Prerequisite Topics

1. 1] Show $-(ab) = (-a)b$ from the field axioms.

2. 2] Show the maximum (minimum) of $ax^2 + bx + c, a \neq 0,$ always occurs when $x = -b/(2a)$.

3. 3] Let $f(t) = t^2 + 4t$ for $t \geq -1$. Find a formula for $f^{-1}(y)$.

4. 4] Suppose

$$f(x,y) = ax^2 + by^2 + 2cxy$$

and suppose $a + b > 0$ and $ab - c^2 > 0$. Show that under these conditions, the above function is nonnegative. This is a function of two variables. You can pick x, y to be any numbers and the above expression under the given conditions is always nonnegative.

5. 5] Start with

$$x = \frac{-b \pm \sqrt{b^2 - 4ca}}{2a}$$

and show directly that x given by either of the two numbers in the above formula satisfies the quadratic equation $ax^2 + bx + c = 0$.

6. 6] Suppose there are two real solutions to $ax^2 + bx + c = 0$, x_1, x_2. Show the maximum (minimum), depending on sign of a, of $f(x) = ax^2 + bx + c$ occurs at $(x_1 + x_2)/2$.

7. 7] For z equal to the following complex number, find z^{-1}.

a. $2 + 5i$

b. $9 - 9i$

c. $-5 + 2i$

d. $9 + 4i$

8. 8] Let $z = 3 + 8i$ and let $w = 6 - 8i$. Find $zw, z + w, z^2,$ and w/z.

9. 9] Give the complete solution to $x^4 + 16 = 0$. Graph these points in the complex plane.

10. 10] Give the complete solution to $x^3 + 64 = 0$. Graph these points in the complex plane.

11. 11] If z is a complex number, show there exists ω a complex number with $|\omega| = 1$ and $\omega z = |z|$.

12. 12] De Moivre's theorem says $[r(\cos t + i \sin t)]^n = r^n(\cos nt + i \sin nt)$ for n a positive integer.

Does this formula continue to hold for all integers, n, even negative integers? Explain.

13. 13] You already know formulas for $\cos(x+y)$ and $\sin(x+y)$ and these were used to prove De Moivre's theorem. Now using De Moivre's theorem, derive a formula for $\sin(4x)$ and one for $\cos(4x)$.

14. 14] If z and w are two complex numbers and the polar form of z involves the angle θ while the polar form of w involves the angle ϕ, show that in the polar form for zw the angle involved is $\theta + \phi$. Also, show that in the polar form of a complex number, z, $r = |z|$.

15. 15] Factor $x^3 + 64$ as a product of linear factors.

16. 16] Write $x^3 + 27$ in the form $(x+3)(x^2 + ax + b)$ where $x^2 + ax + b$ cannot be factored any more using only real numbers.

17. 17] Completely factor $x^4 + 256$ as a product of linear factors.

18. 18] Completely factor $x^4 + 81$ as a product of of two quadratic polynomials each of which cannot be factored further without using complex numbers.

19. 19] If z, w are complex numbers prove $\overline{zw} = \overline{z}\,\overline{w}$ and then show by induction that $\overline{z_1 \cdots z_m} = \overline{z_1} \cdots \overline{z_m}$. Also verify that $\overline{\sum_{k=1}^m z_k} = \sum_{k=1}^m \overline{z_k}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.

20. 20] Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where all the a_k are real numbers. Suppose also that $p(z) = 0$ for some $z \in \mathbf{C}$. Show it follows that $p(\overline{z}) = 0$ also.

21. 21] Show that $-2 + 5i$ and $9 - 10i$ are the only two zeros to the polynomial

$$p(x) = x^2 + (-7 + 5i)x + (32 + 65i)$$

Thus the zeros do not necessarily come in conjugate pairs if the coefficients of the polynomial are not real.

22. 22] I claim that $1 = -1$. Here is why.

$$-1 = i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

This is clearly a remarkable result but is there something wrong with it? If so, what is wrong?

23. 23] De Moivre's theorem is really a grand thing. I plan to use it now for rational exponents, not just integers.

$$1 = 1^{(1/4)} = (\cos 2\pi + i\sin 2\pi)^{1/4} = \cos(\pi/2) + i\sin(\pi/2) = i.$$

Therefore, squaring both sides it follows $1 = -1$ as in the previous problem. What does this tell you about De Moivre's theorem? Is there a profound difference between raising numbers to integer powers and raising numbers to non integer powers?

24. 24] If n is an integer, is it always true that $(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$? Explain.

25. 25] Suppose you have any polynomial in $\cos\theta$ and $\sin\theta$. By this I mean an expression of the form $\sum_{\alpha=0}^{m}\sum_{\beta=0}^{n} a_{\alpha\beta}\cos^{\alpha}\theta\sin^{\beta}\theta$ where $a_{\alpha\beta} \in \mathbf{C}$. Can this always be written in the form $\sum_{\gamma=-(n+m)}^{m+n} b_\gamma \cos\gamma\theta + \sum_{\tau=-(n+m)}^{n+m} c_\tau \sin\tau\theta$? Explain.

26. 26] Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial and it has n zeros,

$$z_1, z_2, \cdots, z_n$$

listed according to multiplicity. (z is a root of multiplicity m if the polynomial $f(x) = (x-z)^m$ divides $p(x)$ but $(x-z)f(x)$ does not.) Show that

$$p(x) = a_n(x-z_1)(x-z_2)\cdots(x-z_n).$$

27. 27] Give the roots of the following quadratic polynomials having real coefficients.

a. $x^2 - 6x + 25$,
b. $3x^2 - 18x + 75$,
c. $9x^2 - 24x + 32$,
d. $16x^2 - 24x + 25$,
e. $16x^2 - 24x + 25$,

28. 28] Give the solutions to the following quadratic equations having possibly complex coefficients.

a. $x^2 - (6-i)x + 39 - 3i = 0$,
b. $x^2 - (6-i)x + 39 - 3i = 0$,
c. $9x^2 - (18-3i)x + 21 - 3i = 0$,
d. $9x^2 + \frac{21}{4}ix + 9 = 0$,

F^n

1. 29] Find $8(9,1,-6,-2)+9(-1,-5,6,8)$.

2. 30] Find parametric equations for the line through $(5,7,0)$ and $(-4,-4,3)$. Recall that to find the parametric line through two points (a,b,c) and (d,e,f), you find a direction vector of the form $(d-a,e-b,f-c)$ and then the equation of the line is $(x,y,z)=(a,b,c)+t(d-a,e-b,f-c)$ where $t\in\mathbf{R}$. You should check that this works in the sense that when $t=0$, you are at (a,b,c) and when $t=1$, the value is (d,e,f).

3. 31] Find parametric equations for the line through the point $(-7,7,0)$ with a direction vector $(1,-5,-3)$. Recall from calculus, the equation of a line through (a,b,c) having direction vector (f,g,h) is given by $(x,y,z)=(a,b,c)+t(f,g,h)$.

4. 32] Parametric equations of the line are $(x,y,z)=\left(\left(\begin{array}{ccc} 2t-6 & 3t+6 & 3t+3 \end{array}\right)\right)$. Find a direction vector for the line and a point on the line. Recall that to find a direction vector, you could subtract one point on the line from the other or else just recall that a parametric equation of a line is of the form $\left(\text{Point on the line}\right)+t\left(\text{direction vector}\right)$ and use this in the above.

5. 33] Find parametric equations for the line through the two points $\left(\left(\begin{array}{ccc} -1 & 9 & 3 \end{array}\right)\right)$, $\left(\left(\begin{array}{ccc} 1 & 6 & 3 \end{array}\right)\right)$. Recall that to find the parametric line through two points (a,b,c) and (d,e,f), you find a direction vector of the form $(d-a,e-b,f-c)$ and then the equation of the line is $(x,y,z)=(a,b,c)+t(d-a,e-b,f-c)$ where $t\in\mathbf{R}$. You should check that this works in the sense that when $t=0$, you are at (a,b,c) and when $t=1$, the value is (d,e,f).

6. 34] The equation of a line in two dimensions is written as $y=2x-2$. Find parametric equations for this line.

7. 35] Find $5(3,-6,0,5)+3(2,-6,0,9)$.

8. 36] Find parametric equations for the line through $(2,4,-1)$ and $(1,7,-6)$. Recall that to find the parametric line through two points (a,b,c) and (d,e,f), you find a direction vector of the form $(d-a,e-b,f-c)$ and then the equation of the line is $(x,y,z)=(a,b,c)+t(d-a,e-b,f-c)$ where $t\in\mathbf{R}$. You should check that this works in the sense that when $t=0$, you are at (a,b,c) and when $t=1$, the value is (d,e,f).

9. 37] Find parametric equations for the line through the point $(-7,2,-4)$ with a direction vector $(1,5,-6)$.

10. 38] Parametric equations of the line are $(x,y,z) = \left(\left(\begin{array}{ccc} 2t+3 & 6t+8 & 3t+6 \end{array}\right)\right)$. Find a direction vector for the line and a point on the line.

11. 39] Find parametric equations for the line through the two points $\left(\left(\begin{array}{ccc} 0 & 4 & 5 \end{array}\right)\right)$, $\left(\left(\begin{array}{ccc} 2 & -5 & -5 \end{array}\right)\right)$. Recall that to find the parametric line through two points (a,b,c) and (d,e,f), you find a direction vector of the form $(d-a, e-b, f-c)$ and then the equation of the line is $(x,y,z) = (a,b,c) + t(d-a, e-b, f-c)$ where $t \in \mathbf{R}$. You should check that this works in the sense that when $t = 0$, you are at (a,b,c) and when $t = 1$, the value is (d,e,f).

12. 40] Find the point on the line segment from $\left(\left(\begin{array}{ccc} -5 & 5 & 4 \end{array}\right)\right)$ to $\left(\left(\begin{array}{ccc} 1 & -6 & -5 \end{array}\right)\right)$ which is $\frac{1}{5}$ of the way from $\left(\left(\begin{array}{ccc} -5 & 5 & 4 \end{array}\right)\right)$ to $\left(\left(\begin{array}{ccc} 1 & -6 & -5 \end{array}\right)\right)$.

13. 41] You have a triangle in $\mathbf{R}^n$ which has vertices at $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$. Consider the lines which are drawn from a vertex to the mid point of the opposite side. Show these three lines intersect in a point and find the coordinates of this point.

14. 42] The wind blows from the South at 50 kilometers per hour and an airplane which flies at 300 kilometers per hour in still air is heading East. What is the actual velocity of the airplane and where will it be after two hours.

15. 43] The wind blows from the West at 50 kilometers per hour and an airplane which flies at 500 kilometers per hour in still air is heading North East. What is the actual velocity of the airplane and where will it be after two hours.

16. 44] The wind blows from the North at 20 kilometers per hour and an airplane which flies at 300 kilometers per hour in still air is supposed to go to the point whose coordinates are at $\left(\left(\begin{array}{cc} 200 & 200 \end{array}\right)\right)$ kilometers. In what direction should the airplane head?

17. 45] Three forces act on an object. Two are $\left(\left(\begin{array}{ccc} -7 & -5 & 1 \end{array}\right)\right)$ and $\left(\left(\begin{array}{ccc} 2 & -1 & -6 \end{array}\right)\right)$ Newtons. Find the third force if the object is not to move.

18. 46] Three forces act on an object. Two are $\left(\left(\begin{array}{ccc} 7 & 2 & -1 \end{array}\right)\right)$ and $\left(\left(\begin{array}{ccc} 1 & -6 & -5 \end{array}\right)\right)$ Newtons. Find the third force if the total force on the object is to be $\left(\left(\begin{array}{ccc} -6 & -6 & -5 \end{array}\right)\right)$.

19. 47] A river flows West at the rate of b miles per hour. A boat can move at the rate of 1 miles per hour. Find the smallest value of b such that it is not possible for the boat to proceed directly across the river.

20. 48] The wind blows from West to East at a speed of 20 miles per hour and an airplane which travels at 300 miles per hour in still air is heading North West. What is the velocity of the airplane relative to the ground? What is the component of this velocity in the direction North?

21. 49] The wind blows from West to East at a speed of 60 miles per hour and an airplane can travel travels at 400 miles per hour in still air. How many degrees West of North should the airplane head in order to travel exactly North?

22. 50] The wind blows from West to East at a speed of 60 miles per hour and an airplane which travels at 300 miles per hour in still air heading somewhat West of North so that, with the wind, it is flying due North. It uses 40.0 gallons of gas every hour. If it has to travel 200.0 miles due North, how much gas will it use in flying to its destination?

23. 51] An airplane is flying due north at 100.0 miles per hour but it is not actually going due North because there is a wind which is pushing the airplane due east at 20.0 miles per hour. After 1 hour, the plane starts flying 30° East of North. Assuming the plane starts at $(0,0)$, where is it after 2 hours? Let North be the direction of the positive y axis and let East be the direction of the positive x axis.

24. 52] City A is located at the origin while city B is located at $(300,500)$ where distances are in miles. An airplane flies at 250 miles per hour in still air. This airplane wants to fly from city A to city B but the wind is blowing in the direction of the positive y axis at a speed of 50 miles per hour. Find a unit vector such that if the plane heads in this direction, it will end up at city B having flown the shortest possible distance. How long will it take to get there?

25. 53] A certain river is one half mile wide with a current flowing at 2.0 miles per hour from East to West. A man takes a boat directly toward the opposite shore from the South bank of the river at a speed of 6.0 miles per hour. How far down the river does he find himself when he has swam across? How far does he end up travelling?

26. 54] A certain river is one half mile wide with a current flowing at 2 miles per hour from East to West. A man can swim at 3 miles per hour in still water. In what direction should he swim in order to travel directly across the river? What would the answer to this problem be if the river flowed at 3 miles per hour and the man could swim only at the rate of 2 miles per hour?

27. 55] Three forces are applied to a point which does not move. Two of the forces are $(3.0)\mathbf{i}+(3.0)\mathbf{j}+(8.0)\mathbf{k}$ Newtons and $(-7.0)\mathbf{i}+(8.0)\mathbf{j}+(3.0)\mathbf{k}$ Newtons. Find the third force.

28. 56] The total force acting on an object is to be $(1)\mathbf{i}+(4)\mathbf{j}+(3)\mathbf{k}$ Newtons. A force of $(-8)\mathbf{i}+(-2)\mathbf{j}+(8)\mathbf{k}$ Newtons is being applied. What other force should be applied to achieve the desired total force?

29. 57] A bird flies from its nest 8 km. in the direction $\frac{3}{2}\pi$ north of east where it stops to rest on a tree. It then flies 5 km. in the direction due southeast and lands atop a telephone pole. Place an xy coordinate system so that the origin is the bird's nest, and the positive x axis points east and the positive y axis points north. Find the displacement vector from the nest to the telephone pole.

