Weakly Holomorphic Modular Forms in Level 64

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ABSTRACT

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Let $M_k^\sharp(64)$ be the space of weakly holomorphic modular forms in level 64 and weight k which can have poles only at infinity, and let $S_k^\sharp(64)$ be the subspace of $M_k^\sharp(64)$ consisting of forms which vanish at all cusps other than infinity. We explicitly construct canonical bases for these spaces and show that the coefficients of these basis elements satisfy Zagier duality. We also compute the generating function for the canonical basis.

Keywords: weakly holomorphic modular forms, Zagier duality

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Contents

Chapter 1. Introduction

A modular form f of level $N \in \mathbb{N}$ and weight $k \in \mathbb{Z}$ is a holomorphic function on the upper half plane $\mathcal{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$ that is holomorphic at the cusps of $\Gamma_0(N)$ and satisfies the equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tag{1.1}$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}.$$

In addition, if f vanishes at the cusps, then f is a cusp form. The set of all modular forms of level N and weight k is denoted by $M_k(N)$. The set of all cusp forms of level N and weight k is denoted by $S_k(N)$. The set $M_k(N)$ is a complex vector space containing $S_k(N)$ as a subspace.

Since $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \Gamma_0(N)$ for any level N we see that $f\left(\frac{-z}{-1}\right) = (-1)^k f(z)$ holds for any modular form f. Therefore k must be even. We note that under a variation of the definition, modular forms may be defined in odd integer and half integer weight. Also notice that $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma_0(N)$ for any level N, so we know that $f(z+1) = (1)^k f(z)$. This shows that f is a periodic function, which therefore guarantees the existence of a Fourier expansion for f in terms of $q = e^{2\pi i z}$. That is, $f(z) = \sum_{n \geq n_0} a_n q^n$ where n_0 is the order of vanishing of f at infinity.

There is a particular relaxation of the preceding definition which yields an interesting space of functions: if f is a holomorphic function on $\mathcal{H}$ that satisfies (1.1) and is meromorphic at the cusps, then f is a weakly holomorphic modular form of level N and weight k. The set of weakly holomorphic modular forms of level N and weight k is denoted by $M_k^!(N)$. The subspace $M_k^\sharp(N)$ of $M_k^!(N)$ is the set of functions $f \in M_k^!(N)$ that are holomorphic at all cusps other than the cusp at infinity. That is, the elements of $M_k^\sharp(N)$ are holomorphic on the upper half plane and at all of the cusps except infinity, where the elements are meromorphic. The subspace of $M_k^\sharp(N)$ comprised by functions which vanish at each cusp other than the cusp at infinity is denoted by $S_k^\sharp(N)$.

There has been a lot of interest in the spaces of weakly holomorphic modular forms for various levels and weights, much of which has been motivated by Duke and Jenkins in [2]. In the aforementioned paper, Duke and Jenkins restrict their attention to a certain canonical basis for the space $M_k^\sharp(1)$ which they denote by the Fourier expansions

$$f_{k,m}(z) = q^{-m} + \sum_{n \geq \ell+1} a_k(m,n) q^n,$$

where $k = 12\ell + k'$ for uniquely determined $\ell \in \mathbb{Z}$ and $k' \in \{ 0, 4, 6, 8, 10, 14 \}$. They then make some statements about these basis elements and the coefficients thereof:

(i) The zeros of the functions $f_{k,m}$ lie on the unit circle for appropriate restrictions on m.

(ii) The generating function is $\displaystyle\sum_{m \geq -\ell} f_{k,m}(\tau) q^m = \frac{f_{k,-\ell}(\tau) f_{2-k,-\ell}(z)}{j(z) - j(\tau)}$.

(iii) The coefficients of the functions $f_{k,m}$ satisfy Zagier duality: $a_k(m,n) = -a_{2-k}(n,m)$.

(iv) If $(m,n) = 1$ and $k \in \{ 4, 6, 8, 10, 14 \}$ then $n^{k-1} \mid a_k(m,n)$.

There has been a lot of effort put into extending these results to higher levels. In levels 2 and 3, Garthwaite and Jenkins [4] give a lower bound on the number of nontrivial zeros of the corresponding canonical basis elements that lie on the lower boundary of a fundamental domain. In doing so they make use of the generating function for these levels as given in El-Guindy's paper [3]. For these same levels, Andersen and Jenkins [1] show that the coefficients of the basis elements satisfy some congruence/divisibility conditions. In level 4, Haddock and Jenkins [5] give a generating function for the basis elements, show that the basis elements satisfy a duality result, and prove a theorem about zeros of the basis elements lying on the lower boundary of the fundamental domain. For levels 2, 3, 5, 7, and 13, Jenkins and Thornton [6] extend the results of Andersen, showing more congruence properties. Jenkins and Thornton [7], in another paper, additionally show congruence properties in levels 8, 9, 16, and 25. These are most of the levels where the congruence subgroup $\Gamma_0(N)$ is genus zero. At the time of this writing several students are working on the remaining genus zero levels and on extensions of the results (i), (ii), (iii), and (iv) from above to several genus one levels, such as 11, 17, and 19.

For levels $N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25\}$ the congruence subgroup $\Gamma_0(N)$ has genus zero. In these levels, one of the most heavily used tools in the study of basis elements is the Hauptmodul. The Hauptmodul in level N (coming from the set above) is a weakly holomorphic modular form φ of weight 0 which has a pole of order one at infinity. This function aids in the construction of the canonical basis of $M_k^\sharp(N)$. In order to describe this canonical basis it is necessary to consider, for given function $f \in M_k^\sharp(N)$, the function $g = f \cdot \varphi$. The function g has some interesting properties. First we notice that $g \in M_k^\sharp(N)$; indeed, g still satisfies (1.1) for weight k, and g is holomorphic at each cusp other than infinity since multiplication by φ cannot lower the order of vanishing at these other cusps. Second, the order of vanishing of g at infinity is exactly one less than the order of vanishing of f at infinity. That is, $\mathrm{ord}_\infty(g) = \mathrm{ord}_\infty(f) - 1$. Since the smallest power of q in the Fourier expansion of f is the order of vanishing at infinity, we often refer to the order of vanishing at infinity by this leading term. With this terminology we see that the given function f has leading term q^m and from that function we create a function g with leading term q^{m-1}. Proceeding in an inductive fashion, we can create a list of weakly holomorphic modular forms indexed by the order of vanishing at infinity. By imposing the condition that we include in this list only one form of each leading power of q, we have in fact written down a basis of $M_k^\sharp(N)$. So far there is nothing that makes the preceding basis canonical, since there is nothing that guarantees uniqueness of its elements. However this issue can be resolved by forcing each basis element to have as large a gap as possible between the leading term and the next nonzero term in its Fourier expansion.

In levels for which $\Gamma_0(N)$ does not have genus zero there does not exist a modular form in $M_0^\sharp(N)$ with leading term q^{-1} and thus there is no Hauptmodul. However, the canonical basis can still be defined. Suppose that B is any basis of $M_0^\sharp(N)$ whose leading terms have coefficient 1 and whose exponents are in decreasing order. Let the first element of B have leading term q^{-n_0} for some $n_0 \in \mathbb{Z}$. We will inductively create a new basis C in the following way. The first element of B will be the first element of C. Assume that for $n \geq 1$ we have already found the first n basis elements of C. Let f be the $(n+1)^{\text{st}}$ element of B. We eliminate from f any terms that happen to be the leading term of one of the elements of C by adding a scalar multiplied by the appropriate element of C. The result is the $(n+1)^{\text{st}}$ element of C. Proceed inductively to construct C. As a consequence the elements of C have as large a gap as possible between the leading term and the next nonzero term. However we note that it is entirely possible that there are finitely many powers $\ell > n_0$ for which $q^{-\ell}$ is not the leading term of an element of B. Therefore, for ℓ in this finite set of powers, there is no element of C with leading term $q^{-\ell}$ (see Section 2.1 for an explicit example). When this happens the elements of C will have as large a gap as possible between the leading term and the first $q^{-\ell}$ term with additional gaps (made as large as possible) between each of the subsequent pairs of $q^{-\ell}$ terms. We note that the basis C will be constructed independent of the choice of initial basis B. This basis C is the canonical basis. In the coming chapter, the bases B and C will be for the appropriate space as understood from context.

There is a weakly holomorphic modular form of level N and weight 0 with a pole of minimal order at infinity, or a minimal $n \in \mathbb{N}$ so that there is a weight 0 form $\varphi_n(z)$ whose leading term is q^{-n}. This function φ_n fulfills the same role that the Hauptmodul did. We inductively create a list of modular forms so that there is exactly one form ordered by decreasing exponent of each possible leading power of q. This list is the basis B of $M_k^\sharp(N)$ that we use to construct the canonical basis.

Notice that for levels N where $\Gamma_0(N)$ has genus zero, there are no skipped powers in the leading terms of the canonical basis elements. However, for levels N where $\Gamma_0(N)$ does not have genus zero, it is often the case that at least one power in the leading terms of the canonical basis for $M_k^\sharp(N)$ is skipped. There are also levels where there is at least one skipped power in the leading terms of the canonical basis for $S_k^\sharp(N)$. We are interested in levels where skipped powers are observed for bases of both $M_k^\sharp(N)$ and $S_k^\sharp(N)$. In particular we will focus on levels when $S_2(N)$ has skipped powers. This happens when level N has a Weierstrass point at infinity.

Definition 1.1. Let $N \in \mathbb{N}$ be given, and let $g(N)$ be the genus of $\Gamma_0(N)$. We say that level N has a Weierstrass point at infinity if there is a non-zero cusp form $f \in S_2(N)$ with

$$\operatorname{ord}_\infty f > g(N).$$

The author ran the SAGE script in appendix A.1 and found that the levels less than 100 with a Weierstrass point at infinity are 54, 64, 72, 96, and 98.

In order to make the computations of basis elements simpler to program, it is desirable to work in a level where the modular forms can be expressed as eta-quotients. An eta-quotient is a function

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}$$

where $r_\delta \in \mathbb{Z}$ and

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta-function. In an introductory text on the theory of modular forms [8], Kilford gives an important theorem due to Newman that states if

$$\begin{aligned} &\text{(i)} \sum_{\delta|N} \delta r_\delta \equiv 0 \mod 24, \\ &\text{(ii)} \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \mod 24, \text{and} \\ &\text{(iii)} \prod_{\delta|N} \delta^{r_\delta} \text{is the square of a rational number;} \end{aligned} \tag{1.2}$$

then $f(z)$ respects the transformation property in (1.1) for weight $k = \frac{1}{2}\sum_{\delta|N} r_\delta$. Ligozat [9] gives the following theorem which allows us to determine at which cusps an eta-quotient vanishes.

Theorem 1.2. *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient, then the order of vanishing of $f(z)$ at the cusp c/d is*

$$\operatorname{ord}_{c/d} f = \frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd\left(d, \frac{N}{d}\right) d\delta}.$$

The results of Newman and Ligozat together provide a way to determine when an eta-quotient is a weakly holomorphic modular form. Rouse and Webb provide a website containing the scripts and output files that are associated to their paper. In particular they provide the file `http://users.wfu.edu/rouseja/eta/etamake9.out` which is useful to us since it shows that for levels $54, 64, 72$, and 96, the space $M_2(N)$ is spanned by eta-quotients.

Armed with this information, we restrict our attention to the level $N = 64$ since it is both spanned by eta-quotients and has a Weierstrass point at infinity. Level 64 was chosen instead of 54, 72, or 96 since 64 has only one prime divisor. This thesis will show several results about the weakly holomorphic modular forms of level 64. First, we will construct canonical bases for both the spaces $M_k^\sharp(64)$ and $S_k^\sharp(64)$ for any even integer weight k. Several explicit examples of what these basis elements look like will be provided along with computer code (see A.4 and A.5) that generates examples for any even integer weight k. The canonical basis elements of $M_k^\sharp(64)$ are denoted by the functions

$$f_{k,m}(z) = q^{-m} + \sum_{n=n_0}^{\infty} a_k(m,n)q^n,$$

and the canonical basis elements of $S_k^\sharp(64)$ are denoted by the functions

$$g_{k,m}(z) = q^{-m} + \sum_{n=n_0}^{\infty} b_k(m,n).$$

Second, we show that the canonical basis elements satisfy a duality property by the following theorem.

Theorem 3.1. *For the basis elements $f_{k,m}(z) \in M_k^\sharp(64)$ and $g_{2-k,n}(z) \in S_{2-k}^\sharp(64)$ we have*

$$a_k(m,n) = -b_{2-k}(n,m).$$

Third, a derivation of the generating function for the basis elements $f_{k,m}(z)$ will be provided, giving a proof of the following theorem.

