Math 343 Lab 2: Markov Chains

Objective

In this lab, we introduce Markov Chains and discuss some of their properties.

Introduction

Fredo the Frog hops around between the three lily pads A, B, and C. If he's on lily pad A and jumps, there is a 25% chance that he will land back on lily pad A, a 25% chance that he will land on lily pad B, and a 50% chance that he will land on lily pad C. In Figure 1, we have a transition diagram that reflects the various probabilities from which Fredo will go from one lily pad to another.
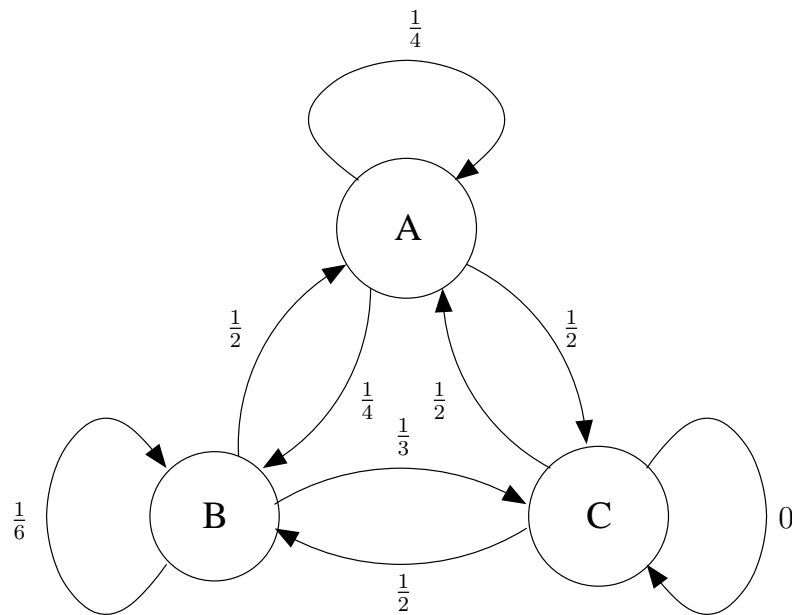

Figure 1: Transition diagram for Fredo the Frog

We can convert our transition diagram into a transition matrix, where the (i, j)-entry of the matrix corresponds to the probability that Fredo jumps

from the jth lily pad to the ith lily pad (where of course A is the first lily pad, B is the second, and so on). In Fredo's case, the transition matrix is

$$A = \begin{pmatrix} 1/4 & 1/2 & 1/2 \\ 1/4 & 1/6 & 1/2 \\ 1/2 & 1/3 & 0 \end{pmatrix}.$$

Note that all of the columns add up to one. This is important.

If Fredo is on lily pad A, where will he be after two jumps? By multiplying the matrix A by itself, we have (approximately)

$$A^2 = \begin{pmatrix} 0.4375 & 0.3750 & 0.3750 \\ 0.3542 & 0.3194 & 0.2083 \\ 0.2083 & 0.3056 & 0.4167 \end{pmatrix}.$$

From this, we infer that there is a 43.75% chance he will still be on lily pad A after two jumps. Note that he might have jumped from A to A to A, denoted $A \rightarrow A \rightarrow A$, or he could have jumped to one of the other lily pads and then back again, that is, either $A \rightarrow B \rightarrow A$ or $A \rightarrow C \rightarrow A$. In addition, there is a 35.42% chance he will be on lily pad B and a 20.83% chance that he will be on lily pad C. Using Matlab, we can type in our transition matrix and see where Fredo will be after 5, 10, 20 or 100 jumps.

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>> A = [1/4 1/2 1/2;1/4 1/6 1/2;1/2 1/3 0]

>> A^5

>> A^10

>> A^20

>> A^100
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Note that in the limit that the number of jumps goes to infinity, we get

$$A^\infty = \begin{pmatrix} 0.4 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

This means that after several jumps, the probability that we will find Fredo on a given lily pad will have nothing to do with where he started initially.

