682 Homework

(1) For any commutative ring R, the group $\mathrm{GL}_2(R)$ is the set of 2×2 matrices with invertible determinant. Let $\mathbb{F}_q$ be the finite field of order q. What is the order of the group $\mathrm{GL}_2(\mathbb{F}_q)$? What is the order of $\mathrm{SL}_2(\mathbb{F}_q)$?

(2) (a) Prove that the only matrices in $\mathrm{SL}_2(\mathbb{Z})$ which act trivially on $\mathcal{H}$ are $\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.
(b) Prove that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $S = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. (See problem 1.1.1 in Diamond and Shurman.)

(3) Let $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$. Define the weight k operator $|[\gamma]_k$ on functions $\mathcal{H} \to \mathbb{C}$ by

$$(f|[\gamma]_k)(z) = (\det\gamma)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

for all $z \in \mathcal{H}$. Show that $f|[\gamma_1\gamma_2]_k = (f|[\gamma_1]_k)|[\gamma_2]_k$ for $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{Q})$. Prove that checking whether a meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is weakly modular can be done by checking $f(z+1)$ and $f(-1/z)$.

(4) Show that the set $M_k(\mathrm{SL}_2(\mathbb{Z}))$ of modular forms of weight k forms a complex vector space. Show that $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is a subspace. Show that the product of a modular form of weight k_1 and a modular form of weight k_2 is a modular form of weight $k_1 + k_2$, so that the space $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_k M_k(\mathrm{SL}_2(\mathbb{Z}))$ of modular forms of all weights forms a graded ring.

(5) (a) What is the kernel of the homomorphism $\mathrm{GL}_2(\mathbb{Z}/p^e\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$?
(b) What is the order of the group $\mathrm{GL}_2(\mathbb{Z}/p^e\mathbb{Z})$?
(c) What is the order of the group $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$?

(6) Let $N = p_1^{e_1} \cdots p_r^{e_r}$. What is the order of the group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$?

(7) (a) Prove the identity

$$\sum_{n=1}^{\infty} \frac{n^a x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_a(n) x^n.$$

(b) Let $k \geq 4$ be an even integer. Let A_k be the coefficient of q in the Fourier expansion of $E_k(z)$. Prove that the Fourier coefficients of $\frac{E_k(z)}{A_k}$ are multiplicative.

(8) Diamond and Shurman, problem 1.1.6, parts (a), (b), (c).

(9) Given that the space $M_8(\mathrm{SL}_2(\mathbb{Z}))$ is one dimensional, prove that, for all positive integers n,

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i).$$

(10) Prove that the following is a complete set of coset representatives $\{\alpha_i\}$ for $\Gamma_0(p^e)$, i.e., $\mathrm{SL}_2(\mathbb{Z}) = \coprod \alpha_i \Gamma_0(p^e)$ is a disjoint union:

$$I; T^{-k}S, k = 0, 1, \ldots, p^e - 1; ST^{-kp}S, k = 1, 2, \ldots, p^{e-1} - 1.$$

(11) Draw a fundamental domain for $\Gamma_0(4)$. Describe a fundamental domain for $\Gamma_0(p)$ and draw a fundamental domain for $\Gamma_0(3)$.

(12) Prove that $\Gamma_0(p)$ has two cusps, which can be taken to be 0 and ∞. Find the three cusps for $\Gamma(2)$.

(13) Fill in the gap in the proof of the calculation of the index of $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ by showing that the map from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is indeed surjective, as we claimed.

(14) Prove that $\Gamma_0(p^2)$ has $p+1$ cusps: $\infty, 0$, and $-1/kp$ for $k = 1, \ldots, p-1$.

(15) Show that $\Gamma_0(4) = \langle \pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)\rangle$. (See Diamond and Shurman, problem 1.2.4.) What conditions must be checked to ensure that a function f is weakly modular for $\Gamma_0(4)$?

(16) Derive relations expressing σ_5 in terms of σ_1 and σ_3, and σ_7 in terms of σ_1 and σ_5.

(17) Kilford, chapter 2, problem 18. Use the fact that

$$E_2(\gamma z) = (cz+d)^2 E_2(z) + \frac{6c}{i\pi}(cz+d).$$

(18) Show that $E_{12} - E_6^2$ is a cusp form. Since $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ has dimension one, it is a multiple of $\Delta(z) = \sum \tau(n) q^n$. Derive an expression for $\tau(n)$ in terms of σ_{11} and σ_5, and show that $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

(19) Show that every modular form for $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients is a polynomial with integral coefficients in E_4, E_6, and Δ. (You may assume that Δ has integral coefficients, although we have not proven this yet.)

