Math 343H Lab 7: Integration by Parts

Objective

In this lab we explore linear algebraic alternatives to integration by parts.

The space of continuous functions

We begin by considering the space $C[a, b]$ of continuous functions defined on the set $[a, b]$. Note that this is a vector space. For example, if $f(x)$ and $g(x)$ are continuous, then so are $f(x) + g(x)$ and $af(x)$, where $a \in \mathbb{R}$. For the same reasons, the space $C^1[a, b]$ of continuously differentiable functions defined on the set $[a, b]$ is also a vector space. Note that

$$
\frac{d}{dx} : C^1[a, b] \rightarrow C[a, b]
$$

is a linear transformation since

$$
\frac{d}{dx}(af(x) + bg(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x).
$$

We remark that both $C[a, b]$ and $C^1[a, b]$ are infinite dimensional vector spaces (sometimes called function spaces), and thus we cannot represent the linear transformation with a matrix representation.

Subspaces of continuous functions

Consider the subspace $W$ of $C^1[a, b]$ spanned by the basis $B = \{e^x, xe^x, x^2e^x\}$. Note that

$$
\frac{d}{dx}e^x = e^x
$$

$$
\frac{d}{dx}xe^x = e^x + xe^x
$$

$$
\frac{d}{dx}x^2e^x = 2xe^x + x^2e^x,
$$
or in other words, the derivatives of the basis functions $B$ of $W$ are in $W$, that is,

$$\frac{d}{dx} : W \rightarrow W.$$ 

Since this is a linear transformation from one finite dimensional vector space to another, it has a matrix representation. Since $B$ is a basis for $W$, a linear combination $f(x) = ae^x + bxe^x + cx^2e^x$ can simply be represented as a column vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$ 

Sometimes we write this as

$$\begin{bmatrix} e^x & xe^x & x^2e^x \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

just to keep the basis $B$ visible. The derivative of $f(x)$ takes the form

$$\frac{d}{dx} (ae^x + bxe^x + cx^2e^x) = ae^x + b(e^x + xe^x) + c(2xe^x + x^2e^x)$$

$$= (a + b)e^x + (b + 2c)xe^x + cx^2e^x$$

$$= \begin{bmatrix} e^x & xe^x & x^2e^x \end{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$ 

Hence the matrix representation of the derivative on $W$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

or in other words

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$
Example

Consider the function $g(x) = 5e^x - 3xe^x + 2x^2e^x$. Note that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Hence, $g'(x) = 2e^x + xe^x + 2x^2e^x$.

Antiderivatives

Recall that the antiderivative is also a linear transformation. Note that the antiderivative of the derivative is the original function\(^1\), and hence the matrix representation of the antiderivative is the inverse of the matrix representation of the derivative. For example, if we wanted to compute

$$\int ae^x + bxe^x + cx^2e^x \, dx,$$

we could simply invert the matrix representation of the derivative

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

This gives us the transformation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$ 

Example

Consider the function $h(x) = 2e^x + xe^x + 2x^2e^x$. Note that

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}.$$ 

\(^1\)This is true up to an additive constant.
Hence the antiderivative of $h(x)$ is $5e^x - 3xe^x + 2x^2e^x + C$. We see from the previous example that we get the right answer. What’s more is we didn’t need to use integration by parts to get the answer!

**Assignment**

**Problem 1.** Apply the above concepts by writing a MATLAB function called `myint` that takes as input the vector $[a_0, a_1, \cdots, a_n]$ and computes the antiderivative of

$$f(x) = a_0e^x + a_1xe^x + a_2x^2e^x + \cdots + a_nx^n e^x.$$  

*Hint: Find the matrix representation of the derivative, then take the inverse.*

**Problem 2.** Let $W$ be the subspace of $C^1[a,b]$ spanned by the basis

$$B = \{\cos(\alpha x)e^{\beta x}, \sin(\alpha x)e^{\beta x}\}.$$  

(a). Find the matrix representation $D$ of the derivative in the basis $B$.

(b). Find the inverse of $D$.

(c). Use your answer above to find the anti-derivative of

$$f(x) = 14 \sin(\alpha x)e^{\beta x}.$$  

*We remark that the traditional way to do this problem requires a special trick where one integrates by parts twice. Doing it this way, one can avoid all that.*