Theorem 4.1.

$$F_k(z,\tau) = \sum_{n=-8k}^{\infty} f_{k,n}(\tau)q^n =$$

$$\frac{f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) + f_{k,-8k}(\tau)g_{2-k,8k+3}(z)}{f_{0,3}(z) - f_{0,3}(\tau)}.$$

The author expects that these three results can be extended to levels 54, 72, and 96 without much difficulty. Lastly, conjectures will be provided about congruences for the coefficients of the basis elements.

In order to construct the canonical bases for $M_k^\sharp(64)$ and $S_k^\sharp(64)$ we need a modular form in $M_0^\sharp(64)$ with a pole at infinity of minimal order. A basis for $S_2(64)$ is given by SAGE:

$$\begin{aligned}
g_{2,-5}(z) &= q^5 - 3q^{13} + 5q^{29} + q^{37} - 3q^{45} - 7q^{53} + O(q^{61}),\\
g_{2,-2}(z) &= q^2 - 2q^{10} - 3q^{18} + 6q^{26} + 2q^{34} - q^{50} + O(q^{58}),\\
g_{2,-1}(z) &= q - 3q^9 + 2q^{17} - q^{25} + 10q^{41} - 7q^{49} + O(q^{65}).
\end{aligned} \tag{2.1}$$

Notice that there are no cusp forms that begin with leading terms q^3 or q^4. Now if φ is a weight 0 weakly holomorphic modular form with leading term q^{-n} then $\varphi(z)g_{2,-5}(z)$ has leading term q^{5-n}. If $n = 1$, then this leading term is q^4, a contradiction. If $n = 2$, then this leading term is q^3, a contradiction. However, there are no problems with $n = 3$ or $n = 4$. The SAGE script A.2 produced the following forms in $M_0^\sharp(64)$ with leading terms q^{-3} and q^{-4} respectively:

$$\begin{aligned}
\frac{\eta^2(16z)\eta(32z)}{\eta(8z)\eta^2(64z)} &= q^{-3} + q^5 + q^{21} + O(q^{29}),\\
\frac{\eta^5(2z)\eta(32z)}{\eta^2(z)\eta^2(4z)\eta^2(64z)} &= q^{-4} + 2q^{-3} + 2 + 2q^5 + 2q^{12} + 2q^{21} + O(q^{28}),\\
\frac{\eta^5(8z)\eta(32z)}{\eta^2(4z)\eta^2(16z)\eta^2(64z)} &= q^{-4} + 2 + 2q^{12} - q^{28} + O(q^{44}),\\
\frac{\eta^6(32z)}{\eta^2(16z)\eta^4(64z)} &= q^{-4} + 2q^{12} - 2q^{28} + O(q^{44}),\\
\frac{\eta^2(4z)\eta(32z)}{\eta(8z)\eta^2(64z)} &= q^{-4} - 2 + 2q^{12} - q^{28} + O(q^{44}),\\
\frac{\eta^2(z)\eta(32z)}{\eta(2z)\eta^2(64z)} &= q^{-4} - 2q^{-3} + 2 - 2q^5 + 2q^{12} - 2q^{21} + (q^{28}).
\end{aligned}$$

It should be noted that the implementation in SAGE that is used for eta quotients is $(r_1, r_2, r_4, r_8, r_{16}, r_{32}, r_{64})$ to represent the eta-quotient

$$\eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_4}(4z)\eta^{r_8}(8z)\eta^{r_{16}}(16z)\eta^{r_{32}}(32z)\eta^{r_{64}}(64z).$$

The SAGE script A.3 provides the code necessary to convert the representation of an eta-quotient as a tuple into its Fourier expansion. We also note that the function `checkNum` in the script A.2 encodes the order of vanishing formula for each possible cusp denominator.

The function which begins with q^{-3} is $f_{0,3}(z)$ and is the form with pole of minimal order. There are also five different forms that begin with q^{-4}. Since the function $f_{0,0}(z) \coloneqq 1$ is an element of $M_0^\sharp(64)$, we can subtract copies of $f_{0,0}(z)$ and $f_{0,3}(z)$ to cancel any constant terms and any q^{-3} terms in each of the forms beginning with q^{-4} from the list above. Each of these forms will reduce to the function $f_{0,4}(z) \coloneqq \dfrac{\eta^6(32z)}{\eta^2(16z)\eta^4(64z)}$ from above. The functions $f_{0,0}(z)$, $f_{0,3}(z)$, and $f_{0,4}(z)$ are the first three nonzero elements of the canonical basis for $M_0^\sharp(64)$.

2.1 The canonical basis for $M_0^\sharp(64)$

We now have all the tools we need to construct the canonical basis for $M_0^\sharp(64)$. Recall that there are no elements of $M_0^\sharp(64)$ beginning with q^{-1} or q^{-2}. There also is no element of $M_0^\sharp(64)$ that begins with q^{-5}; this will be explained in Section 2.3. It is easily noticed that $h_6(z) = f_{0,3}^2(z)$, $h_7 = f_{0,3}(z)f_{0,4}(z)$, and $h_8 = f_{0,4}^2(z)$ are elements of $M_0^\sharp(64)$ beginning with q^{-6}, q^{-7}, and q^{-8} respectively. Taking these three functions multiplied by $f_{0,3}^\ell(z)$ for any $\ell \in \mathbb{N}$ will result in forms $h_n(z)$ that begin with q^{-n} for all $n \geq 6$. That is, for $n \geq 6$ we can write $n = 3\ell + r$ for $r \in \{ 0, 1, 2 \}$ which allows us to define

$$h_n(z) = \begin{cases} h_6(z)\left[f_{0,3}(z)\right]^{\ell-2} & \text{if } r = 0, \\ h_7(z)\left[f_{0,3}(z)\right]^{\ell-2} & \text{if } r = 1, \\ h_8(z)\left[f_{0,3}(z)\right]^{\ell-2} & \text{if } r = 2. \end{cases}$$

The set $B = \{ f_{0,0}(z), f_{0,3}(z), f_{0,4}(z) \} \cup \{ h_n(z) \mid n \geq 6 \}$ is a basis of $M_0^\sharp(64)$. We now use B to construct the canonical basis C in the manner described previously. First notice that the forms $h_6(z)$, $h_7(z)$, $h_8(z)$, $h_9(z)$, and $h_{10}(z)$ do not need any terms canceled since they do not contain any q^{-3}, q^{-4}, or constant terms:

$$\begin{aligned}
h_6(z) &= q^{-6} + 2q^2 + q^{10} + 2q^{18} - 4q^{34} + O(q^{42}),\\
h_7(z) &= q^{-7} + q + 2q^9 + 3q^{17} - 2q^{25} - O(q^{41}),\\
h_8(z) &= q^{-8} + 4q^8 + 2q^{24} + O(q^{40}),\\
h_9(z) &= q^{-9} + 3q^{-1} + 3q^7 + 4q^{15} + 3q^{23} - 6q^{31} + O(q^{39}),\\
h_{10}(z) &= q^{-10} + 2q^{-2} + 3q^6 + 6q^{14} + q^{22} - 2q^{30} + O(q^{38}).
\end{aligned}$$

Therefore $f_{0,6}(z) = h_6(z)$, $f_{0,7}(z) = h_7(z)$, $f_{0,8}(z) = h_8(z)$, $f_{0,9}(z) = h_9(z)$, and $f_{0,10}(z) = h_{10}(z)$ are elements of the canonical basis. Next notice that

$$h_{11}(z) = q^{-11} + q^{-3} + 4q^5 + 5q^{13} + q^{21} + 5q^{29} + O(q^{37})$$

contains a q^{-3} term. We cancel this term by subtracting $f_{0,3}(z)$ to get the next canonical basis element:

$$f_{0,11}(z) = h_{11}(z) - f_{0,3}(z) = q^{-11} + 3q^5 + 5q^{13} + 6q^{29} + O(q^{37}).$$

Similarly we notice that

$$h_{12}(z) = q^{-12} + 4q^{-4} + 6q^4 + 8q^{12} + 9q^{20} - 4q^{28} + O(q^{36})$$

is not an element of the canonical basis since it contains a q^{-4} term. By canceling this term the canonical basis element is obtained:

$$f_{0,12}(z) = h_{12}(z) - 4f_{0,4}(z) = q^{-12} + 6q^4 + 9q^{20} + O(q^{36}).$$

Continuing in this fashion of eliminating the leading terms of previous basis elements we inductively construct the canonical basis for $M_0^\sharp(64)$. It is important to notice that while eliminating terms, we are using *any* of the previous basis elements, not just $f_{0,3}(z)$ and $f_{0,4}(z)$ that appear in the examples.

Explicitly, the canonical basis is given by $\{\, f_{0,m}(z) \mid m \geq 0 \text{ and } m \neq 1, 2, 5\,\}$ where the functions $f_{0,m}(z)$ can be written as

$$f_{0,m}(z) = q^{-m} + a_0(m,-5)q^{-5} + a_0(m,-2)q^{-2} + a_0(m,-1)q^{-1} + \sum_{n=1}^{\infty} a_0(m,n)q^n. \qquad (2.2)$$

With this notation, here and throughout this thesis, it is understood that q^{-m} is the first nonzero term of $f_{0,m}(z)$. If, for example, $m = 4$, then $a_0(4,-5) = 0$ since the q^{-5} term doesn't show up. A rigorous discussion as to why this "alleged" basis does in fact span all of $M_0^\sharp(64)$ will be given in Section 2.3

2.2 The canonical basis for $S_2^\sharp(64)$

Recall that

$$g_{2,-5}(z) = q^5 - 3q^{13} + 5q^{29} + q^{37} - 3q^{45} - 7q^{53} + O(q^{61}),$$
$$g_{2,-2}(z) = q^2 - 2q^{10} - 3q^{18} + 6q^{26} + 2q^{34} - q^{50} + O(q^{58}),$$
$$g_{2,-1}(z) = q - 3q^9 + 2q^{17} - q^{25} + 10q^{41} - 7q^{49} + O(q^{65}).$$

is a basis for $S_2(64)$. Just as with the basis for $M_0^\sharp(64)$, the goal is to construct a list of forms with decreasing power in its leading term. By using the functions $f_{0,3}(z)$ and $f_{0,4}(z)$ we get the following three forms with leading term q^{-1}, q^{-2}, and q^{-3} respectively:

$$h_1'(z) = g_{2,-2}(z)f_{0,3}(z) = q^{-1} - q^7 - 5q^{15} + 4q^{23} + 5q^{31} + O(q^{47}),$$
$$h_2'(z) = g_{2,-1}(z)f_{0,3}(z) = q^{-2} - 2q^6 - q^{14} + 2q^{22} - 5q^{30} + 14q^{38} + O(q^{46}),$$
$$h_3'(z) = g_{2,-1}(z)f_{0,4}(z) = q^{-3} - 3q^5 + 4q^{13} - 7q^{21} + 3q^{29} + 11q^{37} + O(q^{45}).$$

Notice that there is no function $h'_0(z)$; the reason that such a function cannot exist is given in Section 2.3. These functions $h'_1(z)$, $h'_2(z)$, and $h'_3(z)$ can be used in conjunction with $f_{0,3}(z)$ to create, for any $n \in \mathbb{N}$, a modular form in $S_2^\sharp(64)$ with leading term q^{-n}. Let $h'_n(z)$ be the modular form in $S_2^\sharp(64)$ with leading term q^{-n} defined by

$$h'_n(z) = \begin{cases} h'_1(z)\big[f_{0,3}(z)\big]^{\ell} & \text{if } r = 1 \\ h'_2(z)\big[f_{0,3}(z)\big]^{\ell} & \text{if } r = 2 \\ h'_3(z)\big[f_{0,3}(z)\big]^{\ell-1} & \text{if } r = 0 \end{cases}$$

where $n \geq 1$ is written as $n = 3\ell + r$ for $r \in \{\, 0, 1, 2 \,\}$.

Notice that $B = \{\, g_{2,-5}(z), g_{2,-2}(z), g_{2,-1}(z) \,\} \cup \{\, h'_n(z) \mid n \geq 1 \,\}$ is a basis of $S_2^\sharp(64)$, albeit not the canonical one. Just as before, we construct the canonical basis C by eliminating as many terms as possible by using the previous basis elements.