(20) Using the valence formula, find the zeros of the Eisenstein series in the fundamental domain in each of the spaces of modular forms of dimension one. Use the modular transformation equation to check these values.

(21) Find the Fourier expansions, up to the q^5 term, of θE_2, θE_4, θE_6, $E_2^2 - E_4$, $E_2 E_4 - E_6$, and $E_2 E_6 - E_4^2$. What do you notice?

(22) Compute the dimensions of $M_k(\Gamma_0(N))$, $S_k(\Gamma_0(N))$, $M_k(\Gamma_1(N))$, and $S_k(\Gamma_1(N))$ for $1 \le N \le 5$, $0 \le k \le 12$. You may wish to use SAGE to check your work.

(23) Evaluate $\sum_{n=1}^{\infty} \sigma(n) e^{-2\pi n}$.

(24) Show that the differential operator

$$\mathcal{D}_k = \frac{1}{2\pi i}\frac{d}{dz} - \frac{k}{4\pi \mathrm{Im}(z)}$$

satisfies $\mathcal{D}_k(f|[\gamma]_k) = (\mathcal{D}_k f)|[\gamma]_{k+2}$, so it preserves modularity and raises the weight by 2.

(25) Describe a basis for $M_0^!(\mathrm{SL}_2(\mathbb{Z}))$ and $M_2^!(\mathrm{SL}_2(\mathbb{Z}))$. Show that the map $\mathcal{D}_0 : M_0^!(\mathrm{SL}_2(\mathbb{Z})) \to M_2^!(\mathrm{SL}_2(\mathbb{Z}))$ is surjective. (Note that $\mathcal{D}_0 = \theta$ for this weight.) Conclude that if $f \in M_2^!(\mathrm{SL}_2(\mathbb{Z}))$, then the constant term in the Fourier expansion of f is zero.

(26) By induction, prove the formula

$$\mathcal{D}_k^n = \mathcal{D}_{k+2n-2} \circ \cdots \circ \mathcal{D}_k = \sum_{m=0}^{n} \frac{n!}{(n-m)!}\binom{n+k-1}{m}\left(\frac{-1}{4\pi y}\right)^m \theta^{n-m}.$$

Use the fact that if $z = x + iy$, then

$$\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

What happens if the weight is $k = 2 - 2s$ and $n = 2s - 1$?

(27) Recall that if $k = 12\ell + k'$, where $k' \in \{0, 4, 6, 8, 10, 14\}$, then a basis for $M_k^!(\mathrm{SL}_2(\mathbb{Z}))$ is given by the functions $f_{k,m}(z) = q^{-m} + \sum_{n>\ell} a_k(m,n) q^n$ for $m \ge -\ell$. Multiply $f_{k,m}(z)$ by $f_{2-k,n}(z)$ and show that $a_k(m,n) = -a_{2-k}(n,m)$.

(28) Let $k \in \{4, 6, 8, 10, 14\}$. Write $T_p f_{k,m}(z)$ as a sum of basis elements $f_{k,m'}(z)$. Compare Fourier coefficients to show that if $p \nmid m$, then the Fourier coefficient $a_k(m, np)$ is divisible by p^{k-1}.

(29) Find a basis for $S_{32}(\mathrm{SL}_2(\mathbb{Z}))$ consisting of eigenforms. (See Kilford, problem 4.4.1.)

(30) Kilford, problem 4.4.2.

(31) Kilford, problem 4.4.18. Recall that in making a differentiable change of complex variable $z \mapsto u(z)$, the area element $dxdy$ near z is multiplied by $|u'(z)|^2$.

(32) Kilford, problem 4.4.10.

(33) Let $k \ge 4$ be even and let p be prime. Is the form $E_k(z) - E_k(pz)$ a cusp form for $p = 2, 3, 5$? Why or why not?

(34) Kilford, problem 4.4.20.

(35) Kilford, problem 4.4.4, parts (a)-(c).

