Math 371, Exam 2, Study Guide

General information

(1) The exam will cover everything we have done since the last exam. That includes all of sections 4.3–4.5 and chapters 5–6, as well as section 10.4 (3rd ed) or 9.4 (2nd ed).
(2) The exam will be in the testing center on Wednesday–Friday, March 2–4. After 2 pm on Friday the Testing Center will charge you a late fee. We suggest you take it earlier and save your money. Be sure to give yourself several hours to finish.
(3) Books, notes, and calculators will not be allowed.
(4) WARNING: this study guide is not meant to be exhaustive. Just because something is not on the study guide does not mean it will not be on the exam.

Basics

(1) You should know everything that was on the first study guide, especially the basic properties of rings, in Chapter 3.
(2) Ring Definitions:
   - a ring, a field, a domain, an integral domain.
   - a zero divisor, a unit
   - a ring homomorphism and a ring isomorphism
   - the Cartesian product of two rings
   - a monic polynomial and an irreducible polynomial
   - the field of fractions of an integral domain (called field of *quotients* by the book)
   - an ideal
   - the kernel of a homomorphism
   - quotient rings
   - maximal ideals, prime ideals
   - principal ideals, ideals generated by a finite number of elements

(3) Lots of examples of all the things we have discussed, especially:
   - Examples of rings, both commutative and non-commutative, of every order.
   - Examples of subrings and ideals with many different properties (including maximal ideals, non-maximal prime ideals, ideals which are not principal, etc.).
   - A maximal ideal that does not contain all proper ideals in the ring.
   - Examples of many polynomials, such as an irreducible polynomial of degree 3 in $\mathbb{Q}[x]$, or a ring R and a polynomial of degree 2 in $R[x]$ with 4 roots.
   - An infinite ring and an ideal with a finite quotient ring.
   - An infinite ring and an ideal with an infinite quotient ring.
   - A field with 9 elements, and a ring with 9 elements that is not a field.
   - A field F that properly contains the rationals $\mathbb{Q}$ and is properly contained in the reals $\mathbb{R}$ (i.e., $\mathbb{Q} \subset F \subset \mathbb{R}$).

Theorems you should know and be able to use

- If R is an integral domain, then there exists a field (the field of fractions) F consisting of equivalence classes of fractions of the form a/b with $a, b \in R$, and $b \neq 0$.
- If K is a field and $R \subseteq K$ is an integral domain with field of fractions F, then there exists a subfield E of K with $R \subseteq E \subseteq K$ and such that $F \cong E$.
- In $F[x]$, the gcd of $f(x)$ and $g(x)$ can be written as a linear combination of $f(x)$ and $g(x)$.
- The counterpart of the Fundamental Theorem of Arithmetic for $F[x]$
- If F is a field, then $F[x]$ is an integral domain.
- If F is a field and $p(x)$ is a nonconstant polynomial, then $F[x]/(p(x))$ is a commutative ring with identity that contains F.
- $F[x]/(p(x))$ is a field if and only if $p(x)$ is irreducible in $F[x]$.
- $F[x]/p(x)$ is an extension field of F that contains at least one root of $p(x)$.

- If R is a commutative ring with identity and I is an ideal of R, then R/I is an integral domain if and only if I is a prime ideal (Theorem 6.14 in both editions).
- If R is a commutative ring with identity and I is an ideal of R, then R/I is a field if and only if I is a maximal ideal (Theorem 6.15 in both editions).
- In a commutative ring with identity, every maximal ideal is prime.

Things you should be able to prove (and use)

- The remainder theorem
- The factor theorem.
- The simple criterion for checking that a subset is an ideal (Thm 6.1).
- The set of cosets of an ideal forms a ring (the quotient ring). Specifically, addition and multiplication of cosets of an ideal are well defined.
- The kernel of a homomorphism is an ideal.
- for every ring R and every ideal I in R, there is a natural surjective homomorphism $R \to R/I$, given by $r \mapsto r + I$ (Theorem 6.12).
- In a commutative ring with identity, every maximal ideal is prime.
- The First Isomorphism Theorem.

Sample problems

(1) Prove that the set $\{a + b\sqrt{3} | a, b \in \mathbb{Q}\}$ is a field and is isomorphic to $\mathbb{Q}[x]/(x^2 - 3)$.
(2) Give an example of a subring J of a ring R that is not an ideal. Show that multiplication of cosets in R/J is not well defined.
(3) Construct a field of order 4.
(4) Prove that $\mathbb{Z}_4$ is not a field.
(5) Give an example of a maximal ideal in a ring that does not contain all proper ideals of the ring.
(6) Give an example of a prime ideal I in $\mathbb{Z}\times\mathbb{Z}$ that is not maximal. Describe the quotient ring $(\mathbb{Z}\times\mathbb{Z})/I$.
(7) In ring R with multiplicative identity 1_R, prove that any ideal containing a unit must contain the whole ring.
(8) Prove that the only ideals in a field F are (0) and F.
(9) Let $I \subset J$ be ideals of R. Let $J/I = \{j + I \mid j \in J\} \subset R/I$. Prove that J/I is an ideal of R/I.
(10) Prove the first isomorphism theorem.
(11) Find two distinct elements of $\mathbb{Z}_3[x]$ that induce the same function from $\mathbb{Z}_3$ to $\mathbb{Z}_3$.
(12) In the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ find the multiplicative inverse of the congruence class of $x^2 + 1$.
(13) Let T be the space of functions from the real numbers to itself, and I be the set $\{g \in T | g(4) = 0\}$. Prove that I is an ideal and that T/I is isomorphic to $\mathbb{R}$.