Definition 1. *A function $f(x)$ is said to interpolate the set of points $\{(x_j, y_j)\}_{j=0}^n$ if $f(x_j) = y_j$, for all $j = 0, \ldots, n$.*

1. Polynomial Interpolation

Theorem 2. *Given the set $\{(x_j, y_j)\}_{j=0}^n$, where each x_j is distinct, there exists a unique interpolating polynomial $p(x)$ of degree at most n.*

Proof. Define the family of n-degree polynomials

$$L_{n,j}(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}. \tag{1}$$

Note that $L_{n,j}(x_k) = \delta_{jk}$. Hence, the linear combination

$$p(x) = \sum_{j=0}^{n} y_j L_{n,j}(x) \tag{2}$$

is an interpolating polynomial for the given set. To prove uniqueness, suppose there exists another interpolating polynomial $q(x)$ of degree at most n. Then the polynomial $p(x) - q(x)$ is of degree at most n, yet it has $n + 1$ roots at $\{x_i\}_{i=0}^n$. This is a contradiction. □

The functions defined in (**??**) are called Lagrange basis functions, and the process outlined in the proof is called Lagrange interpolation. It is an important method for the theoretical development of polynomial interpolation theory, however, is is generally not a good method for computation. Another problem with lagrange interpolation is that it is not easy to add additional points after computing the interpolating polynomial.

Another way to interpolate is to solve the linear system

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The $(n + 1) \times (n + 1)$ matrix is called the Vandermonde matrix and is provably nonsingular, however, it is also often ill-conditioned. Thus, this approach is likewise not suitable for computation.

The following iterative method of Newton interpolation is computationally practical and it allows for additional points to be added after the fact.

Theorem 3. *Let $p_{n-1}(x)$ be the unique polynomial of degree at most $n-1$ that interpolates the set $\{(x_j, y_j)\}_{i=0}^{n-1}$ and let $p_n(x)$ be the unique polynomial of degree at most n that interpolates the set $\{(x_j, y_j)\}_{i=0}^{n}$. Then $p_n(x)$ is given by*

$$p_n(x) = p_{n-1}(x) + a_n w_n(x), \tag{3}$$

where

$$w_n(x) := \prod_{j=0}^{n-1} (x - x_j) \tag{4}$$

and

$$a_n = \frac{y_n - p_{n-1}(x_n)}{w_n(x_n)}. \tag{5}$$

Proof. Note that $p_n(x) - p_{n-1}(x)$ equals some scalar multiple of $w_n(x)$. Indeed it is a polynomial of degree at most n with all the same zeros as $w_n(x)$. Evaluating (**??**) at the point x_n yields (**??**). □

Corollary 4. *For $\{a_j\}_{j=0}^{n}$ and $w_j(x)$ defined iteratively, as above, we have*

$$p_n(x) = \sum_{j=0}^{n} a_j w_j(x), \tag{6}$$

where $w_0(x) \equiv 1$ and $a_0 = y_0$.

The Corollary introduces an algorithm called divided differences that will allow for fast computation. Let

$$f[x_0, x_1, \ldots, x_k] = a_k$$

for each $k = 0, \ldots, n$. We have the following:

Proposition 5.

$$f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0} \tag{7}$$

Proof. For each k, let $p_k(x)$ denote the unique polynomial of degree at most k that interpolates $f(x)$ at $\{x_j\}_{j=0}^{k}$. Now let $P(x)$ be the unique polynomial of degree at most $k-1$ that interpolates $f(x)$ at $\{x_j\}_{j=1}^{k}$. Then we have

$$p_k(x) = P(x) + \frac{x - x_k}{x_k - x_0}(P(x) - p_{k-1}(x)).$$

By matching the k^{th}-order terms, we have (**??**). □

2. Cubic Spline

Definition 6. *A spline of degree k with knots $\{t_0, \ldots, t_n\}$ is a function $s(x)$ that satisfies the following two properties:*

(i). *On the interval $[t_j, t_{j+1})$, $s(x)$ is a polynomial of degree at most k, that is, $s(x)$ is a polynomial on every subinterval defined by the knots.*

