Boundary-Value Problems

Finite Differences

Consider the boundary-value problem

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(t),$$

where $y(a) = \alpha$ and $y(b) = \beta$.

We discretize the domain into $n+1$ evenly-spaced points $\{x_j\}_{j=0}^{n}$, where $x_0 = a$ and $x_n = b$. Let $y_j = y(x_j)$, $p_j = p(x_j)$, $q_j = q(x_j)$, $r_j = r(x_j)$, and $f_j = f(x_j)$, $j = 0, \ldots, n$. Using finite differences, we approximate the the derivatives

$$\begin{aligned} y_j' &= \frac{y_{j+1} - y_{j-1}}{2h} \\ y_j'' &= \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}, \end{aligned}$$

where $j = 1, \ldots, n-1$. Plugging it all in, we get

Galerkin's Method

Choose the linear independent set of functions $\{\phi_j\}_{j=1}^{n}$. We seek the best linear combination

$$y \approx \sum_{j=1}^{n} c_j \phi_j.$$

Define the residual $r = L(y) - f$ to be perpendicular to the basis $\mathrm{span}\{\phi_j\}$. Hence for all i, we have

$$\begin{aligned} 0 &= \langle \phi_i, r \rangle \\ &= \langle \phi_i, L(y) - f \rangle \\ &= \langle \phi_i, \sum_{j=1}^{n} c_j \phi_j - f \rangle \\ &= \sum_{j=1}^{n} c_j \langle \phi_i, \phi_j \rangle - \langle \phi_i, f \rangle. \end{aligned}$$

Hence we have the following n equations and n unknowns

$$\sum_{j=1}^{n} c_j \langle \phi_i, \phi_j \rangle = \langle \phi_i, f \rangle.$$

This yields the linear system

$$\begin{pmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_1, \phi_2 \rangle & \cdots & \langle \phi_1, \phi_n \rangle \\ \langle \phi_2, \phi_1 \rangle & \langle \phi_2, \phi_2 \rangle & \cdots & \langle \phi_2, \phi_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_n, \phi_1 \rangle & \langle \phi_n, \phi_2 \rangle & \cdots & \langle \phi_n, \phi_n \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \langle \phi_1, f \rangle \\ \langle \phi_2, f \rangle \\ \vdots \\ \langle \phi_n, f \rangle \end{pmatrix}$$

Example

Consider the boundary-value problem

$$y'' + y = -1, \quad 0 \le x \le 1,$$

where $y(0) = y(1) = 0$. The exact solution is given by

$$y(x) = -1 + \cos x + \frac{1 - \cos 1}{\sin 1} \sin x.$$

Divide the interval $[0, 1]$ into n subintervals, each of length $\Delta x = \frac{1}{n}$. Define

$$\phi_j = \begin{cases} 1 - n|x - j/n| & x \in [\frac{j-1}{n}, \frac{j+1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Set

$$y = \sum_{j=1}^{n-1} c_j \phi_j.$$

Then

$$\sum_{j=1}^{n-1} c_j \langle \phi_i, \phi_j'' + \phi_j \rangle + \langle \phi_i, 1 \rangle = 0$$

$$\sum_{j=1}^{n-1} c_j \left(-\langle \phi_i', \phi_j' \rangle + \langle \phi_i, \phi_j \rangle \right) + \langle \phi_i, 1 \rangle = 0$$

Hence, we have the following $n-1$ equations and $n-1$ unknowns

$$\sum_{j=1}^{n-1} c_j \left(\langle \phi_i', \phi_j' \rangle - \langle \phi_i, \phi_j \rangle\right) = \langle \phi_i, 1 \rangle.$$

We can compute the following inner products

$$\langle \phi_i, \phi_j \rangle = \begin{cases} \frac{2}{3n} & i = j \\ \frac{1}{6n} & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\langle \phi_i', \phi_j' \rangle = \begin{cases} 2n & i = j \\ -n & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle \phi_i, 1 \rangle = \frac{1}{n}$$