Math 634 Hwk 1

Problem 1. *Denote the set of k-times continuously differentiable real-valued functions on the interval $J \subset \mathbb{R}$ as*

$$C^k(J;\mathbb{R}) = \{f \in C(J;\mathbb{R}) \mid f^{(k)} \in C(J;\mathbb{R})\},$$

and the Sobolev norm as

$$\|f\|_{\infty,k} = \sum_{i=0}^{k} \sup_{t\in J} |f^{(i)}(t)|.$$

Show that $(C^2(J;\mathbb{R}), \|\cdot\|_{\infty,1})$ is not complete.

Problem 2. *Assume that $(X, \|\cdot\|_X)$ is a Banach space. For an interval J in $\mathbb{R}$, let Lip(J,X) denote the space of all Lipschitz continuous functions in $C(J;X)$, that is, given $f \in$ Lip(J,X), there exists $K > 0$ such that*

$$\|f(y) - f(x)\|_X \le K|y - x| \quad \forall x, y \in X.$$

Show that Lip(J,X) is a Banach space with norm

$$\|f\|_{\mathrm{lip}} = \|f\|_\infty + \sup_{\substack{x,y\in J\\ x\neq y}} \frac{\|f(y) - f(x)\|_X}{|y - x|}$$

Problem 3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $f : X \longrightarrow Y$ be a linear function. Prove that if f is continuous at $x_0 \in X$ then f is uniformly continuous on X.*

Problem 4. *Let X be a finite dimensional vector space over $\mathbb{E}$. If $\|\cdot\|$ is a norm on X, show that $(X, \|\cdot\|)$ is topologically equivalent to the Euclidian space $(X, \|\cdot\|_2)$, where in the standard basis $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$, the Euclidian norm has the form*

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}.$$