Name ______________________________

Section _______________

Math 113 Test 3 Kuttler

Multiple choice. Answer by coloring the appropriate circle on the bubble sheet.

1. Find $\lim_{n\to\infty}\left(\sqrt{2n^2+7n}-\sqrt{2}n\right)$
   (a) 3
   (b) $\dfrac{7}{2}$
   (c) 1
   (d) $\dfrac{7}{4}\sqrt{2}$
   (e) $5\sqrt{2}$
   (f) 0
   (g) 7
   (h) Undefined
   Answer: $=\frac{7}{4}\sqrt{2}$ d

2. $\int_3^\infty \dfrac{2+5\left|\sin\left(\frac{1}{x}\right)\right|}{\sqrt{x^3-2}}dx$
   (a) Converges by a comparison with $\dfrac{1}{\sqrt{x}}$
   (b) Converges by a comparison with $\dfrac{1}{x^2}$
   (c) Converges by a comparison with $\dfrac{1}{x^{3/2}}$
   (d) Diverges by a comparison with $\dfrac{1}{x}$
   (e) Diverges by a comparison with $\dfrac{1}{\sqrt{x^3}}$
   (f) Neither converges nor diverges
   (g) Might either converge or diverge depending on the Jacobsen radical.
   (h) None of the above.
   Answer: $\lim_{x\to\infty}\dfrac{\left(\frac{2+5\left|\sin\left(\frac{1}{x}\right)\right|}{\sqrt{x^3-2}}\right)}{\left(\frac{1}{x^{3/2}}\right)}=2$ c

3. Find the minimum value of the function $f(x)=x+\dfrac{2}{x}$ for $x>0$.
   (a) 1
   (b) $\sqrt{2}$
   (c) $3\sqrt{2}$
   (d) $4\sqrt{2}$
   (e) $2\sqrt{2}$
   (f) None of the preceding.
   $x+\frac{2}{x}$ Candidate(s) for extrema: $\left\{-2\sqrt{2},2\sqrt{2}\right\}$, at $\left\{\left\{x=\sqrt{2}\right\},\left\{x=-\sqrt{2}\right\}\right\}$ e

4. Let $f(x)=\int_{-4}^{x^3}\sqrt{1+t^4}dt$. Find $f'(1)$.
   (a) $\sqrt{2}$
   (b) $2\sqrt{2}$
   (c) $3\sqrt{2}$
   (d) $4\sqrt{2}$
   (e) $5\sqrt{2}$
   (f) 0
   (g) None of the preceding.
   Answer: $D_x\left(\int_{-4}^{x^3}\sqrt{1+t^4}dt\right)=3x^2\sqrt{(1+x^{12})}$ c

5. Find $\lim_{n\to\infty}\left(1-\dfrac{2}{n}\right)^n$
   (a) 1
   (b) ∞
   (c) 0
   (d) e
   (e) π
   (f) $\pi+e$
   (g) e^2
   (h) e^{-1}
   (i) e^{-2}
   (j) None of the preceding.
   Answer: $\lim_{n\to\infty}\left(1-\dfrac{2}{n}\right)^n=e^{-2}$ i

6. A ball rebounds 1/4 of the distance if falls. If dropped from a height of 12 feet, how far will the ball travel in coming to rest? Distances are in feet.
   (a) 40
   (b) 35
   (c) 30
   (d) 20
   (e) 25

(f) 15

(g) None of the preceding.

Answer: 20 d

7. $\sum \frac{(-1)^n}{1+\ln n}$

(a) Converges absolutely by the ratio test.

(b) Converges conditionally by the ratio test.

(c) Diverges by the alternating series test.

(d) Diverges by the ratio test but converges by the alternating series test.

(e) Diverges by the integral test.

(f) Converges by the alternating series test but diverges by the root test.

(g) Converges by the alternating series test.

(h) Diverges by the limit comparison test.

Answer: g

8. $\sum \frac{n!}{n^n}$

(a) Converges absolutely by the ratio test.

(b) Converges conditionally by the ratio test.