Markov Chains

We can generalize this notion beyond that of frogs and lily pads. Let the state of our system be represented by a probability vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where each entry represents the probability of being in that state. Note that each entry is nonnegative and the sum of all the entries adds up to one. For example, in the case of Fredo, if we know initially that he is on lily pad A, then we have the state vector

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

because we know for certainty (100%) that Fredo is in the first state. After one jump, we have

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}.$$

After two jumps, we have

$$\mathbf{x}_2 = A\mathbf{x}_1 = A^2\mathbf{x}_0 = \begin{bmatrix} 0.4375 \\ 0.3542 \\ 0.2083 \end{bmatrix}.$$

After a large number of jumps ($n \gg 1$), we have

$$\mathbf{x}_n = A\mathbf{x}_{n-1} = \cdots = A^n\mathbf{x}_0 \approx \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}.$$

Since all of the columns are the same for A^∞, then for any initial probability vector $\mathbf{x}_0$, we get the same limiting output, or in other words, all initial vectors converge to the same point, call it $\mathbf{x}_\infty$. Moreover, we have that

$$\mathbf{x}_\infty = A\mathbf{x}_\infty.$$

This is called a stable fixed point.

Example

Consider the Markov chain given by

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.3 \end{pmatrix}.$$

We show that it has a stable fixed point by checking that A to a high exponent, say A^{100}, converges to a fixed matrix where all of the columns are the same.

Assignment

Problem 1. *Suppose a basketball player's success at shooting free throws can be described with the following Markov chain*

$$A = \begin{pmatrix} .75 & .50 \\ .25 & .50 \end{pmatrix},$$

where the first state corresponds to success and the second state to failure.

(a). *If the player makes his first free throw, what is the probability that he also makes his third one?*

(b). *What is the player's average free throw percentage?*

Problem 2. *Consider the Markov process given by the transition diagram in Figure 2 below:*

(a). *Find the transition matrix.*

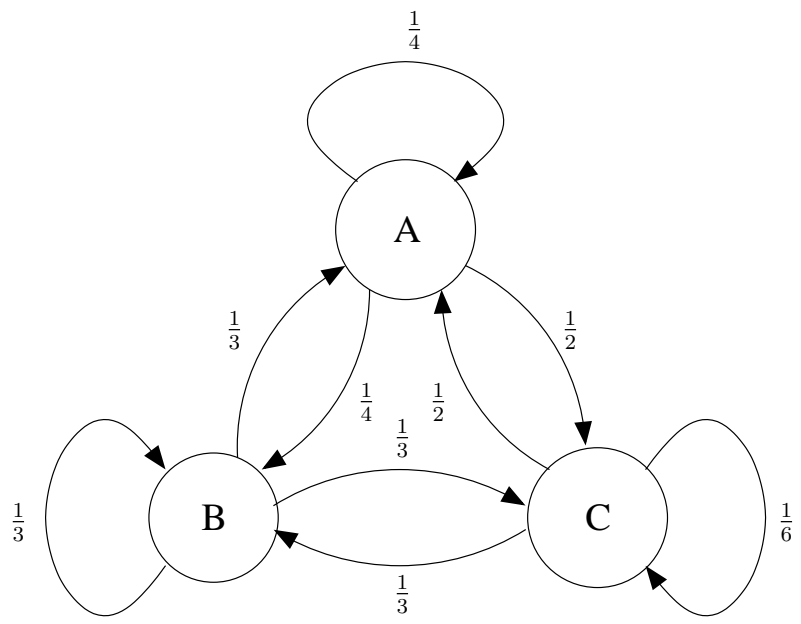(b). *If the Markov process is in state A, initially, find the probability that it is in state B after 2 periods.*

(c). *Find the stable fixed point if it exists.*

Figure 2: Transition diagram