First notice that $h'_1(z)$ and $h'_2(z)$ do not contain any q^5, q^2, or q terms. Therefore $g_{2,1}(z) = h'_1(z)$ and $g_{2,2}(z) = h'_2(z)$. However, $h'_3(z)$ does contain one of these three terms. This means that 3 copies of $g_{2,-5}(z)$ need to be added to it to get the canonical basis element:

$$g_{2,3}(z) = h'_3(z) + 3g_{2,-5}(z) = q^{-3} - 5q^{13} - 7q^{21} + 18q^{29} + 14q^{37} + O(q^{45}).$$

Next, notice that

$$h'_4(z) = h'_1(z)f_{0,3}(z) = q^{-4} - 6q^{12} + 7q^{28} + O(q^{44}),$$
$$h'_5(z) = h'_2(z)f_{0,4}(z) = q^{-5} - q^{3} \quad - 3q^{11} + 2q^{19} - 6q^{27} + 9q^{35} + O(q^{43}).$$

Therefore $g_{2,4}(z) = h'_4(z)$ and $g_{2,5}(z) = h'_5(z)$. Looking at $h'_6(z)$ we see that

$$h'_6(z) = q^{-6} - 2q^{2} + q^{10} - 2q^{18} - 8q^{26} + 20q^{34} + O(q^{42}).$$

The basis element $g_{2,6}(z)$ is obtained by adding 2 copies of $g_{2,-2}(z)$ to $h_6'(z)$:

$$g_{2,6}(z) = h_6'(z) + 2g_{2,-2}(z) = q^{-6} - 3q^{10} - 8q^{18} + 4q^{26} + 24q^{34} + O(q^{42}).$$

Similarly, the function $h_7'(z)$ is not the canonical basis element since it contains a q term:

$$h_7'(z) = q^{-7} + q - 6q^9 - 5q^{17} + 6q^{25} + O(q^{41}).$$

The basis element $g_{2,7}(z)$ is obtained by subtracting $g_{2,-1}(z)$ from $h_7'(z)$:

$$g_{2,7}(z) = h_7'(z) - g_{2,-1}(z) = q^{-7} - 3q^9 - 7q^{17} + 7q^{25} + O(q^{41}).$$

At this point we see exactly how (through the examples) the canonical basis is constructed inductively by eliminating terms that are the leading terms of previous basis elements. Explicitly, this basis is given by the set $\{\, g_{2,m}(z) \mid m \geq -5 \text{ and } m \neq -4, -3, 0 \,\}$ where the modular forms $g_{2,m}(z)$ can be represented by

$$g_{2,m}(z) = q^{-m} + b_k(m,0) + b_k(m,3)q^3 + b_k(m,4)q^4 + \sum_{n=6}^{\infty} b_k(m,n)q^n. \tag{2.3}$$

The next section will give a rigorous explanation for why this "alleged" basis does in fact span all of $S_2^\sharp(64)$.

2.3 The span of the basis elements for $M_0^\sharp(64)$ and $S_2^\sharp(64)$

For the given basis of $M_0^\sharp(64)$, it is clear that there should not be any modular forms that have leading term q^n for $n \geq 1$. If such a form were to exist, it would be an element of $M_0(64)$, which is the set of constant functions. It has already been explained that there cannot be elements of $M_0^\sharp(64)$ with leading term q^{-1} or q^{-2}. However, there has not yet been any justification for why there do not exist elements with leading term q^{-5}. Once this fact has been justified, we know that the set of forms $\{ f_{0,m}(z) \}$ that has been given is a basis. The reason is twofold: this list contains a form of each possible leading term and these elements are linearly independent.

Before we justify the nonexistence of forms with leading term q^{-5} in $M_0^\sharp(64)$ it is necessary to discuss and rigorously explain why the set of forms $\{ g_{2,m}(z) \}$ is a basis of $S_2^\sharp(64)$. We first notice that this set, $\{ g_{2,m}(z) \mid m \geq -5, m \neq -4, -3, 0 \}$, is linearly independent. To show that it spans $S_2^\sharp(64)$, we show that there are no forms in $S_2^\sharp(64)$ that have leading terms differing from those functions $g_{2,m}(z)$. That is, we show that there are no forms in $S_2^\sharp(64)$ with leading term q^4, q^3, or q^0.

Recall that $S_2(64)$ is a subspace of $S_2^\sharp(64)$. If there was a form in $S_2^\sharp(64)$ with leading term q^3 or q^4 then this form would vanish at every cusp (including infinity) and therefore be an element of $S_2(64)$. SAGE has given us a basis of $S_2(64)$ (see equation (2.1)), and it clearly doesn't have any forms with these leading terms. Therefore $S_2^\sharp(64)$ has no forms with leading terms q^3 or q^4. Next we show that there are no forms with constant leading term. This will take a slightly more sophisticated argument since such a form could potentially exist in $S_2^\sharp(64)$ even though it is clear that it would not exist in $S_2(64)$. This potential comes from the fact that the elements of $S_2^\sharp(64)$ are just required to vanish at cusps away from infinity, but not necessarily at infinity itself. Suppose that $f(z)$ is an element of $S_2^\sharp(64)$ with constant leading term. Consider the eta-quotient

$$\psi(z) = \frac{\eta^8(64z)}{\eta^4(32z)} = q^{16} + 4q^{48} + O(q^{80}). \tag{2.4}$$

It is an easy exercise to use the conditions given by Newman in (1.2) to show that $\psi(z)$ respects the transformation given by (1.1) for weight $k = 2$. Additionally one can use the order of vanishing formula given by Ligozat in Theorem 1.2 to show that $\psi(z)$ is holomorphic at every cusp. Hence $\psi(z) \in M_2(64)$. Notice that $\psi(z)f(z)$ is a modular form of weight 4 that vanishes at every cusp: it vanishes at the cusps other than infinity since $f(z)$ does, and vanishes at infinity since its leading term is q^{16}. Next we use SAGE to write down a basis for $S_4(64)$:

$$q - 4q^{25} - q^{33} + O(q^{40}),$$
$$q^2 - 6q^{26} + q^{34} + O(q^{40}),$$
$$q^3 - 5q^{27} - 10q^{35} + O(q^{40}),$$
$$q^4 - 2q^{20} - 11q^{36} + O(q^{40}),$$
$$q^5 - 10q^{29} + 4q^{37} + O(q^{40}),$$
$$q^6 - q^{22} - 8q^{30} + q^{38} + O(q^{40}),$$
$$q^7 - 2q^{23} - 5q^{31} - 5q^{39} + O(q^{40}),$$
$$q^8 - 4q^{24} + O(q^{40}),$$
$$q^9 - 3q^{25} - 7q^{33} + O(q^{40}),$$
$$q^{10} - 3q^{26} - 8q^{34} + O(q^{40}),$$
$$q^{11} - q^{19} - 3q^{27} + 2q^{35} + O(q^{40}),$$
$$q^{12} - 6q^{28} + O(q^{40}),$$
$$q^{13} - 5q^{29} - 7q^{37} + O(q^{40}),$$
$$q^{14} - 2q^{22} - q^{30} + 2q^{38} + O(q^{40}),$$
$$q^{15} - q^{23} - 5q^{31} + 4q^{39} + O(q^{40}),$$
$$q^{17} - 3q^{25} + 2q^{33} + O(q^{40}),$$
$$q^{18} - 2q^{26} - 3q^{34} + O(q^{40}),$$
$$q^{21} - 3q^{29} + O(q^{40}).$$

Notice that none of these functions contain the term q^{16}, therefore the form $\psi(z)f(z)$ doesn't actually exist. Hence the form $f(z)$ doesn't exist, showing that there are no elements of $S_2^\sharp(64)$ with constant leading term. In fact, Lemma 3.3 will show that no element of $S_2^\sharp(64)$ contains a constant term. Finally notice that if $M_0^\sharp(64)$ contains a form $f(z)$ with leading term q^{-5} then $g_{2,-5}(z)f(z)$ will be a modular form in $S_2^\sharp(z)$ with constant leading term. Since this form can't exist, neither can the form $f(z)$. We have now shown that the sets $\{\, f_{0,m}(z) \mid m \geq 0, m \neq 1, 2, 5 \,\}$ and $\{\, g_{2,m}(z) \mid m \geq -5, m \neq -4, -3, 0 \,\}$ are indeed spanning sets, and hence bases, of $M_0^\sharp(64)$ and $S_2^\sharp(64)$ respectively.

2.4 Canonical bases for arbitrary weight $k \in 2\mathbb{Z}$

Now comes the challenge of generalizing the results of the previous sections to the case where k is an arbitrary even integer. Fortunately, there is an incredibly helpful tool that can be used to change weights: the function $\psi(z)$ given in (2.4).

Recall that for a given weight k and the congruence subgroup $\Gamma_0(64)$, the sum of the weighted zeros for a weakly holomorphic modular form is $8k$. In general this weighted sum is equal to $\frac{kM}{12}$ where M is the index of the congruence subgroup in $\mathrm{SL}_2(\mathbb{Z})$. This allows us to see that $\psi(z)$ vanishes only at the cusp at infinity and at no other cusps, since $\mathrm{ord}_\infty \psi = 16 = 8(2)$. Stated another way, $\psi(z)$ has all of its zeros at infinity.

This modular form $\psi(z)$ is useful for its ability to create new functions of differing weights. For example, multiplying the weight 0 form $f_{0,m}(z)$ by $\psi(z)$ results in a weight 2 modular form with leading term q^{-m+16}. In general, multiplication by $\psi(z)$ will produce modular forms whose weight has increased by two.

The function $\psi(z)$ can also be used to create new functions of lower weights. For example, consider the weight 0 form $f_{0,m}(z)$. By dividing this form by $\psi(z)$, the function $f(z)$ with leading term q^{-m-16} is obtained. Therefore the order of vanishing of $f(z)$ at infinity is $-m-16$. Moreover, since $\psi(z)$ does not vanish at any other cusps, the function $\big[\psi(z)\big]^{-1}$ does not have any poles at the cusps away from infinity. Therefore $f(z) = f_{0,m}(z)\big[\psi(z)\big]^{-1}$ also does not have poles at these cusps. This shows that $f(z)$ is holomorphic at the cusps away from infinity. Hence $f(z)$ is a weakly holomorphic modular form of weight -2. In general, division by $\psi(z)$ will produce modular forms whose weight has decreased by two.

In addition, since multiplication and division by $\psi(z)$ only affects the vanishing at infinity, these operations send forms in $S_k^\sharp(N)$ to $S_{k+2}^\sharp(N)$ and $S_{k-2}^\sharp(N)$ respectively. We now understand how $\psi(z)$ will be used. For the rest of the section let $k = 2\ell \in 2\mathbb{Z}$ be fixed.

Notice that for any $m \geq 0$ with $m \neq 1, 2, 5$ the function $h_n(z) = f_{0,m}(z)\big[\psi(z)\big]^\ell$ is a weight k weakly holomorphic modular form with leading term $q^{-n} = q^{-m+16\ell} = q^{-m+8k}$. Since $-n = -m+8k$ we see that $h_n(z)$ is defined for $n \geq -8k$ and $n \neq -8k+1, -8k+2, -8k+5$.

We claim that the set $\{\, h_n(z) \mid n \geq -8k, n \neq -8k+1, -8k+2, -8k+5 \,\}$ is a basis for $M_k^\sharp(64)$. To see this, it suffices to show that the leading term of any modular form must coincide with leading term of $h_n(z)$ for some n. To this end, suppose that $f(z) \in M_k^\sharp(64)$ has leading term q^{-n_0}. Then $f(z)\big[\psi(z)\big]^{-\ell}$ is a weight 0 form with leading term $q^{-n_0-16\ell} = q^{-n_0-8k} = q^{-(n_0+8k)}$. Therefore $n_0 + 8k \geq 0$ with $n_0 + 8k \neq 1, 2, 5$. Hence $n_0 \geq -8k$ and

$n_0 \neq -8k+1, -8k+2, -8k+5$ as desired.

Starting with the set $\{\, h_n(z) \mid n \geq -8k, n \neq -8k+1, -8k+2, -8k+5 \,\}$, we construct the canonical basis for $M_k^\sharp(64)$ inductively just as we did for $M_0^\sharp(64)$. It should be noted that it is necessary to eliminate the terms that are leading terms of previous canonical basis elements. As an example, for weight $k = 2$, the form $h_{16}(z) = f_{0,32}(z)\psi(z) = q^{-16} + 4q^{16} + 282q^{48} + O(q^{80})$ is not an element of the canonical basis since it contains a q^{16} term. The canonical basis element with leading term q^{-16} is

$$f_{2,16}(z) = h_{16}(z) - 4f_{2,-16}(z) = q^{-16} + 266q^{48} + O(q^{80}).$$

The canonical basis for $M_k^\sharp(64)$ is given by

$$\{\, f_{k,m}(z) \mid m \geq -8k \text{ and } m \neq -8k+1, -8k+2, -8k+5 \,\},$$

where

$$\begin{aligned} f_{k,m}(z) = q^{-m} &+ a_k(m, 8k-5)q^{8k-5} + a_k(m, 8k-2)q^{8k-2} \\ &+ a_k(m, 8k-1)q^{8k-1} + \sum_{n=8k+1}^{\infty} a_k(m,n)q^n. \end{aligned} \tag{2.5}$$

We remark that the form $\psi(z)$ is the canonical basis element $f_{2,-16}(z)$.

