Math 343 Lab 5: Integration by Parts

Objective

In this lab we explore linear algebraic alternatives to integration by parts.

The space of continuous functions

We begin by considering the space $C[a,b]$ of continuous functions defined on the set $[a,b]$. Note that this is a vector space. For example, if $f(x)$ and $g(x)$ are continuous, then so are $f(x)+g(x)$ and $af(x)$, where $a \in \mathbb{R}$. For the same reasons, the space $C^1[a,b]$ of continuously differentiable functions defined on the set $[a,b]$ is also a vector space. Note that

$$\frac{d}{dx} : C^1[a,b] \longrightarrow C[a.b]$$

is a linear transformation since

$$\frac{d}{dx}(af(x)+bg(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x).$$

We remark that both $C[a,b]$ and $C^1[a,b]$ are infinite dimensional vector spaces (sometimes called function spaces), and thus we cannot represent the linear transformation with a matrix representation.

Subspaces of continuous functions

Consider the subspace W of $C^1[a,b]$ spanned by the basis $B = \{e^x, xe^x, x^2e^x\}$. Note that

$$\begin{aligned} \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}xe^x &= e^x + xe^x \\ \frac{d}{dx}x^2e^x &= 2xe^x + x^2e^x, \end{aligned}$$

or in other words, the derivatives of the basis functions B of W are in W, that is,

$$\frac{d}{dx} : W \longrightarrow W.$$

Since this is a linear transformation from one finite dimensional vector space to another, it has a matrix representation. Since B is a basis for W, a linear combination $f(x) = ae^x + bxe^x + cx^2e^x$ can simply be represented as a column vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Sometimes we write this as

$$\begin{bmatrix} e^x & xe^x & x^2e^x \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

just to keep the basis B visible. The derivative of $f(x)$ takes the form

$$\begin{aligned} \frac{d}{dx}\left(ae^x + bxe^x + cx^2e^x\right) &= ae^x + b(e^x + xe^x) + c(2xe^x + x^2e^x) \\ &= (a+b)e^x + (b+2c)xe^x + cx^2e^x \\ &= \begin{bmatrix} e^x & xe^x & x^2e^x \end{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \end{aligned}$$

Hence the matrix representation of the derivative on W is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

or in other words

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Example

Consider the function $g(x) = 5e^x - 3xe^x + 2x^2e^x$. Note that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Hence, $g'(x) = 2e^x + xe^x + 2x^2e^x$.

Antiderivatives

Recall that the antiderivative is also a linear transformation. Note that the antiderivative of the derivative is the original function[1], and hence the matrix representation of the antiderivative is the inverse of the matrix representation of the derivative. For example, if we wanted to compute

$$\int ae^x + bxe^x + cx^2e^x \, dx,$$

we could simply invert the matrix representation of the derivative

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives us the transformation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Example

Consider the function $h(x) = 2e^x + xe^x + 2x^2e^x$. Note that

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}.$$

[1] This is true up to an additive constant.

Hence the antiderivative of $h(x)$ is $5e^x - 3xe^x + 2x^2e^x + C$. We see from the previous example that we get the right answer. What's more is we didn't need to use integration by parts to get the answer!

Assignment

Problem 1. *Apply the above concepts by writing a Matlab function called* `myint` *that takes as input the vector* $[a_0, a_1, \cdots, a_n]$ *and computes the antiderivative of*

$$f(x) = a_0e^x + a_1xe^x + a_2x^2e^x + \cdots + a_nx^ne^x.$$

Hint: Find the matrix representation of the derivative, then take the inverse.

Problem 2. *Let* W *be the subspace of* $C^1[a, b]$ *spanned by the basis*

$$B = \{\cos(\alpha x)e^{\beta x}, \sin(\alpha x)e^{\beta x}\}.$$

(a). *Find the matrix representation* D *of the derivative in the basis* B.

(b). *Find the inverse of* D.

(c). *Use your answer above to find the anti-derivative of*

$$f(x) = 14\sin(\alpha x)e^{\beta x}.$$

We remark that the traditional way to do this problem requires a special trick where one integrates by parts twice. Doing it this way, one can avoid all that.