(c) Diverges by the alternating series test.

(d) Diverges by the ratio test but converges by the alternating series test.

(e) Diverges by the integral test.

(f) Converges by the alternating series test but diverges by the root test.

(g) Converges by the alternating series test.

(h) Diverges by the limit comparison test.

Answer: a

9. $\sum \frac{1}{\sqrt{n^2+n}}$

(a) Converges absolutely by the ratio test.

(b) Converges conditionally by the ratio test.

(c) Diverges by the alternating series test.

(d) Diverges by the divergence test.

(e) Diverges by the root test.

(f) Converges by the alternating series test but diverges by the root test.

(g) Converges by the alternating series test.

(h) Diverges by the limit comparison test.

Answer: h

10. $\sum_{h=3}^{\infty} \frac{2+5\sin\left(\frac{1}{n}\right)}{\sqrt{n^2-2}}dx$

(a) Converges by alternating series test.

(b) Converges absolutely by the ratio test.

(c) Converges by a comparison with $\frac{1}{n}$

(d) Converges by the root test.

(e) Converges because the limit of the n^{th} term equals zero.

(f) Diverges by a comparison with $\frac{1}{n}$

(g) Neither converges nor diverges

(h) Converges for certain values of n and diverges for others.

(i) Might either converge or diverge depending on the Jacobsen radical.

(j) None of the above.

Answer: $\lim_{x\to\infty} \frac{\left(\frac{2+5\left|\sin\left(\frac{1}{x}\right)\right|}{\sqrt{x^2-2}}\right)}{\left(\frac{1}{x}\right)} = 2$ so diverges by comparison with $\frac{1}{n}$. f

11. Find the sum of the series $\sum_{n=1}^{\infty} n\,(-1)^n \left(\frac{1}{2}\right)^n$.

(a) 0

(b) -1

(c) 1

(d) $-2/9$

(e) $2/9$

(f) $2/3$

(g) $-2/3$

(h) $-1/4$

(i) $1/4$

(j) None of the above.

$\sum_{n=1}^{\infty} n\,(-1)^n \left(\frac{1}{2}\right)^n = -\frac{2}{9}$ d

12. Find $\lim_{x\to 0} \dfrac{2 - x^4 - 2\cos\left(x^2\right)}{x^8}$. **Hint:** You might want to consider using power series to do this. Of course maybe you wouldn't.

(a) 1
(b) -1
(c) $1/6$
(d) $-1/6$
(e) $1/12$
(f) $-1/12$
(g) 0
(h) $\dfrac{0}{0}$
(i) 6
(j) None of the above.

$\lim_{x\to 0} \frac{2-x^4-2\cos\left(x^2\right)}{x^8} = -\frac{1}{12}$ f

13. Find $\int_0^1 \dfrac{x^2+2+x}{(1+x)\left(x^2+1\right)}dx$

(a) $\ln 5 + \arctan 3 - \ln 4 - \dfrac{1}{2}\pi$
(b) $\ln 6 - \arctan 2 - \ln 3 - \dfrac{\pi}{2}$
(c) $\arctan 2 - \ln 2 - \dfrac{1}{2}\pi$
(d) $\ln 2 + \arctan 2 + \ln 2 - \dfrac{1}{3}\pi$
(e) $\ln 2 + \dfrac{1}{4}\pi$
(f) $\ln 3 - \ln 2 + \pi$
(g) $\ln 3 - \ln 2$
(h) None of the above.

Answer: $\int_0^1 \dfrac{x^2+2+x}{(1+x)\left(x^2+1\right)}dx = \ln 2 + \dfrac{1}{4}\pi$ e

14. What is the correct form for the partial fractions expansion of $\dfrac{4x^4+8x^2+3+7x^3+9x}{(x+2)^2\left(x^2+1\right)^2}$?

