \subset \mathbb{R}^2$ *be an open set and* $f : A \longrightarrow \mathbb{R}$*. If the partial derivatives of* f *exist and are bounded on* A*, then prove that* f *is continuous on* A*.*

Problem 7. *Let* $A \subset \mathbb{R}^2$ *be an open connected set and* $f : A \longrightarrow \mathbb{R}$*. If for each* $a \in A$*, we have* $D_1 f(a) = D_2 f(a) = 0$*, prove that* f *is constant on* A*.*

Problem 8. *Let* $\gamma(t)$ *be a smooth curve in* $\mathbb{R}^3$*. If for some smooth functional* $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$*, we have* $F(\gamma(t)) = C$*, show that* $DF(\gamma(t))^T$ *is orthogonal to the tangent vector* $\gamma'(t)$*.*

Problem 9. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be given by*

$$f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|_2^2,$$

where A *is an* $n \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^n$*. Find* $Df(\mathbf{x}_0)$*.*

Problem 10. *Let* $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be twice continuously differentiable. If* $x = r\cos\theta$ *and* $y = r\sin\theta$*, show that the Laplacian takes the form*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$