30. 58] A car is stuck in the mud. There is a cable stretched tightly from this car to a tree which is 20 feet long. A person grasps the cable in the middle and pulls with a force of 150 pounds perpendicular to the stretched cable. The center of the cable moves 3 feet and remains still. What is the tension in the cable? The tension in the cable is the force exerted on this point by the part of the cable nearer the car as well as the force exerted on this point by the part of the cable nearer the tree. Assume the cable cannot be lengthened so the car moves slightly in moving the center of the cable.

31. 59] A car is stuck in the mud. There is a cable stretched tightly from this car to a tree which is 20 feet long. A person grasps the cable in the middle and pulls with a force of 80 pounds perpendicular to the stretched cable. The center of the cable moves 3 feet and remains still. What is the tension in the cable? The tension in the cable is the force exerted on this point by the part of the cable nearer the car as well as the force exerted on this point by the part of the cable nearer the tree. Assume the cable does lengthen and the car does not move at all.

32. 60] A box is placed on a strong plank of wood and then one end of the plank of wood is gradually raised. It is found that the box does not move till the angle of inclination of the wood equals 60 degrees at which angle the box begins to slide. What is the coefficient of static friction?

33. 61] Recall that the open ball having center at $\mathbf{a}$ and radius r is given by

$$B(\mathbf{a},r) \equiv \{\mathbf{x} : |\mathbf{x}-\mathbf{a}| < r\}$$

Show that if $\mathbf{y} \in B(\mathbf{a},r)$, then there exists a positive number δ such that $B(\mathbf{y},\delta) \subseteq B(\mathbf{a},r)$. (The symbol $\subseteq$ means that every point in $B(\mathbf{y},\delta)$ is also in $B(\mathbf{a},r)$. In words, it states that $B(\mathbf{y},\delta)$ is contained in $B(\mathbf{a},r)$. The statement $\mathbf{y} \in B(\mathbf{a},r)$ says that $\mathbf{y}$ is one of the points of $B(\mathbf{a},r)$.) When you have done this, you will have shown that an open ball is open. This is a fantastically important observation although its major implications will not be explored very much in this book.

Dot Product

1. 62] Find the following dot products.
   a. $(1,4,3,10) \cdot (2,0,-1,3)$
   b. $(2,7,5) \cdot (-1,-3,2)$
   c. $(4,2,5,6,7) \cdot (-1,1,-1,2,0)$

2. 63] Use the geometric description of the dot product to verify the Cauchy Schwartz inequality and to show that equality occurs if and only if one of the vectors is a scalar multiple of the other.

3. 64] For $\mathbf{u}, \mathbf{v}$ vectors in $\mathbf{R}^3$, define the product, $\mathbf{u} * \mathbf{v} \equiv u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$. Show the axioms for a dot product all hold for this funny product. Prove
$$|\mathbf{u} * \mathbf{v}| \leq (\mathbf{u} * \mathbf{u})^{1/2} (\mathbf{v} * \mathbf{v})^{1/2}.$$
   Hint: Do not try to do this with methods from trigonometry.

4. 65] Find the angle between the vectors $4\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$.

5. 66] Find the angle between the vectors $4\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

6. 67] Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{v} = (6.0, 2.0, -2)$ and $\mathbf{u} = (1, 7.0, 3)$.

7. 68] Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{v} = (3.0, 3.0, -2)$ and $\mathbf{u} = (1, 7.0, 3)$.

8. 69] Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{v} = (2.0, -3.0, -2)$ and $\mathbf{u} = (1, 5.0, 3)$.

9. 70] A boy drags a sled for 40 feet along the ground by pulling on a rope which is 45 degrees from the horizontal with a force of 10 pounds. How much work does this force do?

10. 71] A dog drags a sled for 80 feet along the ground by pulling on a rope which is 30 degrees from the horizontal with a force of 20 pounds. How much work does this force do?

11. 72] A girl drags a sled for 70 feet along the ground by pulling on a rope which is 20 degrees from the horizontal with a force of 60 pounds. How much work does this force do?

12. 73] How much work in Newton meters does it take to slide a crate 80.0 meters along a loading dock by pulling on it with a 400.0 Newton force at an angle of 20.0 degrees from the horizontal?

13. 74] How much work in Newton meters does it take to slide a crate 40.0 meters along a loading dock by pulling on it with a 100.0 Newton force at an angle of 45.0 degrees from the horizontal?

14. 75] An object moves 11.0 meters in the direction of $\mathbf{j} + \mathbf{i}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i} + \mathbf{j} + 1.0\mathbf{k}$, and $\mathbf{F}_2 = -5\mathbf{i} + (7.0)\mathbf{j} - 6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint**: You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

15. 76] An object moves 10.0 meters in the direction of $\mathbf{j} + \mathbf{i}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i} + \mathbf{j} + 5.0\mathbf{k}$, and $\mathbf{F}_2 = -5\mathbf{i} + (1.0)\mathbf{j} - 6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint**: You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

16. 77] An object moves 7.0 meters in the direction of $\mathbf{j} + \mathbf{i}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i} + \mathbf{j} + 5.0\mathbf{k}$, and $\mathbf{F}_2 = -5\mathbf{i} + (2.0)\mathbf{j} - 6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint**: You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

17. 78] An object moves 6.0 meters in the direction of $\mathbf{j} + \mathbf{i} - \mathbf{k}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i} + \mathbf{j} + 6.0\mathbf{k}$, and $\mathbf{F}_2 = -5\mathbf{i} + (6.0)\mathbf{j} - 6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint**: You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

18. 79] Does it make sense to speak of $\operatorname{proj}_{\mathbf{0}}(\mathbf{v})$?

19. 80] If $\mathbf{F}$ is a force and $\mathbf{D}$ is a vector, show $\operatorname{proj}_{\mathbf{D}}(\mathbf{F}) = (|\mathbf{F}|\cos\theta)\mathbf{u}$ where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{D}$, $\mathbf{u} = \mathbf{D}/|\mathbf{D}|$ and θ is the included angle between the two vectors, $\mathbf{F}$ and $\mathbf{D}$. $|\mathbf{F}|\cos\theta$ is sometimes called the component of the force, $\mathbf{F}$ in the direction, $\mathbf{D}$.

20. 81] If $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ are vectors. Show that $(\mathbf{b} + \mathbf{c})_{\perp} = \mathbf{b}_{\perp} + \mathbf{c}_{\perp}$ where $\mathbf{b}_{\perp} = \mathbf{b} - \operatorname{proj}_{\mathbf{a}}(\mathbf{b})$.

21. 82] Show that $(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{4}[|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2]$.

22. 83] Prove from the axioms of the dot product the parallelogram identity,

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

Explain what this says geometrically in terms of the diagonals and sides of a parallelogram.

23. 84] Prove the Cauchy Schwarz inequality in $\mathbf{R}^n$ as follows. For $\mathbf{u}, \mathbf{v}$ vectors, note that

$$(\mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u}) \cdot (\mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u}) \geq 0$$

Now simplify using the axioms of the dot product and then put in the formula for the projection. Show that you get equality in the Cauchy Schwarz inequality if and only if the above equals zero which means that $\mathbf{u} = \mathbf{proj}_{\mathbf{v}}\mathbf{u}$. What is the geometric significance of this condition?

24. 85] For $\mathbf{C}^n$, the dot product is often called the inner product. It was given by

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x}, \mathbf{y}) \equiv \sum_{j=1}^{n} x_j \overline{y_j}$$

Why place the conjugate on the y_j? Why is it not possible to simply define this the usual way?

25. 86] For the complex inner product, prove the parallelogram identity.

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

26. 87] For the complex inner product, show that if $\mathrm{Re}(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbf{C}^n$, then $\mathbf{x} = \mathbf{0}$. In fact, show that for all $\mathbf{y}$, you have $(\mathbf{x}, \mathbf{y}) = 0$.

27. 88] For the complex inner product, show that for $(\mathbf{x}, \mathbf{y}) = \mathrm{Re}(\mathbf{x}, \mathbf{y}) + i\,\mathrm{Im}(\mathbf{x}, \mathbf{y})$, it follows that

$$\mathrm{Im}(\mathbf{x}, \mathbf{y}) = \mathrm{Re}(\mathbf{x}, i\mathbf{y})$$

so that

$$(\mathbf{x}, \mathbf{y}) = \mathrm{Re}(\mathbf{x}, \mathbf{y}) + i\,\mathrm{Re}(\mathbf{x}, i\mathbf{y})$$

which shows that the inner product is determined by the real part.

Cross Product

1. 89] Show that if $\mathbf{a} \times \mathbf{u} = \mathbf{0}$ for all unit vectors, $\mathbf{u}$, then $\mathbf{a} = \mathbf{0}$.

2. 90] Find the area of the triangle determined by the three points, $(0,1,-3),(2,2,0)$ and $(-1,1,-1)$.

3. 91] Find the area of the triangle determined by the three points, $(3,-2,2),(4,2,0)$ and $(-1,1,2)$.

4. 92] Find the area of the triangle determined by the three points, $(3,2,0),(4,3,1)$ and $(2,1,-1)$.

5. 93] Find the area of the parallelogram determined by the two vectors, $(3,0,2),(0,-2,0)$.

6. 94] Find the area of the parallelogram determined by the two vectors, $(-2,2,-3),(1,5,0)$.

7. 95] Find the volume of the parallelepiped determined by the vectors,
$\left(\begin{pmatrix} -2 & -3 & -3 \end{pmatrix}\right), \left(\begin{pmatrix} 4 & 2 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 1 & 1 & -3 \end{pmatrix}\right)$.

8. 96] Find the volume of the parallelepiped determined by the vectors,
$\left(\begin{pmatrix} 3 & -1 & 3 \end{pmatrix}\right), \left(\begin{pmatrix} 1 & -3 & 0 \end{pmatrix}\right), \left(\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}\right)$.

9. 97] Suppose $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ are three vectors whose components are all integers. Can you conclude the volume of the parallelepiped determined from these three vectors will always be an integer?

10. 98] What does it mean geometrically if the box product of three vectors gives zero?

11. 99] Using the notion of the box product yielding either plus or minus the volume of the parallelepiped determined by the given three vectors, show that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

In other words, the dot and the cross can be switched as long as the order of the vectors remains the same. **Hint**: There are two ways to do this, by the coordinate description of the dot and cross product and by geometric reasoning. It is better if you use geometric reasoning.

12. 100] Is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$? What is the meaning of $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$? Explain. **Hint**: Try $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$.

13. 101] Discover a vector identity for $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and one for $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

14. 102] Discover a vector identity for $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{z} \times \mathbf{w})$.

15. 103] Simplify $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{v} \times \mathbf{w}) \times (\mathbf{w} \times \mathbf{z})$.

16. 104] Simplify $|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 - |\mathbf{u}|^2|\mathbf{v}|^2$.

17. 105] For $\mathbf{u}, \mathbf{v}, \mathbf{w}$ functions of t, show the product rules

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$
$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$

18. 106] If $\mathbf{u}$ is a function of t, and the magnitude $|\mathbf{u}(t)|$ is a constant, show from the above problem that the velocity $\mathbf{u}'$ is perpendicular to $\mathbf{u}$.

19. 107] When you have a rotating rigid body with angular velocity vector $\mathbf{\Omega}$, then the velocity vector $\mathbf{v} \equiv \mathbf{u}'$ is given by

$$\mathbf{v} = \mathbf{\Omega} \times \mathbf{u}$$

where $\mathbf{u}$ is a position vector. The acceleration is the derivative of the velocity. Show that if $\mathbf{\Omega}$ is a constant vector, then the acceleration vector $\mathbf{a} = \mathbf{v}'$ is given by the formula

$$\mathbf{a} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}).$$

Now simplify the expression. It turns out this is centripetal acceleration.

20. 108] Verify directly that the coordinate description of the cross product, $\mathbf{a} \times \mathbf{b}$ has the property that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Then show by direct computation that this coordinate description satisfies

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2(\theta))$$

where θ is the angle included between the two vectors. Explain why $|\mathbf{a} \times \mathbf{b}|$ has the correct magnitude. All that is missing is the material about the right hand rule. Verify directly from the coordinate description of the cross product that the right thing happens with regards to the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Next verify that the distributive law holds for the coordinate description of the cross product. This gives another way to approach the cross product. First define it in terms of coordinates and then get the geometric properties from this. However, this approach does not yield the right hand rule property very easily.

Systems Of Equations

1. 109] Do the three lines, $2y - x = 2, 3x + 4y = 2,$ and $-24x - 22y = -8$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

2. 110] Do the three lines, $3x + 3y = 3, 2x - y = 3,$ and $16y - 5x = -12$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

3. 111] Do the three lines, $3x + 3y = 4, 2x + y = 2,$ and $2y - 5x = -1$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

4. 112] Do the three planes,

$$x + 2z = 0,$$
$$y - 2x - 3z = -1,$$
$$y - x = 0$$

have a common point of intersection? If so, find one and if not, tell why there is no such point.

5. 113] Do the three planes,

$$x - 5z = 10,$$
$$y + z = 2,$$
$$x + y - 3z = 10$$

have a common point of intersection? If so, find one and if not, tell why there is no such point.

6. 114] Do the three planes,

$$x - z = 3,$$
$$4x + y - 3z = 11,$$
$$5x + y - 3z = 15$$

have a common point of intersection? If so, find one and if not, tell why there is no such point.

7. 115] You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

a. No solution.
b. A unique solution.
c. An infinite number of solutions.

8. 116] Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right)$$

9. 117] Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right)$$

10. 118] Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right)$$

11. 119] Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right)$$

12. 120] Suppose a system of equations has fewer equations than variables. Must such a system be consistent? If so, explain why and if not, give an example which is not consistent.