Notice that for any $m \geq -5$ with $m \neq -4, -3, 0$ the function $h_n'(z) = g_{2,m}(z)\big[\psi(z)\big]^{\ell-1}$ is a weight k weakly holomorphic modular form in $S_k^\sharp(64)$ with leading term $q^{-n} = q^{-m+16(\ell-1)} = q^{-m+8k-16}$. Since $-n = -m + 8k - 16$, we see that $h_n'(z)$ is defined for $n \geq -8k+11$ and $n \neq -8k+12, -8k+13, -8k+16$.

The set $\{\, h_n'(z) \mid n \geq -8k+11, n \neq -8k+12, -8k+13, -8k+16 \,\}$, by an argument similar to that shown above, is a basis for $S_k^\sharp(64)$. From this basis, the canonical basis for $S_k^\sharp(64)$ is constructed inductively, by eliminating as many terms as possible using the previous canonical basis elements.

The canonical basis for $S_k^\sharp(64)$ is given by

$$\{\, g_{k,m}(z) \mid m \geq -8k+11 \text{ and } m \neq -8k+12, -8k+13, -8k+16 \,\},$$

where

$$\begin{aligned} g_{k,m} = q^{-m} &+ b_k(m, 8k-16)q^{8k-16} + b_k(m, 8k-13)q^{8k-13} \\ &+ b_k(m, 8k-12)q^{8k-12} + \sum_{n=8k-10}^{\infty} b_k(m,n)q^n. \end{aligned} \tag{2.6}$$

It should be noted that even though $S_k^\sharp(64)$ is a subspace of $M_k^\sharp(64)$, it is not true that the canonical basis for $S_k^\sharp(64)$ is contained (as a set) in the canonical basis for $M_k^\sharp(64)$. In general, the basis elements $g_{k,m}(z)$ of $S_k^\sharp(N)$ are not equal to any of the basis elements $f_{k,m}(z)$ of $M_k^\sharp(N)$. Additionally, unless stated otherwise, we will refer to the canonical basis elements as just the basis elements, since these canonical bases have been constructed and there will be no ambiguity.

2.5 Recurrence relations

Since the canonical bases for $M_k^\sharp(64)$ and $S_k^\sharp(64)$ are defined inductively, they can be expressed by a recurrence relation. These recurrence relations are important for two reasons. First, they play a critical role in the computation of the generating function whose coefficients are basis elements. Second, they provide the groundwork for an elegant program which computes these basis elements. We begin by first finding the recurrence relation for the basis of $M_k^\sharp(64)$ and then we follow the same procedure to find the recurrence relation for the basis elements of $S_k^\sharp(64)$.

Recall the function (2.4):

$$\psi(z) = f_{2,-16}(z) = \frac{\eta^8(64z)}{\eta^4(32z)} = q^{16} + O(q^{48}).$$

Since we use powers of ψ in the construction of the basis, it is useful to know how large the gap is between the leading term and the next nonzero term in $\left[\psi(z)\right]^{\ell}$.

Lemma 2.1. *For any $\ell \in \mathbb{Z}$ we have $\left[\psi(z)\right]^{\ell} = q^{16\ell} + O(q^{16\ell+32})$.*

Proof.

Let $\ell \in \mathbb{Z}$ be given. If $\ell = 0$, then $\left[\psi(z)\right]^{0} = 1 = 1 + O(q^{32})$.

If $\ell > 0$, then we proceed inductively. Notice that $\left[\psi(z)\right]^{1} = q^{16} + O(q^{16+32})$. Suppose that $\left[\psi(z)\right]^{i} = q^{16i} + O(q^{16i+32})$ for some $i > 0$. Then

$$\begin{aligned}\left[\psi(z)\right]^{i+1} &= \psi(z)\left[\psi(z)\right]^{i} = \left[q^{16} + O(q^{48})\right]\left[q^{16i} + O(q^{16i+32})\right] \\ &= q^{16i+16} + O(q^{16i+48}) = q^{16(i+1)} + O(q^{16(i+1)+32}).\end{aligned}$$

Therefore $\left[\psi(z)\right]^{\ell} = q^{16\ell} + O(q^{16\ell+32})$ holds for all $\ell > 0$.

If $\ell < 0$, then we again proceed inductively. Notice that long division yields $\left[\psi(z)\right]^{-1} = q^{-16} + O(q^{16}) = q^{-16} + O(q^{-16+32})$. Suppose that $\left[\psi(z)\right]^{i} = q^{16i} + O(q^{16i+32})$ for some $i < 0$. Then

$$\begin{aligned}\left[\psi(z)\right]^{i-1} &= \left[\psi(z)\right]^{-1}\left[\psi(z)\right]^{i} = \left[q^{-16} + O(q^{16})\right]\left[q^{16i} + O(q^{16i+32})\right] \\ &= q^{16i-16} + O(q^{16i+16}) = q^{16(i-1)} + O(q^{16(i-1)+32}).\end{aligned}$$

Therefore $\left[\psi(z)\right]^{\ell} = q^{16\ell} + O(q^{16\ell+32})$ holds for all $\ell < 0$. □

2.5.1 Recurrence for $M_k^\sharp(64)$. Notice that since the basis element $f_{k,n}(z)$ has leading term q^{-n}, the function $f_{k,n}(z)f_{0,3}(z)$ is a modular form with leading term q^{-n+3}. This observation inspires us to establish a relationship between the function $f_{k,n}(z)f_{0,3}(z)$ and the basis element $f_{k,n+3}(z)$ since they both have the same leading term. We need to have, as initial values of the recurrence relation, three basis elements with leading terms whose exponents are consecutive integers. Consider the first 6 basis elements of $M_k^\sharp(64)$:

$$f_{k,-8k}(z), \quad f_{k,-8k+3}(z), \quad f_{k,-8k+4}(z), \quad f_{k,-8k+6}(z), \quad f_{k,-8k+7}(z), \quad f_{k,-8k+8}(z).$$

These basis elements are the initial values of the recurrence relation. We have the following theorem which shows how these first six basis elements can be computed without having to subtract any previous basis elements.

Theorem 2.2.

(i) $f_{k,-8k}(z) = q^{-8k} + O(q^{8k+32})$

(ii) $f_{k,-8k+3}(z) = q^{-8k+3} + O(q^{8k+5})$

(iii) $f_{k,-8k+4}(z) = q^{-8k+4} + O(q^{8k+12})$

(iv) $f_{k,-8k+6}(z) = q^{-8k+6} + O(q^{8k+2})$

(v) $f_{k,-8k+7}(z) = q^{-8k+7} + O(q^{8k+1})$

(vi) $f_{k,-8k+8}(z) = q^{-8k+8} + O(q^{8k+8})$

Proof.

Recall that $k = 2\ell$, so that $\left[\psi(z)\right]^{\ell} = q^{16\ell} + O(q^{16\ell+32}) = q^{8k} + O(q^{8k+32})$ by Lemma 2.1. Now consider the following computations:

$$
\begin{aligned}
f_{0,0}(z)\left[\psi(z)\right]^{\ell} &= \left[1\right]\left[q^{8k} + O(q^{8k+32})\right] = q^{8k} + O(q^{8k+32}),\\
f_{0,3}(z)\left[\psi(z)\right]^{\ell} &= \left[q^{-3} + O(q^{5})\right]\left[q^{8k} + O(q^{8k+32})\right] = q^{8k-3} + O(q^{8k+5}),\\
f_{0,4}(z)\left[\psi(z)\right]^{\ell} &= \left[q^{-4} + O(q^{12})\right]\left[q^{8k} + O(q^{8k+32})\right] = q^{8k-4} + O(q^{8k+12}),\\
f_{0,6}(z)\left[\psi(z)\right]^{\ell} &= \left[q^{-6} + O(q^{2})\right]\left[q^{8k} + O(q^{8k+32})\right] = q^{8k-6} + O(q^{8k+2}),\\
f_{0,7}(z)\left[\psi(z)\right]^{\ell} &= \left[q^{-7} + O(q)\right]\left[q^{8k} + O(q^{8k+32})\right] = q^{8k-7} + O(q^{8k+1}),\\
f_{0,8}(z)\left[\psi(z)\right]^{\ell} &= \left[q^{-8} + O(q^{8})\right]\left[q^{8k} + O(q^{8k+32})\right] = q^{8k-8} + O(q^{8k+8}).
\end{aligned}
$$

Since each of the exponents in the big-O terms above are greater than or equal to $8k + 1$, we know that those functions written above are elements of the canonical basis. Hence

$$\begin{aligned}
f_{k,-8k}(z) &= f_{0,0}(z)\big[\psi(z)\big]^{\ell} = q^{8k} + O(q^{8k+32}),\\
f_{k,-8k+3}(z) &= f_{0,3}(z)\big[\psi(z)\big]^{\ell} = q^{8k-3} + O(q^{8k+5}),\\
f_{k,-8k+4}(z) &= f_{0,4}(z)\big[\psi(z)\big]^{\ell} = q^{8k-4} + O(q^{8k+12}),\\
f_{k,-8k+6}(z) &= f_{0,6}(z)\big[\psi(z)\big]^{\ell} = q^{8k-6} + O(q^{8k+2}),\\
f_{k,-8k+7}(z) &= f_{0,7}(z)\big[\psi(z)\big]^{\ell} = q^{8k-7} + O(q^{8k+1}),\\
f_{k,-8k+8}(z) &= f_{0,8}(z)\big[\psi(z)\big]^{\ell} = q^{8k-8} + O(q^{8k+8}).
\end{aligned} \tag{2.7}$$

□

In particular, we have shown that for any weight $k = 2\ell$, the first six basis elements of $M_k^\sharp(64)$ are defined in terms of just the first six basis elements of $M_0^\sharp(64)$ and the function $\psi(z)$. Moreover, the equations in 2.7 imply that

$$\begin{aligned}
f_{k,-8k+3}(z) &= f_{0,3}(z) f_{k,-8k}(z),\\
f_{k,-8k+6}(z) &= f_{0,3}(z) f_{k,-8k+3}(z),\\
f_{k,-8k+7}(z) &= f_{0,3}(z) f_{k,-8k+4}(z).
\end{aligned} \tag{2.8}$$

We will now let $n \geq -8k+6$ be given. Notice that $f_{k,n}(z)$ is defined since $n \neq -8k+1$, $-8k+2$, or $-8k+5$. To simplify notation, we let $\varphi_3(z) \coloneqq f_{0,3}(z) = q^{-3} + \sum_{i=5}^{\infty} c_i q^i$. Next we multiply $f_{k,n}(z)$ by $\varphi_3(z)$ and expand the result as shown below.

$$\begin{aligned}
\varphi_3(z)\cdot f_{k,n}(z) = \left(q^{-3} + \sum_{i=5}^{\infty} c_i q^i\right)\Bigg(&q^{-n} + a_k(n, 8k-5)q^{8k-5} + a_k(n, 8k-2)q^{8k-2}\\
&+ a_k(n, 8k-1)q^{8k-1} + \sum_{i=8k+1}^{\infty} a_k(n,i) q^i\Bigg)
\end{aligned}$$

$$= q^{-n-3} + a_k(n,8k-5)q^{8k-8} + a_k(n,8k-2)q^{8k-5} + a_k(n,8k-1)q^{8k-4}$$
$$+ \sum_{i=8k+1}^{\infty} a_k(n,i)q^{i-3} + \sum_{i=5}^{\infty} c_i q^{i-n} + \sum_{i=5}^{\infty} c_i a_k(n,8k-5)q^{8k-5+i}$$
$$+ \sum_{i=5}^{\infty} c_i a_k(n,8k-2)q^{8k-2+i} + \sum_{i=5}^{\infty} c_i a_k(n,8k-1)q^{8k-1+i}$$
$$+ \sum_{i=8k+6}^{\infty} \left(\sum_{j=5}^{i-8k-1} c_j a_k(n,i-j) \right) q^i$$

$$= q^{-n-3} + a_k(n,8k-5)q^{8k-8} + a_k(n,8k-2)q^{8k-5} + a_k(n,8k-1)q^{8k-4}$$
$$+ a_k(n,8k+1)q^{8k-2} + a_k(n,8k+2)q^{8k-1} + a_k(n,8k+3)q^{8k}$$
$$+ \sum_{i=8k+4}^{\infty} a_k(n,i)q^{i-3} + \sum_{i=5}^{n+8k} c_i q^{i-n} + \sum_{i=n+8k+1}^{\infty} c_i q^{i-n} + c_5 a_k(n,8k-5)q^{8k}$$
$$+ \sum_{i=6}^{\infty} c_i a_k(n,8k-5)q^{8k-5+i} + \sum_{i=5}^{\infty} c_i a_k(n,8k-2)q^{8k-2+i}$$
$$+ \sum_{i=5}^{\infty} c_i a_k(n,8k-1)q^{8k-1+i} + \sum_{i=8k+6}^{\infty} \left(\sum_{j=5}^{i-8k-1} c_j a_k(n,i-j) \right) q^i$$

Notice that the function above has several terms that are colored red. These terms are the leading terms of basis elements in $M_k^\sharp(64)$. Therefore the function $\varphi_3(z)f_{k,n}(z)$ must be a linear combination of these basis elements. This linear combination is shown below.

$$\begin{aligned} f_{k,n+3}(z) = \varphi_3(z)f_{k,n}(z) - a_k(n,8k-5)f_{k,-8k+8}(z) - a_k(n,8k-1)f_{k,-8k+4}(z) \\ - a_k(n,8k+3)f_{k,-8k}(z) - a_k(n,8k-5)f_{k,-8k}(z) - \sum_{i=5}^{n+8k} c_i f_{k,n-i}(z). \end{aligned} \tag{2.9}$$

The astute reader will notice that there are basis elements contained in $\sum_{i=5}^{n+8k} c_i f_{k,n-i}(z)$ that are not actually defined. This is resolved by using the convention that $f_{k,m}(z) = 0$ for $m = -8k+1$, $-8k+2$, or $-8k+5$. This recurrence relation is used in the SAGE script A.4 which computes the basis elements of $M_k^\sharp(64)$.

