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Math 290
Fall 2018
Practice Final Exam

Note that the first 10 questions are true–false. Mark A for true, B for false. Questions 11 through 20 are multiple choice. Mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly. Questions 1 to 20 are worth 2.5 points each, and questions 21 to 25 are worth 10 points each.

1. For any three sets A, B, C, $A - (B \cup C) = (A - B) \cup (A - C)$.
2. The statement $P \Rightarrow Q$ is logically equivalent to the statement $(\neg P) \vee Q$.
3. If $x, y \in \mathbb{R}$ then $|x + y| \geq |x| + |y|$.
4. The negation of the statement "There exist irrational numbers a, b such that a^b is rational." is the statement "For all irrational numbers a, b, it is true that a^b is irrational."
5. Let R be an equivalence relation on a nonempty set A, and denote the equivalence class of $a \in A$ by $[a]$. For $a, b \in A$, we have aRb if and only if $[a] = [b]$.
6. Let A be a nonempty set, and suppose $f : A \to A$ is a surjective function. Then f is injective.
7. Every nonempty subset of the positive integers has a least element.
8. If S is an uncountable set and $T \subseteq S$ is uncountable, then $|T| = |S|$.
9. The GCD of 1357 and 1633 is between 50 and 100.
10. Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $a|bc$ and $a \nmid b$, then $a|c$.

Multiple Choice Questions

11. Which of the following definitions is incorrect?

    (a) An equivalence relation is a relation on a nonempty set which is reflexive, symmetric and transitive.
    (b) A function $f : A \to B$ is injective if for every $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$ implies that $a_1 \neq a_2$.
    (c) Two sets A and B have the same cardinality if there is a bijection $f : A \to B$.
    (d) A partition of a set A is a collection of nonempty pairwise disjoint subsets of A whose union is A.
    (e) Two integers a and b are relatively prime if their greatest common divisor is 1.
    (f) None of the above. All of these definitions are correct.

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12. Which of the following would be the best method for proving the statement

$$P \Rightarrow (Q \vee R).$$

    (a) Assume P and Q and prove that R is true.
    (b) Assume P and $\neg Q$ and prove that R is true.
    (c) Assume P and Q and prove that R is false.
    (d) Assume $\neg Q$ or $\neg R$ and prove $\neg P$.
    (e) Assume Q and R and prove that P is true.

13. Which of the following is the contrapositive of the statement

If every $x \in S$ is prime, then every $x \in S$ is odd.

(a) If there is an $x \in S$ which is prime, then there is an $x \in S$ that is odd.
(b) If there is an $x \in S$ that is odd, then there is an $x \in S$ that is prime.
(c) If there is an $x \in S$ that is even, then there is an $x \in S$ that is not prime.
(d) If every $x \in S$ is odd, then every $x \in S$ is prime.
(e) If every x in S is not prime, then every $x \in S$ is odd.
(f) If every $x \in S$ is odd, then there is an $x \in S$ that is prime.
(g) If every $x \notin S$ is even, then every $x \notin S$ is not prime.

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14. Which of the following statements is true.

(a) Let $x \in \mathbb{Z}$. If $4x + 7$ is odd then x is even.
(b) There exists a real number x such that $x^2 < x < x^3$.
(c) If $x \in \mathbb{Z}$ is odd, then $x^2 + x$ is even.
(d) Every odd integer is a sum of four odd integers.
(e) Let $x, y, z \in \mathbb{Z}$. If $z = x - y$ and z is even, then x and y are odd.
(f) For every two sets A and B, $(A \cup B) - B = A$.

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15. Which of the following is **not** an equivalence relation on the set $\mathbb{Z}$ of integers?

(a) xRy if $7|(x - y)$.
(b) xRy if $6|(x^2 - y^2)$
(c) xRy if $x + y \geq 0$.
(d) xRy if $x + y = 2x$.
(e) xRy if $x^2 - 2xy + y^2 \geq 0$.

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16. Evaluate the proposed proof of the following result. Choose the most complete correct answer.

Theorem: If two functions $f : A \to B$ and $g : B \to C$ are bijective, then $g \circ f$ is bijective.

