RED

Name:____________________________

Instructor: ***

Math 290
Sample Exam 2

Note that the first 10 questions are true-false. Mark A for true, B for false. Questions 11 through 20 are multiple choice–mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly.

True-false questions

1. For every natural number n, the integer $2n^2 - 4n + 31$ is prime.
2. For every real number x, it is not the case that $x^4 < x < x^2$.
3. Let R be the relation defined on $\mathbb{Z}$ by aRb if $a \neq b$. Then R is symmetric and transitive.
4. The intersection of two equivalence relations on a set A is an equivalence relation.
5. If you expand $(7a + 3b)^{11}$, the coefficient of ab^{10} is $7 \cdot 3^{10} \cdot 11$.
6. If proving a statement for all integers, by induction, then you don't need a base case.
7. To prove the statement "There exists a unique integer n, such that $7 < n < 9$" you just need to say "The integer 8 works."
8. The number of subsets of cardinality 6, from a 9 element set, is $\binom{9}{6}$.
9. If A and B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.
10. When proving $A \subseteq B$, you assume $x \in A$ and $x \in B$.

Multiple choice section

On all problems, choose the most complete correct answer.

11. Let $A = \{1, 2, 3, 4, 5\}$, and let

$$R = \{(1,1),(1,3),(1,4),(2,2),(2,5),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4),(5,2),(5,5)\}$$

be an equivalence relation on A. Which of the following is an equivalence class?

a) $\{1, 2, 3\}$. b) $\{2, 3, 5\}$. c) $\{1, 3, 4\}$. d) $\{1, 2\}$. e) $\{1, 2, 3, 4, 5\}$.

12. Define a relation R on the integers by aRb if $a^2 - b^2 \leq 3$. Choose the most complete correct statement below:

a) R is reflexive. b) R is symmetric. c) R is transitive.
d) R is reflexive and transitive. e) R is symmetric and transitive.
f) R is reflexive and symmetric. g) R is an equivalence relation. h) None of the above.

13. The value of $\binom{10}{5}$ is:

a) 10. b) 15. c) 72. d) 252.
e) 30240. f) 100000. g) 3628800.

---

14. Which of the following is an equivalence relation?

a) The relation R on $\mathbb{Z}$ defined by aRb if $a^2 - b^2 \leq 7$.

b) The relation R on $\mathbb{Z}$ defined by aRb if $2a + 5b \equiv 0 \pmod 7$.

c) The relation R on $\mathbb{Z}$ defined by aRb if $a + b \equiv 0 \pmod 5$.

d) The relation R on $\mathbb{Z}$ defined by aRb if $a^2 + b^2 = 0$.

---

15. If R is an equivalence relation on $\mathbb{Z}$, which of the following does not hold?

a) The equivalence classes of R will partition $\mathbb{Z}$.

b) The relation must be antisymmetric.

c) The relation R is non-empty.

d) The relation R can be described from a partition of $\mathbb{Z}$.

e) Exactly one of the previous answers is false.

---

16. Let A be a set with $|A| = 10$, and let R be an equivalence relation on A. Let $a, b, c \in A$, with $|[a]| = 3$, $|[b]| = 5$, and $|[c]| = 1$. How many equivalence classes does A comtain?

a) 2. b) 3. c) 4. d) 5. e) 6. f) 10. g) Infinitely many.

---

17. Which of the following sets equals $\{2x + 3y \ : \ x, y \in \mathbb{N}\}$.

a) $\mathbb{Z}$ b) $\mathbb{N}$ c) $2\mathbb{Z}$ d) $\{2, 3, 4, 5, \ldots\}$ e) $\{5, 7, 8, 9, \ldots\}$ f) None of the above.

---

18. Evaluate the following proof that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Proof. Take $x \in \overline{A \cap B}$. Hence $x \notin A \cap B$. So $x \notin A$ and $x \notin B$. Therefore, $x \in \overline{A}$ and $x \in \overline{B}$. Hence $x \in \overline{A} \cap \overline{B}$.

a) The theorem and proof are correct.
b) The theorem is correct, but the proof is in error.
c) The proof is correct, but the theorem is false.
d) The theorem is false, and the proof is in error.
e) None of the above.

19. Evaluate the proof of the given theorem:

Theorem: Let $a, b, c \in \mathbb{Z}$ be nonzero. If $a|bc$ and $a \nmid b$ then $a|c$.

Proof: Assume $a|bc$. This implies $bc = ax$ for some $x \in \mathbb{Z}$. Since a doesn't divide b, we must have a divides c.

a) The theorem and proof are correct.

b) The theorem is correct, but the proof makes an unwarranted implication.

c) The theorem is incorrect, but the proof is correct and proves something else.

d) The theorem is false, and the proof makes an unjustified step.

e) all of the above

f) none of the above

---

20. Evaluate the proof of the following statement:

Result: For all $n \in \mathbb{N}$, $4|(3^{2n} + 7)$.

Proof. Let S_n be the statement $4|(3^{2n} + 7)$.

S_1 is true, since $3^2 + 7 = 16$ is divisible by 4.

For $k \geq 1$, assume that S_k is true, so that $4|(3^{2k} + 7)$. Then $3^{2k} + 7 = 4\ell$ for some $\ell \in \mathbb{Z}$.

Now,

$$\begin{aligned} 3^{2(k+1)} + 7 &= 9(3^{2k}) + 7 \\ &= 8(3^{2k}) + 3^{2k} + 7 \\ &= 8(3^{2k}) + 4\ell \\ &= 4(2(3^{2k}) + \ell), \end{aligned}$$

so $4|(3^{2(k+1)} + 7)$, and we see that S_{k+1} is true. Hence S_k implies S_{k+1}.

Therefore, by the principle of mathematical induction, S_n is true for all $n \in \mathbb{N}$. $\square$

a) The theorem is false but the proof is correct.

b) The proof contains arithmetic mistakes which make it incorrect.

c) The proof incorrectly assumes what it is trying to prove.

d) The proof is a correct proof of the stated result.

e) None of the above.

Essay Section

21. Write out the first seven rows of Pascal's triangle, and use them to expand the binomial $(x-y)^7$.

---

22. Prove that $3^n > n^2$ for all $n \in \mathbb{N}$.

---

23. Define five of the following seven terms (written in boldface) by completing the sentences. Only the first five defined terms will be graded.

Given two nonzero integers $a, b \in \mathbb{Z}$, their **greatest common divisor** is

A relation R on a set A is **transitive** if

Let R be an equivalence relation on a set A. An **equivalence class** of R is

An integer n is **composite** if

Given two integers $a, b \in \mathbb{Z}$, a **linear combination** of them is

Euclid's lemma states

A **partition** of a set A is a collection P of subsets of A satisfying

---

24. Prove that the relation $\sim$ defined on $\mathbb{R} - \{0\}$ by the rule $a \sim b$ if and only if $ab > 0$, is an equivalence relation, and then give the partition corresponding to this equivalence relation.

---

25. Let R be an equivalence relation on a nonempty set A, and let $a, b \in A$. Prove that $[a] = [b]$ if and only if aRb.