Math 634 Hwk 2

Problem 1. *Show that the space* C^ℓ *of all continuous real-valued functions on the non-negative reals* $\mathbb{R}_+$ *and with finite limit as* $t \to \infty$*, is a closed subspace of* $(C(\mathbb{R}_+, \mathbb{R}), \| \cdot \|_\infty)$ *(and is thus a Banach space)*

Problem 2. *Let* $D \subset X \oplus \mathbb{R}$ *be open and* f, g *continuous maps of* D *into* X*. Assume that* f *and* g *are both Lipschitz continuous in the first variable with constant* $K > 0$*. Prove that if* $x(t)$ *and* $y(t)$ *are, respectively, solutions of the initial-value problems*

$$\begin{aligned} x'(t) &= f(x(t), t) \\ x(t_0) &= x_0 \end{aligned}$$

and

$$\begin{aligned} y'(t) &= g(y(t), t) \\ y(t_0) &= y_0, \end{aligned}$$

then

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{K|t-t_0|} + \frac{M}{K}\left(e^{K|t-t_0|} - 1\right),$$

where

$$M = \sup_{(x,t)\in D} \|f(x,t) - g(x,t)\|.$$

Problem 3. *Let* $a, b, u : [0, T] \longrightarrow \mathbb{R}_+$ *be continuous. Suppose that* $u(t)$ *satisfies*

$$u(t) \leq a(t) + \int_0^t b(s)u(s)ds.$$

Show that

$$u(t) \leq a(t) + \int_0^t a(s)b(s)\exp\left(\int_s^t b(r)dr\right)ds.$$