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Math 290
Fall 2016
Sample Exam 3

Note that the first 10 questions are true–false. Mark A for true, B for false. Questions 11 through 20 are multiple choice. Mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly. Questions 1 to 20 are worth 2.5 points each, and questions 21 to 25 are worth 10 points each.

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1. Let R be a relation defined on the set $\mathbb{Z}$ by aRb if $a \neq b$. Then R is symmetric and transitive.

   Answer. False. It is symmetric but not transitive as shown by the following counter-example. Let $a = 1$, $b = 2$, and $c = 1$. Then aRb and bRc, but $a \not R c$.

2. The relation R on $\mathbb{Z}$ given by aRb if $a \geq b$ is reflexive and transitive but not symmetric.

   Answer. True. [You should be able to prove this statement.]

3. Let $f : A \to B$ be an injective function and let $C \subseteq A$. Let $g : C \to B$ be the restriction of f to C. Then g is injective.

   Answer. True. Let $x, y \in C$ and assume $g(x) = g(y)$. Since $g = f|_C$, we have $f(x) = f(y)$. Since f is injective, $x = y$.

4. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) = n + 3$. Then f is surjective.

   Answer. False. Note that $f(n) \neq 0$ for any $n \in \mathbb{N}$.

5. If $|A| = 4$ and $|B| = 5$, then there cannot be a surjective function from A to B.

   Answer. True. There are not enough elements of A to hit every element of B.

6. Let $a, b \in \mathbb{R}$ with $a \neq 0$. Then $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is a bijection.

   Answer. True. First, we prove injectivity. Let $x, y \in \mathbb{R}$ and assume $f(x) = f(y)$. Then $ax + b = ay + b$. Subtracting b from both sides, $ax = ay$. Then dividing both sides by a (using the fact $a \neq 0$), we have $x = y$. Thus, f is injective. Next we prove surjectivity. Let $y \in \mathbb{R}$ and fix $x = (c - b)/a \in \mathbb{R}$. We have $f(x) = f((c - b)/a) = a((c - b)/a) + b = c - b + b = c$. Thus, f is surjective.

7. Let $A = \{-1, 0, 1\}$. Define $f : A \to \mathbb{Z}$ by the rule $f(a) = a^4$, and define $g : A \to \mathbb{Z}$ by the rule $g(a) = a^2$. Then f is equal to g.

   Answer. True. The functions have the same domains and same codomains. Further, since $f(x) = g(x)$ for all $x \in A$, then f is equal to g.

8. Let A be a nonempty set. If a function $f : A \to A$ is surjective, then it is injective.

   Answer. False. Here is a counter-example with $A = \mathbb{N}$. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$ and $f(n) = n - 1$ when $n > 1$. This is not injective since $f(1) = f(2)$. It is surjective since for any $b \in \mathbb{N}$ we can take $a = b+1 \in \mathbb{N}$ and $f(a) = f(b+1) = b$. [Note: This statement would be true if A were assumed to be a finite set, by the pigeon-hole principle.]

9. If A is an uncountable set and $|B| < |A|$, then B is countable.

   Answer. False. Consider the case $B = \mathbb{R}$ and A is the power set of $\mathbb{R}$.

10. If two sets A and B are both uncountable, then they have the same cardinality.

    Answer. False. Consider the case $B = \mathbb{R}$ and A is the power set of $\mathbb{R}$.

Multiple choice section

11. Let $A = \{1, 2, 3, 4, 5\}$, and let

$$R = \{(1,1), (1,3), (1,4), (2,2), (2,5), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4), (5,2), (5,5)\}$$

be an equivalence relation on A. Which of the following is an equivalence class?
a) $\{1, 2, 3\}$ b) $\{2, 3, 5\}$ c) $\{1, 3, 4\}$ d) $\{1, 2\}$ e) $\{1, 2, 3, 4, 5\}$

Answer. (c). The equivalence classes are $\{1, 3, 4\}$ and $\{2, 5\}$.

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12. Define a relation R on the integers by aRb if $a^2 - b^2 \leq 3$. Choose the most complete correct statement below:
    a) R is reflexive. b) R is symmetric. c) R is transitive.
    d) R is reflexive and transitive. e) R is symmetric and transitive. f) R is reflexive and symmetric.
    g) R is an equivalence relation. h) None of the above.