(36) Kilford, problem 4.4.4, parts (d)-(e).
(37) Prove Theorem 4.19. Note that for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we have $\langle f, g\rangle = \langle f|[\alpha]_k, g|[(\det \alpha^{-1})\cdot \alpha]_k\rangle$.
(38) Kilford, problem 4.4.15.
(39) Kilford, problem 4.4.16.
(40) Kilford, problem 4.4.21.
(41) Show that $|g_1(\chi)| = \sqrt{p}$ by summing $\sum_{a\in(\mathbb{Z}/pZ)^\times} g_a(\chi)\overline{g_a(\chi)}$ in two ways.
(42) Let G_N be the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$ and let $\widehat{G_N}$ be the group of all Dirichlet characters $G_N \to \mathbb{C}^\times$. Show that $\sum_{n\in G_N} \chi(n)$ is $\phi(N)$ if χ is trivial and 0 otherwise. For $(n, N) = 1$, show that $\sum_{\chi\in\widehat{G_N}} \chi(n)$ is $\phi(N)$ if $n \equiv 1 \pmod N$ and 0 otherwise.
(43) Let $t \in \mathbb{Z}^+$. Fix $r \in \mathbb{Z}$ with $(r, t) = 1$, and let $n \in \mathbb{Z}$ be arbitrary. Use the previous problem to show that

$$\sum_{\varepsilon\in\widehat{G_t}} \varepsilon(r)\overline{\varepsilon}(n) = \begin{cases} \varphi(t) & \text{if } r \equiv n \pmod t \\ 0 & \text{if } \gcd(r,t) > 1 \text{ or } r \not\equiv n \pmod t. \end{cases}$$

(44) Let $f(z) = \sum a(n)q^n \in M_k(\Gamma_0(N), \chi)$. Suppose that $r \in \mathbb{Z}$ with $\gcd(r, t) = 1$. Show that

$$\sum_{\substack{n=0 \\ n\equiv r \pmod t}}^{\infty} a(n)q^n \in M_k(\Gamma_1(Nt^2)).$$

This holds also for $\gcd(r, t) > 1$, which you are not required to prove. Hint: Use Theorem 4.29 to twist f by $\overline{\varepsilon}$ and sum $\varepsilon(r)(f_{\overline{\varepsilon}})$ over all $\varepsilon \in \widehat{G_t}$.
(45) Find a holomorphic modular form of weight 6 on $\Gamma_0(3)$ with a zero of order 2 at ∞. Find a weakly holomorphic modular form of weight 0 on $\Gamma_0(3)$ with a pole at ∞. Find a basis for $M_6(\Gamma_0(3))$.
(46) Find the space of modular forms that

$$f(z) = \frac{\eta(5z)^5}{\eta(z)} = q + q^2 + 2q^3 + 3q^4 + \dots$$

is an element of. Repeat for

$$g(z) = \eta(4z)^2\eta(8z)^2,$$

$$F(z) = \frac{\eta^8(4z)}{\eta^4(2z)}.$$

(47) Prove that

$$\eta\left(z + \frac{1}{2}\right) = \frac{e^{2\pi i/48}\eta^3(2z)}{\eta(z)\eta(4z)}.$$

(48) The cusps of $\Gamma_0(4)$ are represented by $\{\infty, 0, 1/2\}$. Show that the values of $F(z)$ (given above) at the cusps are $F(\infty) = 0$, $F(0) = -1/64$, and $F(1/2) = 1/16$.
Hint: Recall the transformation properties for $\eta(z)$ for the matrices S, T. Note that

$$F(1/2) = \lim_{z\to i\infty} F|[\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)]_2,$$

and use the transformation laws to show that

$$F|[\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)]_2 = \frac{-1}{64z^2}\left(e^{-\frac{2\pi i}{24}}\right)^4 \frac{\eta^8\left(\frac{-1}{4z} + \frac{1}{2}\right)}{\eta^4\left(\frac{-1}{2z}\right)}.$$

(49) Let
$$E_2^*(z) = E_2(z) - \frac{3}{\pi \mathrm{Im}(z)}$$
be the nonholomorphic Eisenstein series of weight 2. Show that $E_2^*(z)$ is weight 2 invariant for $\mathrm{SL}_2(\mathbb{Z})$. You may want to prove first that
$$\frac{1}{(cz+d)^2 \mathrm{Im}(\gamma z)} = \frac{1}{\mathrm{Im}(z)} - \frac{2ic}{cz+d}.$$
(50) Let $N \geq 2$ be an integer. Suppose that for all $d|N$, there exist $c_d \in \mathbb{C}$ with $\sum_{d|N} \frac{c_d}{d} = 0$. Show that
$$\sum_{d|N} c_d E_2^*(dz) = \sum_{d|N} c_d E_2(dz) \in M_2(\Gamma_0(N)).$$
Conclude that $E_2(z) - N E_2(Nz) \in M_2(\Gamma_0(N))$.