13. 121] If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

14. 122] Find x such that the following augmented matrix is the augmented matrix of an inconsistent system. Then find values of x such that the augmented matrix pertains to a consistent system.

$$\left(\left(\begin{array}{ccc} 1 & -2 & x^2+4 \\ 1 & -2 & 2x^2+3 \end{array}\right)\right)$$

15. 123] Find x such that the following augmented matrix is the augmented matrix of an inconsistent system. Then find values of x such that the augmented matrix pertains to a consistent system.

$$\left(\left(\begin{array}{ccc} 1 & 2 & x^2 \\ 1 & 2 & 2x^2-4 \end{array}\right)\right)$$

16. 124] Find x such that the following augmented matrix is the augmented matrix of an inconsistent system. Then find values of x such that the augmented matrix pertains to a consistent system.

$$\left(\left(\begin{array}{ccc} 1 & x+1 & -1 \\ 2 & x+2 & 3 \end{array}\right)\right)$$

17. 125] Find x such that the following augmented matrix is the augmented matrix of an inconsistent system. Then find values of x such that the augmented matrix pertains to a consistent system.

$$\left(\left(\begin{array}{cccc} 1 & 3 & 1 & x+4 \\ 1 & 3 & 1 & x+4 \\ 1 & 3 & 1 & 2x+2 \end{array}\right)\right)$$

18. 126] Find x such that the following augmented matrix is the augmented matrix of an inconsistent system. Then find values of x such that the augmented matrix pertains to a consistent system.

$$\left(\left(\begin{array}{cccc} 1 & 1 & 1 & x^2+6 \\ 1 & 1 & 1 & x^2+6 \\ 1 & 1 & 1 & 2x^2+5 \end{array}\right)\right)$$

19. 127] Find x such that the following augmented matrix is the augmented matrix of an inconsistent system. Then find values of x such that the augmented matrix pertains to a consistent system.

$$\left(\left(\begin{array}{cccc} 1 & 3 & 4 & x^2+2 \\ 1 & 3 & 4 & x^2+2 \\ 1 & 3 & 4 & 2x^2-2 \end{array}\right)\right)$$

20. 128] Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.

$$\left(\left(\begin{array}{ccc} 1 & h & 2 \\ 1 & 2 & k \end{array}\right)\right).$$

21. 129] Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.

$$\left(\left(\begin{array}{ccc} 4 & h & 2 \\ 28 & 56 & k \end{array}\right)\right).$$

22. 130] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{array}{cccc} -1 & 2 & 3 & 5 \\ 1 & -2 & -3 & -5 \\ -2 & 4 & 6 & 10 \end{array}\right)\right)$$

23. 131] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{array}{cccc} 3 & 8 & 3 & 46 \\ 1 & 3 & 1 & 17 \\ 2 & 7 & 3 & 37 \end{array}\right)\right)$$

24. 132] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{array}{cccc} 3 & -6 & 3 & -18 \\ 1 & -2 & 1 & -6 \\ 2 & -4 & 2 & -12 \end{array}\right)\right)$$

25. 133] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{array}{cccc} -1 & -4 & -10 & -27 \\ 1 & 3 & 8 & 22 \\ -2 & -5 & -14 & -38 \end{array}\right)\right)$$

26. 134] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{matrix} 0 & -1 & 0 & 4 \\ 1 & 3 & 1 & -8 \\ -1 & -2 & 0 & 8 \end{matrix}\right)\right)$$

27. 135] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{matrix} 0 & -1 & -1 & -4 \\ 1 & 3 & 6 & 19 \\ -1 & -2 & -5 & -15 \end{matrix}\right)\right)$$

28. 136] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{matrix} 1 & 2 & -2 & 15 \\ 1 & 3 & -4 & 20 \\ 0 & 1 & -2 & 5 \end{matrix}\right)\right)$$

29. 137] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{matrix} 0 & -1 & 0 & 1 \\ 1 & 3 & 1 & -6 \\ -1 & -2 & 0 & 0 \end{matrix}\right)\right)$$

30. 138] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\left(\begin{matrix} 1 & 2 & 1 & 5 \\ 1 & 3 & 1 & 8 \\ 0 & 1 & 1 & 7 \end{matrix}\right)\right)$$

31. 139] Give the complete solution to the system of equations.

$$\left(\begin{matrix} 3x_1 + 8x_2 + 3x_3 - 5x_4 - 11x_5 - 16 = 0 \\ x_1 + 3x_2 + x_3 - 2x_4 - 4x_5 - 6 = 0 \\ 2x_1 + 7x_2 + 3x_3 - 7x_4 - 8x_5 - 15 = 0 \\ 3x_1 + 7x_2 + 3x_3 - 4x_4 - 10x_5 - 15 = 0 \end{matrix}\right)$$

32. 140] Give the complete solution to the system of equations whose augmented matrix is.

$$\begin{pmatrix} 2 & 0 & 5 & -8 & 15 \\ 1 & 0 & 3 & -5 & 9 \\ 1 & 0 & 4 & -7 & 12 \\ -2 & 0 & 4 & -10 & 12 \end{pmatrix}$$

33. 141] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\begin{pmatrix} -1 & -4 & -1 & 13 & -25 \\ 1 & 3 & 1 & -11 & 19 \\ -2 & -5 & -1 & 15 & -27 \\ 6 & -5 & -1 & 15 & -51 \end{pmatrix}\right)$$

34. 142] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\begin{pmatrix} 2 & -4 & 5 & 15 & 14 \\ 1 & -2 & 3 & 9 & 8 \\ 1 & -2 & 4 & 12 & 10 \\ 6 & -12 & 4 & 12 & 20 \end{pmatrix}\right)$$

35. 143] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\begin{pmatrix} 1 & 1 & 0 & -1 & 4 \\ 1 & 3 & 1 & -2 & 5 \\ 3 & 10 & 4 & -5 & 17 \\ -1 & 10 & 4 & -3 & 3 \end{pmatrix}\right)$$

36. 144] Give the complete solution to the system of equations whose augmented matrix is.

$$\left(\begin{pmatrix} 1 & 3 & 2 & -5 & 5 \\ 1 & 3 & 3 & -8 & 8 \\ 0 & 0 & 1 & -3 & 3 \\ 4 & 12 & 1 & 1 & -1 \end{pmatrix}\right)$$

37. 145] Give the complete solution to the system of equations.

$$\begin{pmatrix} 4x_1 + 11x_2 + 4x_3 - 33x_4 - 29x_5 - 29 = 0 \\ x_1 + 3x_2 + x_3 - 9x_4 - 8x_5 - 8 = 0 \\ 3x_1 + 10x_2 + 4x_3 - 32x_4 - 23x_5 - 29 = 0 \\ 3x_1 + 10x_2 + 4x_3 - 32x_4 - 23x_5 - 29 = 0 \end{pmatrix}$$

38. 146] Give the complete solution to the system of equations.

$$\begin{pmatrix} 6x_5 - 4x_2 - x_3 - x_1 + 6 = 0 \\ x_1 + 3x_2 + x_3 + x_4 - 5x_5 - 5 = 0 \\ 2x_4 - 5x_2 - x_3 - 2x_1 + 5x_5 + 9 = 0 \\ 4x_4 - 5x_2 - x_3 - 4x_1 + x_5 + 13 = 0 \end{pmatrix}$$

39. 147] Give the complete solution to the system of equations whose augmented matrix is.

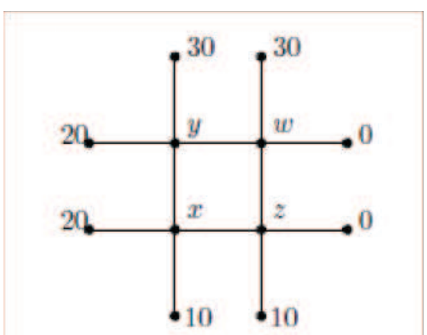
$$\left(\begin{pmatrix} 3 & 8 & 3 & -19 & -19 & 10 \\ 1 & 3 & 1 & -7 & -7 & 4 \\ 2 & 7 & 3 & -16 & -15 & 10 \\ 1 & 7 & 3 & -15 & -13 & 12 \end{pmatrix}\right)$$

40. 148] Give the complete solution to the system of equations whose augmented matrix is.

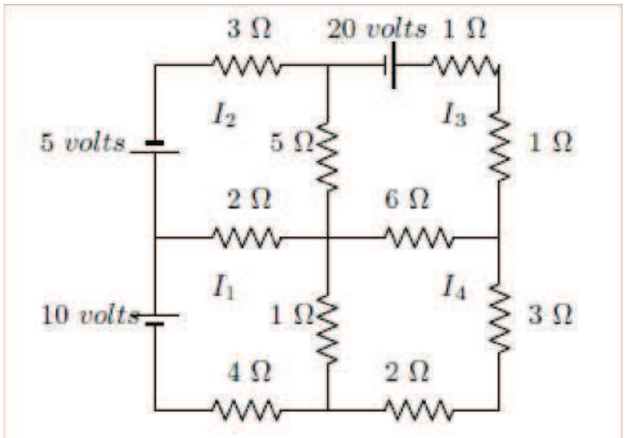
$$\left(\begin{pmatrix} -1 & -4 & -1 & 8 & -6 \\ 1 & 3 & 1 & -7 & 4 \\ -2 & -5 & -1 & 10 & -8 \\ 4 & -5 & -1 & 4 & -8 \end{pmatrix}\right)$$

41. 149] Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four sisters.

42. 150] The steady state temperature, u in a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution which is often used is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. This procedure is justified by the mean value property of harmonic functions. In the following picture, the numbers represent the observed temperature at the indicated nodes. Your task is to find the temperature at the interior nodes, indicated by $x, y, z,$ and w. One of the equations is $z = \frac{1}{4}(10 + 0 + w + x)$.



43. 151] Consider the following diagram of four circuits.



Those jagged places denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The breaks in the lines having one short line and one long line denote a voltage source which causes the current to flow in the direction which goes from the longer of the two lines toward the shorter along the unbroken part of the circuit. The current in amps in the four circuits is denoted by I_1, I_2, I_3, I_4 and it is understood that the motion is in the counter clockwise direction if I_k ends up being negative, then it just means it moves in the clockwise direction. Then Kirchhoff's law states that

The sum of the resistance times the amps in the counter clockwise direction around a loop equals the sum of the voltage sources in the same direction around the loop.

In the above diagram, the top left circuit should give the equation

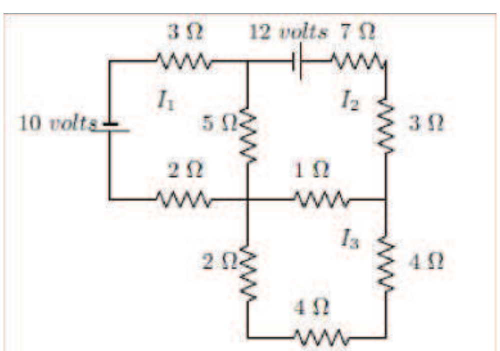
$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5$$

For the circuit on the lower left, you should have

$$4I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -10$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations. You might use a computer algebra system to find the solution. It might be more convenient than doing it by hand.

44. 152] Consider the following diagram of four circuits.



Those jagged places denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The breaks in the lines having one short line and one long line denote a voltage source which causes the current to flow in the direction which goes from the longer of the two lines toward the shorter along the unbroken part of the circuit. The current in amps in the four circuits is denoted by I_1, I_2, I_3 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction. Then Kirchhoff's law states that

The sum of the resistance times the amps in the counter clockwise direction around a loop equals the sum of the voltage sources in the same direction around the loop. Find I_1, I_2, I_3.

Matrices

1. 153] Here are some matrices:

$$A = \left(\left(\begin{array}{ccc} -2 & 2 & -3 \\ -3 & 1 & 2 \end{array}\right)\right), B = \left(\left(\begin{array}{ccc} -3 & 2 & 2 \\ -1 & 0 & 1 \end{array}\right)\right),$$

$$C = \left(\left(\begin{array}{cc} 2 & -2 \\ 3 & 1 \end{array}\right)\right), D = \left(\left(\begin{array}{cc} 2 & 0 \\ 2 & 1 \end{array}\right)\right), E = \left(\left(\begin{array}{c} -2 \\ 0 \end{array}\right)\right).$$

Find if possible $3A, 3B - A, AC, CB, AE, EA$. If it is not possible explain why.

2. 154] Here are some matrices:

$$A = \left(\left(\begin{array}{ccc} 2 & 2 & -3 \\ -1 & 1 & 2 \end{array}\right)\right), B = \left(\left(\begin{array}{ccc} -3 & 2 & 2 \\ -1 & -1 & 1 \end{array}\right)\right),$$

$$C = \left(\left(\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array}\right)\right), D = \left(\left(\begin{array}{cc} 2 & -1 \\ 2 & 1 \end{array}\right)\right), E = \left(\left(\begin{array}{c} 2 \\ -1 \end{array}\right)\right).$$

Find if possible $A, B - A, AC, CB, AE, EA$. If it is not possible explain why.

3. 155] Here are some matrices:

$$A = \left(\left(\begin{array}{ccc} 0 & 2 & -2 \\ 1 & 1 & 1 \end{array}\right)\right), B = \left(\left(\begin{array}{ccc} -2 & 2 & 2 \\ -1 & 2 & 1 \end{array}\right)\right),$$

$$C = \left(\left(\begin{array}{cc} 2 & 0 \\ 2 & 1 \end{array}\right)\right), D = \left(\left(\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right)\right), E = \left(\left(\begin{array}{c} 0 \\ 2 \end{array}\right)\right).$$

Find if possible $AB^T, BA^T - C, CA, E^TA, E^TE, EE^T$. If it is not possible explain why.

4. 156] Here are some matrices:

$$A = \left(\left(\begin{array}{ccc} -2 & 3 & 3 \\ -2 & 1 & -1 \end{array}\right)\right), B = \left(\left(\begin{array}{ccc} 3 & 3 & 2 \\ -1 & -3 & 1 \end{array}\right)\right),$$

$$C = \left(\left(\begin{array}{cc} 3 & -2 \\ 3 & 1 \end{array}\right)\right), D = \left(\left(\begin{array}{cc} -1 & -3 \\ 2 & 1 \end{array}\right)\right), E = \left(\left(\begin{array}{c} -2 \\ -3 \end{array}\right)\right).$$

Find if possible $AB^T, BA^T - C, CA, E^TA, E^TE, EE^T$. If it is not possible explain why.

5. 157] Here are some matrices:

$$A = \left(\left(\begin{array}{ccc} 4 & 1 & 0 \\ -3 & 1 & 2 \end{array}\right)\right), B = \left(\left(\begin{array}{ccc} 0 & 1 & 2 \\ -1 & -2 & 1 \end{array}\right)\right),$$

$$C = \left(\left(\begin{array}{cc} 1 & 4 \\ 2 & 1 \end{array}\right)\right), D = \left(\left(\begin{array}{cc} 2 & -2 \\ 2 & 1 \end{array}\right)\right), E = \left(\left(\begin{array}{c} 4 \\ -2 \end{array}\right)\right).$$

Find if possible $AB^T, BA^T - C, CA, E^TA, E^TE, EE^T$. If it is not possible explain why.

6. 158] Suppose A and B are square matrices of the same size. Which of the following are correct?

- **a**. $(A-B)^2 = A^2 - 2AB + B^2$
- **b**. $(AB)^2 = A^2B^2$
- **c**. $(A+B)^2 = A^2 + 2AB + B^2$
- **d**. $(A+B)^2 = A^2 + AB + BA + B^2$
- **e**. $A^2B^2 = A(AB)B$
- **f**. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
- **g**. $(A+B)(A-B) = A^2 - B^2$

7. 159] Let

$$A = \left(\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array}\right)$$

Find all 2×2 matrices B such that $AB = 0$.

8. 160] Let $A = \left(\begin{array}{cc} 2 & 3 \\ -1 & -3 \end{array}\right), B = \left(\begin{array}{cc} 2 & 3 \\ 1 & k \end{array}\right)$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

9. 161] Let $A = \left(\begin{array}{cc} 2 & 3 \\ 0 & 1 \end{array}\right), B = \left(\begin{array}{cc} 2 & 3 \\ 0 & k \end{array}\right)$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

10. 162] Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix. (M is skew symmetric if $M = -M^T$. M is symmetric if $M^T = M$.) **Hint**: Show that $\frac{1}{2}(A^T + A)$ is symmetric and then consider using this as one of the matrices.