2.5.2 Recurrence for $S_k^\sharp(64)$. Similar to the recurrence relation for $M_k^\sharp(64)$, we need the first six basis elements of $S_k^\sharp(64)$:

$$g_{k,-8k+11}(z), \quad g_{k,-8k+14}(z), \quad g_{k,-8k+15}(z), \quad g_{k,-8k+17}(z), \quad g_{k,-8k+18}(z), \quad g_{k,-8k+19}(z).$$

We also need a theorem that shows that these first six basis elements can be computed without subtracting any previous basis elements.

Theorem 2.3.

(i) $g_{k,-8k+11}(z) = q^{-8k+11} + O(q^{8k-3})$

(ii) $g_{k,-8k+14}(z) = q^{-8k+14} + O(q^{8k-6})$

(iii) $g_{k,-8k+15}(z) = q^{-8k+15} + O(q^{8k-7})$

(iv) $g_{k,-8k+17}(z) = q^{-8k+17} + O(q^{8k-9})$

(v) $g_{k,-8k+18}(z) = q^{-8k+18} + O(q^{8k-10})$

(vi) $g_{k,-8k+19}(z) = q^{-8k+19} + O(q^{8k-3})$

Proof. Recall that $\left[\psi(z)\right]^{\ell-1} = q^{16(\ell-1)} + O(q^{16(\ell-1)+32}) = q^{8k-16} + O(q^{8k+16})$ by Lemma 2.1 when $k = 2\ell$. Now consider the following computations:

$$g_{2,-5}(z)\left[\psi(z)\right]^{\ell-1} = \left[q^5 + O(q^{13})\right]\left[q^{8k-16} + O(q^{8k+16})\right] = q^{8k-11} + O(q^{8k-3}),$$

$$g_{2,-2}(z)\left[\psi(z)\right]^{\ell-1} = \left[q^2 + O(q^{10})\right]\left[q^{8k-16} + O(q^{8k+16})\right] = q^{8k-14} + O(q^{8k-6}),$$

$$g_{2,-5}(z)\left[\psi(z)\right]^{\ell-1} = \left[q^1 + O(q^{9})\right]\left[q^{8k-16} + O(q^{8k+16})\right] = q^{8k-15} + O(q^{8k-7}),$$

$$g_{2,-5}(z)\left[\psi(z)\right]^{\ell-1} = \left[q^{-1} + O(q^{7})\right]\left[q^{8k-16} + O(q^{8k+16})\right] = q^{8k-17} + O(q^{8k-9}),$$

$$g_{2,-5}(z)\left[\psi(z)\right]^{\ell-1} = \left[q^{-2} + O(q^{6})\right]\left[q^{8k-16} + O(q^{8k+16})\right] = q^{8k-18} + O(q^{8k-10}),$$

$$g_{2,-5}(z)\left[\psi(z)\right]^{\ell-1} = \left[q^{-3} + O(q^{13})\right]\left[q^{8k-16} + O(q^{8k+16})\right] = q^{8k-19} + O(q^{8k-3}).$$

Since each of the exponents in the big-O terms above are greater than or equal to $8k - 10$, we know that those functions written above are elements of the canonical basis. Hence

$$\begin{aligned}
g_{k,-8k+11}(z) &= g_{2,-5}(z)\left[\psi(z)\right]^{\ell-1} = q^{8k-11} + O(q^{8k-3}),\\
g_{k,-8k+14}(z) &= g_{2,-2}(z)\left[\psi(z)\right]^{\ell-1} = q^{8k-14} + O(q^{8k-6}),\\
g_{k,-8k+15}(z) &= g_{2,-1}(z)\left[\psi(z)\right]^{\ell-1} = q^{8k-15} + O(q^{8k-7}),\\
g_{k,-8k+17}(z) &= g_{2,1}(z)\left[\psi(z)\right]^{\ell-1} = q^{8k-17} + O(q^{8k-9}),\\
g_{k,-8k+18}(z) &= g_{2,2}(z)\left[\psi(z)\right]^{\ell-1} = q^{8k-18} + O(q^{8k-10}),\\
g_{k,-8k+19}(z) &= g_{2,3}(z)\left[\psi(z)\right]^{\ell-1} = q^{8k-19} + O(q^{8k-3}).
\end{aligned}$$

□

In particular, we have shown that for any weight $k = 2\ell$, the first six basis elements of $S_k^\sharp(64)$ are defined in terms of just the first six basis elements of $S_2^\sharp(64)$ and the function $\psi(z)$. We will now let $n \geq -8k+17$ be given. Notice that $g_{k,n}(z)$ is defined since $n \neq -8k+12$, $-8k+13$, or $-8k+16$. Next, we multiply $g_{k,n}(z)$ by $\varphi_3(z)$ and expand the result as shown below.

$$\begin{aligned}
\varphi_3(z)\cdot g_{k,n}(z) = \left(q^{-3} + \sum_{i=5}^{\infty} c_i q^i\right)\Bigg(q^{-n} &+ b_k(n,8k-16)q^{8k-16} + b_k(n,8k-13)q^{8k-13}\\
&+ b_k(n,8k-12)q^{8k-12} + \sum_{i=8k-10}^{\infty} b_k(n,i)q^i\Bigg)
\end{aligned}$$

$$\begin{aligned}
&= q^{-n-3} + b_k(n,8k-16)q^{8k-19} + b_k(n,8k-13)q^{8k-16} + b_k(n,8k-12)q^{8k-15}\\
&\quad + \sum_{i=8k-10}^{\infty} b_k(n,i)q^{i-3} + \sum_{i=5}^{\infty} c_i q^{i-n} + \sum_{i=5}^{\infty} c_i b_k(n,8k-16)q^{8k-16+i}\\
&\quad + \sum_{i=5}^{\infty} c_i b_k(n,8k-13)q^{8k-13+i} + \sum_{i=5}^{\infty} c_i b_k(n,8k-12)q^{8k-12+i}\\
&\quad + \sum_{i=8k-5}^{\infty}\left(\sum_{j=5}^{i-8k+10} c_j b_k(n,i-j)\right)q^i
\end{aligned}$$

$$= q^{-n-3} + b_k(n, 8k-16)q^{8k-19} + b_k(n, 8k-13)q^{8k-16} + b_k(n, 8k-12)q^{8k-15}$$
$$+ b_k(n, 8k-10)q^{8k-13} + b_k(n, 8k-9)q^{8k-12} + b_k(n, 8k-8)q^{8k-11}$$
$$+ \sum_{i=8k-7}^{\infty} b_k(n, i)q^{i-3} + \sum_{i=5}^{n+8k-11} c_i q^{i-n} + \sum_{i=n+8k-10}^{\infty} c_i q^{i-n} + c_5 b_k(n, 8k-16)q^{8k-11}$$
$$+ \sum_{i=6}^{\infty} c_i b_k(n, 8k-16)q^{8k-16+i} + \sum_{i=5}^{\infty} c_i b_k(n, 8k-13)q^{8k-13+i}$$
$$+ \sum_{i=5}^{\infty} c_i b_k(n, 8k-12)q^{8k-12+i} + \sum_{i=8k-5}^{\infty} \left(\sum_{j=5}^{i-8k+10} c_j b_k(n, i-j) \right) q^i$$

Notice that the function above has several terms that are colored red. These terms are the leading terms of basis elements in $S_k^\sharp(64)$. Therefore the function $\varphi_3(z)g_{k,n}(z)$ must be a linear combination of these basis elements. This linear combination is shown below.

$$g_{k,n+3}(z) = \varphi_3(z)g_{k,n}(z) - b_k(n, 8k-16)g_{k,-8k+19}(z) - b_k(n, 8k-12)g_{k,-8k+15}(z)$$
$$- b_k(n, 8k-8)g_{k,-8k+11}(z) - b_k(n, 8k-16)g_{k,-8k+11}(z) - \sum_{i=5}^{n+8k-11} c_i g_{k,n-i}(z) \tag{2.10}$$

Just as before, there are basis elements contained in $\sum_{i=5}^{n+8k-11} c_i g_{k,n-i}(z)$ that are not actually defined. This is resolved by using the convention that $g_{k,m}(z) = 0$ for $m = -8k+12$, $-8k+13$, or $-8k+16$. This recurrence relation is used in the SAGE script A.5 which computes the basis elements of $S_k^\sharp(64)$.

Now that the treatment of the canonical bases is complete, we turn our attention to examining these basis elements to determine what properties their coefficients have. Consider the following example involving the first two elements of $S_2^\sharp(64)$ and several elements of $M_0^\sharp(64)$:

$$g_{2,-5}(z) = q^5 - 3q^{13} + 5q^{29} + 1q^{37} - 3q^{45} + O(q^{53}),$$
$$g_{2,-2}(z) = q^2 - 2q^{10} - 3q^{18} + 6q^{26} + 2q^{34} + O(q^{50}),$$

$$f_{0,10}(z) = q^{-10} + 2q^{-2} + 3q^6 + O(q^{14}),$$
$$f_{0,13}(z) = q^{-13} + 3q^{-5} + 3q^3 + O(q^{11}),$$
$$f_{0,18}(z) = q^{-18} + 3q^{-2} + 8q^6 + O(q^{14}),$$
$$f_{0,26}(z) = q^{-26} - 6q^{-2} - 4q^6 + O(q^{14}),$$
$$f_{0,29}(z) = q^{-29} - 5q^{-5} - 18q^3 + O(q^{11}),$$
$$f_{0,34}(z) = q^{-34} - 2q^{-2} - 24q^6 + O(q^{14}),$$
$$f_{0,37}(z) = q^{-37} - 1q^{-5} - 14q^3 + O(q^{11}),$$
$$f_{0,45}(z) = q^{-45} + 3q^{-5} + 20q^3 + O(q^{11}).$$

Notice that the coefficient $b_2(-5, 13) = -3$ while the coefficient $a_0(13, -5) = 3$. Also notice that the coefficient $b_2(-2, 34) = 2$ while the coefficient $a_0(34, -2) = -2$. There are six similar observations that can be made. This is not a coincidence. The following theorem generalizes this phenomenon.

Theorem 3.1. *For the basis elements $f_{k,m}(z) \in M_k^\sharp(64)$ and $g_{2-k,n}(z) \in S_{2-k}^\sharp(64)$ we have*

$$a_k(m, n) = -b_{2-k}(n, m).$$

We say that the coefficients $a_k(m, n)$ and $b_{2-k}(n, m)$ are dual. Alternatively, it may be stated that the basis elements of $M_k^\sharp(64)$ and $S_{2-k}^\sharp(64)$ satisfy duality. This property is also called Zagier duality, after Zagier's publication of a similar result for half integral weight modular forms in [10].

In order to show that Zagier duality holds we need the following two operators and then the subsequent two lemmas. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. We define the weight k operator on functions $f : \mathcal{H} \to \mathbb{C}$ by

$$\big(f \,|\, [\gamma]_k\big)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

This operator is called the slash operator. Let $\theta := q\dfrac{d}{dq} = \dfrac{1}{2\pi i} \cdot \dfrac{d}{dz}$ be the Ramanujan operator. We note that θ sends forms in $M_0^!(N)$ to $M_2^!(N)$ (c.f. Theorem 2.26 in [8]).