(51) Let $k \in \mathbb{Z}$ and define the weight k hyperbolic Laplacian by
$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$
It acts on functions $f : \mathcal{H} \to \mathbb{C}$ for which $f(x+iy) = u(x,y) + iv(x,y)$ has u and v with continuous partial derivatives of all orders.
  (a) Show that if $f : \mathcal{H} \to \mathbb{C}$ is holomorphic, then $\Delta_k(f) = 0$. (Remember the Cauchy-Riemann equations.)
  (b) Show that $\Delta_2(E_2^*(z)) = 0$.

(52) Let $\Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$. It is true that $\Theta^4(z) \in M_2(\Gamma_0(4))$. Show that
$$\Theta(z) + \Theta\left(z + \frac{1}{2}\right) = 2\Theta(4z).$$
(53) Show that the values of Θ^4 at the cusps of $\Gamma_0(4)$ are $\Theta^4(\infty) = 1$, $\Theta^4(0) = -1/4$, and $\Theta^4(1/2) = 0$. You will need the transformation laws from class, and the fact that
$$\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) z = \frac{-1}{4\left(\frac{-1}{4z} - \frac{1}{2}\right)}.$$
(54) Integrating around a fundamental domain for $\Gamma_0(4)$, we obtain, for every nonzero modular function of weight k on $\Gamma_0(4)$, the valence formula
$$v_\infty(f) + v_0(f) + v_{1/2}(f) + \sum_{z \in \Gamma_0(4) \backslash \mathcal{H}} v_z(f) = \frac{k}{2}.$$
Assuming this, find a basis for $M_k(\Gamma_0(4))$ for all integers $k \leq 2$. (Look at previous problems for forms that could be basis elements.)

(55) Show that if $k \geq 0$ is even, then a basis for $M_k(\Gamma_0(4))$ is given by the modular forms $\Theta^{4a} F^b$, where $2a + 2b = k$. What is the dimension of $M_k(\Gamma_0(4))$?

(56) Let $G = \frac{\eta^{20}(2z)}{\eta^8(z)\eta^8(4z)}$. Prove that $\Theta^4 = G$, and use this to obtain an eta-quotient expression for $\Theta(z)$.

(57) Kilford, problem 5.11.6: Find a formula for $r_8(n)$.

(58) Show that
$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sigma_1(n) q^n \in M_2(\Gamma_0(8))$$
by twisting a certain modular form by an appropriate character. Given that this form is actually in $M_2(\Gamma_0(4))$, write it in terms of basis elements for this space.

(59) Does the θ-operator commute with the U_p operators? How is $U_p(\theta f)$ related to $\theta(U_p f)$? Repeat for the V_p operator.
(60) Find the dimension of $M_k(\Gamma_0(4), \psi_k)$ for all $k \in \frac{1}{2}\mathbb{Z}$. Recall that this is generated by $\Theta^a F^b$, where $2b + \frac{a}{2} = k$, and that ψ_k is equal to $\left(\frac{-4}{\bullet}\right)$ if k is odd and the trivial character otherwise.
(61) Write $T_{25}\Theta^7$ and $T_9\Theta^{13}$ as polynomials in F and Θ.
(62) Compare the action of T_{p^2} on the Fourier expansion of a modular form of half integral weight $k/2$ for $\Gamma_0(4N)$ with the action of T_{p^2} on the Fourier expansion of a modular form of integral weight k for $\Gamma_0(N)$.
(63) Suppose that $f(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ has integer Fourier coefficients. Show that if $f(z_0) = 0$ for some $z_0 \in \mathcal{H}$, then $j(z_0)$ is an algebraic number.
(64) Kilford, problem 5.11.17: Find the Fourier expansion of the unique normalized eigenform $h \in S_{10}(\Gamma_0(2))$ and represent it in terms of F and Θ.
(65) For primes $p \geq 5$, there is a modular form (specifically, the Eisenstein series) for $\mathrm{SL}_2(\mathbb{Z})$ of weight $p-1$ which is congruent to $1 \pmod p$. For $p = 3$, there are no modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight $p-1$, so this cannot be true. Find a modular form of higher level and weight 2 which is congruent to $1 \pmod 3$.
(66) Calculate $e^{\pi\sqrt{163}}$ to at least twenty decimal places. It is true that if the quadratic imaginary field $\mathbb{Q}(\sqrt{d})$ has class number one, then the j-function, evaluated at imaginary quadratic points in the fundamental domain of discriminant d, has integer values. It is also true that the q^{-1} term of the Fourier expansion of $j(z)$ is the dominant term. How does this explain the answer you found? Find a list of discriminants with class number one and find more such "almost integers". (See Kilford section 5.3.2.)
(67) Evaluate

$$\sum_{\substack{1 \leq k \leq 10^{100} \\ k \,\text{odd}}} \sigma_1(k)\sigma_1(10^{100} - k).$$