11. 163] Show every skew symmetric matrix has all zeros down the main diagonal. The main diagonal consists of every entry of the matrix which is of the form a_{ii}. It runs from the upper left down to the lower right.

12. 164] Suppose M is a 3×3 skew symmetric matrix. Show there exists a vector $\boldsymbol{\Omega}$ such that for all $\mathbf{u} \in \mathbf{R}^3$

$$M\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{u}$$

Hint: Explain why, since M is skew symmetric it is of the form

$$M = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where the ω_i are numbers. Then consider $\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$.

13. 165] Using only the vector space axioms, show that 0 is unique.

14. 166] Using only the vector space axioms, show that $-A$ is unique.

15. 167] Using only the vector space axioms, show that $0A = 0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero for $m \times n$ matrices.

16. 168] Using only the vector space axioms, show that $(-1)A = -A$.

17. 169] Prove that for A, B $m \times n$ matrices, and a, b scalars, it follows that

$$(aA + bB)^T = aA^T + bB^T$$

18. 170] Prove that $I_m A = A$ where A is an $m \times n$ matrix.

19. 171] Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

20. 172] Give an example of matrices, A, B, C such that $B \neq C$, $A \neq 0$, and yet $AB = AC$.

21. 173] Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not. What if A were a non invertible $n \times n$ matrix?

22. 174] Find your own examples:

a. 2×2 matrices, A and B such that $A \neq 0, B \neq 0$ with $AB \neq BA$.
b. 2×2 matrices, A and B such that $A \neq 0, B \neq 0$, but $AB = 0$.
c. 2×2 matrices, A, D, and C such that $A \neq 0, C \neq D$, but $AC = AD$.

23. 175] Give an example of a matrix, A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

24. 176] Let

$$A = \begin{pmatrix} 4 & 2 \\ -4 & -2 \end{pmatrix}$$

Find A^{-1} if it exists and if it does not, explain why.

25. 177] Let

$$A = \begin{pmatrix} 0 & -4 \\ -1 & -1 \end{pmatrix}$$

Find A^{-1} if it exists and if it does not, explain why.

26. 178] Let

$$A = \begin{pmatrix} 3 & 1 \\ -3 & -2 \end{pmatrix}$$

Find A^{-1} if it exists and if it does not, explain why.

27. 179] Let

$$A = \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix}$$

Find A^{-1} if it exists and if it does not, explain why.

28. 180] Let

$$A = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$$

Find A^{-1} if it exists and if it does not, explain why.

29. 181] Let A be a 2×2 matrix which has an inverse. Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find a formula for A^{-1} in terms of a, b, c, d. Assume $ad - bc \neq 0$.

30. 182] Prove that if A^{-1} exists and $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.

31. 183] Let

$$A = \begin{pmatrix} 3 & 5 & 12 \\ 1 & 2 & 5 \\ 1 & 2 & 6 \end{pmatrix}$$

Find A^{-1} if possible. If not possible, explain why.

32. 184] Let

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Find A^{-1} if possible. If not possible, explain why.

33. 185] Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 7 \\ 2 & 4 & 15 \end{pmatrix}$$

Find A^{-1} if possible. If not possible, explain why.

34. 186] Let

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & 8 \end{pmatrix}$$

Find A^{-1} if possible. If not possible, explain why.

35. 187] Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

Find A^{-1} if possible. If not possible, explain why.

36. 188] Let

$$A = \begin{pmatrix} -1 & -3 & 8 \\ 1 & 2 & -3 \\ -3 & -6 & 10 \end{pmatrix}$$

Find A^{-1} if possible. If not possible, explain why.

37. 189] Write $\begin{pmatrix} -2x_1 - 3x_2 - 3x_3 \\ 2x_1 + 3x_3 \\ 0 \\ x_1 - 2x_2 + 2x_4 \end{pmatrix}$ in the form $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where A is an appropriate matrix.

38. 190] Write $\begin{pmatrix} 4x_1 + 4x_2 - x_3 \\ 2x_1 + x_3 \\ 3x_3 \\ x_1 + x_2 + x_4 \end{pmatrix}$ in the form $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where A is an appropriate matrix.

39. 191] Show that if A is an $n \times n$ invertible matrix and $\mathbf{x}$ is a $n \times 1$ matrix such that $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ an $n \times 1$ matrix, then $\mathbf{x} = A^{-1}\mathbf{b}$.

40. 192] Using the inverse of the matrix, find the solution to the systems

$$\begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

Now give the solution in terms of $l, m,$ and n to

$$\begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

41. 193] Using the inverse of the matrix, find the solution to the systems

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & -5 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & -5 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & -5 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & -5 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

Now give the solution in terms of $l, m,$ and n to

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & -5 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

42. 194] Using the inverse of the matrix, find the solution to the system

$$\begin{pmatrix} -8 & 3 & 0 & 1 \\ -21 & 3 & 2 & 5 \\ -5 & 1 & 1 & 1 \\ -8 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \\ r \end{pmatrix}.$$

43. 195] Using the inverse of the matrix, find the solution to the system

$$\begin{pmatrix} 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 2 & -1 & 1 & 1 \\ 3 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \\ r \end{pmatrix}.$$

44. 196] Show that if A is an invertible $n \times n$ matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

45. 197] Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by verifying that $(ABC)(C^{-1}B^{-1}A^{-1}) = I$ and $(C^{-1}B^{-1}A^{-1})\,(ABC) = I$.

46. 198] If A is invertible, show $(A^2)^{-1} = (A^{-1})^2$.

47. 199] If A is invertible, show $(A^{-1})^{-1} = A$.

48. 200] Let A and be a real $m \times n$ matrix and let $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{y} \in \mathbf{R}^m$. Show $(A\mathbf{x},\mathbf{y})_{\mathbf{R}^m} = (\mathbf{x},A^T\mathbf{y})_{\mathbf{R}^n}$ where $(\cdot,\cdot)_{\mathbf{R}^k}$ denotes the dot product in $\mathbf{R}^k$. In the notation above, $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y}$. Use the definition of matrix multiplication to do this.

49. 201] Use the fact that $(A\mathbf{x} \cdot \mathbf{y}) = (\mathbf{x} \cdot A^T\mathbf{y})$ to verify directly that $(AB)^T = B^TA^T$ without making any reference to subscripts.

50. 202] A matrix A is called a projection if $A^2 = A$. Here is a matrix.

$$\begin{pmatrix} 3 & -6 & -2 \\ -1 & 4 & 1 \\ 6 & -18 & -5 \end{pmatrix}$$

Show that this is a projection. Show that a vector in the column space of a projection is left unchanged by multiplication by A.

51. 203] Suppose A is an $n \times n$ matrix and for each j,

$$\sum_{i=1}^{n} |A_{ij}| < 1$$

Show that the infinite series $\sum_{k=0}^{\infty} A^k$ converges in the sense that the ij^{th} entry of the partial sums converge for each ij. **Hint**: Let $R \equiv \max_j \sum_{i=1}^{n} |A_{ij}|$. Thus $R < 1$. Show that $\left|(A^2)_{ij}\right| \leq R^2$. Then generalize to show that $\left|(A^m)_{ij}\right| \leq R^m$. Use this to show that the ij^{th} entry of the partial sums is a Cauchy sequence. From calculus, these converge by completeness of the real or complex numbers. Next show that $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$. The Leontief model in economics involves solving an equation for $\mathbf{x}$ of the form

$$\mathbf{x} = A\mathbf{x} + \mathbf{b}, \text{ or } (I - A)\mathbf{x} = \mathbf{b}$$

The vector $A\mathbf{x}$ is called the intermediate demand and the vectors $A^k\mathbf{x}$ have economic meaning as does the series which is also called the Neuman series.

Determinants

1. 1] Find the determinant of the following matrix.

$$\begin{pmatrix} -5 & -3 & -4 \\ 0 & -4 & 0 \\ 4 & 6 & 3 \end{pmatrix}$$

2. 2] Find the determinant of the following matrix.

$$\begin{pmatrix} -12 & -10 & -7 \\ 0 & -3 & 0 \\ 9 & 10 & 4 \end{pmatrix}$$

3. 3] Find the determinant of the following matrix.

$$\begin{pmatrix} 10 & 18 & 9 \\ -3 & -6 & -3 \\ -11 & -16 & -10 \end{pmatrix}$$

4. 4] Find the determinant of the following matrix.

$$\begin{pmatrix} 2 & -1 & 1 \\ -4 & -1 & -4 \\ 5 & 5 & 6 \end{pmatrix}$$

5. 5] Find the determinant of the following matrix.

$$\begin{pmatrix} -2 & 0 & 2 & -3 \\ 6 & 11 & 12 & 6 \\ -2 & -4 & -4 & -2 \\ 1 & -4 & -7 & 2 \end{pmatrix}$$

6. 6] Find the determinant of the following matrix.

$$\begin{pmatrix} -5 & -14 & -16 & -6 \\ 7 & 10 & 7 & 7 \\ -6 & -9 & -9 & -6 \\ 2 & 11 & 16 & 3 \end{pmatrix}$$

7. 7] Find the determinant of the following matrix.

$$\begin{pmatrix} 20 & 31 & 34 & 17 \\ -3 & -6 & -5 & -3 \\ 5 & 7 & 7 & 5 \\ -21 & -31 & -35 & -18 \end{pmatrix}$$

8. 8] Find the determinant of the following matrix.

$$\begin{pmatrix} -6 & -9 & -12 & -7 \\ 11 & 18 & 21 & 11 \\ -5 & -7 & -7 & -5 \\ 3 & 1 & 1 & 4 \end{pmatrix}$$

9. 9] Find the determinant of the following matrix.

$$\begin{pmatrix} 3 & 3 & 6 & 1 \\ 6 & 13 & 14 & 6 \\ 0 & -3 & -3 & 0 \\ -8 & -12 & -16 & -6 \end{pmatrix}$$

10. 10] Find the determinant of the following matrix.

$$\begin{pmatrix} -6 & -7 & -10 & -5 \\ -3 & -9 & -11 & -3 \\ -2 & 0 & 0 & -2 \\ 10 & 15 & 20 & 9 \end{pmatrix}$$

11. 11] Find the determinant of the following matrix.

$$\begin{pmatrix} -5 & -8 & -5 & -6 \\ 11 & 23 & 25 & 11 \\ -3 & -8 & -8 & -3 \\ 0 & -4 & -9 & 1 \end{pmatrix}$$

12. 12] Find the following determinant by expanding along the first row and then by expanding along the second column.

$$\begin{pmatrix} -7 & -14 & 2 \\ 8 & 13 & 2 \\ -6 & -4 & -9 \end{pmatrix}$$

13. 13] Find the following determinant by expanding along the first row and then by expanding along the second column.

$$\begin{pmatrix} 40 & 52 & 20 \\ -20 & -26 & -10 \\ -20 & -26 & -10 \end{pmatrix}$$

14. 14] Find the following determinant by expanding along the first row and then by expanding along the second column.

$$\begin{pmatrix} 23 & 29 & 11 \\ -8 & -10 & -3 \\ -20 & -24 & -13 \end{pmatrix}$$

15. 15] Find the following determinant by expanding along the first row and then by expanding along the second column.

$$\begin{pmatrix} -34 & -40 & -20 \\ 17 & 20 & 10 \\ 20 & 23 & 13 \end{pmatrix}$$

16. 16] Find the following determinant by cofactor expansion. Pick whatever is easiest.

$$\begin{pmatrix} -2 & -7 & -4 & 1 \\ 1 & 10 & 13 & 1 \\ 2 & -1 & -1 & 2 \\ 0 & -1 & -7 & -3 \end{pmatrix}$$

17. 17] Find the following determinant by cofactor expansion. Pick whatever is easiest.

$$\begin{pmatrix} -18 & -26 & -38 & -23 \\ 20 & 25 & 28 & 20 \\ -11 & -12 & -12 & -11 \\ 10 & 14 & 23 & 15 \end{pmatrix}$$

18. 18] Find the following determinant by cofactor expansion. Pick whatever is easiest.

$$\begin{pmatrix} -23 & -32 & -40 & -25 \\ 19 & 28 & 31 & 19 \\ -11 & -14 & -14 & -11 \\ 18 & 21 & 26 & 20 \end{pmatrix}$$

19. 19] Find the following determinant by cofactor expansion. Pick whatever is easiest.

$$\begin{pmatrix} -14 & -23 & -24 & -9 \\ 1 & 8 & 11 & 1 \\ -1 & -3 & -3 & -1 \\ 15 & 19 & 17 & 10 \end{pmatrix}$$

20. 20] An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

21. 21] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = 2$$

Find

$$\det \begin{pmatrix} u+x & v+y & w+z \\ q+3x & r+3y & s+3z \\ x & y & z \end{pmatrix}$$

22. 22] Suppose that A, B are $n \times n$ matrices and $\det(A) = 4$ while $\det(B) = 2$. Find $\det(AB^{-1})$.

23. 23] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = -9$$

Find

$$\det \begin{pmatrix} u & v & w \\ q+3x & r+3y & s+3z \\ x & y & z \end{pmatrix}$$

24. 24] Suppose that A is an $n \times n$ matrix and $A^2 = A$. Find the possible values for $\det(A)$.

25. 25] An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

26. 26] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = -9$$

Find

$$\det\begin{pmatrix} u & v & w \\ x & y & z \\ q & r & s \end{pmatrix}$$

27. 27] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = 10$$

Find

$$\det\begin{pmatrix} u-3x & v-3y & w-3z \\ x & y & z \\ q & r & s \end{pmatrix}$$

28. 28] An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix}$$

29. 29] Suppose that A, B are $n \times n$ matrices and $\det(A) = 5$ while $\det(B) = 3$. Find $\det(AB^T)$.

30. 30] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = -7$$

Find

$$\det\begin{pmatrix} v & u & w-3u \\ r & q & s-3q \\ y & x & z-3x \end{pmatrix}$$

31. 31] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = 3$$

Find

$$\det\begin{pmatrix} -3u & v+3w & w \\ -3q & r+3w & s \\ -3x & 3w+y & z \end{pmatrix}$$

32. 32] An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$$

33. 33] Suppose that A is a complex $n \times n$ matrix and $A^m = I$. Find the possible values for $\det(A)$.

34. 34] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = -9$$

Find

$$\det\begin{pmatrix} u & v & 3u+w \\ q & r & 3q+s \\ x & y & 3x+z \end{pmatrix}$$

35. 35] An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

36. 36] Suppose

$$\begin{vmatrix} u & v & w \\ q & r & s \\ x & y & z \end{vmatrix} = 10$$

Find

$$\det\begin{pmatrix} u-x & v-y & w-z \\ -q & -r & -s \\ x & y & z \end{pmatrix}$$

37. 37] Verify $\det(AB) = \det(A)\det(B), \det(A^T) = \det(A)$, and that det is multilinear in the rows (columns) for 2×2 matrices.

