Math 371, Midterm Exam #3 Study Guide

General information

(1) The exam will cover sections 7.1 to 7.9 (second edition) or 7.1-7.5 and 8.1-8.4 (third edition).
(2) Books and notes will not be allowed.
(3) WARNING: this study guide is not meant to be exhaustive. Just because something is not on the study guide does not mean it will not be on the exam.

Basics

(1) Be able to do all homework problems.
(2) Know all the definitions discussed in the book, especially the definitions of
   - a group, an abelian group, a subgroup, a cyclic group
   - the center of a group, direct products of groups, a simple group
   - group isomorphism and group homomorphism
   - left and right cosets
   - congruence modulo a subgroup
   - index of a subgroup
   - order of a group and an element
   - subgroup generated by a finite number of elements
   - quotient group
   - kernel of a group homomorphism
(3) Lots of examples of all the things we have discussed, especially:
   - Non-abelian groups: S_n, A_n, D_n, matrix groups.
   - Abelian groups: $\mathbb{Z}$, $\mathbb{Z}_n$, U_n.
   - An element of finite order contained in a group of infinite order.
   - Cyclic groups of all orders—both infinite and finite.
   - Groups which are not cyclic, including a (sub)group generated by two elements which is not cyclic.
   - A group with a non-trivial center.
   - A subgroup of an infinite group that has finite index.

Theorems you should know and be able to state and prove and use

- The First Isomorphism Theorem for groups (Theorem 7.42 in Ed.2, Theorem 8.20 in Ed.3).
- The center of a group is a subgroup (Theorem 7.12 in Ed.2, Theorem 7.13 in Ed.3).
- Lagrange's theorem: the order of a subgroup of a finite group divides the order of the group (Theorem 7.26 in Ed.2, Theorem 8.5 in Ed.3).

Theorems you should be able to use

- The simple criterion for checking that a subset is a subgroup (Theorem 7.10 in Ed.2, Theorem 7.11 in Ed.3).
- Every ring is an abelian group under the addition of the ring.
- If R is a ring with identity, then the set of units of R is a group under multiplication of the ring.
- The identity element of a group is unique.
- Cancellation holds in a group; that is, $ab = ac$ or $ba = ca$ implies that $b = c$.
- In a group, inverses are unique.
- Every subgroup of a cyclic group is cyclic.
- Every permutation is either even or odd, but not both.
- Every subgroup of a cyclic group is cyclic.
- Disjoint cycles in S_n commute.
- Every permutation in S_n is the product of disjoint cycles.
- Every k-cycle in S_n has order k.
- Every permutation is the product of transpositions.
- Groups of prime order are cyclic.

- The kernel of a homomorphism is a normal subgroup of the source group.
- If N is a normal subgroup of G, then the set G/N of all cosets of N in G forms a group with the product $(Na)(Nc) = N(ac)$ (the induced operation).
- If N is a normal subgroup of G, then there is a (canonical) surjective homomorphism $G \to G/N$.
- For a finite group G and N a normal subgroup of G, the equality $|G/N| = |G|/|N|$ holds.
- If N is a normal subgroup of G and K is a subgroup of G containing N, then K/N is a subgroup of G/N.

Sample problems

(1) Prove that $H := \left\{ \left(\begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) \middle| a = \pm 1, b \in \mathbb{Z} \right\}$ is a subgroup of $GL(2, \mathbb{Q})$, the group of invertible 2 by 2 matrices with entries from $\mathbb{Q}$. Is H a normal subgroup?
(2) Show that the group $\mathbb{Z}_5 \times \mathbb{Z}_2$ is cyclic, and that $\mathbb{Z}_6 \times \mathbb{Z}_2$ is not cyclic but is generated by two elements.
(3) Prove that the identity is unique in a group.
(4) Prove that inverses are unique in a group.
(5) Give an example of a non-abelian group G of order 24 and identify its center $Z(G)$.
(6) Are there any groups of order 3 which are not cyclic? If so, give an example, if not, prove it.
(7) Find a non-cyclic normal subgroup N of D_4 and determine what D_4/N is isomorphic to.
(8) List all subgroups of S_3 and show whether each is normal.