    Answer. (a). Clearly $a^2 - a^2 \leq 3$. If $a = 2$ and $b = 0$, then $b^2 - a^2 \leq 3$, but $a^2 - b^2 \not\leq 3$. Thus, R is not symmetric. If $a = 2$, $b = 1$, and $c = 0$, then $a^2 - b^2 \leq 3$ and $b^2 - c^2 \leq 3$, but $a^2 - c^2 = 4 \not\leq 3$. Thus R is not transitive.

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13. Which of the following is an equivalence relation?
    a) The relation R on $\mathbb{Z}$ defined by aRb if $a^2 - b^2 \leq 7$.
    b) The relation R on $\mathbb{Z}$ defined by aRb if $2a + 5b \equiv 0 \pmod 7$.
    c) The relation R on $\mathbb{Z}$ defined by aRb if $a + b \equiv 0 \pmod 5$.
    d) The relation R on $\mathbb{Z}$ defined by aRb if $a^2 + b^2 = 0$.

    Answer. (b). The relation in (a) is not an equivalence relation for the same reasons as in question #12. (c) is neither reflexive nor transitive, and (d) is not reflexive. (b) can be shown to be reflexive, symmetric, and transitive.

14. How many different equivalence relations are there on the set $A = \{a, b, c\}$?
    a) 1. b) 2. c) 3. d) 4. e) 5. f) 6. g) 256. h) 512.

    Answer. (e). Think about how many ways there are to partition the set A.

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15. Which of the following functions from $\mathbb{Z}_6 \to \mathbb{Z}_6$ is injective? (They are all well defined.)
    a) $f([x]) = [2x + 3]$ b) $f([x]) = [x^3 + 1]$ c) $f([x]) = [x^2 + 3]$
    d) All of the above e) None of the above

    Answer. (b). This can be verified directly by plugging the 6 equivalence classes into $\mathbb{Z}_6$. The function in (a) satisfies $f([0]) = f([3])$, while the function in (c) satisfies $f([2]) = f([4])$.

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16. Let $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{2\}$ be given by

$$f(x) = \frac{4x-7}{2x-6}.$$

Which of the following is true:
 a) f is not injective.
 b) f is not surjective.
 c) f is bijective but has no inverse.
 d) f is bijective, and the inverse of f is $f^{-1}(x) = \frac{2x-6}{4x-7}$.
 e) f is bijective, and the inverse of f is $f^{-1}(x) = \frac{6x-7}{2x-4}$.
 f) f is not a function from $\mathbb{R} - \{3\}$ to $\mathbb{R} - \{2\}$.
 g) None of a-f are true.

Answer. (e).

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17. The number of functions from $A = \{1, 2, 3\}$ to $B = \{1, 2, 3, 4, 5\}$ is
 a) 0 b) 1 c) 8 d) 15 e) 125 f) 243 g) 6561 h) Infinite

Answer. (e).

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18. Let $f : (0, 1) \to [0, 1]$ be defined by $f(x) = x$. Which of the following is true? (Choose the most complete correct answer.)
a) f is a surjective function b) f is an injective function c) f is a bijective function
d) f is an invertible function e) f is not a function

Answer. (b).

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19. Evaluate the proposed proof of the following result:
 Theorem: The sets $(0, \infty)$ and $[0, \infty)$ have the same cardinality.

Proof. Let $f : (0, \infty) \to [0, \infty)$ be defined by $f(x) = x$. Then f is clearly injective, so $|(0, \infty)| \leq |[0, \infty)|$. Define $g : [0, \infty) \to (0, \infty)$ by $g(x) = 1/x$. Since $g(x) = g(y)$ implies that $1/x = 1/y$, which implies that $x = y$, we see that g is injective. Hence, $|[0, \infty)| \leq |(0, \infty)$. By the Schröder-Bernstein theorem, we see that $|(0, \infty)| = |[0, \infty)|$. □

a) The theorem and its proof are true.
b) The stated theorem is false.
c) The proof is incorrect because $f(x)$ is not an injective function from $(0, \infty)$ to $[0, \infty)$.
d) The proof is incorrect because $g(x)$ is not an injective function from $[0, \infty)$ to $(0, \infty)$.
e) The proof is incorrect because $f(x)$ is not a surjective function from $(0, \infty)$ to $[0, \infty)$.
f) The proof is incorrect because $[0, \infty)$ is larger than $(0, \infty)$.

Answer. (d). The function $g(x) = 1/x$ is not defined at $x = 0$, so it is not a function from $[0, \infty)$ to $(0, \infty)$.

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20. Which of the following sets have equal cardinality?

$$A = (0, 1), \qquad B = \mathcal{P}(\mathbb{N}), \qquad C = \mathbb{R}$$

 a) A and B b) B and C c) A and C d) All three have equal cardinality
 e) None of them have equal cardinality.

