PRACTICE EXAM 1 SOLUTIONS

Problem 1. *For any set A, the empty set is an element of the power set of A.*

Proof. This is true. The empty set is a subset of A, hence it is an element of the power set of A. $\square$

Problem 2. *For any sets A and B, we have $A - B \subseteq A$.*

Proof. This is true. If $x \in A - B$ then $x \in A$ (and not in B). $\square$

Problem 3. *Let I be the set of natural numbers, and for each $i \in I$ let A_i be the closed interval in the real numbers $[1/i, i^2 + 1]$. Then*

$$\bigcap_{i \in I} A_i = [1, 2].$$

Proof. This is true. The intervals are growing bigger as i increases, so their intersection is just $A_1 = [1, 2]$. $\square$

Problem 4. *Let $A = \{1, 2, 3\}$. Then A is a subset of the power set of A.*

Proof. This is false. No element of A is a set, so they cannot belong to the power-set. $\square$

Problem 5. *If $a \equiv 3 \pmod 5$, then $a^2 \equiv 4 \pmod 5$.*

Proof. This is true. Squaring both sides, we have $a^2 \equiv 3^2 = 9 \equiv 4 \pmod 5$ since $5 | (9 - 4)$. $\square$

Problem 6. *Let A, B, and C be sets. Then $A - (B \cup C) = (A - B) \cap (A - C)$.*

Proof. This is true. You can use Venn diagrams to see the equality. $\square$

Problem 7. *The converse of the statement "If x is even, then $x + 1$ is odd," is the statement "If $x + 1$ is even, then x is odd."*

Proof. This is false. The given statement is the contrapositive, not the converse. $\square$

Problem 8. *The negation of the statement "There exists $x \in \mathbb{R}$, $x^2 - 1 < 0$," is the statement "For all $x \in \mathbb{R}$, $x^2 - 1 < 0$."*

Proof. This is false. It should read "For all $x \in \mathbb{R}$, $x^2 - 1 \geq 0$." $\square$

Problem 9. *The statement $P \wedge (\neg P)$ is a tautology.*

Proof. This is false. You can see this using truth tables; this is a contradiction! $\square$

Problem 10. *Let A and B be sets. If A has seven elements, $A \cup B$ has ten elements, and $A - B$ has two elements, then B must contain eight elements.*

Proof. This is true. Venn diagrams might help show you how many elements are in each set. $\square$

Problem 11. *For the following proof, determine which of the statements given below is being proved.*

Proof. Assume a and b are odd integers. Then $a = 2k + 1$ and $b = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Then $ab^2 = (2k + 1)(2\ell + 1)^2 = 8kl^2 + 8kl + 2k + 4l^2 + 4l + 1 = 2(4kl^2 + 4kl + k + 2l^2 + 2l) + 1$. Since $4kl^2 + 4kl + k + 2l^2 + 2l \in \mathbb{Z}$, we see that ab^2 is odd. $\square$

a) If a or b is even, then ab^2 is even.
b) If a and b are even, then ab^2 is even.
c) If ab^2 is even, then a and b are even.
d) If ab^2 is even, then a is even or b is even.
e) None of the above.

Proof. The answer is (d). They are using the contrapositive. □

Problem 12. *Let A be a set with 5 elements. Which of the following cannot exist:*
a) A subset of the power set of A with six elements.
b) An element of the power set of A with six elements.
c) An element of A containing six elements.
d) Any of the above can exist, for suitable sets A.
e) None of (a) through (c) can exist, no matter what A is.

Proof. The answer is (b) because elements of the power set are subsets of A, and subsets of A can have only elements of A. A subset of A can have at most 5 elements. □

Problem 13. *Which of the following has a vacuous proof?*
a) Let $n \in \mathbb{Z}$. If $|n| < 1$ then $5n + 3$ is odd.
b) Let $n \in \mathbb{Z}$. If $2n + 1$ is odd, then $n^2 + 1 > 0$.
c) Let $x \in \mathbb{R}$. If $x^2 - 2x + 3 < 0$, then $2x + 3 < 5$.
d) Let $x \in \mathbb{R}$. If $-x > 0$, then $x^2 + 3 > 3$.
e) None of the above.

Proof. The answer is (c), because $x^2 - 2x + 3 = x^2 - 2x + 1 + 2 = (x - 1)^2 + 2 > 0$, so the premise is bogus. □

Problem 14. *Which of the following statements has a trivial proof.*
a) Let $x \in \mathbb{N}$. If $x > 0$ then $x^2 > x$. *b) Let $x \in \mathbb{N}$. If $x > 3$ then $2x$ is even.*
c) Let $x \in \mathbb{N}$. If $x < 2$ then $x^2 + 1$ is even. *d) Let $x \in \mathbb{N}$. If $2x$ is even, then x is odd.*

Proof. The answer is (b), since $2x$ is even, so the Q is true. □

Problem 15. *Evaluate the following proof:*
Theorem: *Let $n \in \mathbb{Z}$. If $3n - 8$ is odd, then n is odd.*

Proof. Let $n \in \mathbb{Z}$. Assume that n is odd. Then $n = 2k + 1$ for some integer k. Then

$$3n - 8 = 3(2k + 1) - 8 = 6k + 3 - 8 = 6k - 5 = 2(3k - 3) + 1.$$

Since $3k - 3 \in \mathbb{Z}$, we know that $3n - 8$ is odd. □

a) The proof and the theorem are correct.
b) The proof proves the converse of the given statement.
c) The proof proves the contrapositive of the given statement.
d) The proof contains arithmetic mistakes, which make it incorrect.
e) None of the above.