Lemma 3.2. *Let $f(z) \in M_0^\sharp(64)$. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(64)$ we have*

$$\theta\big(f \,|\, [\gamma]_0\big) = \big(\theta f\big) \,|\, [\gamma]_2.$$

Proof. This proof is a direct computation.

$$\begin{aligned}
\Big[\theta\big(f \,|\, [\gamma]_0\big)\Big](z) &= \theta\left((cz+d)^0 f\left(\frac{az+b}{cz+d}\right)\right) \\
&= \frac{1}{2\pi i}\,\frac{d}{dz}\left(f\left(\frac{az+b}{cz+d}\right)\right) \\
&= \frac{1}{2\pi i}\,\frac{df}{dz}\left(\frac{az+b}{cz+d}\right)\cdot\left(\frac{a(cz+d)-c(az+b)}{(cz+d)^2}\right) \\
&= \frac{(ac-ac)z+(ad-bc)}{(cz+d)^2}\,\frac{1}{2\pi i}\,\frac{df}{dz}\left(\frac{az+b}{cz+d}\right) \\
&= (cz+d)^{-2}\Big(\theta f\Big)\left(\frac{az+b}{cz+d}\right) \\
&= \Big(\big(\theta f\big) \,|\, [\gamma]_2\Big)(z).
\end{aligned}$$

□

Lemma 3.3. *If $f(z) \in S_2^\sharp(64)$, then $f(z)$ has no constant term.*

Proof. We show that the basis elements of $S_2^\sharp(64)$ do not contain constant terms. Notice that the first eight basis elements do not have constant term:

$$g_{2,-5}(z) = q^5 \quad - 3q^{13} + 5q^{29} + q^{37} \quad - 3q^{45} \quad + O(q^{53}),$$

$$g_{2,-2}(z) = q^2 - 2q^{10} - 3q^{18} + 6q^{26} + 2q^{34} + O(q^{50}),$$

$$g_{2,-1}(z) = q - 3q^9 + 2q^{17} - q^{25} + 10q^{41} + O(q^{49}),$$

$$g_{2,1}(z) = q^{-1} - q^7 - 5q^{15} + 4q^{23} + 5q^{31} + O(q^{47}),$$

$$g_{2,2}(z) = q^{-2} - 2q^6 - q^{14} + 2q^{22} - 5q^{30} + O(q^{38}),$$

$$g_{2,3}(z) = q^{-3} - 5q^{13} - 7q^{21} + 18q^{29} + 14q^{37} + O(q^{45}),$$

$$g_{2,4}(z) = q^{-4} - 6q^{12} + 7q^{28} + 22q^{44} - 45q^{60} + O(q^{74}),$$

$$g_{2,5}(z) = q^{-5} - q^3 - 3q^{11} + 2q^{19} - 6q^{27} + O(q^{35}).$$

Now apply θ to the basis elements $f_{0,m}(z)$ of $M_0^\sharp(64)$ for $m \geq 6$:

$$\left(\theta f_{0,m}\right)(z) = q\frac{d}{dq}\left[q^{-m} + O(q^{-5})\right] = -mq^{-m} + O(q^{-5}).$$

The function $\theta f_{0,m}$ is a weight 2 weakly holomorphic modular form that doesn't have a constant term, since the derivative was taken. Moreover, we know that $\theta f_{k,m}$ vanishes at every cusp because of Lemma 3.2. The idea is this: slash $f_{0,m}$ with a matrix so that the Fourier expansion at a cusp (choose the appropriate matrix for each cusp) is holomorphic. Now apply θ so that the constant term of $\theta(f_{0,m}\,|\,[\gamma]_0)$ disappears. This function is equal to $(\theta f_{0,m})\,|\,[\gamma]_2$. Since $(\theta f_{0,m})\,|\,[\gamma]_2$ has no constant term and no negative powers, $\theta f_{0,m}$ must vanish at the cusp.

Thus $\theta f_{0,m} \in S_2^\sharp(64)$ for all $m \geq 6$. Since $\theta f_{0,m}$ has leading term q^{-m} there is a linear combination of functions $\theta f_{0,m}$ and $g_{2,i}$, where $i < m$, that yields the basis element $g_{2,m}$. The functions in the linear combination do not have constant terms, so $g_{2,m}$ also doesn't. Proceeding inductively shows that the rest of the basis elements in $S_2^\sharp(64)$ do not have constant terms. Since the basis elements for $S_2^\sharp(64)$ do not have constant terms, neither does $f(z)$. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1.

Recall that $f_{k,m}(z)$ is a basis element for $M_k^\sharp(64)$ and that $g_{2-k,n}(z)$ is a basis element for $S_{2-k}^\sharp(64)$. Explicitly we have

$$\begin{aligned} f_{k,m}(z) &= q^{-m} + a_k(m, 8k-5)q^{8k-5} + a_k(m, 8k-2)q^{8k-2} \\ &\quad + a_k(m, 8k-1)q^{8k-1} + \sum_{i=8k+1}^{\infty} a_k(m,i)q^i, \\ g_{k,n}(z) &= q^{-n} + b_k(n, 8k-16)q^{8k-16} + b_k(n, 8k-13)q^{8k-13} \\ &\quad + b_k(n, 8k-12)q^{8k-12} + \sum_{i=8k-10}^{\infty} b_k(n,i)q^i. \end{aligned}$$

By replacing k with $2-k$ in $g_{k,n}(z)$ we get

$$\begin{aligned} g_{2-k,n}(z) &= q^{-n} + b_{2-k}(n, -8k)q^{-8k} + b_{2-k}(n, -8k+3)q^{-8k+3} \\ &\quad + b_{2-k}(n, -8k+4)q^{-8k+4} + \sum_{i=-8k+6}^{\infty} b_{2-k}(n,i)q^i. \end{aligned} \tag{3.1}$$

Now let $F(z) = f_{k,m}(z)g_{2-k,n}(z)$. The constant term of this function is $a_k(m,n)+b_{2-k}(n,m)$. This is not a trivial observation; for example, we need to notice that the q^{8k-5} term in $f_{k,m}(z)$ doesn't contribute to the constant term since there is no q^{-8k+5} in $g_{2-k,n}(z)$. Similar observations can be made about the q^{8k-2} and q^{8k-1} terms of $f_{k,m}(z)$. On the other hand, the same thing can be observed for the q^{-8k}, q^{-8k+3} and q^{-8k+4} terms of $g_{2-k,n}(z)$. Also any term from $\sum_{i=8k+1}^{\infty} a_k(m,i)q^i$ when multiplied by any term in $\sum_{i=-8k+6}^{\infty} b_{2-k}(n,i)q^i$ will be a power of q greater than 5 and therefore will not contribute to the constant term. Hence the only term from $f_{k,m}(z)$ that contributes to the constant term of $F(z)$ is q^{-m} and the only term from $g_{2-k,n}(z)$ that contributes is q^{-n}.

Now notice that $F(z)$ vanishes at all cusps away from infinity since $g_{2-k,n}(z)$ does. Therefore $F(z) \in S_2^\sharp(64)$. By Lemma 3.3, we know that $F(z)$ has no constant term. Therefore $a_k(m,n) + b_{2-k}(n,m) = 0$. □

We can now compute the generating function for the basis elements of $M_k^\sharp(64)$. The generating function is given in the following theorem.

Theorem 4.1.

$$F_k(z,\tau) = \sum_{n=-8k}^{\infty} f_{k,n}(\tau)q^n =$$

$$\frac{f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) + f_{k,-8k}(\tau)g_{2-k,8k+3}(z)}{f_{0,3}(z) - f_{0,3}(\tau)}.$$

Before we prove that this is in fact the generating function, we will state a few preliminary results. First, now that we have proven duality, we can restate the recurrence relation (2.9) as follows:

$$\begin{aligned} f_{k,n+3}(z) &= \varphi_3(z)f_{k,n}(z) + b_{2-k}(8k-5,n)f_{k,-8k+8}(z) + b_{2-k}(8k-1,n)f_{k,-8k+4}(z) \\ &\quad + b_{2-k}(8k+3,n)f_{k,-8k}(z) + b_{2-k}(8k-5,n)f_{k,-8k}(z) - \sum_{i=5}^{n+8k} c_i f_{k,n-i}(z). \end{aligned} \tag{4.1}$$

Second, we also notice that the equation (3.1) can be rewritten as

$$\begin{aligned} \sum_{i=-8k+6}^{\infty} b_{2-k}(n,i)q^i &= g_{2-k,n}(z) - b_{2-k}(n,-8k)q^{-8k} - b_{2-k}(n,-8k+3)q^{-8k+3} \\ &\quad - b_{2-k}(n,-8k+4)q^{-8k+4}. \end{aligned} \tag{4.2}$$

Third, we notice that by definition of the generating function, we have the summation

$$\sum_{n=8k+6}^{\infty} f_{k,n}(\tau)q^n = F_k(z,\tau) - f_{k,-8k}(z) - f_{k,-8k+3}(z) - f_{k,-8k+4}(z). \tag{4.3}$$

Lastly, we notice that by applying duality we get the following corollary of Theorem 2.2:

Corollary 4.2.

(i) $b_{2-k}(8k-5,-8k) = b_{2-k}(8k-1,-8k) = b_{2-k}(8k+3,-8k) = 0,$

(ii) $b_{2-k}(8k-5,-8k+3) = b_{2-k}(8k-1,-8k+3) = b_{2-k}(8k+3,-8k+3) = 0,$

(iii) $b_{2-k}(8k-5,-8k+4) = b_{2-k}(8k-1,-8k+4) = b_{2-k}(8k+3,-8k+4) = 0.$

We now prove Theorem 4.1.

Proof of Theorem 4.1.

We proceed by direct computation appealing to the previous results of this thesis when needed.

$$F_k(z,\tau) = \sum_{n=-8k}^{\infty} f_{k,n}(\tau)q^n$$

$$= f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} + \sum_{n=-8k+6}^{\infty} f_{k,n+3}(\tau)q^{n+3}.$$

We use the recurrence relation (4.1) to get

$$\begin{aligned} F_k(z,\tau) &= f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\ &\quad + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} + q^3 \sum_{n=-8k+6}^{\infty} \Bigg[\varphi_3(\tau)f_{k,n}(\tau) \\ &\quad + b_{2-k}(8k-5,n)f_{k,-8k+8}(\tau) + b_{2-k}(8k-1,n)f_{k,-8k+4}(\tau) \\ &\quad + b_{2-k}(8k+3,n)f_{k,-8k}(\tau) + b_{2-k}(8k-5,n)f_{k,-8k}(\tau) - \sum_{i=5}^{n+8k} c_i f_{k,n-i}(\tau)\Bigg] q^n. \end{aligned}$$

Now we split the summation into six summations.

$$
\begin{aligned}
F_k(z,\tau) = {} & f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\
& + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} + q^3\varphi_3(\tau)\sum_{n=-8k+6}^{\infty} f_{k,n}(\tau)q^n \\
& + q^3 f_{k,-8k+8}(\tau)\sum_{n=-8k+6}^{\infty} b_{2-k}(8k-5,n)q^n + q^3 f_{k,-8k+4}(\tau)\sum_{n=-8k+6}^{\infty} b_{2-k}(8k-1,n)q^n \\
& + q^3 f_{k,-8k}(\tau)\sum_{n=-8k+6}^{\infty} b_{2-k}(8k+3,n)q^n + q^3 f_{k,-8k}(\tau)\sum_{n=-8k+6}^{\infty} b_{2-k}(8k-5,n)q^n \\
& - q^3\sum_{n=-8k+6}^{\infty}\left[\sum_{i=5}^{n+8k} c_i f_{k,n-i}(\tau)\right]q^n.
\end{aligned}
$$

We rearrange the terms so that like summations are grouped.

$$
\begin{aligned}
F_k(z,\tau) = {} & f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\
& + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} \\
& + q^3\varphi_3(\tau)\sum_{n=-8k+6}^{\infty} f_{k,n}(\tau)q^n \\
& + q^3\Big[f_{k,-8k+8}(\tau) + f_{k,-8k}(\tau)\Big]\sum_{n=-8k+6}^{\infty} b_{2-k}(8k-5,n)q^n \\
& + q^3 f_{k,-8k+4}(\tau)\sum_{n=-8k+6}^{\infty} b_{2-k}(8k-1,n)q^n \\
& + q^3 f_{k,-8k}(\tau)\sum_{n=-8k+6}^{\infty} b_{2-k}(8k+3,n)q^n \\
& - q^3\sum_{n=-8k+6}^{\infty}\left[\sum_{i=5}^{n+8k} c_i f_{k,n-i}(\tau)\right]q^n.
\end{aligned}
$$

On line three we apply equation 4.3. On lines 4, 5, and 6 we apply equation 4.2. On line 7 we add and subtract the term $c_5 f_{k,-8k}(\tau)q^{-8k+5}$.

