Math 290 Practice Exam 2 Solutions

Problem 1. False. The smallest value of n for which $2n^2 - 4n + 31$ is not prime is n = 30, for which the polynomial gives 1711 which is the product of 29 and 59. But the easiest value of n to find for which the polynomial $2n^2 - 4n + 31$ is not prime is n = 31, for then $2(31)^2 - 4(31) + 31$ factors as (31)(2(31) - 4 + 1).

Problem 2. True. The universally quantified statement is $\forall x \in \mathbb{R}$, $\sim (x^4 < x < x^2)$, which carrying out the negation gives $\forall x \in \mathbb{R}$, $x^4 \ge x$ or $x \ge x^2$. [The statement $x^4 < x < x^2$ is logically the same as $x^4 < x$ AND $x < x^2$.] For $x \ge 1$ or $x \le 0$ we have $x^4 \ge x$, while for 0 < x < 1, we have $x \ge x^2$.

Problem 3. False. The relation is symmetric (since if $x \neq y$ then $y \neq x$). However it is not transitive. Take x = 1, y = 0, and z = 1. Then xRy and yRz but x = z so they are not related.

Problem 4. True. You can prove this directly.

Problem 5. True. With x = 7a and b = 3b, the Binomial Theorem states that

$$(x+y)^{11} = \sum_{k=0}^{11} {\binom{11}{k}} x^{11-k} y^k.$$

The coefficient of ab^{10} corresponds to k = 10. The coefficient of ab^{10} is

$$(7)(3^{10})\binom{11}{10} = (7)(3^{10})(11).$$

Problem 6. False. A proof by induction always starts with a base case.

Problem 7. False. By stating "The integer 8 works" only gives existence, not uniqueness.

Problem 8. True. The definition of $\binom{n}{k}$ is the number of subsets having k elements taken from a set of n elements.

Problem 9. False. One inclusion is true, but taking $A = \{1\}$ and $B = \{2\}$ we see that $\{1,2\} \in \mathcal{P}(A \cup B)$ and yet $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Problem 10. False. In proving $A \subset B$ one starts with $x \in A$ and proceeds to prove that $x \in B$.

Problem 11. (c) The equivalence classes are $\{1, 3, 4\}$ and $\{2, 5\}$.

Problem 12. (a) The relation is reflexive since $x^2 - x^2 = 0 \le 3$, for every $x \in \mathbb{Z}$. It is not symmetric since 0R10 but 10R0. It is not transitive since 2R1 and 1R0 but 2R0.

Problem 13. (d) We have

$$\binom{10}{5} = \frac{10!}{5!5!} = \frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)(1)} = (2)(9)(2)(7) = (18)(14) = 252.$$

Problem 14. (b) The relation in (a) is not symmetric or transitive. The relation in (c) is not reflexive or transitive. The relation in (d) is not reflexive. The relation in (b) is secretly just $a \equiv b \pmod{7}$, which we already know is an equivalence relation.

Problem 15. (b) There are many equivalence relations which are not antisymmetric on \mathbb{Z} . Part (a) is a theorem, (c) comes from the fact R is reflexive, and (d) is true by a theorem in the book. Thus part (e) is also true.

Problem 16. (c) There are classes of sizes 5, 3, and 1. Thus, the only element of A not accounted for also lives in its own equivalence class.

Problem 17. (e) We have 2(1) + 3(1) = 5, 2(2) + 3(1) = 7, 2(1) + 3(2) = 8, and 2(3) + 3(1) = 9. It is easy to get all other bigger numbers.

Problem 18. (d) A counterexample is $A = \{1, 2\}, B = \{2, 3\}$ are subsets of the universal set $U = \{1, 2, 3, 4, 5\}$. Here $\overline{A \cap B} = \overline{\{2\}} = \{1, 3, 4, 5\}$ while $\overline{A} \cap \overline{B} = \{3, 4, 5\} \cap \{1, 4, 5\} = \{4, 5\}$, and hence $\overline{A \cap B} \not\subset \overline{A} \cap \overline{B}$. The proof fails to use DeMorgan's law correctly (to change from "and" to "or").

Problem 19. (d) A counterexample is a = 6, b = 8, and c = 15. For then a divides bc = 120 and $a \nmid b$, but $a \nmid c$. Can you find the unjustified step?

Problem 20. (d) There are no errors in the proof by induction for the statement given.Problem 21. By the Binomial Theorem we have

$$(a+b)^7 = \sum_{k=1}^7 \binom{7}{k} a^{n-k} b^k$$

= $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

With a = x and b = -y we have

$$(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7.$$

Problem 22. For n = 1 we have $3^1 = 3 > 1^2 = 1$, and for n = 2 we have $3^2 = 9 > 4 = 2^2$ (we see why in a moment we needed to check n = 2).

Now suppose that $3^k > k^2$ for some $k \ge 2$.

Then

$$3^{k+1} = 3(3^k) > 3(k^2) = k^2 + k^2 + k^2$$

With $k \ge 2$ we have $k^2 = k(k) \ge 2k$, and with $k \ge 2$ we have $k^2 \ge 1$. Thus

$$3^{k+1} > k^2 + 2k + 1 = (k+1)^2.$$

Therefore by induction $3^n > n^2$ for all $n \ge 1$.

Problem 23. Found in the book and listed on the review sheet.

Problem 24. (Reflexive): Given $a \in \mathbb{R} - \{0\}$ then $a^2 > 0$ so $a \sim a$.

(Symmetric): Let $a, b \in \mathbb{R} - \{0\}$. Assume $a \sim b$. Thus ab > 0. Hence ba > 0. Therefore $b \sim a$.

(Transitive): Let $a, b, c \in \mathbb{R} - \{0\}$. Assume $a \sim b$ and $b \sim c$. Thus ab > 0 and bc > 0. Multiplying we have $ab^2c > 0$. Since $b^2 > 0$ we can divide by it, and get ac > 0. Thus, $a \sim c$. (Equivalence classes): We find $[1] = \mathbb{R}_{>0}$ and $[-1] = \mathbb{R}_{<0}$. As these partition the set $\mathbb{R} - \{0\}$, they are the only two equivalence classes. Thus, the partition is $\{\mathbb{R}_{>0}, \mathbb{R}_{<0}\}$. **Problem 25**. Found in the book.