Proof. Suppose that f and g are both bijective. Note that f and g are each both injective and surjective.

Let $c \in C$. Then, since g is surjective, $c = g(b)$ for some $b \in B$. Since f is surjective, $b = f(a)$ for some $a \in A$. Then $c = g(b) = g(f(a)) = (g \circ f)(a)$. Hence, $g \circ f$ is surjective.

Suppose that $a_1, a_2 \in A$, and $a_1 = a_2$. Then $f(a_1) = f(a_2)$, since f is injective, and $g(f(a_1)) = g(f(a_2))$ since g is injective. Hence $(g \circ f)(a_1) = (g \circ f)(a_2)$, so $g \circ f$ is injective.

Since $g \circ f$ is both injective and surjective, it is bijective. □

(a) The theorem and the proof are correct.
(b) The proof is correct but the theorem is false.
(c) The proof does not successfully prove that $g \circ f$ is surjective.
(d) The proof does not successfully prove that $g \circ f$ is injective.
(e) The proof proves neither that $g \circ f$ is surjective, nor that it is injective.
(f) The proof is irrelevant because injectivity and surjectivity have nothing to do with proving a function to be bijective.

17. Which of the following is the negation of the statement

$$\lim_{x \to a} f(x) = L.$$

a) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

b) There exists an $\epsilon \leq 0$ such that for all $\delta \leq 0$, there is an $x \in \mathbb{R}$ with $0 \geq |x - a| \geq \delta$ such that $|f(x) - L| \geq \epsilon$.

c) There exists an $\epsilon > 0$ such that for all $\delta > 0$, there is an $x \in \mathbb{R}$ with $0 < |x - a| < \delta$ such that $|f(x) - L| \geq \epsilon$.

d) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| \geq \epsilon$.

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18. Which of the following definitions is correct:

(a) A prime number is any integer whose only integer divisors are itself and 1.
(b) A function $f : A \to B$ is injective if every element of A maps to exactly one element of B.
(c) A relation R on a nonempty set A is transitive if for $a, b, c \in A$, aRb and bRc imply cRa.
(d) A set A is countably infinite if it is contained in the set $\mathbb{N}$ of natural numbers.
(e) A sequence $\{a_n\}$ diverges to infinity if it does not converge to any finite limit.
(f) None of the above–all of these definitions are incorrect.

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19. In an ϵ-δ proof that

$$\lim_{x \to 1} 3x + 5 = 8,$$

which of the following is the largest δ that we can associate with a given $\epsilon > 0$.

(a) $\delta = 3$	(b) $\delta = 1/3$	(c) $\delta = \epsilon$	(d) $\delta = 3\epsilon$
(e) $\delta = \epsilon/3$	(f) $\delta = \epsilon/5$	(g) $\delta = \min(\epsilon, 1)$	(h) $\delta = 0$

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20. Let A be a nonempty set, and let R be an equivalence relation on A. Let E be the set of equivalence classes of R on A, with the equivalence class of $a \in A$ denoted by $[a]$. Define $f : A \to E$ by $f(a) = [a]$. Choose the most complete correct answer below.

(a) f is not a well defined function.
(b) f must be an injective function.
(c) f must be a surjective function.
(d) f must be a bijective function.
(e) f may be neither surjective nor injective, but it is a function.

Written Answer Section

21. Prove that if $a|b$ and $b|c$, then $a|c$, for integers a, b, c.

22. Find the smallest integer that cannot be created from a (nonnegative) number of stamps of size 4 and 7. Then prove that all larger numbers can be so represented, by strong induction.

23. Prove/disprove: If $f : A \to B$ and $g : B \to C$ are functions and $g \circ f$ is injective, then f is injective. (Now change injective to surjective everywhere. Prove/disprove that f is surjective. Also prove/disprove that g is surjective.)

24. Prove for sets S and T that $S \subseteq T$ if and only if $S \cup T = T$.

25. Prove that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Other problems we could have put on the test include important theorems from the book like:

Prove that $(0, 1)$ is uncountable.

Prove that $|S| < |\mathcal{P}(S)|$ for any set S.

Prove there are infinitely many prime numbers.

and so forth...