38. 38] Find $\det\begin{pmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{pmatrix}$ and explain why.

39. 39] *Find $\det\begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}$, $\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$. Then conjecture a formula for

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}$$

Finally prove your formula. This is called a Vandermonde determinant. **Hint**: The formula you should get is

$$\prod_{0 \le i < j \le n} (a_j - a_i)$$

which means to take all possible products of terms of the form $a_j - a_i$ where $i < j$. To verify this formula, consider

$$p(t) \equiv \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & t \\ \vdots & \vdots & & \vdots \\ a_0^{n-1} & a_1^{n-1} & \cdots & t^{n-1} \\ a_0^n & a_1^n & \cdots & t^n \end{pmatrix}$$

Explain why, using properties of determinants that $p(a_j) = 0$ for all $j < n$. Therefore, explain why

$$p(t) = c\prod_{i=0}^{n-1}(t - a_i)$$

Now note that $p(a_n)$ is the thing which is wanted. Of course c is the coefficient of t^n and you know what this is by induction. It equals the determinant of the upper left corner of the matrix whose determinant equals $p(t)$ in the above formula.

40. 40] *Suppose $a_0 < a_1 < a_2 < \cdots < a_n$ and numbers $b_0, b_1, \cdots, b_n$ are numbers. Show that there exists a unique polynomial, $p(t)$ of degree n such that $p(a_i) = b_i$. This is known as Lagrange interpolation.

41. 41] *Suppose $a_0 < a_1 < a_2 < \cdots < a_n$ and numbers $b_0, b_1, \cdots, b_n$ are numbers. Show that there exists a unique polynomial, $p(t)$ of degree n such that $p(a_i) = b_i$. In fact, show that the following polynomial works.

$$\sum_{i=0}^{n} b_i \frac{\prod_{j:j\neq i}(t-a_j)}{\prod_{j:j\neq i}(a_i-a_j)} = p(t)$$

42. 42] An integration scheme is a formula

$$\frac{1}{n}\sum_{i=0}^{n} c_i f_i \approx \int_x^{x+h} f(t)dt$$

where $\approx$ is $=$ in case $f(t) = 1, t, \cdots, t^n$. Here f_i is the value $f(x_i)$ for $x = x_0 < x_1 < \cdots < x_n = x+h$, these points being equally spaced. Find the integration scheme for $n = 1$. You need to find the c_i. Then give a technique based on this which will integrate every function which is linear on the subintervals determined by the partition $a = x_0 < x_1 < \cdots < x_n = b$ where $x_k - x_{k-1} = h > 0$.

43. 43] Let A be an $r \times r$ matrix and suppose there are $r-1$ rows (columns) such that all rows (columns) are linear combinations of these $r-1$ rows (columns). Show $\det(A) = 0$.

44. 44] Show $\det(aA) = a^n \det(A)$ where here A is an $n \times n$ matrix and a is a scalar.

45. 45] Is it true that $\det(A+B) = \det(A) + \det(B)$? If this is so, explain why it is so and if it is not so, give a counter example.

46. 46] An $n \times n$ matrix is called **nilpotent** if for some positive integer, k it follows $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

47. 47] A matrix is said to be **orthogonal** if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

48. 48] Fill in the missing entries to make the matrix orthogonal. A is orthogonal if $A^T A = I$.

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{12}}{6} \\ \frac{1}{\sqrt{2}} & - & - \\ - & \frac{\sqrt{6}}{3} & - \end{pmatrix}.$$

49. 49] Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix, S such that $A = S^{-1}BS$. Show that if $A \sim B$, then $B \sim A$. Show also that $A \sim A$ and that if $A \sim B$ and $B \sim C$, then $A \sim C$.

50. 50] Show that if $A \sim B$, then $\det(A) = \det(B)$. To say $A \sim B$ is to say there exists S such that $A = S^{-1}BS$. If $\det(A) = \det(B)$, does it follow that $A \sim B$?

51. 51] Two $n \times n$ matrices, A and B, are similar if $B = S^{-1}AS$ for some invertible $n \times n$ matrix, S. Show that if two matrices are similar, they have the same characteristic polynomials. The characteristic polynomial of an $n \times n$ matrix, M is the polynomial, $\det(\lambda I - M)$.

52. 52] Recall that the volume of a parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbf{R}^3$ is the absolute value of the box product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. Show that this box product is the same as the determinant

$$\det \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix} = \det \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$$

Also show that for $\mathbf{a}, \mathbf{b}$ two vectors in $\mathbf{R}^2$ the area of the parallelogram determined by these two vectors is the absolute value of the determinant

$$\det \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{pmatrix} = \det \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}$$

It can be shown that the only reasonable way to define the volume of a parallelepiped determined by n vectors $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ in $\mathbf{R}^n$ is to take the absolute value of the determinant

$$\det \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$$

53. 53] Suppose you have a box in three dimensions $[a,b] \times [c,d] \times [e,f] = R$. Also suppose A is a 3×3 matrix. Show that the volume of AR is the absolute value of $\det(A)\left(\text{volume of } R\right)$. Say $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}$. Then if you have an ellipsoid determined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$, show that the volume of the ellipsoid is $\frac{4}{3}\pi abc$. Here each of a, b, c is positive.

54. 54] Tell whether the statement is true or false.

a. If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column.

b. If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.

c. For A and B two $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$.

d. For A an $n \times n$ matrix, $\det(3A) = 3\det(A)$

e. If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.

f. If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4\det(A)$.

g. For A an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$.

h. If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$.

i. Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions.

j. If $A^k = 0$ for some positive integer, k, then $\det(A) = 0$.

k. If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\det(A) = 0$.

55. 55] Use Cramer's rule to find the solution to

$$\begin{aligned} x - 3y &= 3 \\ x - 2y &= 1 \end{aligned}$$

56. 56] Use Cramer's rule to find the solution to

$$\begin{aligned} x - 3y &= -3 \\ 4x - 11y &= -11 \end{aligned}$$

57. 57] Use Cramer's rule to find the solution to

$$\begin{aligned} x - 4y &= 0 \\ 5x - 19y &= 1 \end{aligned}$$

58. 58] Use Cramer's rule to find the solution to

$$\begin{aligned} -3x - 2y - 2z - 3 &= 0 \\ -4x - 3y - 3z - 3 &= 0 \\ 2x + 2y + 3z + 2 &= 0 \end{aligned}$$

59. 59] Use Cramer's rule to find the solution to

$$\begin{aligned} 6x + 7y + 7z + 16 &= 0 \\ 5x + 6y + 6z + 14 &= 0 \\ 3x + 3y + 4z + 6 &= 0 \end{aligned}$$

60. 60] Use Cramer's rule to find the solution to

$$\begin{aligned} 4x + 5y + 5z - 31 &= 0 \\ 3x + 4y + 4z - 24 &= 0 \\ 3x + 3y + 4z - 20 &= 0 \end{aligned}$$

61. 61] Here is a matrix,

$$\begin{pmatrix} 4 & 5 & 5 \\ 6 & 8 & 8 \\ 5 & 5 & 6 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

62. 62] Here is a matrix,

$$\begin{pmatrix} 2 & 3 & 3 \\ 4 & 5 & 8 \\ 1 & 1 & 2 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

63. 63] Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 3 & 3 & 4 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

64. 64] Here is a matrix,

$$\begin{pmatrix} -2 & -1 & -1 \\ -3 & -2 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

65. 65] Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -2 \\ 5 & 5 & 6 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

66. 66] Here is a matrix,

$$\begin{pmatrix} 3 & 4 & 4 \\ 2 & 3 & 3 \\ 2 & 2 & 3 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

67. 67] Here is a matrix,

$$\begin{pmatrix} e^{2t} & \cos t & \sin t \\ 2e^{2t} & -\sin t & \cos t \\ 4e^{2t} & -\cos t & -\sin t \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

68. 68] Here is a matrix,

$$\begin{pmatrix} e^{t} & \cos 2t & -\sin 2t \\ e^{t} & -2\sin 2t & -2\cos 2t \\ e^{t} & -4\cos 2t & 4\sin 2t \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

69. 69] Here is a matrix,

$$\begin{pmatrix} e^{t} & \cosh t & \sinh t \\ e^{t} & \sinh t & \cosh t \\ e^{t} & \cosh t & \sinh t \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

70. 70] Here is a matrix,

$$\begin{pmatrix} -1 & t & t^2 \\ 0 & -1 & 2t \\ t & 0 & 4 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

71. 71] Here is a matrix,

$$\begin{pmatrix} e^{5t} & (\cos 2t)e^{-2t} & -e^{-2t}\sin 2t \\ 5e^{5t} & -2e^{-2t}(\cos 2t+\sin 2t) & -2e^{-2t}(\cos 2t-\sin 2t) \\ 25e^{5t} & 8e^{-2t}\sin 2t & 8(\cos 2t)e^{-2t} \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero.

72. 72] Here is a matrix,

$$\begin{pmatrix} 1 & t & e^{-2t} \\ 0 & 1 & -2e^{-2t} \\ 0 & 0 & 4e^{-2t} \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

73. 73] Here is a matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (\cos 3t)e^{-3t} & -e^{-3t}\sin 3t \\ 0 & -3e^{-3t}(\cos 3t+\sin 3t) & -3e^{-3t}(\cos 3t-\sin 3t) \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

74. 74] Here is a matrix,

$$\begin{pmatrix} e^{t} & (\cos 4t)e^{2t} & -e^{2t}\sin 4t \\ e^{t} & 2e^{2t}(\cos 4t-2\sin 4t) & -2e^{2t}(2\cos 4t+\sin 4t) \\ e^{t} & -12(\cos 4t)e^{2t}-16e^{2t}\sin 4t & 12e^{2t}\sin 4t-16(\cos 4t)e^{2t} \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero.

75. 75] Here is a matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (\cos t)e^{-t} & (\sin t)e^{-t} \\ 0 & -e^{-t}(\cos t+\sin t) & e^{-t}(\cos t-\sin t) \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

76. 76] Here is a matrix,

$$\begin{pmatrix} e^{-3t} & te^{-3t} & 1 \\ -3e^{-3t} & -e^{-3t}(3t-1) & 0 \\ 9e^{-3t} & 3e^{-3t}(3t-2) & 0 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. Then find its inverse if it has one.

77. 77] Show that if $\det(A) \neq 0$ for A an $n \times n$ matrix, it follows that if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$.

78. 78] Suppose A, B are $n \times n$ matrices and that $AB = I$. Show that then $BA = I$. **Hint**: You might do something like this: First explain why $\det(A), \det(B)$ are both nonzero. Then $(AB)A = A$ and then show $BA(BA - I) = 0$. From this use what is given to conclude $A(BA - I) = 0$.

79. 79] Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Is everything the same for lower triangular matrices?

80. 80] If A, B, and C are each $n \times n$ matrices and ABC is invertible, why are each of A, B, and C invertible.

81. 81] Let $F(t) = \det\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$. Verify

$$F'(t) = \det\begin{pmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{pmatrix} + \det\begin{pmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{pmatrix}.$$

Now suppose

$$F(t) = \det\begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix}.$$

Use Laplace expansion and the first part to verify $F'(t) =$

$$\det\begin{pmatrix} a'(t) & b'(t) & c'(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix} + \det\begin{pmatrix} a(t) & b(t) & c(t) \\ d'(t) & e'(t) & f'(t) \\ g(t) & h(t) & i(t) \end{pmatrix}$$

$$+ \det\begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g'(t) & h'(t) & i'(t) \end{pmatrix}.$$

Conjecture a general result valid for $n \times n$ matrices and explain why it will be true. Can a similar thing be done with the columns?

82. 82] Let $Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$ where the a_i are given continuous functions defined on a closed interval, (a, b) and y is some function which has n derivatives so it makes sense to write Ly. Suppose $Ly_k = 0$ for $k = 1, 2, \cdots, n$. The **Wronskian** of these functions,

y_i is defined as

$$W(y_1, \cdots, y_n)(x) \equiv \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

Show that for $W(x) = W(y_1, \cdots, y_n)(x)$ to save space,

$$W'(x) = \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{pmatrix}.$$

Now use the differential equation, $Ly = 0$ which is satisfied by each of these functions, y_i and properties of determinants presented above to verify that $W' + a_{n-1}(x)W = 0$. Give an explicit solution of this linear differential equation, **Abel's formula**, and use your answer to verify that the Wronskian of these solutions to the equation, $Ly = 0$ either vanishes identically on (a, b) or never. **Hint**: To solve the differential equation, let $A'(x) = a_{n-1}(x)$ and multiply both sides of the differential equation by $e^{A(x)}$ and then argue the left side is the derivative of something.

83. 83] Find the determinant of the following matrix.

$$\begin{pmatrix} -2+i & 1+4i \\ 3i & 0 \end{pmatrix}$$

84. 84] Find the determinant of the following matrix.

$$\begin{pmatrix} 22-24i & 32-34i & 16-10i \\ -13+14i & -19+20i & -9+6i \\ -8+10i & -12+14i & -6+4i \end{pmatrix}$$

85. 85] Find the determinant of the following matrix.

$$\begin{pmatrix} 19-36i & 21-51i & 5-15i \\ -5+21i & -3+30i & -1+9i \\ -14+15i & -18+21i & -4+6i \end{pmatrix}$$

Rank, Dimension, Subspaces

1. 86] Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be vectors in $\mathbf{R}^n$. The parallelepiped determined by these vectors $P(\mathbf{u}_1, \cdots, \mathbf{u}_n)$ is defined as

$$P(\mathbf{u}_1, \cdots, \mathbf{u}_n) \equiv \left\{ \sum_{k=1}^{n} t_k \mathbf{u}_k \, : \, t_k \in [0,1] \text{ for all } k \right\}.$$

Now let A be an $n \times n$ matrix. Show that

$$\{A\mathbf{x} \, : \, \mathbf{x} \in P(\mathbf{u}_1, \cdots, \mathbf{u}_n)\}$$

is also a parallelepiped.

2. 87] Draw $P(\mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors for $\mathbf{R}^2$. Thus $\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)$. Now suppose

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

where E is the elementary matrix which takes the third row and adds to the first. Draw

$$\{E\mathbf{x} \, : \, \mathbf{x} \in P(\mathbf{e}_1, \mathbf{e}_2)\}.$$

In other words, draw the result of doing E to the vectors in $P(\mathbf{e}_1, \mathbf{e}_2)$. Next draw the results of doing the other elementary matrices to $P(\mathbf{e}_1, \mathbf{e}_2)$.

3. 88] Either draw or describe the result of doing elementary matrices to $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, the parallelepiped determined by the three listed vectors. Describe geometrically the fact proved in the chapter that every invertible matrix is a product of elementary matrices.