(ii). *The function $s(x)$ has a continuous $(k-1)^{th}$ derivative at each knot.*

For example, consider the cubic spline $s(x)$, which takes the form

$$s(x) = \begin{cases} s_0(x) & x \in [t_0, t_1) \\ s_1(x) & x \in [t_1, t_2) \\ \vdots & \\ s_{n-1}(x) & x \in [t_{n-1}, t_n], \end{cases} \tag{8}$$

where

$$s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3.$$

and

(i). $s_j(t_{j+1}) = s_{j+1}(t_{j+1}) = a_{j+1}$
(ii). $s'_j(t_{j+1}) = s'_{j+1}(t_{j+1}) = b_{j+1}$
(iii). $s''_j(t_{j+1}) = s''_{j+1}(t_{j+1}) = 2c_{j+1}$

Let $h_j = t_{j+1} - t_j$, and assume that the knots are interpolation points $\{(x_j, y_j)\}_{j=0}^n$. We can solve everything in terms of y_j and x_j. Note that (i)–(iii) reduce to:

$$\begin{aligned} y_{j+1} &= y_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \\ b_{j+1} &= b_j + 2c_j h_j + 3d_j h_j^2 \\ 2c_{j+1} &= 2c_j + 6d_j h_j \end{aligned}$$

Solving for d_j we have

$$d_j = \frac{c_{j+1} - c_j}{3h_j}.$$

Then we can reduce to two equations

$$\begin{aligned} b_j h_j + \frac{c_{j+1} + 2c_j}{3} h_j^2 &= y_{j+1} - y_j \\ (c_{j+1} + c_j) h_j &= b_{j+1} - b_j. \end{aligned}$$

Solving for b_j we have

$$b_j = \frac{y_{j+1} - y_j}{h_j} - \frac{c_{j+1} + 2c_j}{3} h_j,$$

which we can use to reduce to one equation

$$h_{j+1}c_{j+2} + 2(h_{j+1} + h_j)c_{j+1} + h_j c_j = \Delta_{j+1},$$

where $j = 0, 1, \ldots, n-2$ and

$$\Delta_{j+1} = 3\left(\frac{y_{j+2} - y_{j+1}}{h_{j+1}} - \frac{y_{j+1} - y_j}{h_j}\right).$$

Write as a linear system

$$\begin{bmatrix} * & * & * & \cdots & * & * \\ h_0 & 2(h_0+h_1) & h_1 & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & \\ & & h_2 & 2(h_2+h_3) & h_3 & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ * & * & \cdots & * & * & * \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \vdots \\ \Delta_{n-1} \\ \Delta_n \end{bmatrix},$$

where the top and bottom rows are yet to be determined and depend on the boundary conditions at t_0 and t_n.

The most common boundary conditions are:

Natural: Assume $s''(x_0) = s''(x_n) = 0$ or $c_0 = c_n = 0$. Hence set $\Delta_0 = \Delta_n = 0$ and the top row to be all zeros except the first entry, which is a one, and the last row to be all zeros, except the last entry, which is also a one.

Clamped: $s'(x_0) = \alpha$ and $s'(x_n) = \beta$. Hence set

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(y_1 - y_0) - 3\alpha$$

and

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3\beta - \frac{3}{h_{n-1}}(y_n - y_{n-1}).$$

Not a Knot: Assume $s_0'''(x_1) = s_1'''(x_1)$ or equivalently $d_0 = d_1$ and $s_{n-2}'''(x_{n-1}) = s_{n-1}'''(x_{n-1})$ or equivalently $d_{n-2} = d_{n-1}$. Hence,

$$h_1c_0 - (h_0 + h_1)c_1 + h_0c_2 = 0$$

and

$$h_{n-1}c_{n-2} - (h_{n-2} + h_{n-1})c_{n-1} + h_{n-2}c_n = 0.$$

3. Bernstein Polynomials

In 1912, Bernstein proved that the sequence of polynomials

$$f_m(t) = \sum_{i=0}^{m} f(i/n)\binom{m}{i} t^i(1-t)^{m-i}$$

converges uniformly to the function $f(x) \in C[0,1]$, thus providing a nice proof of the Weierstrass Approximation Theorem.

