RED

Name:____________________________

Math 290
Fall 2016
Sample Exam 3

Note that the first 10 questions are true–false. Mark A for true, B for false. Questions 11 through 20 are multiple choice. Mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly. Questions 1 to 20 are worth 2.5 points each, and questions 21 to 25 are worth 10 points each.

---

1. Let R be a relation defined on the set $\mathbb{Z}$ by aRb if $a \neq b$. Then R is symmetric and transitive.

2. The relation R on $\mathbb{Z}$ given by aRb if $a \geq b$ is reflexive and transitive but not symmetric.

3. Let $f : A \to B$ be an injective function and let $C \subseteq A$. Let $g : C \to B$ be the restriction of f to C. Then g is injective.

4. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) = n + 3$. Then f is surjective.

5. If $|A| = 4$ and $|B| = 5$, then there can not be a surjective function from A to B.

6. Let $a, b \in \mathbb{R}$ with $a \neq 0$. Then $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is a bijection.

7. Let $A = \{-1, 0, 1\}$. Define $f : A \to \mathbb{Z}$ by the rule $f(a) = a^4$, and define $g : A \to \mathbb{Z}$ by the rule $g(a) = a^2$. Then f is equal to g.

8. Let A be a nonempty set. If a function $f : A \to A$ is surjective, then it is injective.

9. If A is an uncountable set and $|B| < |A|$, then B is countable.

10. If two sets A and B are both uncountable, then they have the same cardinality.

Multiple choice section

11. Let $A = \{1, 2, 3, 4, 5\}$, and let

    $R = \{(1,1), (1,3), (1,4), (2,2), (2,5), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4), (5,2), (5,5)\}$

    be an equivalence relation on A. Which of the following is an equivalence class?

    a) $\{1, 2, 3\}$ b) $\{2, 3, 5\}$ c) $\{1, 3, 4\}$ d) $\{1, 2\}$ e) $\{1, 2, 3, 4, 5\}$

---

12. Define a relation R on the integers by aRb if $a^2 - b^2 \leq 3$. Choose the most complete correct statement below:

    a) R is reflexive. b) R is symmetric. c) R is transitive.
    d) R is reflexive and transitive. e) R is symmetric and transitive. f) R is reflexive and symmetric.
    g) R is an equivalence relation. h) None of the above.

---

13. Which of the following is an equivalence relation?

    a) The relation R on $\mathbb{Z}$ defined by aRb if $a^2 - b^2 \leq 7$.
    b) The relation R on $\mathbb{Z}$ defined by aRb if $2a + 5b \equiv 0 \pmod 7$.
    c) The relation R on $\mathbb{Z}$ defined by aRb if $a + b \equiv 0 \pmod 5$.
    d) The relation R on $\mathbb{Z}$ defined by aRb if $a^2 + b^2 = 0$.

14. How many different equivalence relations are there on the set $A = \{a, b, c\}$?
   a) 1. b) 2. c) 3. d) 4. e) 5. f) 6. g) 256. h) 512.

---

15. Which of the following functions from $\mathbb{Z}_6 \to \mathbb{Z}_6$ is injective? (They are all well defined.)
   a) $f([x]) = [2x+3]$ b) $f([x]) = [x^3+1]$ c) $f([x]) = [x^2+3]$
   d) All of the above e) None of the above

---

16. Let $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{2\}$ be given by
$$f(x) = \frac{4x-7}{2x-6}.$$
Which of the following is true:
   a) f is not injective.
   b) f is not surjective.
   c) f is bijective but has no inverse.
   d) f is bijective, and the inverse of f is $f^{-1}(x) = \frac{2x-6}{4x-7}$.
   e) f is bijective, and the inverse of f is $f^{-1}(x) = \frac{6x-7}{2x-4}$.
   f) f is not a function from $\mathbb{R} - \{3\}$ to $\mathbb{R} - \{2\}$.
   g) None of a-f are true.

---

17. The number of functions from $A = \{1, 2, 3\}$ to $B = \{1, 2, 3, 4, 5\}$ is
   a) 0 b) 1 c) 8 d) 15 e) 125 f) 243 g) 6561 h) Infinite

---

18. Let $f : (0, 1) \to [0, 1]$ be defined by $f(x) = x$. Which of the following is true? (Choose the most complete correct answer.)
   a) f is a surjective function b) f is an injective function c) f is a bijective function
   d) f is an invertible function e) f is not a function

---

19. Evaluate the proposed proof of the following result:
   Theorem: The sets $(0, \infty)$ and $[0, \infty)$ have the same cardinality.

   Proof. Let $f : (0, \infty) \to [0, \infty)$ be defined by $f(x) = x$. Then f is clearly injective, so $|(0, \infty)| \leq |[0, \infty)|$. Define $g : [0, \infty) \to (0, \infty)$ by $g(x) = 1/x$. Since $g(x) = g(y)$ implies that $1/x = 1/y$, which implies that $x = y$, we see that g is injective. Hence, $|[0, \infty)| \leq |(0, \infty)$. By the Schröder-Bernstein theorem, we see that $|(0, \infty)| = |[0, \infty)|$. □

   a) The theorem and its proof are true.
   b) The stated theorem is false.
   c) The proof is incorrect because $f(x)$ is not an injective function from $(0, \infty)$ to $[0, \infty)$.
   d) The proof is incorrect because $g(x)$ is not an injective function from $[0, \infty)$ to $(0, \infty)$.
   e) The proof is incorrect because $f(x)$ is not a surjective function from $(0, \infty)$ to $[0, \infty)$.
   f) The proof is incorrect because $[0, \infty)$ is larger than $(0, \infty)$.

---

20. Which of the following sets have equal cardinality?
$$A = (0, 1), \qquad B = \mathcal{P}(\mathbb{N}), \qquad C = \mathbb{R}$$
   a) A and B b) B and C c) A and C d) All three have equal cardinality
   e) None of them have equal cardinality.

21. Let $A = \{1, 2, 3, 4, 5\}$. Then $P = \{\{1, 3\}, \{2\}, \{4, 5\}\}$ is a partition of A. Write down (as a set of ordered pairs) the equivalence relation R whose equivalence classes are the elements of P.

---

22. Let $f : A \to B$ and $g : B \to C$ be functions.

    (a) Prove or disprove: If $g \circ f$ is injective then f is injective.
    (b) Prove or disprove: If f is surjective, then $g \circ f$ is surjective.

---

23. Let $a, b \in \mathbb{R}$ with $b \neq 0$. Define $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{a\}$ by

$$f(x) = a + \frac{b}{x}.$$

Prove that f is bijective.

---

24. Let $A = (0, 1)$ be the open interval of real numbers between 0 and 1. Let $B = (-2, 3)$ be the open interval of real numbers between -2 and 3. Prove that $|A| = |B|$.

---

25. Let A be a nonempty set. Prove that $|A| < |\mathcal{P}(A)|$.