This exam is being given under the guidelines of the **Honor Code**. You are expected to respect those guidelines and to report those who do not.

Answer each question carefully and completely. Make your work clear. If you have questions you may ask your instructor, but do not work with anyone else on this assignment. You may use your text and your notes.

There are 4 questions for a total of 15 points.

Name: ____________________________

In Chapters 3 and 4 we have learned about the theory of vector spaces, subspaces, bases, and linear transformations. We have learned how to represent vectors spaces in terms of a basis and coordinate vectors. In this exam we will look at another way to represent and deal with vectors spaces. Specifically, we will show how to decompose a vector space into two subspaces and how to generalize the idea of a projection. These concepts are a very powerful result of the theory we have developed and play a central role in applications of orthogonality (chapter 5) and spectral theory (chapter 6).

Throughout this exam, $V$ will be a finite-dimensional vector space with dimension $\dim V = n$. Recall that $U \leq V$ means that $U$ is a subspace of $V$. We will also use $R(L)$ to denote the range of a linear transformation $L$ and $N(L)$ to denote the kernel (or null space).

In your homework, you showed that, for any two subspaces $U$ and $W$ of $V$, the set

$$U + W = \{ u + w \mid u \in U \text{ and } w \in W \}$$

is a subspace of $V$. In fact, it is the smallest subspace of $V$ that contains both $U$ and $W$ (at this point you should convince yourself that $U \leq U + W$ and $W \leq U + W$). This exam is about how to decompose a vector space $V$ into a sum of subspaces $U$ and $W$ called *complementary subspaces*. Essentially, we want to find subspaces $U$ and $W$ that are big enough so that their sum is the entire space, that is, $V = U + W$. However, we don’t want them to overlap.

**Definition 1.** Two subspaces $U$ and $W$ of $V$ are said to be *complementary* if $U + W = V$ and $U \cap W = \{0\}$. In this case we write $V = U \oplus W$.

Recall that one of the main benefits of a basis is that we can represent each vector in $V$ uniquely as a linear combination of the basis. With complementary spaces we get a similar property.

**Theorem 2.** If $V = U \oplus W$ then for each $v \in V$ there exists a unique pair $u \in U$ and $w \in W$ such that $v = u + w$.

**Proof.** Let $v \in V$ be given. Since $V = U + W$, we may write $v = u + w$ where $u \in U$ and $w \in W$. Now suppose that $v = u_1 + w_1 = u_2 + w_2$. Then $u_1 - u_2 = w_2 - w_1$. However, $u_1 - u_2 \in U$ and $w_2 - w_1 \in W$ which means that

$$u_1 - u_2 = w_2 - w_1 \in U \cap W = \{0\}.$$

Therefore, we see that $u_1 - u_2 = w_2 - w_1 = 0$ which implies that $u_1 = u_2$ and $w_1 = w_2$.  

\[ \square \]
As an example of complementary subspaces, let \( V = \mathbb{R}^2, U = \text{Span}(e_1) \) and \( W = \text{Span}(e_2) \). Then \( V = U \oplus W \). For example, the vector \((2, -3) \in V \) can be written (uniquely) as the sum \((2, -3) = (2, 0) + (0, -3) \) where \((2, 0) \in U \) and \((0, -3) \in W \). Notice that we do not allow scalar multiples. We are not taking linear combinations, just sums. But since \( U \) and \( W \) are subspaces (not just collections of vectors like with a basis) scalar multiples are already vectors in \( U \) and \( W \).

In your homework you showed that if \( U \cap W = \{0\} \) then \( \dim U + \dim W = \dim U + W \). If \( U \) and \( W \) are complementary subspaces, this implies that \( \dim U + \dim W = \dim V \). Your first problem is to go the other direction. That is, to show that if \( U \cap W = \{0\} \) and \( \dim U + \dim W = \dim V \) then \( U \) and \( W \) are complementary.

**Problem 1.** (5 points) Let \( U \) and \( W \) be subspaces of \( V \) such that \( \dim U + \dim W = \dim V \) and \( U \cap W = \{0\} \). Prove that \( U \oplus W = V \). [Hint: You are given that \( U \cap W = \{0\} \) so all you need to show is that \( U + W = V \). Take a basis for \( U \) and one for \( W \), combine them and show that you have a linearly independent set in \( V \) (use \( U \cap W = \{0\} \)). You want to show that this is a basis for \( V \), in other words that it spans \( V \) (if it doesn’t, you could add something else, what would be wrong with that? Think \( \dim U + \dim W = \dim V \)). Once you have that it is a basis, you should be able to prove that every \( v \in V \) can be written in the form \( v = u + w \).