(a) $\dfrac{A}{(x+2)^2} + \dfrac{C}{x^2+1} + \dfrac{D}{\left(x^2+1\right)^2}$
(b) $\dfrac{A}{x+2} + \dfrac{Cx+D}{\left(x^2+1\right)^2}$
(c) $\dfrac{A}{x+2} + \dfrac{Bx}{x^2+1} + \dfrac{Cx}{\left(x^2+1\right)^2}$
(d) $\dfrac{A}{x+2} + \dfrac{E}{(x+2)^2} + \dfrac{Bx+C}{x^2+1} + \dfrac{Mx+N}{\left(x^2+1\right)^2}$
(e) $\dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+1} + \dfrac{E}{(x+2)^2} + \dfrac{Cx^2+Dx+E}{\left(x^2+1\right)^2}$
(f) None of the above.

Answer: d

15. Find the area enclosed by the two curves, $x = 5 - y^2$ and $x = -4y$.

(a) $\dfrac{122}{11}$
(b) $\dfrac{121}{4}$
(c) 36
(d) $\dfrac{100}{3}$
(e) 4
(f) None of the preceding.

Answer: $\int_{-1}^5 \left(5 - y^2 + 4y\right) = 36$ c

16. Let $p > 0$ then $\int_0^1 \frac{1}{\sqrt{x^p}}dx$

(a) Diverges if $p \geq 2$ and converges if $p < 2$
(b) Converges if $p \geq 2$ and diverges if $p < 2$
(c) Both converges and diverges if $p = 1$.
(d) Converges if $p > 2$ and diverges if $p \leq 2$
(e) Diverges if $p > 2$ and converges if $p \leq 2$.
(f) Never converges for any value of p.
(g) Converges for all values of p
(h) None of the above.

Answer: Diverges if $p \geq 2$ and converges if $p < 2$ a

17. Let A denote the region between the graph of $y = \cos x$ and the x axis for $x \in \left[0, \dfrac{\pi}{2}\right]$. A solid is obtained by revolving A about the line $x = -1$. Find the volume of this solid.

(a) π
(b) $1 + \pi$
(c) π^2
(d) $2 + \pi$
(e) $\pi^2 - 2\pi$
(f) None of the preceding.

Answer: $\int_0^{\pi/2} (1+x)\, 2\pi \cos(x)\, dx = \pi^2$ c

18. $\int_2^\infty e^{-t}\sin(2t)\,dt =$

   (a) $\frac{1}{2}e^{-2}\cos 2 + \frac{1}{2}e^{-2}\sin 3$
   (b) $e^{-2}\cos 2 + e^{-2}\sin 2$
   (c) 3
   (d) $\frac{2}{5}e^{-2}\cos 4 + \frac{1}{5}e^{-2}\sin 4$
   (e) $\sin(2)\,e^2$
   (f) Does not converge.
   (g) Neither converges nor diverges.
   (h) None of the above.
   Answer: $\frac{2}{5}e^{-2}\cos 4 + \frac{1}{5}e^{-2}\sin 4$ d

19. $\sum_{k=1}^\infty \frac{1}{k(k+3)} =$

   (a) $\frac{11}{19}$
   (b) $\frac{7}{16}$
   (c) 3
   (d) $\frac{11}{18}$
   (e) $\frac{11}{21}$
   (f) $\frac{11}{23}$
   (g) The series diverges by the root test.
   (h) The sum cannot be determined because the subgroup is not normal.
   (i) None of the above.
   Answer: $\frac{11}{18}$ d

20. Find the length of the graph of $y = x^{1/2} - \dfrac{x^{3/2}}{3}$ for $x \in [0,1]$.

   (a) $\sqrt{3}$
   (b) $\dfrac{4}{3}$
   (c) $2\sqrt{3}$
   (d) $\dfrac{5}{3}\sqrt{3}$
   (e) $\dfrac{2}{3}$
   (f) None of the preceding
   Answer: $D_x\left(x^{1/2} - \frac{x^{3/2}}{3}\right) = -\frac{1}{2}\frac{-1+x}{\sqrt{x}}$ $\sqrt{1+\left(-\frac{1}{2}\frac{-1+x}{\sqrt{x}}\right)^2} = \frac{1}{2}\sqrt{\left(4+\frac{(-1+x)^2}{x}\right)} = \frac{1}{2}\frac{x+1}{\sqrt{x}}$ $\int_0^1 \frac{1}{2}\frac{x+1}{\sqrt{x}} = \frac{4}{3}$ b