Proof. The answer is (b). □

Problem 16. *Let $A = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. The number of elements in the power set of A is*
a) 3 b) 4 c) 6 d) 8 e) 16 f) 64

Proof. The answer is (d). □

Problem 17. *Let $x \in \mathbb{Z}$. The contrapositive of the open sentence "If x is even then $3x+7$ is odd." is the statement*

a) If x is odd then $3x+7$ is even. *b) If $3x+7$ is odd then x is even.*
c) If $3x+7$ is even then x is odd. *d) If $3x+7$ is even, then x is even.*
e) x is odd or $3x+7$ is odd. *f) x is odd or $3x+7$ is even.*

Proof. The answer is (c). □

Problem 18. *Let x and y be integers. The negation of the statement "If xy is even then x is even or y is even" is*

a) If x is odd and y is odd, then xy is odd. *b) If x is even or y is even, then xy is even.*
c) If xy is odd, then x is even and y is even. *d) xy is even and x is odd and y is odd.*
e) xy is even and (x is odd or y is odd). *f) xy is odd and (x is even or y is even).*
g) xy is odd and (x is odd and y is odd).

Proof. The answer is (d). Remember that the negation of an implication $P \Rightarrow Q$ is the statement $P \wedge Q$. Also, the negation of an "or" is an "and". □

Problem 19. *If you wish to prove a statement of the form "If P then (Q or R),", which of the following would* **not** *be a good way to begin.*

a) Assume P
b) Assume $(\neg P) \wedge (Q \vee R)$
c) Assume $(\neg Q) \wedge (\neg R)$.
d) Assume $P \wedge (\neg Q) \wedge (\neg R)$.
e) None of the above: all of these would be acceptable ways to begin.

Proof. The answer is (b). We never assume the negation of the premise when proving an implication. □

Problem 20. *The following is a theorem proved in "Cohomology of number fields" (pg. 75) by J. Neukirch.*

Theorem*: Let G be a finite group, and let A, B be G-modules. If A is cohomologically trivial or B is divisible, then* $\hom(A, B)$ *is cohomologically trivial.*

Suppose that we know that G is a finite group, A and B are G-modules, and that $\hom(A, B)$ *is not cohomologically trivial. Which of the following must be true? (Think about the contrapositive.)*

a) A is cohomologically trivial and B is divisible.
b) A is cohomologically trivial or B is divisible.
c) A is not cohomologically trivial or B is divisible.
d) A is not cohomologically trivial or B is not divisible.
e) A is not cohomologically trivial and B is not divisible.

Proof. The answer is (e). □

Problem 21. *Truth table.*

Proof. Come see me if you need help on this one. □

Problem 22. *Prove that if n is an even integer, then $3n+2$ is even in each of the following three ways: (i) a direct proof, (ii) a contrapositive proof, and (iii) a proof by contradiction.*

Direct. Let $n \in \mathbb{Z}$. We work directly. Assume n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Thus we find $3n + 2 = 3(2k) + 2 = 2(3k + 1)$. Since $3k + 1 \in \mathbb{Z}$, we see that $3n + 2$ is even. □

Contrapositive. Let $n \in \mathbb{Z}$. We work contrapositively. Assume $3n + 2$ is odd. Then $3n + 2 = 2k + 1$ for some $k \in \mathbb{Z}$. Thus we find $n = 3n+2-2n-2 = 2k+1-2n-2 = 2(k-n-1)+1$. Since $k-n-1 \in \mathbb{Z}$, we see that n is odd. □

Contradiction. Assume, by way of contradiction, there is some integer n such that n is even and $3n+2$ is odd. Then $n = 2k$ and $3n + 2 = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. We then find

$$2\ell + 1 = 3n + 2 = 3(2k) + 2 = 2(3k + 1).$$

Since $3k+1 \in \mathbb{Z}$, this shows the right side is even. But the left side is odd. This gives us a contradiction. □

Problem 23. *Prove the following statement. If x and y are rational, $x \neq 0$, and z is irrational, then $\frac{y+z}{x}$ is irrational.*

Proof. Assume, by way of contradiction that $x, y \in \mathbb{Q}$, $x \neq 0$, z is irrational, and $\frac{y+z}{x} \in \mathbb{Q}$.

Since $x, \frac{y+z}{x} \in \mathbb{Q}$ their product $y + z = x\frac{y+z}{x}$ is rational.

Since $y + z, y \in \mathbb{Q}$, their difference $z = y + z - y$ is rational. This contradicts that fact that z is irrational. □

Problem 24. *Let A, B, C be sets. Prove (with justification for every step) that $A - B = A \cap \overline{B}$.*

Proof. Let A, B be sets.

($\subseteq$) : Let $x \in A - B$. Thus, by definition of set difference, $x \in A$ and $x \notin B$. Thus, by definition of complements, $x \in A$ and $x \in \overline{B}$. By definition of intersection $x \in A \cap \overline{B}$. We have now shown $A - B \subseteq A \cap \overline{B}$.

($\supseteq$): This containment is similar, just reverse the steps above. □

Problem 25. *Give examples of three sets A, B and C such that $A \in B$, $B \subseteq C$, and $A \notin C$.*

Proof. Let $A = \{1\}$, $B = \{\{1\}\}$, and $C = B$. □