Math 371, Midterm Exam #2 Study Guide

General information

(1) The exam will cover Chapters 4, 5, and 6.
(2) Books and notes will not be allowed.
(3) WARNING: this study guide is not meant to be exhaustive. Just because something is not on the study guide does not mean it will not be on the exam.

Basics

(1) You should know everything that was on the first study guide, especially the basic properties of rings, in Chapter 3.
(2) Ring Definitions:
- a ring, a field, an integral domain
- a zero divisor, a unit
- a ring homomorphism and a ring isomorphism
- the Cartesian product of two rings
- a monic polynomial and an irreducible polynomial
- the gcd of two polynomials
- an ideal
- the kernel of a homomorphism
- maximal ideals, prime ideals
- principal ideals, ideals generated by a finite number of elements
- the quotient ring of a ring by an ideal

(3) Lots of examples of all the things we have discussed, especially:
- Examples of rings, both commutative and non-commutative, of every kind.
- Examples of polynomials, such as "an irreducible polynomial of degree 3 in $\mathbb{Q}[x]$" or "A ring R and a polynomial of degree 2 in $R[x]$ with 4 roots".
- Examples of subrings and ideals with many different properties (including maximal ideals, non-maximal prime ideals, ideals which are not principal, etc.).
- A maximal ideal that does not contain all proper ideals in the ring.
- An infinite ring and an ideal with a finite quotient ring.
- An infinite ring and an ideal with an infinite quotient ring.
- A field with 4 elements, and a ring with 4 elements that is not a field.
- A field F that properly contains the rationals $\mathbb{Q}$ and is properly contained in the reals $\mathbb{R}$ (i.e., $\mathbb{Q} \subset F \subset \mathbb{R}$).

Theorems you should know and be able to state and prove and use

- The First Isomorphism Theorem for rings (Theorem 6.13 in both editions).
- For a field F and an irreducible $p(x) \in F[x]$, the extension field $F[x]/(p(x))$ contains a root of $p(x)$ (Theorem 5.11 in both editions).
- Remainder and factor theorems (Theorems 4.14 and 4.15 in the second edition, or Theorems 4.15 and 4.16 in the third edition).

Theorems you should be able to use

- In $F[x]$, the gcd of $f(x)$ and $g(x)$ can be written as a linear combination of $f(x)$ and $g(x)$.
- The counterpart of the Fundamental Theorem of Arithmetic for $F[x]$
- If F is a field, then $F[x]$ is an integral domain.
- If F is a field and $p(x)$ is a nonconstant polynomial, then $F[x]/(p(x))$ is a commutative ring with identity that contains F.
- $F[x]/(p(x))$ is a field if and only if $p(x)$ is irreducible in $F[x]$.
- The simple criterion for checking that a subset is an ideal (Theorem 6.1 in both editions).
- If R is a commutative ring with identity and I is an ideal of R, then R/I is an integral domain if and only if I is a prime ideal (Theorem 6.14 in both editions).

- If R is a commutative ring with identity and I is an ideal of R, then R/I is a field if and only if I is a maximal ideal (Theorem 6.15 in both editions).
- The set of cosets of an ideal forms a ring (the quotient ring). Specifically, addition and multiplication of cosets of an ideal are well defined.
- The kernel of a homomorphism is an ideal.
- for every ring R and every ideal I in R, there is a natural surjective homomorphism R to R/I, given by $r \mapsto r + I$ (Theorem 6.12).
- In a commutative ring with identity, every maximal ideal is prime.

Sample problems

(1) Find the gcd of $4x^4 + 2x^3 + 6x^2 + 4x + 5$ and $3x^3 + 5x^2 + 6x$ in $\mathbb{Z}_7[x]$.
(2) Find the roots of the polynomial $x^3 + x^2 + 1$ in the field $\mathbb{Z}_2[x]/(x^3 + x + 1)$.
(3) Prove that the set $\{a + b\sqrt{3} | a, b \in \mathbb{Q}\}$ is a field and is isomorphic to $\mathbb{Q}[x]/(x^2 - 3)$.
(4) Explain why multiplication of cosets in R/J makes sense only if J is an ideal.
(5) Construct a field of order 4.
(6) Prove that $\mathbb{Z}_4$ is not a field.
(7) Give an example of a maximal ideal in a ring that does not contain all proper ideals of the ring.
(8) Give an example of a prime ideal I in $\mathbb{Z} \times \mathbb{Z}$ that is not maximal. Describe the quotient ring $(\mathbb{Z} \times \mathbb{Z})/I$.
(9) Let T be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let I be the set $\{g \in T : g(-2) = 0\}$. Prove that I is an ideal and that $T/I \cong \mathbb{R}$.