21. $\int_1^\infty \dfrac{1}{x(x+3)}dx$

   (a) $\ln 3$
   (b) $3\ln 4$
   (c) $\dfrac{1}{2}\ln 3$
   (d) $\dfrac{2}{3}\ln 2$
   (e) $-\ln 3$
   (f) Does not converge.
   (g) Converges by comparison with $\dfrac{1}{x^4}$.
   (h) Neither converges nor diverges.
   (i) None of the above.
   Answer: $\int_1^\infty \dfrac{1}{x(x+3)}dx = \dfrac{2}{3}\ln 2$ d

22. Find the power series expansion for the function $\arctan(x)$ expanded about 0. **Hint:** Recall $\arctan(x) = \int_0^x \dfrac{1}{1+t^2}dt$.

   (a) $\sum_{k=0}^\infty \dfrac{x^{2k}}{(2k)!}$
   (b) $\sum_{k=1}^\infty x^k$
   (c) $\sum_{k=0}^\infty \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$
   (d) $\sum_{k=0}^\infty \dfrac{x^{2k}}{2k+1}$
   (e) $\sum_{k=0}^\infty \dfrac{(-1)^k x^k}{(2k+1)!}$
   (f) $\sum_{k=0}^\infty \dfrac{(-1)^k x^{2k+1}}{2k+1}$
   (g) The given function does not have a power series.
   (h) None of the above.
   Answer: $\sum_{k=0}^\infty \dfrac{(-1)^k x^{2k+1}}{2k+1}$ f

23. Give the first three terms of the power series for $f(x) = \sqrt[3]{1+x}$ expanded about 0.

   (a) $1 + x - \dfrac{1}{3}x^2$
   (b) $1 - \dfrac{2}{3}x - \dfrac{2}{9}x^2$
   (c) $1 + x^{1/3}$
   (d) $3 + x - \dfrac{1}{3}x^2$
   (e) $1 + \dfrac{1}{3}x - \dfrac{1}{9}x^2$

(f) $1+\dfrac{2}{3}x+\dfrac{2}{9}x^2$

(g) This function has no power series.

(h) None of the above.

Answer: $\sqrt[3]{1+x} = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\frac{22}{729}x^5-\frac{154}{6561}x^6+\frac{374}{19\,683}x^7-\frac{935}{59\,049}x^8+\frac{21\,505}{1594\,323}x^9+O\left(x^{10}\right)$ e

Short Answer. Give your answer in the space provided. All responses on scratch paper will be ignored.

1. Recall $\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k$. Therefore, replacing x with $1/x$, we also have $\frac{1}{1-1/x}=\frac{x}{x-1}=\sum_{k=0}^{\infty}\left(\frac{1}{x}\right)^k$. Therefore,

$$1=\frac{1}{1-x}+\frac{x}{x-1}=\sum_{k=0}^{\infty}x^k+\sum_{k=0}^{\infty}\left(\frac{1}{x}\right)^k.$$

   Letting $x=1$, it follows $1>\sum_{k=0}^{\infty}1>\sum_{k=1}^{100}1=100$. What is wrong with this argument? Give your answer in the space provided. There is more than enough room.

   The problem here is that at least one of the series does not converge. Therefore, it makes no sense.

2. Find the radius of convergence of the following series.

   (a) $\sum_{n=1}^{\infty}\frac{n^n}{n!}x^n$

   $1/e$

   (b) $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}$

   1

   (c) $\sum\frac{nx^n}{3^n}$

   3

   (d) $\sum_{n=1}^{\infty}nx^n$

   1

   (e) $\sum_{n=0}^{\infty}\frac{n^2+2}{2^n}x^n$

   2

3. Find the Taylor polynomial of degree 3 for $f(x)=x^3+2x^2-x+7$ which is centered at 1.

   $x^3+2x^2-x+7=9+6(x-1)+5(x-1)^2+(x-1)^3$ *Just a suggestion: For something this easy, you should check your work. If what you get does not agree with the original function, it must be wrong!*

4. Prove that if $\{a_n\}$ is bounded and if $\{b_n\}$ converges to 0, then $\{a_nb_n\}$ also converges to 0. To receive credit, you must avoid saying anything which is false and your answer must be well formed. No one will try to speculate about what you mean. Give your answer in the space provided. Plenty of room is provided to do so.