No sigma functions should appear in your answer.
(68) Kilford, problem 6.6.4: Show that $\Delta(z) \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and $f = \eta^2(z)\eta^2(11z) \in S_2(\Gamma_0(11))$ are congruent modulo 11.
(69) Kilford, problem 6.6.6: Check that

$$\theta(\Delta) \equiv \Delta E_4^2 E_6 \pmod{13}$$

and find another congruence $\theta(f) \equiv g$ where both f and g are eigenforms.
(70) Find the partition with odd parts that Sylvester's hook bijection maps to the partition $13 + 10 + 9 + 8 + 7 + 4$. Repeat for $12 + 11 + 8 + 6 + 3(+0)$.
(71) Prove that the number of partitions of n in which only odd parts may be repeated equals the number of partitions of n in which no part appears more than three times.
(72) Apply the θ-operator to the log of the generating function for $p(n)$ to prove that $np(n) = \sum_{k=1}^{n} \sigma_1(k)p(n-k)$.
(73) Prove that the number of partitions of n with distinct parts that are all odd equals the number of partitions of n that are self-conjugate.
(74) Prove that

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n (q;q)_{n+1}} = \frac{1}{(q;q)_\infty}.$$

(75) Let $p^{\pm}(n)$ denote the number of partitions of n into an even or odd number of parts, respectively. Show that $|p^-(n) - p^+(n)|$ equals the number of partitions of n into distinct odd parts. (Hint: Replace q by something else in the generating function for $p(n)$.)
(76) Modify the proof that $p(5n + 4) \equiv 0 \pmod 5$ to show that $p(7n + 5) \equiv 0 \pmod 7$.

(77) Show that $\tau(n)$ is 0 $\pmod 7$ when $n \equiv 0, 3, 5, 6 \pmod 7$ by writing $\Delta(z) = q(q;q)_\infty^{24}$ and working modulo 7.

(78) Calculate a large number of consecutive values of $p(n)$. What proportion of them are even? Odd? What proportion of them are $0, 1, 2 \pmod 3$?

(79) Define the Dirichlet character χ_{12} modulo 12 by letting $\chi_{12}(n)$ be equal to 1 if $n \equiv \pm 1 \pmod{12}$, -1 if $n \equiv \pm 5 \pmod{12}$, and 0 otherwise. Use infinite product identities from class to show that

$$\eta(z) = \sum_{n=1}^{\infty} \chi_{12}(n) q^{\frac{n^2}{24}}.$$

(80) Show that

$$\Delta(z)^2 \equiv \left(\prod_{m=1}^{\infty} (1 - q^{7m})^7 \right) \sum_{n=2}^{\infty} p(n-2) q^n \pmod 7.$$

(81) Show that

$$f(z) = q \prod_{n=1}^{\infty} \frac{(1 - q^{8n})^{10} (1 + q^{4n})^{14}}{(1 + q^{8n})^{14}} = \sum_{n=1}^{\infty} a(n) q^n$$

is a cusp form, and find its weight, level and character. Compute the Sturm bound for this space, and check that $f(z)$ is an eigenform for the Hecke operator T_5 by computing enough terms of $U_5 f + 5^{k-1} V_5 f$. (It is actually a Hecke eigenform.)

(82) Let $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$, as usual. Assuming Deligne's result that $|\tau(p)| \leq 2p^{11/2}$, prove that $|\tau(n)| \leq d(n) n^{11/2}$, where $d(n)$ is the number of divisors of n. (Hint: You may need results on solutions for linear recurrence relations.)

(83) Let $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix} \right)$ be the Fricke involution. Show that the trace operator given by $f \to f + p^\alpha f | W_p | U_p$ sends a form $f \in M_k(\Gamma_0(p))$ to a form in $M_k(\mathrm{SL}_2(\mathbb{Z}))$ if α is chosen correctly.

(84) Find a basis of eigenforms for $M_{24}(\mathrm{SL}_2(\mathbb{Z}))$ in terms of the canonical basis $f_{24,m}(z)$ by computing the matrix of at least one Hecke operator.