This exam is being given under the guidelines of the Honor Code. You are expected to respect those guidelines and to report those who do not.

Select the best answer for each multiple choice question and mark your selection on the provided bubble sheet. The free response questions should be answered directly on this exam.

No notes, books, or calculators, and no time limit. There are 22 questions for a total of 39 points. Good luck!

SOLUTIONS

Name: ________________________________

1. (1 point) Let $V = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integers. Define vector addition and scalar multiplication as

$$ (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) $$

$$ c(x_1, x_2) = (cx_1, cx_2). $$

Then $V$ is a vector space over $\mathbb{R}$.

(a) True
(b) False CORRECT

2. (1 point) The set of all continuous functions $f : [-1, 1] \to \mathbb{R}$ satisfying $f(x) \geq 0$ for all $x \in [-1, 1]$ is a subspace of $C[-1, 1]$.

(a) True
(b) False CORRECT

3. (1 point) Which of the following is a basis for the subspace of $\mathbb{R}^4$ spanned by

$$ \{(1, 2, -1, 3), (3, -2, 1, 1), (2, 4, -2, 6)\}? $$

(a) $\{(1, 2, -1, 3), (2, 4, -2, 6)\}$
(b) $\{(3, -2, 1, 1)\}$
(c) $\{(1, 2, -1, 3), (3, -2, 1, 1), (2, 4, -2, 6)\}$
(d) $\{(3, -2, 1, 1), (2, 4, -2, 6)\}$ CORRECT
4. (1 point) What is the dimension of the vector space of $3 \times 3$ upper triangular matrices.

(a) 3
(b) 6 CORRECT
(c) 7
(d) 9
(e) This is not a vector space

Problems 5-8 refer to the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & -2 & 2 & 1 \\ 2 & 10 & 2 & -5 \end{bmatrix}.$$

5. (1 point) Which of the following is a basis for the row space of $A$?

(a) $\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$ CORRECT

(c) $\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 10 \\ 2 \\ -5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

6. (1 point) What is the rank of $A$?

(a) 0
(b) 1
(c) 2 CORRECT
(d) 3
(e) 4
7. (1 point) Which of the following is a basis for the null space of $A$?

(a) \[
\begin{pmatrix}
4 \\
2 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
-1 \\
1 \\
-5
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
0 \\
1 \\
-1 \\
-1
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
0 \\
1 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
-6 \\
1 \\
1 \\
0
\end{pmatrix}
\]
CORRECT

(d) \[
\begin{pmatrix}
-6 \\
2 \\
1 \\
2
\end{pmatrix}
\]

8. (1 point) Which of the following is a basis for the column space of $A$?

(a) \[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
2 \\
-2 \\
10
\end{pmatrix}
\]
CORRECT

(c) \[
\begin{pmatrix}
2 \\
-2 \\
10
\end{pmatrix}, \quad
\begin{pmatrix}
4 \\
2 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
-1 \\
1 \\
5
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
4 \\
2 \\
2
\end{pmatrix}
\]

9. (1 point) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with matrix representation $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ with respect to the standard basis $[e_1, e_2]$. What is the matrix representation of $L$ with respect to the basis $B = [(1, -1), (3, -2)]$?

(a) \[
\begin{pmatrix}
-1 & -1 \\
-4 & -9
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & -13 \\
0 & 5
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
14 & 29 \\
-5 & -10
\end{pmatrix}
\]
CORRECT

(d) \[
\begin{pmatrix}
15 & 17 \\
-10 & -11
\end{pmatrix}
\]
10. (1 point) If $A$ is a $5 \times 3$ matrix and $N(A) = \text{Span}(e_1)$ what is the rank of $A$?
   (a) 1
   (b) 2 CORRECT
   (c) 3
   (d) 4
   (e) 5

11. (1 point) Consider the system $Ax = b$. Which of the following statements is true?
   (a) If $N(A) = \{0\}$ then $Ax = b$ has a unique solution.
   (b) If $\text{col}(A) = \{0\}$ then $Ax = b$ has a unique solution.
   (c) If $N(A) \neq \{0\}$ then no solution is unique. CORRECT
   (d) If $b \in \text{col}(A)$ then $Ax = b$ has a unique solution.

12. (1 point) Which of the following is NOT true of the vector space $P_n$?
   (a) Any subset of size $m > n$ is linearly dependent.
   (b) Any subset of size $m < n$ does not span $P_n$.
   (c) $P_m$ is a subspace of $P_n$ if $m \leq n$.
   (d) Any subset of size $m < n$ is linearly independent. CORRECT

13. (1 point) If $A$ is a $3 \times 4$ matrix of rank 2, what is the rank of $A^T$?
   (a) 1
   (b) 2 CORRECT
   (c) 3
   (d) 4
   (e) Cannot be determined

14. (1 point) If $[p(x)]_B = \begin{bmatrix} 6 \\ -5 \\ -4 \end{bmatrix}$ where $B$ is the ordered basis $B = [1 + x + x^2, 1 + x, x + x^2]$ for $P_3$, what is $p(x)$?
   (a) $p(x) = -4x^2 - 5x + 6$
   (b) $p(x) = 6x^2 - 5x - 4$
   (c) $p(x) = 2x^2 - 3x - 9$
   (d) $p(x) = 2x^2 - 3x + 1$ CORRECT