   Let $|a_n|<L$. *You can't conclude that* $\lim_{n\to\infty}a_n$ *exists! Just consider* $(-1)^n=a_n$. *Therefore, if you say* $\lim_{n\to\infty}a_n$ *exists, your answer is totally wrong! However,* $|a_nb_n|\le L|b_n|$ *and there exists* N *such that if* $n>N$, *then* $|b_n|<\varepsilon/L$. *Therefore, if* $n>N$, $|a_nb_n-0|=|a_nb_n|<(\varepsilon/L)L=\varepsilon$. *Since* $\varepsilon>0$ *is arbitrary, this says* $\lim_{n\to\infty}a_nb_n=0$.

5. In each of the following examples, tell whether the series converges and give a reason. To get any credit, both the answer and the reason must be valid.

   (a) $\sum_{n=2}^{\infty}\frac{1}{\ln n}$

   This diverges because $\frac{1}{\ln n}>\frac{1}{n}$ *for all* $n>3$ *and so, since* $\sum\frac{1}{n}$ *diverges, so does* $\sum\frac{1}{\ln n}$

(b) $\sum_{n=5}^{\infty} \left(\frac{1}{\ln(n)} \right)^n$

This converges by the root test. $\lim_{n\to\infty} \left(\left(\frac{1}{\ln(n)} \right)^n \right)^{1/n} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0 < 1.$

(c) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.01}}$

For large n, $\ln n \leq n^{.0001}$. *This follows from what you were supposed to learn in math 112. Therefore, for large* n *it follows* $\frac{\ln n}{n^{1.001}} < \frac{n^{.0001}}{n^{1.01}} = \frac{1}{n^{1.0099}}$. *Now* $\sum \frac{1}{n^{1.0099}}$ *converges and so by the comparison test do does* $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.01}}$ *Note: This is **not** a* p *series!*

(d) $\sum_{n=3}^{\infty} \frac{n}{n^2 + 5^{-n}}$

This diverges by a limit comparison test with $1/n$.

(e) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

This also diverges by a limit comparison with $1/n$.

6. Prove that for f having $n+1$ continuous derivatives,

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

You will get 1 point for leaving it blank and will lose one point for every incorrect or meaningless statement.

$f(x) - f(a) = \int_a^x f'(t)\,dt = (t-x) f'(t)|_a^x + \int_a^x (x-t) f''(t)\,dt$

$= f'(a)(x-a) + \int_a^x (x-t) f''(t)\,dt$

$= f'(a)(x-a) + -\frac{(x-t)^2}{2} f''(t)|_a^x + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt$. *This proves the formula for* $n = 1, 2$. *Suppose the formula is true for* n. *Then*

$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$. *Integrating by parts in the last integral again, you obtain* $f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + -\frac{1}{n!}\frac{(x-t)^{n+1}}{n+1} f^{(n+1)}(t)|_a^x + \frac{1}{n!}\int_a^x \frac{(x-t)^{n+1}}{n+1} f^{(n+2)}(t)\,dt$

$= f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!}\frac{(x-a)^{n+1}}{n+1} f^{(n+1)}(a) + \frac{1}{(n+1)!}\int_a^x (x-t)^{n+1} f^{(n+2)}(t)\,dt$

$= f(a) + \sum_{k=1}^{n+1} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{(n+1)!}\int_a^x (x-t)^{n+1} f^{(n+2)}(t)\,dt$. *Thus, when the formula holds for* n, *it also holds for* $n+1$ *and so this proves the formula for as long as you have continuous derivatives for* f.