4. 89] Explain why the elementary matrix which comes from the row operation which replaces a row with a constant times another row added to it does not change the volume of a box which has the sides parallel to the coordinate axes.

5. 90] Determine which matrices are in row reduced echelon form.

a. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$

6. 91] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} -1 & -1 & -2 & -6 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

7. 92] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} -1 & 1 & 2 & 1 \\ 1 & -2 & -7 & -3 \\ 1 & -3 & -12 & -5 \end{pmatrix}$$

8. 93] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} 3 & 11 & -24 & 11 \\ 1 & 4 & -9 & 4 \\ 1 & 3 & -6 & 3 \end{pmatrix}$$

9. 94] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} -2 & -4 & -1 & -1 \\ 1 & 2 & 0 & -1 \\ 1 & 2 & -1 & -4 \end{pmatrix}$$

10. 95] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} -2 & -9 & 27 & -29 \\ 1 & 4 & -12 & 13 \\ 1 & 3 & -9 & 10 \end{pmatrix}$$

11. 96] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} 3 & -4 & 21 & -5 \\ 1 & -1 & 6 & -1 \\ 1 & -2 & 9 & -3 \end{pmatrix}$$

12. 97] Row reduce the following matrix to obtain the row reduced echelon form.

$$\begin{pmatrix} -1 & 1 & -1 & -5 \\ 1 & -1 & 0 & 2 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

13. 98] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 1 & -4 & 1 & -1 \\ 1 & -4 & 2 & 1 \\ 1 & -4 & 1 & -1 \end{pmatrix}$$

14. 99] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} -2 & 4 & 4 & -5 & -18 \\ 1 & -2 & -2 & 2 & 7 \\ 1 & -2 & -2 & 1 & 3 \\ -3 & 6 & 6 & -3 & -9 \end{pmatrix}$$

15. 100] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 1 & -2 & 0 & -2 \\ 1 & -2 & 1 & 0 \\ 1 & -2 & 0 & -2 \end{pmatrix}$$

16. 101] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 3 & 9 & -4 & -14 \\ 1 & 3 & -1 & -4 \\ 1 & 3 & -2 & -6 \end{pmatrix}$$

17. 102] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 4 & 8 & -16 & -9 & 6 \\ 1 & 2 & -4 & -2 & 2 \\ 1 & 2 & -4 & -3 & 0 \\ 3 & 6 & -12 & -9 & 0 \end{pmatrix}$$

18. 103] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} -1 & -1 & 1 & 5 & 33 \\ 1 & 1 & -2 & -5 & -32 \\ 1 & 1 & -3 & -4 & -24 \\ -2 & -2 & 6 & 8 & 48 \end{pmatrix}$$

19. 104] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 2 & -5 & 23 \\ 1 & -2 & 10 \\ 1 & -3 & 13 \\ 1 & -3 & 13 \end{pmatrix}$$

20. 105] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 1 & 2 & -1 & 0 & -4 \\ 1 & 2 & 0 & 0 & -5 \\ 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

21. 106] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} -1 & -5 & -15 \\ 1 & 4 & 12 \\ 1 & 3 & 9 \\ -2 & -6 & -18 \end{pmatrix}$$

22. 107] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} 2 & 4 & 5 & 8 & 35 \\ 1 & 2 & 3 & 4 & 16 \\ 1 & 2 & 2 & 5 & 26 \\ 1 & 2 & 2 & 5 & 26 \end{pmatrix}$$

23. 108] Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{pmatrix} -2 & 2 & 2 & -1 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 & 3 \\ -3 & 3 & 3 & 3 & -9 \end{pmatrix}$$

24. 109] Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

25. 110] Let H denote span$\left(\begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 14 \\ 3 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 33 \\ 7 \\ 5 \\ 20 \end{pmatrix}, \begin{pmatrix} 15 \\ 3 \\ 3 \\ 12 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \\ 8 \end{pmatrix} \right)$. Find the dimension of H and determine a basis.

26. 111] Let H denote span$\left(\begin{pmatrix}5\\1\\1\\4\end{pmatrix},\begin{pmatrix}19\\4\\3\\12\end{pmatrix},\begin{pmatrix}62\\13\\10\\40\end{pmatrix},\begin{pmatrix}5\\1\\2\\8\end{pmatrix}\right)$. Find the dimension of H and determine a basis.

27. 112] Let H denote span$\left(\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\begin{pmatrix}-1\\5\\0\\-4\end{pmatrix},\begin{pmatrix}-1\\3\\0\\-2\end{pmatrix},\begin{pmatrix}0\\1\\0\\-2\end{pmatrix},\begin{pmatrix}-2\\7\\0\\-5\end{pmatrix}\right)$. Find the dimension of H and determine a basis.

28. 113] Let H denote span$\left(\begin{pmatrix}3\\1\\1\\2\end{pmatrix},\begin{pmatrix}-7\\-2\\-3\\-6\end{pmatrix},\begin{pmatrix}27\\8\\11\\22\end{pmatrix},\begin{pmatrix}-8\\-2\\-4\\-8\end{pmatrix}\right)$. Find the dimension of H and determine a basis.

29. 114] Let H denote span$\left(\begin{pmatrix}-2\\1\\1\\-3\end{pmatrix},\begin{pmatrix}-9\\4\\3\\-9\end{pmatrix},\begin{pmatrix}-33\\15\\12\\-36\end{pmatrix},\begin{pmatrix}-2\\1\\1\\-3\end{pmatrix}\right)$. Find the dimension of H and determine a basis.

30. 115] Let H denote span$\left(\begin{pmatrix}2\\1\\2\\1\end{pmatrix},\begin{pmatrix}27\\15\\24\\12\end{pmatrix},\begin{pmatrix}7\\4\\6\\3\end{pmatrix},\begin{pmatrix}6\\3\\8\\4\end{pmatrix},\begin{pmatrix}16\\9\\14\\7\end{pmatrix}\right)$. Find the dimension of H and determine a basis.

31. 116] Let H denote span$\left(\begin{pmatrix}4\\1\\1\\3\end{pmatrix},\begin{pmatrix}3\\1\\0\\0\end{pmatrix},\begin{pmatrix}12\\3\\3\\9\end{pmatrix},\begin{pmatrix}12\\3\\4\\12\end{pmatrix}\right)$. Find the dimension of H and determine a basis.

32. 117] Let H denote span$\left(\begin{pmatrix}1\\2\\1\\0\end{pmatrix},\begin{pmatrix}2\\6\\2\\0\end{pmatrix},\begin{pmatrix}6\\16\\6\\0\end{pmatrix},\begin{pmatrix}2\\4\\2\\0\end{pmatrix}\right)$. Find the dimension of H

and determine a basis.

33. 118] Let $M = \{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : \sin(u_1) = 1\}$. Is M a subspace? Explain.

34. 119] Let $M = \left\{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : u_i \geq 0 \text{ for each } i = 1,2,3,4\right\}$. Is M a subspace? Explain.

35. 120] Let $\mathbf{w} \in \mathbf{R}^4$ and let $M = \{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0\}$. Is M a subspace? Explain.

36. 121] Let $M = \{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : |u_1| \leq 4\}$. Is M a subspace? Explain.

37. 122] Let $\mathbf{w}, \mathbf{w}_1$ be given vectors in $\mathbf{R}^4$ and define

$$M = \left\{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0 \text{ and } \mathbf{w}_1 \cdot \mathbf{u} = 0\right\}.$$

Is M a subspace? Explain.

38. 123] Prove the following theorem: If A,B are $n \times n$ matrices and if $AB = I$, then $BA = I$ and $B = A^{-1}$. **Hint**: First note that if $AB = I$, then it must be the case that A is onto. Explain why this requires $\text{span}\left(\text{columns of } A\right) = \mathbf{F}^n$. Now explain why, using the corollary that this requires A to be one to one. Next explain why $A(BA - I) = 0$ and why the fact that A is one to one implies $BA = I$.

39. 124] Let $M = \{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : u_3 \geq u_1\}$. Is M a subspace? Explain.

40. 125] Suppose $\{\mathbf{x}_1,\cdots,\mathbf{x}_k\}$ is a set of vectors from $\mathbf{F}^n$. Show that $\mathbf{0}$ is in $\text{span}(\mathbf{x}_1,\cdots,\mathbf{x}_k)$.

41. 126] Explain why if A is $m \times n$ where $m < n$, then A cannot be one to one.

42. 127] Let $M = \{\mathbf{u} = (u_1,u_2,u_3,u_4) \in \mathbf{R}^4 : u_3 = u_1 = 0\}$. Is M a subspace? Explain

43. 128] Prove the following theorem: If A,B are $n \times n$ matrices and if $AB = I$, then $BA = I$ and $B = A^{-1}$. **Hint**: First note that if $AB = I$, then it must be the case that A is onto. Explain why this requires $\text{span}\left(\text{columns of } A\right) = \mathbf{F}^n$. Now explain why, using the corollary that this requires A to be one to one. Next explain why $A(BA - I) = 0$ and why the fact that A is one to one implies $BA = I$.

44. 129] Study the definition of span. Explain what is meant by the span of a set of vectors. Include pictures.

45. 130] Let $\{\mathbf{v}_1,\cdots,\mathbf{v}_m\}$ be vectors. Prove that $\text{span}(\mathbf{v}_1,\cdots,\mathbf{v}_m)$ is a subspace.

46. 131] Let A be an $m \times n$ matrix. Then

$$N(A) \equiv \ker(A) \equiv \{\mathbf{x} \in \mathbf{F}^n : A\mathbf{x} = \mathbf{0}\}$$

Show that $N(A)$ is a subspace of $\mathbf{F}^n$. It is in the book but you need to do it on your own.

47. 132] Let A be an $m \times n$ matrix. Then the image of A is defined as

$$\text{Im}(A) \equiv A(\mathbf{F}^n) \equiv \{A\mathbf{x} : \mathbf{x} \in \mathbf{F}^n\}$$

Show that $N(A)$ is a subspace of $\mathbf{F}^m$. Note that this is really the column space. It is done in the book but you should do it here without dealing with columns.

48. 133] *Suppose you have a subspace V of $\mathbf{F}^n$. Is it the case that V equals $N(A) = \ker(A)$ for some $m \times n$ matrix?

49. 134] Suppose you have a subspace V of $\mathbf{F}^n$. Is it the case that V equals $\operatorname{Im}(A)$ for some matrix A?

50. 135] Consider the vectors of the form

$$\left\{ \begin{pmatrix} v \\ u - v - w \\ w \end{pmatrix} : u, v, w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

51. 136] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2u + v - 5w \\ 4v - 2u + 10w \\ 2v - 4u + 14w \end{pmatrix} : u, v, w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

52. 137] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 3u + v + 5w \\ 7u + 3v + 13w \\ 15u + 6v + 27w \end{pmatrix} : u, v, w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

53. 138] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2u + v \\ w - v - u \\ w - 2v - 2u \end{pmatrix} : u, v, w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

54. 139] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 4u + 12v + 15w \\ u + 3v + 4w \\ 4u + 12v + 12w \\ 3u + 9v + 9w \end{pmatrix} : u, v, w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

55. 140] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 3u+v \\ 5u+v-w \\ w-2v-7u \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

56. 141] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 3u+v-w \\ 3u+2v+w \\ u+v+w \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

57. 142] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2u+v+7w \\ 2u-4v+2w \\ -12v-12w \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

58. 143] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 3u+v \\ 6u+6v+2w \\ 2u+4v+2w \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

59. 144] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 3u+v+5w \\ 0 \\ -6u-4v-14w \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

60. 145] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 4u+4v+7w \\ 0 \\ 4u+4v+4w \\ 3u+3v+3w \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

61. 146] Consider the vectors of the form

$$\left\{ \begin{pmatrix} -w \\ 2u+2v+4w \\ 0 \\ -u-v-w \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

62. 147] Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2u+v \\ w-2v-3u \\ 2w-6v-8u \end{pmatrix} : u,v,w \in \mathbf{R} \right\}.$$

Is this set of vectors a subspace of $\mathbf{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

63. 148] If you have 5 vectors in $\mathbf{F}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbf{F}^5$? Explain.

64. 149] If you have 6 vectors in $\mathbf{F}^5$, is it possible they are linearly independent? Explain.

65. 150] Suppose A is an $m \times n$ matrix and $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is a linearly independent set of vectors in $A(\mathbf{F}^n) \subseteq \mathbf{F}^m$. Now suppose $A(\mathbf{z}_i) = \mathbf{w}_i$. Show $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is also independent.

66. 151] Suppose V, W are subspaces of $\mathbf{F}^n$. Show $V \cap W$ defined to be all vectors which are in both V and W is a subspace also.

67. 152] Suppose V and W both have dimension equal to 7 and they are subspaces of $\mathbf{F}^{10}$. What are the possibilities for the dimension of $V \cap W$? **Hint**: Remember that a linear independent set can be extended to form a basis.

68. 153] Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$? **Hint**: Remember that a linear independent set can be extended to form a basis.

69. 154] If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin? Explain.

70. 155] Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

71. 156] Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

72. 157] Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

73. 158] Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix.

74. 159] State whether each of the following sets of data are possible for the matrix equation $A\mathbf{x} = \mathbf{b}$. If possible, describe the solution set. That is, tell whether there exists a unique solution no solution or infinitely many solutions.

a. A is a 5×6 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 4$. **Hint**: This says **b** is in the span of four of the columns. Thus the columns are not independent.

b. A is a 3×4 matrix, $\text{rank}(A) = 3$ and $\text{rank}(A|\mathbf{b}) = 2$.

c. A is a 4×2 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 4$. **Hint**: This says **b** is in the span of the columns and the columns must be independent.

d. A is a 5×5 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 5$. **Hint**: This says **b** is not in the span of the columns.

e. A is a 4×2 matrix, $\text{rank}(A) = 2$ and $\text{rank}(A|\mathbf{b}) = 2$.

75. 160] Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that A maps $\mathbf{F}^n$ onto $\mathbf{F}^m$. **Hint**: The vectors $\mathbf{e}_1, \cdots, \mathbf{e}_m$ occur as columns in the row reduced echelon form for A.

76. 161] Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. **Hint**: If not, there exists a vector, $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and this implies at least one column of A is a linear combination of the others. Show this would require the column rank to be less than n.

77. 162] Explain why an $n \times n$ matrix, A is both one to one and onto if and only if its rank is n.

78. 163] Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Hint: Consider the subspace, $B(\mathbf{F}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$. Now suppose $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is a basis for $\ker(B)$. Let $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ be such that $B\mathbf{z}_i = \mathbf{w}_i$ and argue that

$$\ker(AB) \subseteq \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_r, \mathbf{z}_1, \cdots, \mathbf{z}_k).$$

79. 164] Explain why $A\mathbf{x} = \mathbf{0}$ always has a solution even when A^{-1} does not exist. Here A is $n \times n$.

a. What can you conclude about A if the solution is unique?

b. What can you conclude about A if the solution is not unique?