Answer. (d).

21. Let $A = \{1, 2, 3, 4, 5\}$. Then $P = \{\{1, 3\}, \{2\}, \{4, 5\}\}$ is a partition of A. Write down (as a set of ordered pairs) the equivalence relation R whose equivalence classes are the elements of P.

Solution: $R = \{(1, 1), (1, 3), (3, 1), (3, 3), (2, 2), (4, 4), (4, 5), (5, 4), (5, 5)\}$.

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22. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove or disprove: If $g \circ f$ is injective then f is injective.
(b) Prove or disprove: If f is surjective, then $g \circ f$ is surjective.

Solution: (a) We will prove this directly. Assume $g \circ f$ is injective. We will show f is injective. Let $x, y \in A$ and assume $f(x) = f(y)$. Plugging this value into g we have $g(f(x)) = g(f(y))$. In other words, $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is injective, we obtain $x = y$ as desired.

We can also prove this contrapositively. Assume that f is not injective. Hence there exists some $a, b \in A$, such that $a \neq b$ and $f(a) = f(b)$. Then applying g to both sides of this equation gives $g(f(a)) = g(f(b))$, or in other words $(g \circ f)(a) = (g \circ f)(b)$, where $a \neq b$. Hence $g \circ f$ is not injective.

(b) We will disprove this statement with a counterexample. Fix $A = B = C = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = x$, and define $g : \mathbb{R} \to \mathbb{R}$ by the rule $g(x) = x^2$. Then f is clearly surjective, but $(g \circ f)(x) = g(f(x)) = x^2$ is not surjective.

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23. Let $a, b \in \mathbb{R}$ with $b \neq 0$. Define $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{a\}$ by

$$f(x) = a + \frac{b}{x}.$$

Prove that f is bijective.

Solution: We will prove that f is both surjective and injective. To prove injectivity, assume that $x, y \in \mathbb{R} - \{0\}$, with $f(x) = f(y)$. Then $a + \frac{b}{x} = a + \frac{b}{y}$. Subtracting a and dividing by b (noting $b \neq 0$) we obtain $x = y$, and hence that f is injective.

To prove injectivity, let $y \in \mathbb{R} - \{a\}$ be arbitrary. Doing scratch-work we see that we want to fix $x = \frac{b}{y-a} \in \mathbb{R}$. We also must check $x \neq 0$. Assume, by way of contradiction $x = 0$. Then $\frac{b}{y-a} = 0$. So $b = 0$, a contradiction.

Finally we check that $f(x) = f\left(\frac{b}{y-a}\right) = y$, and thus f is surjective. As f is both injective and surjective, it is thus a bijection.

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24. Let $A = (0, 1)$ be the open interval of real numbers between 0 and 1. Let $B = (-2, 3)$ be the open interval of real numbers between -2 and 3. Prove that $|A| = |B|$.

Solution: We will construct a bijection f from A to B. Let f be defined by

$$f(x) = -2 + 5x.$$

It is straightforward to check that f is indeed a function from A to B, and that the function

$$f^{-1}(x) = \frac{2}{5} + \frac{1}{5}x$$

is the inverse function of f from B to A. Thus f is a bijection, and $|A| = |B|$.

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25. Let A be a nonempty set. Prove that $|A| < |\mathcal{P}(A)|$.

Solution: We must show that there is an injection from A to $\mathcal{P}(A)$, but no surjection. Let $f : A \to \mathcal{P}(A)$ be the function defined by $f(a) = \{a\}$ for all $a \in A$. Clearly f is an injection, and hence $|A| \leq |\mathcal{P}(A)|$.

To show that no surjection exists from A to $\mathcal{P}(A)$, assume that $g : A \to \mathcal{P}(A)$ is any function. Define the set B as follows:

$$B = \{a \in A : a \notin g(a)\}.$$

Clearly B is a subset of A, and hence $B \in \mathcal{P}(A)$. Let $a \in A$ be arbitrary. We need to show that $B \neq g(a)$. We consider two cases:

Case 1: $a \in g(a)$. Then by definition of B, we must have that $a \notin B$. Thus $g(a) \neq B$.

Case 2: $a \notin g(a)$. Then $a \in B$ by the definition of B, but again this gives $g(a) \neq B$.

We conclude that no function from A to $\mathcal{P}(A)$ is surjective, and hence $|A| < |\mathcal{P}(A)|$.