Definition 7. *The Bernstein polynomials take the form*

$$B_i^m(t) = \binom{m}{i} t^i(1-t)^{m-i}.$$

Theorem 8.

$$B_j^m(t) = \sum_{i=j}^{m} (-1)^{i-j}\binom{m}{i}\binom{i}{j} t^i. \tag{9}$$

Thus, we have the matrix representation

$$\begin{bmatrix} B_0^m & B_1^m & \cdots & B_m^m \end{bmatrix} = \begin{bmatrix} 1 & t & \cdots & t^m \end{bmatrix} \begin{pmatrix} \mu_{00} & 0 & \cdots & 0 \\ \mu_{10} & \mu_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m0} & \mu_{m1} & \cdots & \mu_{mm} \end{pmatrix}$$

where

$$\mu_{ij} = (-1)^{i-j}\binom{m}{i}\binom{i}{j}.$$

Proof.

$$\begin{aligned} B_j^m(t) &= \binom{m}{j} t^j(1-t)^{m-j} \\ &= \binom{m}{j} t^j \sum_{i=0}^{m-j} \binom{m-j}{i} (-1)^i t^i \\ &= \sum_{i=0}^{m-j} (-1)^i \binom{m}{j}\binom{m-j}{i} t^{i+j} \\ &= \sum_{i=j}^{m} (-1)^{i-j} \binom{m}{j}\binom{m-j}{i-j} t^i \\ &= \sum_{i=j}^{m} (-1)^{i-j} \binom{m}{i}\binom{i}{j} t^i \end{aligned}$$

$\square$

Example 9. *The following equality provides the transition from the power basis into the Bernstein basis for* $\mathbb{R}_3[x]$.

$$\begin{bmatrix} B_0^3 & B_1^3 & B_2^3 & B_3^3 \end{bmatrix} = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

Theorem 10. $\{B_i^m(t)\}_{i=0}^m$ *is a basis for* $\mathbb{R}_m[x]$.

Lemma 11. *The following hold:*

(i). $B_i^m(t) \geq 0$ *on the interval* $[0, 1]$.
(ii). *The set* $\{B_i^m(t)\}_{i=0}^m$ *forms a partition of unity, that is, satisfies*

$$\sum_{i=0}^{m} B_i^m(t) = 1.$$

Theorem 12. *Any set* $\{B_i^m(t)\}_{i=0}^m$ *that is a partition of unity has the property that any linear combination* $\sum_{i=0}^m P_i B_i^m(t)$ *is in the convex hull of* $\{P_i\}_{i=0}^m$, *denoted* $co(\{P_i\}_{i=0}^m)$.

Theorem 13. *The Bernstein polynomials can be recursively defined as follows:*

$$\begin{aligned} B_k^n(t) &= (1-t)B_k^{n-1}(t) + tB_{k-1}^{n-1}(t) \\ B_k^n(t) &= 0 \quad (\text{if } k < 0 \text{ or } k > n) \\ B_0^0(t) &= 1. \end{aligned}$$

Proof.

$$\begin{aligned} (1-t)B_k^{n-1} + tB_{k-1}^{n-1} &= \binom{n-1}{k} t^k (1-t)^{n-k} + \binom{n-1}{k-1} t^k (1-t)^{n-k} \\ &= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] t^k (1-t)^{n-k} \\ &= \left[\frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!k}{k!(n-k)!}\right] t^k (1-t)^{n-k} \\ &= \frac{n!}{k!(n-k)!} t^k (1-t)^{n-k}. \end{aligned}$$

□

4. Bezier Curves

Definition 14. *Given control points* $\{P_i\}_{i=0}^m$*, we define the corresponding Bezier curve*

$$P(t) = \sum_{i=0}^{m} P_i B_i^m(t).$$

Proposition 15. *The Bezier curve* $R(t)$ *is uniquely determined by its control points.*