80. 165] Suppose $\det(A - \lambda I) = 0$. Show there exists $\mathbf{x} \neq \mathbf{0}$ such that $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

81. 166] Let A be an $n \times n$ matrix and let $\mathbf{x}$ be a nonzero vector such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. When this occurs, the vector $\mathbf{x}$ is called an **eigenvector** and the scalar, λ is called an **eigenvalue**. It turns out that not every number is an eigenvalue. Only certain ones are. Why? **Hint**: Show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Explain why this shows that $(A - \lambda I)$ is not one to one and not onto. Now argue $\det(A - \lambda I) = 0$. What sort of equation is this? How many solutions does it have?

82. 167] Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one. **Hint**: Consider the $n \times n$ matrix, A_1 which is of the form

$$A_1 \equiv \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where the 0 denotes an $(n - m) \times n$ matrix of zeros. Thus $\det A_1 = 0$ and so A_1 is not one to one. Now observe that $A_1\mathbf{x}$ is the vector,

$$A_1\mathbf{x} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

which equals zero if and only if $A\mathbf{x} = \mathbf{0}$. Do this using the Fredholm alternative.

83. 168] Let A be an $m \times n$ real matrix and let $\mathbf{b} \in \mathbf{R}^m$. Show there exists a solution, $\mathbf{x}$ to the system

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

Next show that if $\mathbf{x}, \mathbf{x}_1$ are two solutions, then $A\mathbf{x} = A\mathbf{x}_1$. **Hint**: First show that $(A^TA)^T = A^TA$. Next show if $\mathbf{x} \in \ker(A^TA)$, then $A\mathbf{x} = \mathbf{0}$. Finally apply the Fredholm alternative. This will give existence of a solution.

84. 169] Show that in the context of the above problem that if $\mathbf{x}$ is the solution there, then $|\mathbf{b} - A\mathbf{x}| \leq |\mathbf{b} - A\mathbf{y}|$ for every $\mathbf{y}$. Thus $A\mathbf{x}$ is the point of $A(\mathbf{R}^n)$ which is closest to $\mathbf{b}$ of every point in $A(\mathbf{R}^n)$.

85. 170] Let A be an $n \times n$ matrix and consider the matrices $\{I, A, A^2, \cdots, A^{n^2}\}$. Explain why there exist scalars, c_i not all zero such that

$$\sum_{i=1}^{n^2} c_i A^i = 0.$$

Then argue there exists a polynomial, $p(\lambda)$ of the form

$$\lambda^m + d_{m-1}\lambda^{m-1} + \cdots + d_1\lambda + d_0$$

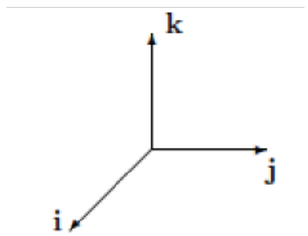
such that $p(A) = 0$ and if $q(\lambda)$ is another polynomial such that $q(A) = 0$, then $q(\lambda)$ is of the form $p(\lambda)l(\lambda)$ for some polynomial, $l(\lambda)$. This extra special polynomial, $p(\lambda)$ is called the **minimal polynomial**. **Hint**: You might consider an $n \times n$ matrix as a vector in $\mathbf{F}^{n^2}$.

Linear Transformations

1. 171] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{3}\pi$.
2. 172] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $-\frac{1}{4}\pi$.
3. 173] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $5\pi/12$. **Hint**: Note that $5\pi/12 = 2\pi/3 - \pi/4$.
4. 174] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of π.
5. 175] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{3}{2}\pi$.
6. 176] Show the map $T : \mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix and $\mathbf{x}$ is an $m \times 1$ column vector is a linear transformation.
7. 177] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $-\frac{1}{6}\pi$.
8. 178] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{12}\pi$. **Hint**: Note that $\frac{\pi}{4} - \frac{\pi}{6} = \frac{1}{12}\pi$.

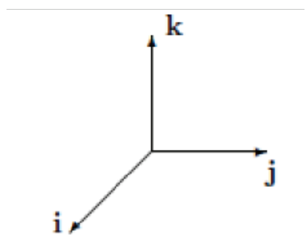
9. 179] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{6}\pi$ and then reflects through the y axis.
10. 180] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{2}{3}\pi$ and then reflects through the x axis.
11. 181] Find the matrix for the linear transformation which reflects through the y axis and then rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{2}\pi$.
12. 182] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{2}{3}\pi$ and then reflects through the y axis.
13. 183] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $-\frac{1}{2}\pi$ and then reflects through the x axis.
14. 184] Find the matrix for the linear transformation which reflects through the x axis and then rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{3}{4}\pi$.
15. 185] Find the matrix for the linear transformation which reflects through the x axis and then rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{3}\pi$.

16. 186] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



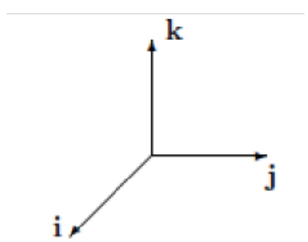
Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ through an angle of $-\frac{1}{6}\pi$ about the positive z axis when viewed from high on the positive z axis and then reflects through the xy plane.

17. 187] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



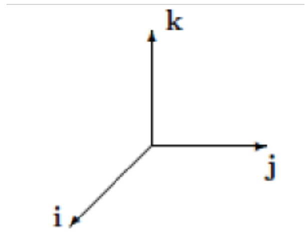
Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ through an angle of $\frac{1}{3}\pi$ about the positive x axis when viewed from the positive x axis and then reflects through the yz plane.

18. 188] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



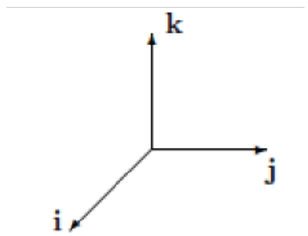
Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ through an angle of $\frac{1}{2}\pi$ about the positive z axis when viewed from high on the positive z axis and then reflects through the xy plane.

19. 189] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



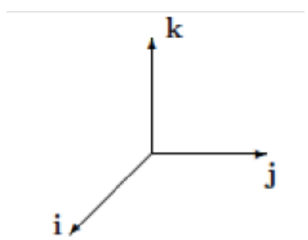
Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ through an angle of $\frac{3}{2}\pi$ about the positive z axis when viewed from high on the positive z axis and then reflects through the xy plane.

20. 190] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



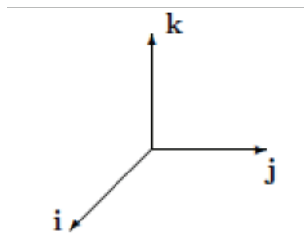
Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ through an angle of $\frac{1}{2}\pi$ about the positive z axis when viewed from high on the positive z axis and then reflects through the yz plane.

21. 191] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



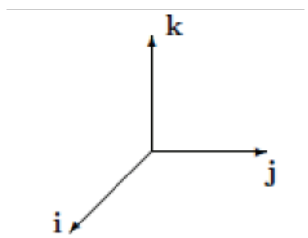
Find the matrix of the linear transformation which reflects every vector in $\mathbf{R}^3$ through the xy plane and then rotates every vector in $\mathbf{R}^3$ through an angle of $\frac{1}{4}\pi$ about the positive y axis when viewed from the positive y axis and then reflects through the xz plane.

22. 192] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ through an angle of $\frac{1}{2}\pi$ about the positive y axis when viewed from the positive y axis and then reflects through the xz plane.

23. 193] Recall the special vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}$ points in the direction of the positive x axis, $\mathbf{j}$ in the direction of the positive y axis and $\mathbf{k}$ in the direction of the positive z axis. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right handed system as shown in the following picture.



Find the matrix of the linear transformation which reflects every vector in $\mathbf{R}^3$ through the xz plane and then rotates every vector through an angle of $\frac{1}{4}\pi$ about the positive z axis when viewed from high on the positive z axis.

24. 194] Find the matrix for $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = \begin{pmatrix} 2 & 2 & 6 \end{pmatrix}^T$.

25. 195] Find the matrix for $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}^T$.

26. 196] Find the matrix for $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = \begin{pmatrix} -2 & -1 & 3 \end{pmatrix}^T$.

27. 197] Find the matrix for $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = \begin{pmatrix} 2 & -1 & -2 \end{pmatrix}^T$.

28. 198] Let M be the matrix for $\text{proj}_{\mathbf{u}}(\mathbf{v}), \mathbf{u} \in \mathbf{R}^n, n > 1$. Explain why $\det(M) = 0$.

29. 199] Show that the function $T_{\mathbf{u}}$ defined by $T_{\mathbf{u}}(\mathbf{v}) \equiv \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})$ is also a linear transformation.

30. 200] Show that $(\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}), \mathbf{u}) = 0$ and conclude every vector in $\mathbf{R}^n$ can be written as the sum of two vectors, one which is perpendicular and one which is parallel to the given vector.

31. 201] Here are some descriptions of functions mapping $\mathbf{R}^n$ to $\mathbf{R}^n$.

a. T multiplies the j^{th} component of $\mathbf{x}$ by a nonzero number b.

b. T replaces the i^{th} component of $\mathbf{x}$ with b times the j^{th} component added to the i^{th} component.

c. T switches two components.

Show these functions are linear and describe their matrices.

32. 202] Let $\mathbf{u} = (a, b)$ be a unit vector in $\mathbf{R}^2$. Find the matrix which reflects all vectors across this vector. **Hint**: You might want to notice that $(a, b) = (\cos\theta, \sin\theta)$ for some θ. First rotate through $-\theta$. Next reflect through the x axis which is easy. Finally rotate through θ.

33. 203] Let $\mathbf{u}$ be a unit vector. Show the linear transformation of the matrix $I - 2\mathbf{u}\mathbf{u}^T$ preserves all distances and satisfies

$$(I - 2\mathbf{u}\mathbf{u}^T)^T(I - 2\mathbf{u}\mathbf{u}^T) = I.$$

This matrix is called a Householder reflection. More generally, any matrix Q which satisfies $Q^TQ = QQ^T$ is called an orthogonal matrix. Show the linear transformation determined by an orthogonal matrix always preserves the length of a vector in $\mathbf{R}^n$. **Hint**: First show that for any matrix A,

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle$$

34. 204] Suppose $|\mathbf{x}| = |\mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. The problem is to find an orthogonal transformation Q ($Q^TQ = I$) which has the property that $Q\mathbf{x} = \mathbf{y}$ and $Q\mathbf{y} = \mathbf{x}$. Show

$$Q \equiv I - 2\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^2}(\mathbf{x} - \mathbf{y})^T$$

does what is desired.

35. 205] Let $\mathbf{a}$ be a fixed vector. The function $T_{\mathbf{a}}$ defined by $T_{\mathbf{a}}\mathbf{v} = \mathbf{a} + \mathbf{v}$ has the effect of translating all vectors by adding $\mathbf{a}$. Show this is not a linear transformation. Explain why it is not possible to realize $T_{\mathbf{a}}$ in $\mathbf{R}^3$ by multiplying by a 3×3 matrix.

36. 206] In spite of the above problem, we can represent both translations and rotations by matrix multiplication at the expense of using higher dimensions. This is done by the homogeneous coordinates. I will illustrate in $\mathbf{R}^3$ where most interest in this is found. For each vector $\mathbf{v} = (v_1, v_2, v_3)^T$, consider the vector in $\mathbf{R}^4$ $(v_1, v_2, v_3, 1)^T$. What happens when you do

$$\begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix}?$$

Describe how to consider both rotations and translations all at once by forming appropriate 4×4 matrices.

37. 207] You want to add $\begin{pmatrix} 3 & -1 & -2 \end{pmatrix}$ to every point in $\mathbf{R}^3$ and then rotate about the z axis counter clockwise through an angle of $\frac{1}{4}\pi$. Find what happens to the point $\begin{pmatrix} 1 & 3 & 4 \end{pmatrix}$.

38. 208] You want to add $\begin{pmatrix} -1 & -1 & -3 \end{pmatrix}$ to every point in $\mathbf{R}^3$ and then rotate about the z axis counter clockwise through an angle of $\frac{2}{3}\pi$. Find what happens to the point $\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$.

39. 209] You want to add $\begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$ to every point in $\mathbf{R}^3$ and then rotate about the z axis counter clockwise through an angle of $\frac{1}{2}\pi$. Find what happens to the point $\begin{pmatrix} 3 & 2 & 3 \end{pmatrix}$.

40. 210] You want to rotate about the z axis counter clockwise through an angle of $\frac{1}{3}\pi$ and then add $\begin{pmatrix} 3 & 3 & -2 \end{pmatrix}$ to every point in $\mathbf{R}^3$. Find what happens to the point $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$.

41. 211] You want to rotate about the z axis counter clockwise through an angle of $\frac{5}{4}\pi$ and then add $\begin{pmatrix} -3 & 0 & -1 \end{pmatrix}$ to every point in $\mathbf{R}^3$. Find what happens to the point $\begin{pmatrix} 2 & 1 & 4 \end{pmatrix}$.

