Math 290 Homework 24

(1) Let a, b, n be integers with $n > 1$ and let $d = \gcd(a, n)$. If the equation $\overline{a}x = \overline{b}$ has a solution $x = \overline{r}$ in $\mathbb{Z}_n$, prove that $d|b$.
(2) In the subset $S = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}$ of $\mathbb{Z}_{10}$, is there an element e of S that satisfies $x \cdot e = x$ for all $x \in S$?
(3) Prove or disprove: If $a^2 \equiv b^2 \pmod{n}$, then $a \equiv b \pmod{n}$ or $a \equiv -b \pmod{n}$.
(4) Repeat the previous problem when n is prime.
(5) If $p \geq 5$ is prime, prove that $p^2 + 2$ is composite. (Hint: Which equivalence classes in $\mathbb{Z}_6$ could p be in?)
(6) Show that $10^n \equiv 1 \pmod{9}$ for every positive integer n.
(7) Prove that every positive integer is congruent to the sum of its digits $\pmod{9}$.
(8) For $p = 2, 3, 5$ compute $(a + b)^p$ in $\mathbb{Z}_p$. State and prove a general conjecture for all primes p that includes these results.