$$\begin{aligned}
F_k(z,\tau) = {} & f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\
& + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} \\
& + q^3\varphi_3(\tau)\Big[F_k(z,\tau) - f_{k,-8k}(\tau)q^{-8k} - f_{k,-8k+3}(\tau)q^{-8k+3} - f_{k,-8k+4}(\tau)q^{-8k+4}\Big] \\
& + q^3\Big[f_{k,-8k+8}(\tau) + f_{k,-8k}(\tau)\Big]\Big[g_{2-k,8k-5}(z) - q^{-8k+5} - b_{2-k}(8k-5,-8k)q^{-8k} \\
& \qquad - b_{2-k}(8k-5,-8k+3)q^{-8k+3} - b_{2-k}(8k-5,-8k+4)q^{-8k+4}\Big] \\
& + q^3 f_{k,-8k+4}(\tau)\Big[g_{2-k,8k-1}(z) - q^{-8k+1} - b_{2-k}(8k-1,-8k)q^{-8k} \\
& \qquad - b_{2-k}(8k-1,-8k+3)q^{-8k+3} - b_{2-k}(8k-1,-8k+4)q^{-8k+4}\Big] \\
& + q^3 f_{k,-8k}(\tau)\Big[g_{2-k,8k+3}(z) - q^{-8k-3} - b_{2-k}(8k+3,-8k)q^{-8k} \\
& \qquad - b_{2-k}(8k+3,-8k+3)q^{-8k+3} - b_{2-k}(8k+3,-8k+4)q^{-8k+4}\Big] \\
& - q^3\left[-c_5 f_{k,-8k}(\tau)q^{-8k+5} + c_5 f_{k,-8k}(\tau)q^{-8k+5} + \sum_{n=-8k+6}^{\infty}\left(\sum_{i=5}^{n+8k} c_i f_{k,n-i}(\tau)\right)q^n\right].
\end{aligned}$$

We use Corollary 4.2 to cancel out all of the $b_{2-k}(\cdot,\cdot)$ terms. We also combine the positive $c_5 f_{k,-8k}(\tau)q^{-8k+5}$ term with the double summation on line 10.

$$\begin{aligned}
F_k(z,\tau) = {} & f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\
& + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} \\
& + q^3\varphi_3(\tau)\Big[F_k(z,\tau) - f_{k,-8k}(\tau)q^{-8k} - f_{k,-8k+3}(\tau)q^{-8k+3} - f_{k,-8k+4}(\tau)q^{-8k+4}\Big] \\
& + q^3\Big[f_{k,-8k+8}(\tau) + f_{k,-8k}(\tau)\Big]\Big[g_{2-k,8k-5}(z) - q^{-8k+5}\Big] \\
& + q^3 f_{k,-8k+4}(\tau)\Big[g_{2-k,8k-1}(z) - q^{-8k+1}\Big] \\
& + q^3 f_{k,-8k}(\tau)\Big[g_{2-k,8k+3}(z) - q^{-8k-3}\Big] \\
& - q^3\left[-c_5 f_{k,-8k}(\tau)q^{-8k+5} + \sum_{n=-8k+5}^{\infty}\left(\sum_{i=5}^{n+8k} c_i f_{k,n-i}(\tau)\right)q^n\right].
\end{aligned}$$

Next we distribute the products on lines 3, 4, 5, and 6. On line 7 we notice that $c_5 = 1$ and factor the double summation.

$$\begin{aligned}
F_k(z,\tau) = {} & f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\
& + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} + q^3\varphi_3(\tau)F_k(z,\tau) \\
& - \varphi_3(\tau)f_{k,-8k}(\tau)q^{-8k+3} - \varphi_3(\tau)f_{k,-8k+3}(\tau)q^{-8k+6} - \varphi_3(\tau)f_{k,-8k+4}(\tau)q^{-8k+7} \\
& + q^3 f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + q^3 f_{k,-8k}(\tau)g_{2-k,8k-5}(z) - f_{k,-8k+8}(\tau)q^{-8k+8} \\
& - f_{k,-8k}(\tau)q^{-8k+8} + q^3 f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) - f_{k,-8k+4}(\tau)q^{-8k+4} \\
& + q^3 f_{k,-8k}(\tau)g_{2-k,8k+3}(z) - f_{k,-8k}(\tau)q^{-8k} \\
& - q^3\left[- f_{k,-8k}(\tau)q^{-8k+5} + \left(\sum_{i=5}^{\infty} c_i q^i\right)\left(\sum_{n=-8k}^{\infty} f_{k,n}(\tau)q^n\right)\right].
\end{aligned}$$

On line 7 we replace the first summation with $\varphi_3(z) - q^{-3}$ and the second with the generating function. We also distribute the $-q^3$.

$$\begin{aligned}
F_k(z,\tau) = {} & f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6} \\
& + f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} + q^3\varphi_3(\tau)F_k(z,\tau) \\
& - \varphi_3(\tau)f_{k,-8k}(\tau)q^{-8k+3} - \varphi_3(\tau)f_{k,-8k+3}(\tau)q^{-8k+6} - \varphi_3(\tau)f_{k,-8k+4}(\tau)q^{-8k+7} \\
& + q^3 f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + q^3 f_{k,-8k}(\tau)g_{2-k,8k-5}(z) - f_{k,-8k+8}(\tau)q^{-8k+8} \\
& - f_{k,-8k}(\tau)q^{-8k+8} + q^3 f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) - f_{k,-8k+4}(\tau)q^{-8k+4} \\
& + q^3 f_{k,-8k}(\tau)g_{2-k,8k+3}(z) - f_{k,-8k}(\tau)q^{-8k} \\
& + f_{k,-8k}(\tau)q^{-8k+8} - q^3\Big(\varphi_3(z) - q^{-3}\Big)\Big(F_k(z,\tau)\Big).
\end{aligned}$$

We finish distributing on line 7.

$$F_k(z,\tau) = f_{k,-8k}(\tau)q^{-8k} + f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+4}(\tau)q^{-8k+4} + f_{k,-8k+6}(\tau)q^{-8k+6}$$
$$+ f_{k,-8k+7}(\tau)q^{-8k+7} + f_{k,-8k+8}(\tau)q^{-8k+8} + q^3\varphi_3(\tau)F_k(z,\tau)$$
$$- \varphi_3(\tau)f_{k,-8k}(\tau)q^{-8k+3} - \varphi_3(\tau)f_{k,-8k+3}(\tau)q^{-8k+6} - \varphi_3(\tau)f_{k,-8k+4}(\tau)q^{-8k+7}$$
$$+ q^3 f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + q^3 f_{k,-8k}(\tau)g_{2-k,8k-5}(z) - f_{k,-8k+8}(\tau)q^{-8k+8}$$
$$- f_{k,-8k}(\tau)q^{-8k+8} + q^3 f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) - f_{k,-8k+4}(\tau)q^{-8k+4}$$
$$+ q^3 f_{k,-8k}(\tau)g_{2-k,8k+3}(z) - f_{k,-8k}(\tau)q^{-8k}$$
$$+ f_{k,-8k}(\tau)q^{-8k+8} - q^3\varphi_3(z)F_k(z,\tau) + F_k(z,\tau).$$

The $f_{k,-8k}(\tau)q^{-8k}$, $f_{k,-8k+4}(\tau)q^{-8k+4}$, $f_{k,-8k+8}(\tau)q^{-8k+8}$, and $f_{k,-8k}(\tau)q^{-8k+8}$ terms cancel.

$$F_k(z,\tau) = f_{k,-8k+3}(\tau)q^{-8k+3} + f_{k,-8k+6}(\tau)q^{-8k+6} + f_{k,-8k+7}(\tau)q^{-8k+7}$$
$$+ q^3\varphi_3(\tau)F_k(z,\tau) - \varphi_3(\tau)f_{k,-8k}(\tau)q^{-8k+3} - \varphi_3(\tau)f_{k,-8k+3}(\tau)q^{-8k+6}$$
$$- \varphi_3(\tau)f_{k,-8k+4}(\tau)q^{-8k+7} + q^3 f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + q^3 f_{k,-8k}(\tau)g_{2-k,8k-5}(z)$$
$$+ q^3 f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) + q^3 f_{k,-8k}(\tau)g_{2-k,8k+3}(z) - q^3\varphi_3(z)F_k(z,\tau) + F_k(z,\tau).$$

We rearrange the terms so that the instances of the generating function are written at the beginning of the expression.

$$F_k(z,\tau) = F_k(z,\tau) + q^3\varphi_3(\tau)F_k(z,\tau) - q^3\varphi_3(z)F_k(z,\tau) + f_{k,-8k+3}(\tau)q^{-8k+3}$$
$$+ f_{k,-8k+6}(\tau)q^{-8k+6} + f_{k,-8k+7}(\tau)q^{-8k+7} - \varphi_3(\tau)f_{k,-8k}(\tau)q^{-8k+3}$$
$$- \varphi_3(\tau)f_{k,-8k+3}(\tau)q^{-8k+6} - \varphi_3(\tau)f_{k,-8k+4}(\tau)q^{-8k+7} + q^3 f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z)$$
$$+ q^3 f_{k,-8k}(\tau)g_{2-k,8k-5}(z) + q^3 f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) + q^3 f_{k,-8k}(\tau)g_{2-k,8k+3}(z).$$

By subtracting $F_k(z,\tau)$ from each side and then dividing by q^3 we get:

$$0 = F_k(z,\tau)\big[\varphi_3(\tau) - \varphi_3(z)\big] + f_{k,-8k+3}(\tau)q^{-8k} + f_{k,-8k+6}(\tau)q^{-8k+3} + f_{k,-8k+7}(\tau)q^{-8k+4}$$
$$- \varphi_3(\tau)f_{k,-8k}(\tau)q^{-8k} - \varphi_3(\tau)f_{k,-8k+3}(\tau)q^{-8k+3} - \varphi_3(\tau)f_{k,-8k+4}(\tau)q^{-8k+4}$$
$$+ f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z)$$
$$+ f_{k,-8k}(\tau)g_{2-k,8k+3}(z).$$

We group the like powers of q.

$$0 = F_k(z,\tau)\big[\varphi_3(\tau) - \varphi_3(z)\big] + q^{-8k}\big[f_{k,-8k+3}(\tau) - \varphi_3(\tau)f_{k,-8k}(\tau)\big]$$
$$+ q^{-8k+3}\big[f_{k,-8k+6}(\tau) - \varphi_3(\tau)f_{k,-8k+3}(\tau)\big] + q^{-8k+4}\big[f_{k,-8k+7}(\tau) - \varphi_3(\tau)f_{k,-8k+4}(\tau)\big]$$
$$+ f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z)$$
$$+ f_{k,-8k}(\tau)g_{2-k,8k+3}(z).$$

We now use the equations in 2.8 to eliminate the q^{-8k}, q^{-8k+3}, and q^{-8k+4} terms.

$$0 = F_k(z,\tau)\big[\varphi_3(\tau) - \varphi_3(z)\big] + f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k}(\tau)g_{2-k,8k-5}(z)$$
$$+ f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) + f_{k,-8k}(\tau)g_{2-k,8k+3}(z).$$

Now we can isolate $F_k(z,\tau)$ to get

$$F_k(z,\tau) = \frac{f_{k,-8k+8}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k}(\tau)g_{2-k,8k-5}(z) + f_{k,-8k+4}(\tau)g_{2-k,8k-1}(z) + f_{k,-8k}(\tau)g_{2-k,8k+3}(z)}{\varphi_3(z) - \varphi_3(\tau)}.$$

□

We briefly discuss congruences for the coefficients $a_0(m,n)$. Jenkins and Thornton, in [7], provide results about congruences for levels 2, 4, 8, and 16 in Theorems 2.2, 2.3, and 3.1. The analogous results for level 64 are stated in the following conjecture.

Conjecture 5.1. *Let $f_{0,m}(z)$ be an element of the canonical basis for $M_0^\sharp(64)$. Let $m = 2^\alpha m'$ and $n = 2^\beta n'$ where $(m',2) = (n',2) = 1$. For $\alpha \neq \beta$ the following congruences hold.*

$$a_0(2^\alpha m', 2^\beta n') \equiv 0 \mod 2^{4(\alpha-\beta)+8} \qquad \text{if } \alpha > \beta,$$
$$a_0(2^\alpha m', 2^\beta n') \equiv 0 \mod 2^{3(\beta-\alpha)+8} \qquad \text{if } \beta > \alpha.$$

There are plenty of examples to look at. We first consider

$$f_{0,32}(z) = q^{-32} + 276q^{32} - 2048q^{64} + 11202q^{96} - 49152q^{128} + O(q^{160}).$$

We notice that for the q^{64} term we have $\alpha = 5$, $\beta = 6$ and $a_0(32,64) = -2^{11}$. Notice that $3(7-5)+8 = 11$ and so $a_0(2^5, 2^6) = -2^{11} \equiv 0 \mod 2^{11}$ as expected.
We notice that for the q^{128} term we have $\alpha = 5$, $\beta = 7$ and $a_0(32,128) = -2^{14}\cdot 3$. Notice that $3(7-5)+8 = 14$ and so $a_0(2^5, 2^7) = -2^{14}\cdot 3 \equiv 0 \mod 2^{14}$ as expected.