Proposition 16. $R([0,1]) \subset co(\{P_i\}_{i=0}^m)$

Lemma 17.

$$\frac{d^k}{dt^k} B_i^m(t) = (-1)^k \frac{m!}{(m-k)!} \delta_-^k B_i^{m-k}(t).$$

Proof. This follows straightforwardly from the case $k = 1$. Note that

$$\begin{aligned}
\frac{d}{dt} B_i^m(t) &= \frac{d}{dt}\left[\binom{m}{i} t^i (1-t)^{m-i}\right] \\
&= m\binom{m-1}{i-1} t^{i-1}(1-t)^{m-i} - m\binom{m-1}{i} t^i (1-t)^{m-i-1} \\
&= m[B_{i-1}^{m-1}(t) - B_i^{m-1}(t)] \\
&= -m\delta_- B_i^{m-1}(t).
\end{aligned}$$

□

Theorem 18.

$$\frac{d^k}{dt^k} P(t) = \frac{m!}{(m-k)!} \sum_{i=0}^{m-k} (\delta_+^k P_i) B_i^{m-k}(t).$$

Lemma 19. *A degree* m *Bernstein polynomial can be expressed as a linear combination of degree* $m+1$ *Bernstein polynomials. Indeed, we have the following recursion relation*

$$B_i^m(t) = \frac{i+1}{m+1} B_{i+1}^{m+1}(t) + \frac{m+1-i}{m+1} B_i^{m+1}(t). \tag{10}$$

Proof. Note that

$$tB_i^m(t) = \binom{m}{i} t^{i+1}(1-t)^{m-i} = \frac{\binom{m}{i}}{\binom{m+1}{i+1}} B_{i+1}^{m+1}(t) = \frac{i+1}{m+1} B_{i+1}^{m+1}(t)$$

and

$$(1-t)B_i^m(t) = \binom{m}{i} t^i (1-t)^{m+1-i} = \frac{\binom{m}{i}}{\binom{m+1}{i}} B_i^{m+1}(t) = \frac{m+1-i}{m+1} B_i^{m+1}(t).$$

Thus

$$B_i^m(t) = tB_i^m(t) + (1-t)B_i^m(t) = \frac{i+1}{m+1}B_{i+1}^{m+1}(t) + \frac{m+1-i}{m+1}B_i^{m+1}(t).$$

$\square$

Theorem 20. *The Bezier curve $P(t) = \sum_{i=0}^{m} P_i B_i^m(t)$ of degree m can be expressed as a Bezier curve $P(t) = \sum_{i=0}^{m+1} Q_i B_i^{m+1}(t)$ of degree $m+1$ via the relationship*

$$\begin{cases} Q_0 = P_0 \\ Q_i = \dfrac{i}{m+1}P_{i-1} + \left(1 - \dfrac{i}{m+1}\right)P_i, \quad i = 1, \dots, m \\ Q_{m+1} = P_m. \end{cases}$$

Bezier curves are symmetric, which means that the curve is the same if you reverse the order of the control points. Bezier curves are affine invariant. Also the tangent vectors at the end points are given by the segments $\overline{P_0P_1}$ and $\overline{P_mP_{m-1}}$, respectively.

5. Composite Bezier Curves

Let $\gamma_1(t) = \sum_{i=0}^{m} P_i B_i^m(t)$ and $\gamma_2(t) = \sum_{j=0}^{n} Q_j B_j^n(t)$ be two Bezier curves. The composite curve

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0,1] \\ \gamma_2(1-t) & t \in [1,2] \end{cases}$$

is called geometrically continuous, denoted G^0, if $\gamma_1(1) = \gamma_2(0)$. This connection point at $t = 1$ is called a knot or a joint. In terms of Bezier control points, this means that $P_m = Q_0$. Note that a G^0 composite curve can have corners or cusps. Indeed all that is required is continuity at $t = 1$.