15. (1 point) Define $L : C^2(\mathbb{R}) \to C(\mathbb{R})$ by $L(f) = f'' + 3f' + 2f$ (this is linear). Which of the following functions is in the kernel of $L$?
   (a) $f(x) = \cos(x)$
   (b) $f(x) = e^{-2x}$ CORRECT
   (c) $f(x) = x^2 - 2$
   (d) $f(x) = 3$
16. Using complete sentences, give precise, complete definitions for any four of the following terms: Vector Space, Subspace, Span, Linear Independence, Basis, Dimension, Null Space, Column Space, Row Space, Rank, Function, Linear Transformation, Kernel, Injective, Surjective, Bijective, Similar Matrices.

(a) (2 points) Definition:

(See Study Guide for definitions)

(b) (2 points) Definition:

(See Study Guide for definitions)

(c) (2 points) Definition:

(See Study Guide for definitions)

(d) (2 points) Definition:

(See Study Guide for definitions)
17. (2 points) Let \( B = [(-1, 2), (-1, 1)] \) and \( C = [(1, 3), (1, 4)] \) be ordered bases for \( \mathbb{R}^2 \). Find the transition matrix for the change of basis from \( C \) to \( B \). That is, find \( T \) such that \( [v]_B = T[v]_C \) for all \( v \in \mathbb{R}^2 \).

The transition from \( B \) to the standard basis \( E \) is
\[
R = \begin{bmatrix}
-1 & -1 \\
2 & 1
\end{bmatrix}.
\]

The transition from \( E \) to \( B \) is the inverse,
\[
R^{-1} = \begin{bmatrix}
1 & 1 \\
-2 & -1
\end{bmatrix}.
\]

The transition from \( C \) to \( E \) is
\[
S = \begin{bmatrix}
1 & 1 \\
3 & 4
\end{bmatrix}.
\]

The transition from \( C \) to \( B \) is the product of the transition from \( E \) to \( B \) with the transition from \( C \) to \( E \):
\[
T = R^{-1}S = \begin{bmatrix}
1 & 1 \\
-2 & -1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
3 & 4
\end{bmatrix} = \begin{bmatrix}
4 & 5 \\
-5 & -6
\end{bmatrix}.
\]
18. (2 points) Let

\[
E = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

be the standard ordered basis for \( \mathbb{R}^{2 \times 2} \). Define \( L : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2} \) by

\[
L(X) = X + 2X^T.
\]

Note that this is linear (you don’t need to prove this). Find the matrix representation of \( L \), that is, find a matrix \( A \) such that \( [L(X)]_E = A [X]_E \).

The matrix representation is a matrix \( A \in \mathbb{R}^{m \times n} \) where \( m = n \) is the dimension of \( \mathbb{R}^{2 \times 2} \), so \( n = 4 \). Also, the \( j^{th} \) column of \( A \) is \( [L(e_j)]_E \). Therefore,

\[
L(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
L(e_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}
\]

\[
L(e_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}
\]

\[
L(e_4) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}
\]

The coordinate vectors are

\[
[L(e_1)]_E = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [L(e_2)]_E = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad [L(e_3)]_E = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad [L(e_4)]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}.
\]

Therefore, \( A \) is

\[
A = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]
19. (3 points) Prove either of the following two theorems (DO NOT DO BOTH - ONLY ONE WILL BE GRADED).

(a) The zero element of a vector space is unique.

(See the Study Guide for proofs)

(b) Let $x, y, z \in V$ be given, where $V$ is vector space. If $x + y = x + z$ then $y = z$. 
20. (3 points) Prove either of the following two theorems (DO NOT DO BOTH - ONLY ONE WILL BE GRADED).

(a) Let $A \in \mathbb{R}^{m \times n}$ be given. Then $N(A)$ is a subspace of $\mathbb{R}^n$.

(See the Study Guide for proofs)

(b) Let $L : V \to W$ be a linear transformation. Then $\ker L$ is a subspace of $V$. 

21. (3 points) Prove either of the following two theorems (DO NOT DO BOTH - ONLY ONE WILL BE GRADED).

(a) Let \( L : V \rightarrow W \) be a linear transformation and \( S \) be a subspace of \( V \). Then \( L(S) \) is a subspace of \( W \).

(See the Study Guide for proofs)

(b) Let \( L : V \rightarrow W \) be a linear transformation and let \( T \) be a subspace of \( W \). Then \( L^{-1}(T) \) is a subspace of \( V \).
22. (3 points) Prove either of the following two theorems (DO NOT DO BOTH - ONLY ONE WILL BE GRADED).

(a) Let $L : V \to W$ be a linear transformation. Then $L$ is injective if and only if $\ker L = \{0\}$.

(See the Study Guide for proofs)

(b) Let $A$ and $B$ be similar matrices. Then $\det(A) = \det(B)$. 