As a second example consider

$$f_{0,64}(z) = q^{-64} - 4096q^{32} + 98580q^{64} - 1228800q^{96} + 10745856q^{128} + O(q^{160}).$$

Notice that for the q^{96} term we have $\alpha = 6$, $\beta = 5$ and $a_0(64,96) = -2^{14}\cdot 3\cdot 5^2$. Notice that $4(6-5)+8 = 12$ and so $a_0(2^6, 2^5\cdot 3) \equiv 0 \mod 2^{12}$ as expected.
Notice that for the q^{128} term we have $\alpha = 6$, $\beta = 7$ and $a_0(64,128) = 2^{11}\cdot 3^2\cdot 11\cdot 53$. Notice that $3(7-6)+8 = 11$ and so $a_0(2^6, 2^7) \equiv 0 \mod 2^{11}$ as expected.

While looking for the previous examples, we noticed that the condition $32 \mid m$ needed to be met for α and β to be different. This observation is expressed in the following conjecture.

Conjecture 5.2. *Let $f_{0,m}(z)$ be an element of the canonical basis for $M_0^\sharp(64)$. Let $m = 2^\alpha m'$ and $n = 2^\beta n'$ where $(m', 2) = (n', 2) = 1$. Suppose that $a_0(m, n) \neq 0$. Then $\alpha = \beta$ if and only if $\alpha \leq 4$.*

Appendix A. Computer Code

A.1 Weierstrass points at infinity

```
for i in range(1,100):
    S = CuspForms(i,2,prec=30)
    b = S.basis()
    g = Gamma0(i).genus()
    # if the basis is nonempty then display the information
    if b != []:
        print "Level: {0} \nGenus: {1} \n{2}\n".format(i,g,b)
```

A.2 Find eta-quotients: q^{-3} and q^{-4}

```
# The checkNum function
# Purpose:      This funtion will determine whether or not an eta-
#               quotient is a modular form with conditions on the
#               order of vanishing at each cusp. This function
#               assumes we are working in level 64.
# Parameters:   The 7 exponents that define the given eta-quotient.
#               p is the desired leading power of the eta-quotient.
def checkNum(r1,r2,r4,r8,r16,r32,r64,p):
    # check for trivial character
    if (mod(r2+3*r8 + 5*r32,2) == 1): return False

    # determine the leading power of the eta-quotient
    leadCoef=(r1 + 2*r2 + 4*r4 + 8*r8 + 16*r16 + 32*r32 + 64*r64)/24

    # check the order of vanishing at infinity
    if (leadCoef != p): return False

    # check the order of vanishing for cusps with denominator 1
    ordVan = (64*r1 + 32*r2 + 16*r4 + 8*r8 + 4*r16 + 2*r32 + r64)/24
    if (ordVan < 0 or ordVan.is_integer() == False ): return False

    # check the order of vanishing for cusps with denominator 2
    ordVan = (16*r1 + 32*r2 + 16*r4 + 8*r8 + 4*r16 + 2*r32 + r64)/24
    if (ordVan < 0 or ordVan.is_integer() == False ): return False

    # check the order of vanishing for cusps with denominator 4
    ordVan = (4*r1 + 8*r2 + 16*r4 + 8*r8 + 4*r16 + 2*r32 + r64)/24
    if (ordVan < 0 or ordVan.is_integer() == False ): return False

    # check the order of vanishing for cusps with denominator 8
    ordVan = (r1 + 2*r2 + 4*r4 + 8*r8 + 4*r16 + 2*r32 + r64)/24
    if (ordVan < 0 or ordVan.is_integer() == False ): return False

    # check the order of vanishing for cusps with denominator 16
    ordVan = (r1 + 2*r2 + 4*r4 + 8*r8 + 16*r16 + 8*r32 + 4*r64)/24
    if (ordVan < 0 or ordVan.is_integer() == False ): return False

    # check the order of vanishing for cusps with denominator 32
    ordVan = (r1 + 2*r2 + 4*r4 + 8*r8 + 16*r16 + 32*r32 + 16*r64)/24
    if (ordVan < 0 or ordVan.is_integer() == False ): return False

    # the only pole is at infinity and is equal to p as desired;
    # the order of vanishing at each cusp is an integer: This is
    # a valid eta-quotient.
    return True

```

```
# The tuple_generator function
# Purpose:      Find eta-quotients that begin with a particular power
#               and are holomorphic at each cusp other than infinity.
# Parameters:   p is the desired leading power (and also the order of
#               vanishing at infinity).
#               MAX is the value which will bound the values of the
#               exponents. This algorithm runs in O(n^6) time, so
#               keep MAX small.
#               For p = -3, MAX = 2 works; For p = -4, MAX 6 works.
# Returns:      A list containing all of the eta quotients found.
def tuple_generator(p,MAX):
    # initialize lists for the bounds of the search.
    numbers = range(-MAX,MAX+1)
    evens = [i for i in numbers if mod(i,2)==0]

    # store the results in myList
    myList=[]

    # the conditions given by Newman for an eta-quotient to be a
    # modular form imply that r1 and r64 are even.
    for r1 in evens:
        for r2 in numbers:
            for r4 in numbers:
                for r8 in numbers:
                    for r16 in numbers:
                        for r64 in evens:
                            # since the weight is 0, r32 is
                            # determined by the other 6 exponents.
                            r32 = -r1-r2-r4-r8-r16-r64
                            if (checkNum(r1,r2,r4,r8,r16,r32,r64,p) == true):
                                # the eta-quotient represented by
                                # (r1,r2,r4,r8,r16,r32,r64) works!
                                myList.append((r1,r2,r4,r8,r16,r32,r64))
    return myList

tuple_generator(-3,2)
tuple_generator(-4,6)
```

A.3 The Fourier expansion of an eta-quotient

```
R.<q> = LaurentSeriesRing(QQ)

# The eta_from_tuple function
# Purpose:       Compute the Fourier expansion of an eta-quotient
# Parameters:    myTuple is a tuple containing the exponents for
#                the eta-quotient in order of increasing divisors
#                N is the level
#                prec is the precision to which the Fourier expansion
#                will be computed, so that it ends with O(q^prec)
def eta_from_tuple(myTuple, N, prec):
    # w is the eta-function not including the q^1/24
    w = qexp_eta(QQ[[q]],prec)

    # e will represent the exponent of the q^1/24 part
    e = 0

    # prod will store everything in the \prod_{n=1}^\infty portion
    prod = 1
    divList = divisors(N)

    # Compute the eta function associated to each divisor
    for i in range(len(myTuple)):
        e += divList[i]*myTuple[i]
        prod *= w(q^divList[i])^myTuple[i]
    return q^(int(e/24))*prod + O(q^prec)
```

A.4 Basis for $M_k^\sharp(64)$

```
# the m_basis function
# Purpose:       This function will compute the first t basis
#                elements of M_k^#(64). Note: The function
#                eta_from_tuple(myTuple,k,prec) is needed.
# Parameters:    k is the weight of the space of modular forms
#                t is the number of basis elements that this
#                function will compute. Note that t can be any
#                integer, however if t < 10 then the default
#                output will be the first nine elements.
# Returns:       A list containing the Fourier expansions of the
#                basis elements.
def m_basis(k,t):
    # guarantee that k is even
    assert(k%2 == 0)
    # set the precision
    prec = t+30

    # define the functions needed to build the basis.
    f3_tup = (0,0,0,-1,2,1,-2)
    f4_tup = (0, 0, 0, 0, -2, 6, -4)
    g_tup = (0,0,0,0,0,-4,8)
    # Compute the Fourier expansion of each of these functions.
    f3 = eta_from_tuple(f3_tup,64,prec)     # This is f_{0,3}(z)
    f4 = eta_from_tuple(f4_tup,64,prec)     # This is f_{0,4}(z)
    g = eta_from_tuple(g_tup,64,prec)       # This is $\psi(z)

    # Explicitly compute the first 9 basis elements.
    thing = g^(k/2)
    b0 = 1*thing
    b3 = f3*thing
    b4 = f4*thing
    b6 = f3^2*thing
    b7 = f3*f4*thing
    b8 = f4^2*thing
    # Put them in a list.
    myList=[b0,0,0,b3,b4,0,b6,b7,b8]

    for m in [6..t-4]:
        theSum = 0
        for i in [5..m]:
            theSum += f3[i]*myList[m-i]
        # Use the recurrence relation to compute the next function
        h = f3*myList[m] - theSum - myList[m][8*k-5]*b8 - myList[m][8*k-1]*b4 - (myList[m][8*k+3] + myList[m][8*k-5])*b0
        myList.append(h)
    return myList
```

A.5 Basis for $S_k^\sharp(64)$

```
# the s_basis function
# Purpose:       This function will compute the first t basis
#                elements of S_k^#(64). Note: The function
#                eta_from_tuple(myTuple,k,prec) is needed.
# Parameters:    k is the weight of the space of modular forms
#                t is the number of basis elements that this
#                function will compute. Note that t can be any
#                integer, however if t < 10 then the default
#                output will be the first nine elements.
# Returns:       A list containing the Fourier expansions of the
#                basis elements.
def s_basis(k,t):
    # guarantee that k is even
    assert(k%2 == 0)
    # set the precision
    prec = t+30

    # define the functions needed to build the basis.
    f3_tup = (0,0,0,-1,2,1,-2)
    f4_tup = (0, 0, 0, 0, -2, 6, -4)
    g_tup = (0,0,0,0,0,-4,8)

    # Compute the Fourier expansion of each of these functions.
    f3 = eta_from_tuple(f3_tup,64,prec)     # This is f_{0,3}(z)
    f4 = eta_from_tuple(f4_tup,64,prec)     # This is f_{0,4}(z)
    g = eta_from_tuple(g_tup,64,prec)       # This is $\psi(z)

    # This is a list of eta-quotients that make a basis of M_2(64)
    # This list was produced and given by Rouse and Webb in
    # http://users.wfu.edu/rouseja/eta/etamake9.data
    basisTuples = [(-2,9,-4,-1,2,0,0), (0,0,2,-1,-4,9,-2),
   (0,-4,8,0,0,0,0), (0,-2,9,-4,-1,2,0), (0,0,0,0,8,-4,0),
   (8,-4,0,0,0,0,0), (0,0,0,0,0,-4,8), (0,2,-3,4,-1,2,0),
   (0,0,-4,8,0,0,0), (0,0,0,8,-4,0,0), (0,2,-1,4,-3,2,0),
   (2,-1,0,0,0,-3,6), (6,-3,0,0,0,-1,2), (-4,10,-4,-1,2,-1,2)]

    # This is the linear combination of basis elements from
    # basisTuples that give the same Fourier Expansion as the
    # basis given by SAGE for S_2(64).
    combo = Matrix([
        [ 3/4, 11, 1/12, 0, 59/24, -1/48, 224/3, 2, -7/2, -39/16,
   5/2, 0, 1/2, 1],
        [ 0, 4, -2/3, 0, 4/3, -1/12, 128/3, 0, 2, -5/4, 1, 0, 0, 0],
        [ -1/8, 11/2, -31/24, 0, 91/48, -17/96, 176/3, -1, 13/4,
   -55/32, 5/4, 0, 1/4, 1/2]])

```

```
    # Now use the matrix/basis above to compute the Fourier
    # expansion of the basis for S_2(64).
    g5 = 0
    g2 = 0
    g1 = 0
    for i in range(len(basisTuples)): g5 += combo[2][i]*eta_from_tuple(basisTuples[i],64,prec)
    for i in range(len(basisTuples)): g2 += combo[1][i]*eta_from_tuple(basisTuples[i],64,prec)
    for i in range(len(basisTuples)): g1 += combo[0][i]*eta_from_tuple(basisTuples[i],64,prec)

    # Now compute the first 9 basis elements of S_2^#(64)
    thing = g^(k/2-1)
    b0 = g5*thing
    b3 = g2*thing
    b4 = g1*thing
    b6 = g2*f3*thing
    b7 = g1*f3*thing
    b8 = (g1*f4+3*g5)*thing
    #Put them in a list.
    myList = [b0,0,0,b3,b4,0,b6,b7,b8]

    for m in [6..t-4]:
        theSum = 0
        for i in [5..m]:
            theSum += f3[i]*myList[m-i]
        # Use the recurrence relation to compute the next function
        h = f3*myList[m] - theSum - myList[m][8*k-16]*b8 - myList[m][8*k-12]*b4 - (myList[m][8*k-16]+myList[m][8*k-8])*b0
        myList.append(h)
    return myList
```

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