**Proof.** Let \( \{u_1, \ldots, u_k\} \) and \( \{w_1, \ldots, w_r\} \) be bases for \( U \) and \( W \) respectively. If

\[
\alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 w_1 + \cdots + \beta_r w_r = 0
\]

then since \( U \) and \( W \) are subspaces (closed under linear combinations),

\[
\alpha_1 u_1 + \cdots + \alpha_k u_k = -\beta_1 w_1 - \cdots - \beta_r w_r \in U \cap W.
\]

Since \( U \cap W = \{0\} \),

\[
\alpha_1 u_1 + \cdots + \alpha_k u_k = -\beta_1 w_1 - \cdots - \beta_r w_r = 0.
\]

However, \( \{u_1, \ldots, u_k\} \) is a linearly independent set, as is \( \{w_1, \ldots, w_r\} \). Therefore, \( \alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_r = 0 \) and

\[
S = \{u_1, \ldots, u_k, w_1, \ldots, w_r\}
\]

is a linearly independent set. Notice that there are \( k + r = \dim U + \dim W = \dim V \) vectors in \( S \), which means that \( S \) is a basis for \( V \). Therefore, we may write any \( v \in V \) as

\[
v = a_1 u_1 + \cdots + a_k u_k + b_1 w_1 + \cdots + b_r w_r.
\]

Set \( u = a_1 u_1 + \cdots + a_k u_k \) and \( w = b_1 w_1 + \cdots + b_r w_r \). Then \( v = u + w \). Since \( v \) is arbitrary, \( V = U + W \). Combining this with \( U \cap W = \{0\} \) we have \( V = U \oplus W \). \( \square \)
In class we mentioned that \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by
\[
L(x_1, x_2) = (x_1, 0)
\]
is the projection onto the \( x_1 \) axis. More precisely, we say that \( L \) is the projection onto the \( x_1 \) axis along the \( x_2 \) axis since the \( x_2 \) axis is what we are projecting out (or what we are forcing to be zero). Recall from your homework that we can take powers, \( L^k \), of a linear operator, \( L : V \to V \), using function composition. If you think of this in terms of the (square) matrix representation, you are taking powers of the matrix.

**Definition 3.** A linear operator \( L : V \to V \) is said to be idempotent if \( L^2 = L \).

**Problem 2.** (3 points) Let \( L : V \to V \) be idempotent. Show that \( L(x) = x \) for all \( x \in R(L) \).

[Hint: \( x \in R(L) \) means that \( x = L(v) \) for some \( v \in V \).]

**Proof.** The definition of \( R(L) \) is
\[
R(L) = \{ x \in V \mid x = L(v) \text{ for some } v \in V \}.
\]
Therefore, if \( x \in R(L) \), there exists some \( v \in V \) such that \( L(v) = x \). Therefore,
\[
L(x) = L(L(v)) = L^2(v) = L(v) = x.
\]
\( \square \)
Problem 3. (3 points) Let $L : V \rightarrow V$ be idempotent. Prove that $V = R(L) \oplus N(L)$. That is, show that $R(L)$ and $N(L)$ are complementary subspaces in $V$. [Hint: Use problem (2) to show that $N(L) \cap R(L) = \{0\}$ and then use the Rank-Nullity Theorem and problem (1) to conclude that they are complementary.]

Proof. First we show that $R(L) \cap N(L) = \{0\}$. Suppose $x \in R(L) \cap N(L)$. Then since $x \in R(L)$, the previous problem implies that $L(x) = x$. However, by definition, if $x \in N(L)$ then $L(x) = 0$. Therefore, $x = L(x) = 0$ so $R(L) \cap N(L) = \{0\}$. Furthermore, the Rank-Nullity Theorem for linear operators implies that

$$\dim V = \dim R(L) + \dim N(L).$$

By Problem 1, we have that $V = R(L) \oplus N(L)$.  \qed
Now that we know that \( R(L) \) and \( N(L) \) are complementary subspaces for any idempotent operator, we can define what we mean by a projection.

**Definition 4.** Let \( V = U \oplus W \). The projection onto \( U \) along \( W \) is the (unique) idempotent operator \( L : V \to V \) satisfying \( R(L) = U \) and \( N(L) = W \).

We claimed that \( L \) is unique. We should verify this, which we will do now (it is pretty easy based on what we know so far).

**Theorem 5.** If \( V = U \oplus W \), then there is a unique idempotent \( L : V \to V \) satisfying \( R(L) = U \) and \( N(L) = W \).

