Math 371, Exam 1, Study Guide

General information

(1) The exam will cover all of chapters 1–3 and section 9.4 (2nd edition) or 10.4 (3rd edition).
(2) Be sure to check the hours of the testing center, and give yourself enough time to take the exam.
(3) Books and notes will not be allowed.
(4) WARNING: this study guide is not meant to be exhaustive. Just because something is not on the study guide does not mean it will not be on the exam.

Basics

- Be able to do all homework problems.
- Know all the definitions discussed in the book, especially the definitions of
  - a prime number
  - a ring, a (non)commutative ring with or without identity, an integral domain, a field
  - a zero divisor
  - a subring
  - a ring homomorphism and a ring isomorphism
  - the direct (Cartesian) product of two rings
  - a unit
  - the gcd of a pair of integers
  - the definition of $\mathbb{Z}_n$ and the definition of $+$ and $\cdot$ for $\mathbb{Z}_n$
  - the field of quotients of an integral domain
- Know lots of examples of all the things we talked about, especially:
  (1) Examples of rings, both commutative and non-commutative, with or without identity
  (2) Examples of all sorts of homomorphisms
    (a) A ring homomorphism that is injective but not surjective.
    (b) A ring homomorphism that is surjective but not injective.
    (c) A surjective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_n$.
    (d) A non-trivial ring isomorphism.
    (e) A map that preserves addition, but not multiplication.
    (f) A map that preserves multiplication, but not addition.

Theorems or axioms you should know and be able to use

(1) The Well-Ordering Axiom
(2) The division algorithm for $\mathbb{Z}$
(3) The Euclidean algorithm for $\mathbb{Z}$
(4) In $\mathbb{Z}$, the gcd of f and g can be written as $uf + vg$ for some u and v.
(5) The Fundamental Theorem of Arithmetic for $\mathbb{Z}$
(6) Every field is an integral domain.
(7) $\mathbb{Z}_p$ is a field.
(8) The equation $ax = b$ has solutions in $\mathbb{Z}_n$ if and only if $(a, n)|b$.
(9) The operations of $+$ and $\cdot$ in $\mathbb{Z}_n$ are well-defined.
(10) In any ring R, and for any $a \in R$, we have $0 \cdot a = a \cdot 0 = 0$.
(11) In any ring R, and for any $a \in R$, we have $-(-a) = a$
(12) In any ring R, and for any $a, b \in R$, we have $(-a) \cdot b = -(ab)$
(13) To check that a nonempty subset $S \subseteq R$ is a subring, it suffices to check that S is closed under subtraction and multiplication.
(14) The image of a homomorphism $f : R \to S$ is a subring of S.

Theorems you should know and be able to prove and be able to use

(1) Cancellation is valid in any integral domain R: if $a \neq 0_R$ and $ab = ac$, then $b = c$. (Theorem 3.10 in the second edition, and Theorem 3.7 in the third edition).

(2) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$. (Theorem 2.2 in the second edition, and Theorem 2.6 in the third edition.)
(3) Every finite integral domain is a field. (Theorem 3.11 in the second edition, and Theorem 3.9 in the third edition.)

Sample problems

(1) Prove or disprove: $\mathbb{Z}_{15}$ is a field.
(2) Prove or disprove: $\mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5$.
(3) Prove or disprove: $\mathbb{Z}_8 \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.
(4) Prove or disprove: If A and B are both subrings of C, then $A \cap B$ is a subring of A and of B and of C.
(5) Prove or disprove: If A and B are both subrings of C, then $A \cup B$ is a subring of C.
(6) What is the last digit of the number 7^{2010}? (Hint: Use congruence $\pmod{10}$.)
(7) Prove or disprove: The set of functions which are differentiable on all of $\mathbb{R}$ forms a subring of the ring of all functions with domain $\mathbb{R}$, with the standard definitions of $+$ and $\cdot$.
(8) If there exists a ring isomorphism $A \to B$ we write $A \cong B$. Prove that $\cong$ is an equivalence relation on the class of all rings.
(9) Let T be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $\theta : T \to \mathbb{R}$ be the function defined by $\theta(f) = f(5)$. Prove that θ is a surjective homomorphism. Is θ an isomorphism?