By adding additional conditions on the control points, we can improve the degree of smoothness of the resulting composite curve $\gamma(t)$. Specifically, we say that $\gamma(t)$ is geometrically continuously differentiable, denoted G^1, if it is G^0 and the tangent vectors of γ_1 and γ_2 at the knot are in the same direction–specifically

$$\gamma_1(1) = \gamma_2(0) \quad \text{and} \quad \frac{\gamma_1'(1)}{\|\gamma_1'(1)\|} = \frac{\gamma_2'(0)}{\|\gamma_2'(0)\|}.$$

In terms of the control points, this means that both $P_m = Q_0$ and $P_m - P_{m-1} = c_1(Q_1 - Q_0)$, where $c_1 = \|P_m - P_{m-1}\|/\|Q_1 - Q_0\|$. This means that the control points P_{m-1},$P_m = Q_0$, and Q_1 are colinear.

Geometric continuity is a weaker condition than parametric continuity. For a curve to be parametrically C^1, we would require that both the direction and magnitudes of the tangent vectors at the knot be the same.

It is sometimes necessary for a composite curve to have continuous curvature. Such curves are said to be geometrically twice differentiable, denoted G^2, if they are G^1 and $\gamma_1''(1) = c_2\gamma_2'(0) + c_1^2\gamma_2''(0)$. In terms of the control points, this last condition reduces to

$$P_m - 2P_{m-1} + P_{m-2} = \frac{c_2}{m-1}(Q_1 - Q_0) + c_1^2(Q_0 - 2Q_1 + Q_2).$$

6. Difference Equations

Definition 21. *Given the sequence $\{y_n\}_{n=0}^m$, we define the difference operators*

$$\begin{aligned} \delta_+ y_n &= y_{n+1} - y_n \\ \delta_- y_n &= y_n - y_{n-1} \end{aligned}$$

and the shift operators

$$\begin{aligned} E_+ y_n &= y_{n+1} \\ E_- y_n &= y_{n-1}. \end{aligned}$$

Remark. *Note that $E_\pm = I \pm \delta_\pm$ and $\delta_\pm = \pm(E - I)$.*

Lemma 22.

(i). $\delta_\pm^m = \pm\sum_{k=0}^m \binom{m}{k} E_\pm^k$.
(ii). $E_\pm^m = \sum_{k=0}^m \binom{m}{k}(-1)^k \delta_\pm^k$.

Lemma 23.

(i). $\delta_\pm^k(\delta_\pm^m y_n) = \delta_\pm^{m+k} y_n$
(ii). $\delta_\pm(y_n + z_n) = \delta_\pm y_n + \delta_\pm z_n$
(iii). $\delta_\pm(a y_n) = a\delta_\pm y_n$
(iv). $\delta_\pm(y_n z_n) = (\delta_\pm y_n)(E_\pm z_n) + y_n(\delta_\pm z_n)$.
(v).

$$\delta_\pm\left(\frac{y_n}{z_n}\right) = \frac{(\delta_\pm y_n)z_n - y_n(\delta_\pm z_n)}{z_n E_\pm z_n}.$$

Lemma 24 (Fundumental Theorem)**.**

$$\sum_{i=m}^n \delta_\pm(y_i z_i) = \begin{cases} y_{n+1}z_{n+1} - y_m z_m \\ y_n z_n - y_{m-1}z_{m-1} \end{cases}$$

Theorem 25 (Summation by Parts)**.**

$$\sum_{i=m}^{n} y_i(\delta_+ z_i) = y_{n+1}z_{n+1} - y_m z_m - \sum_{i=m}^{n}(\delta_+ z_i)E_+ z_i. \tag{11}$$

$$\sum_{i=m}^{n} y_i(\delta_- z_i) = y_n z_n - y_{m-1}z_{m-1} - \sum_{i=m}^{n}(\delta_- z_i)E_- z_i. \tag{12}$$

Proof. By summing the product rule and applying the Fundumental Theorem, we have

$$\sum_{i=m}^{n} y_i(\delta_\pm z_i) = \sum_{i=m}^{n} \delta_\pm(y_i z_i) - \sum_{i=m}^{n}(\delta_\pm y_i)(E_\pm z_i)$$

□