**Proof.** For any \( v \in V \), decompose \( v \) into \( v = u + w \). Now define \( L : V \to V \) by

\[
L(v) = L(u + w) = u.
\]

Therefore, \( L(v) \in U \) for all \( v \) so \( R(L) \subseteq U \). Furthermore, for every \( u \in U \), \( L(u) = u \) so that \( U \subseteq R(L) \). Therefore, \( R(L) = U \). Similarly, \( N(L) = W \) since \( L(v) = u = 0 \) if and only if \( u = 0 \), in other words, \( L(v) = 0 \) if and only if \( v \in W \).

Now we will show that \( L \) is (1) linear, (2) idempotent, and (3) unique.

(1) Given \( x, y \in V \) and \( \alpha, \beta \in \mathbb{F} \), write \( x = u_x + w_x \) and \( y = u_y + w_y \). Then it is clear that the unique decomposition of \( \alpha x + \beta y \) is given by

\[
\alpha x + \beta y = (\alpha u_x + \beta u_y) + (\alpha w_x + \beta w_y).
\]

Therefore,

\[
L(\alpha x + \beta y) = \alpha u_x + \beta u_y = \alpha L(x) + \beta L(y).
\]

(2) We have

\[
L^2(v) = L(L(v)) = L(L(u + w)) = L(u) = u = L(v)
\]

for all \( v \in V \). Therefore \( L^2 = L \).

(3) Suppose there exists some \( L_2 : V \to V \) which is idempotent and satisfies \( R(L) = U \) and \( N(L) = W \). Then for any \( v \in V \)

\[
L_2(v) = L_2(u + w) = L_2(u) + L_2(w).
\]

Since \( N(L_2) = W \) we must have that \( L_2(w) = 0 \) and since \( u \in R(L_2) \) and \( L_2 \) is idempotent, problem 2 implies that \( L_2(u) = u \). Therefore,

\[
L_2(v) = L_2(u) + L_2(w) = u = L(v).
\]

Hence, \( L = L_2 \) is unique. \( \square \)

Notice that in the previous proof, we needed the fact that the decomposition \( v = u + w \) was unique for the definition of our linear operator to even make sense. If we could write \( v = u_1 + w_1 \) and \( v = u_2 + w_2 \) in two different ways, then it wouldn’t make sense to say that \( L(v) = u \).

Let’s summarize before we get to the last part. We have two “important” complementary subspaces \( U \oplus W = V \). Because they are complementary, we can split each vector \( v \in V \) into two unique parts: one in \( U \) and one in \( W \). Furthermore, there exists a unique linear operator \( L : V \to V \) (and, given a basis, a matrix representation) such that \( L(v) = u \). In other words, this linear operator gives the part of \( v \) that lies in \( U \). This is why we call it the projection onto \( U \) along
W. It projects the W part out of v and returns just the U part. Now, I am sure you are asking yourself at this point: “How do I get the other part?”. Well, the other part is \( w = v - u = v - L(v) \). This leads us to the last problem on the exam.

**Problem 4.** (4 points) Let \( V = U \oplus W \), let \( L \) be the projection onto \( U \) along \( W \), and let \( I : V \to V \) be the identity linear operator. That is, \( I(v) = v \) for all \( v \in V \). Prove that \( I - L \) is the projection onto \( W \) along \( U \). [Hint: It is obvious that \( I - L \) is linear (the difference of two linear transformations is always linear), so don’t worry about that. First show that \( (I - L)(v) = w \). From there you can easily show that \( I - L \) is idempotent, \( R(I - L) = W \) and \( N(I - L) = U \). See the proof of Theorem 5 if you get stuck.]

**Proof.** Recall that for any \( v \in V \) there is a unique decomposition \( v = u + w \) because \( V = U \oplus W \). If we write \( v = u + w \), then we see that

\[
(I - L)(v) = I(u + w) - L(u + w) = u + w - L(u) + L(w).
\]

Since \( w \in W = N(L) \), \( L(w) = 0 \). Also, since \( u \in U - R(L) \), \( L(u) = u \). Therefore,

\[
(I - L)(v) = u + w - u = w.
\]

To verify that \( I - L \) is idempotent,

\[
(I - L)^2(v) = (I - L)(w) = w - L(w) = w = (I - L)(v).
\]

Also, \( (I - L)(v) = w \) implies that \( R(I - L) \subseteq W \). Furthermore, for \( w \in W \), \( (I - L)(w) = w \) which implies \( R(I - L) = W \). Also, if \( u \in U \) then \( (I - L)(u) = u - L(u) = u - u = 0 \) so \( U \subseteq N(I - L) \).

What’s more, if \( v \in N(I - L) \), then

\[
0 = (I - L)(v) = w
\]

where \( v = u + w \). This implies that \( v = u \in U \) so that \( N(I - L) = U \). \qed