42. 212] Give the general solution to the following system of equations

$$\begin{aligned} w + x - z &= 0 \\ x - w + y &= 0 \\ x - w + y &= 0 \end{aligned}$$

by considering it as finding the solution to a system of the form

$$A\mathbf{x} = \mathbf{0}$$

Now, using what you just did, find the general solution to the system

$$\begin{aligned} w + x - z + 3 &= 0 \\ x - w + y + 2 &= 0 \\ x - w + y + 2 &= 0 \end{aligned}$$

43. 213] Give the general solution to the following system of equations

$$\begin{aligned} x - 3w + 3z &= 0 \\ x - 4w + y + 7z &= 0 \\ x - 4w + y + 7z &= 0 \end{aligned}$$

by considering it as finding the solution to a system of the form

$$A\mathbf{x} = \mathbf{0}$$

Now, using what you just did, find the general solution to the system

$$x - 3w + 3z + 2 = 0$$
$$x - 4w + y + 7z + 1 = 0$$
$$x - 4w + y + 7z + 1 = 0$$

44. 214] Give the general solution to the following system of equations

$$x - w - 2z = 0$$
$$x - 2w + y + 2z = 0$$
$$x - 2w + y + 2z = 0$$

by considering it as finding the solution to a system of the form

$$A\mathbf{x} = \mathbf{0}$$

Now, using what you just did, find the general solution to the system

$$x - w - 2z + 1 = 0$$
$$x - 2w + y + 2z - 2 = 0$$
$$x - 2w + y + 2z - 2 = 0$$

45. 215] Give the general solution to the following system of equations

$$3w + x - z = 0$$
$$8w + x + y + 3z = 0$$
$$8w + x + y + 3z = 0$$

by considering it as finding the solution to a system of the form

$$A\mathbf{x} = \mathbf{0}$$

Now, using what you just did, find the general solution to the system

$$3w + x - z - 2 = 0$$
$$8w + x + y + 3z - 4 = 0$$
$$8w + x + y + 3z - 4 = 0$$

46. 216] Give the general solution to the following system of equations

$$3w + x - 2z = 0$$
$$8w + x + y - 4z = 0$$
$$8w + x + y - 4z = 0$$

by considering it as finding the solution to a system of the form

$$A\mathbf{x} = \mathbf{0}$$

Now, using what you just did, find the general solution to the system

$$3w + x - 2z - 1 = 0$$
$$8w + x + y - 4z - 2 = 0$$
$$8w + x + y - 4z - 2 = 0$$

47. 217] Give the general solution to the following system of equations

$$\begin{aligned} 3w+x-z &= 0 \\ 5w+x+y &= 0 \\ 5w+x+y &= 0 \end{aligned}$$

by considering it as finding the solution to a system of the form

$$A\mathbf{x} = \mathbf{0}$$

Now, using what you just did, find the general solution to the system

$$\begin{aligned} 3w+x-z-1 &= 0 \\ 5w+x+y-3 &= 0 \\ 5w+x+y-3 &= 0 \end{aligned}$$

48. 218] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & 3 & -3 & 6 \\ 3 & 2 & 7 & -4 & 10 \\ 2 & 1 & 3 & -3 & 6 \\ 3 & 1 & 2 & -5 & 8 \end{pmatrix}$$

49. 219] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & 11 & -2 & 6 \\ 3 & 2 & 19 & -2 & 10 \\ 2 & 1 & 11 & -2 & 6 \\ 3 & 1 & 14 & -4 & 8 \end{pmatrix}$$

50. 220] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & 5 & 9 & 0 \\ 3 & 2 & 7 & 15 & 1 \\ 2 & 1 & 5 & 9 & 1 \\ 3 & 1 & 8 & 12 & 1 \end{pmatrix}$$

51. 221] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & 0 & 10 & 0 \\ 3 & 2 & 1 & 17 & 1 \\ 2 & 1 & 0 & 10 & 1 \\ 3 & 1 & -1 & 13 & 1 \end{pmatrix}$$

52. 222] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & -1 & -3 & 0 \\ 3 & 2 & 1 & -3 & 1 \\ 2 & 1 & -1 & -3 & 1 \\ 3 & 1 & -4 & -6 & 1 \end{pmatrix}$$

53. 223] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & 8 & -5 & 7 \\ 3 & 2 & 15 & -7 & 9 \\ 2 & 1 & 8 & -5 & 7 \\ 3 & 1 & 9 & -8 & 12 \end{pmatrix}$$

54. 224] Find $\ker(A) = N(A)$ where

$$A = \begin{pmatrix} 2 & 1 & 10 & -2 & 0 \\ 3 & 2 & 17 & -1 & -1 \\ 2 & 1 & 10 & -2 & 0 \\ 3 & 1 & 13 & -5 & 1 \end{pmatrix}$$

55. 225] Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution.

56. 226] Show that if A is an $m \times n$ matrix, then $\ker(A)$ is a subspace.

57. 227] Give an example of a 2×3 matrix which maps $\mathbf{R}^3$ onto $\mathbf{R}^2$. Is it possible that there could exist an example of such a matrix which is one to one?

Factorizations

1. 228] Find an LU factorization for the matrix

$$\begin{pmatrix} 5 & 1 & 2 & 2 \\ 5 & 3 & 7 & 3 \\ -5 & 3 & 9 & 1 \\ 10 & 10 & 29 & 12 \end{pmatrix}$$

and use to compute the determinant of the matrix.

2. 229] Find an LU factorization for the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ -3 & -1 & -1 & -2 & -1 \\ -3 & -1 & -4 & -1 & -4 \\ 2 & 10 & 9 & 8 & -4 \end{pmatrix}$$

3. 230] Find an LU factorization for the matrix

$$\begin{pmatrix} 5 & 1 & 2 & -3 \\ -5 & -2 & 5 & 4 \\ 20 & 7 & -14 & -14 \\ -5 & -5 & 19 & 18 \end{pmatrix}$$

4. 231] Find an LU factorization for the matrix

$$\begin{pmatrix} 5 & 1 & 2 & 1 & 1 \\ 5 & 4 & 7 & 2 & 3 \\ -15 & 0 & 0 & -1 & -4 \\ 15 & 9 & 21 & 7 & -7 \end{pmatrix}$$

5. 232] Find an LU factorization for the matrix

$$\begin{pmatrix} 4 & 1 & 2 & 3 \\ -8 & -4 & 2 & -5 \\ -4 & -7 & 14 & 1 \\ -8 & -12 & 14 & 4 \end{pmatrix}$$

and use to compute the determinant of the matrix.

6. 233] Find an LU factorization for the matrix

$$\begin{pmatrix} 3 & 1 & 2 & -3 \\ 6 & 2 & 9 & -5 \\ 18 & 6 & -1 & -20 \\ 0 & 0 & 30 & 15 \end{pmatrix}$$

7. 234] Find an LU factorization for the matrix

$$\begin{pmatrix} 4 & 1 & 2 & -3 \\ -4 & -4 & 2 & 4 \\ 4 & 10 & -11 & -5 \\ -12 & -21 & 14 & 20 \end{pmatrix}$$

8. 235] Find an LU factorization for the matrix

$$\begin{pmatrix} 4 & 1 & 2 & 3 \\ -4 & -1 & 5 & -2 \\ -12 & -3 & 14 & -5 \\ 0 & 0 & 42 & 11 \end{pmatrix}$$

9. 236] Is there only one LU factorization for a given matrix? **Hint**: Consider the equation

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

10. 237] Computer algebra systems are very good at finding PLU factorizations. Describe how to use a PLU factorization to compute the determinant of a matrix.

11. 238] Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$.

12. 239] Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.

13. 240] Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ and use it to solve the systems

a. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

14. 241] Find a QR factorization for the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

15. 242] Find a QR factorization for the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

16. 243] Find a QR factorization for the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

17. 244] Find a QR factorization for the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

18. 245] If you had a QR factorization, $A = QR$, describe how you could use it to solve the equation $A\mathbf{x} = \mathbf{b}$. This is not usually the way people solve this equation. However, the QR factorization is of great importance in certain other problems, especially in finding eigenvalues and eigenvectors.

19. 246] Starting with an independent set of n vectors in $\mathbf{R}^n \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, form the matrix and QR factorization

$$\begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{pmatrix} R$$

where $Q \equiv \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{pmatrix}$ is the orthogonal matrix. Thus $Q^T Q = I$ and the columns of Q form an orthonormal set of vectors. Show that for each $k \leq n$,
$\text{span}\begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_k \end{pmatrix} = \text{span}\begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix}$. Finding this orthonormal set of vectors is a very important problem and this shows how to do it with the QR factorization.

Linear Programming

1. 247] Maximize and minimize $z = x_1 - 2x_2 + x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 2$, and $x_1 + 2x_2 + x_3 \leq 7$ if possible. All variables are nonnegative.

2. 248] Maximize and minimize the following if possible. All variables are nonnegative.
   a. $z = x_1 - 2x_2$ subject to the constraints $x_1 + x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$
   b. $z = x_1 - 2x_2 - 3x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 8$, $x_1 + x_2 + 3x_3 \geq 1$, and $x_1 + x_2 + x_3 \leq 7$
   c. $z = 2x_1 + x_2$ subject to the constraints $x_1 - x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$.
   d. $z = x_1 + 2x_2$ subject to the constraints $x_1 - x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$.

3. 249] Consider contradictory constraints, $x_1 + x_2 \geq 12$ and $x_1 + 2x_2 \leq 5$. You know these two contradict but show they contradict using the simplex algorithm.

4. 250] Find a solution to the following inequalities for $x, y \geq 0$ if it is possible to do so. If it is not possible, prove it is not possible.

   a. $$\begin{aligned} 6x + 3y &\geq 4 \\ 8x + 4y &\leq 5 \end{aligned}$$

   b. $$\begin{aligned} 6x_1 + 4x_3 &\leq 11 \\ 5x_1 + 4x_2 + 4x_3 &\geq 8 \\ 6x_1 + 6x_2 + 5x_3 &\leq 11 \end{aligned}$$

   c. $$\begin{aligned} 6x_1 + 4x_3 &\leq 11 \\ 5x_1 + 4x_2 + 4x_3 &\geq 9 \\ 6x_1 + 6x_2 + 5x_3 &\leq 9 \end{aligned}$$

   d. $$\begin{aligned} x_1 - x_2 + x_3 &\leq 2 \\ x_1 + 2x_2 &\geq 4 \\ 3x_1 + 2x_3 &\leq 7 \end{aligned}$$

   e. $$\begin{aligned} 5x_1 - 2x_2 + 4x_3 &\leq 1 \\ 6x_1 - 3x_2 + 5x_3 &\geq 2 \\ 5x_1 - 2x_2 + 4x_3 &\leq 5 \end{aligned}$$

5. 251] Minimize $z = x_1 + x_2$ subject to $x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 20,\ x_1 + x_2 \leq 18.$ Change to a maximization problem and solve as follows: Let $y_i = M - x_i$. Formulate in terms of y_1, y_2.

Spectral Theory

1. 1] State the eigenvalue problem from a geometric perspective.

2. 2] If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA.

3. 3] If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

4. 4] State the eigenvalue problem from an algebraic perspective.

5. 5] If A is the matrix of a linear transformation which rotates all vectors in $\mathbf{R}^2$ through 60°, explain why A cannot have any real eigenvalues. Is there an angle such that rotation through this angle would have a real eigenvalue. What eigenvalues would be obtainable in this way?

6. 6] Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose $\mathbf{x}$ is an eigenvector of B. Show that then $A\mathbf{x}$ must also be an eigenvector for B.

7. 7] Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

8. 8] Show that if $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$, then whenever a, b are scalars,

$$A(a\mathbf{x} + b\mathbf{y}) = \lambda(a\mathbf{x} + b\mathbf{y}).$$

Does this imply that $a\mathbf{x} + b\mathbf{y}$ is an eigenvector? Explain.

9. 9] Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

10. 10] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -17 & 17 & 16 \\ 28 & -37 & -32 \\ -50 & 59 & 53 \end{pmatrix}$$

One eigenvalue is 5. Determine whether the matrix is defective.

11. 11] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ -3 & -91 & 2 \end{pmatrix}$$

One eigenvalue is 2. Determine whether the matrix is defective.

12. 12] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -23 & -66 & -88 \\ -4 & -13 & -16 \\ 8 & 24 & 31 \end{pmatrix}$$

One eigenvalue is –3. Determine whether the matrix is defective.

13. 13] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -12 & 45 & 15 \\ -2 & 9 & 2 \\ -8 & 24 & 11 \end{pmatrix}$$

One eigenvalue is 2. Determine whether the matrix is defective.

14. 14] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -11 & -26 & -8 \\ 6 & 14 & 4 \\ -4 & -17 & -10 \end{pmatrix}$$

One eigenvalue is 1. Determine whether the matrix is defective.

15. 15] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 41 & 12 & 78 \\ 8 & -3 & 12 \\ -20 & -4 & -37 \end{pmatrix}$$

One eigenvalue is –3. Determine whether the matrix is defective.

16. 16] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 53 & 20 & 160 \\ 8 & -1 & 22 \\ -18 & -6 & -54 \end{pmatrix}$$

One eigenvalue is –3. Determine whether the matrix is defective.

17. 17] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 50 & 114 & -60 \\ -10 & -23 & 12 \\ 22 & 51 & -26 \end{pmatrix}$$

One eigenvalue is –2. Determine whether the matrix is defective.

18. 18] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -2 & -24 & -12 \\ 0 & 6 & 4 \\ 2 & -19 & -14 \end{pmatrix}$$

One eigenvalue is -2. Determine whether the matrix is defective.

19. 19] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -15 & -56 & 0 \\ 4 & 15 & 0 \\ -12 & -48 & -1 \end{pmatrix}$$

One eigenvalue is 1. Determine whether the matrix is defective.

20. 20] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -7 & -361 & 8 \\ 0 & 1 & 0 \\ -4 & -181 & 5 \end{pmatrix}$$

One eigenvalue is -3. Determine whether the matrix is defective.

21. 21] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -9 & -6 & -12 \\ -4 & 1 & -4 \\ 12 & 6 & 15 \end{pmatrix}$$

One eigenvalue is 3. Determine whether the matrix is defective.

22. 22] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -13 & -716 & -48 \\ 0 & -1 & 0 \\ 4 & 239 & 15 \end{pmatrix}$$

One eigenvalue is 3. Determine whether the matrix is defective.

23. 23] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -8 & -20 & 5 \\ 5 & 27 & -5 \\ 5 & 85 & -13 \end{pmatrix}$$

One eigenvalue is 2. Determine whether the matrix is defective. **Hint**: This one has some complex eigenvalues.

24. 24] Let A be a real (has all real entries) 3×3 matrix which has a complex eigenvalue of the form $a + ib$ where $b \neq 0$. Could A be defective? Explain. Either give a proof or an example.

25. 25] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 7 & -8 & 3 \\ -14 & 21 & -5 \\ -66 & 92 & -24 \end{pmatrix}$$

One eigenvalue is 2. Determine whether the matrix is defective. **Hint**: This one has some complex eigenvalues.

26. 26] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 7 & -15 & 6 \\ -3 & 1 & 0 \\ -18 & 24 & -8 \end{pmatrix}$$

One eigenvalue is −2. Determine whether the matrix is defective. **Hint**: This one has some complex eigenvalues.

27. 27] Let T be the linear transformation which reflects all vectors in $\mathbf{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

28. 28] Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

29. 29] Let A be the 2×2 matrix of the linear transformation which rotates all vectors in $\mathbf{R}^2$ through an angle of θ. For which values of θ does A have a real eigenvalue?

30. 30] Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 0 & -10 & 3 \\ -12 & 32 & -7 \\ -61 & 175 & -39 \end{pmatrix}$$

One eigenvalue is −1. Determine whether the matrix is defective. **Hint**: This one has some complex eigenvalues.

31. 31] Let T be the linear transformation which rotates all vectors in $\mathbf{R}^2$ counterclockwise through an angle of $\pi/2$. Find a matrix of T and then find eigenvalues and eigenvectors.

32. 32] Suppose A is a 3×3 matrix and the following information is available.

$$A\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} -3 \\ -4 \\ -3 \end{pmatrix} = -3\begin{pmatrix} -3 \\ -4 \\ -3 \end{pmatrix}$$

Find $A\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

33. 33] Suppose A is a 3×3 matrix and the following information is available.

$$A\begin{pmatrix}-2\\-3\\-3\end{pmatrix}=1\begin{pmatrix}-2\\-3\\-3\end{pmatrix}, A\begin{pmatrix}1\\1\\1\end{pmatrix}=-3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A\begin{pmatrix}-5\\-6\\-5\end{pmatrix}=-1\begin{pmatrix}-5\\-6\\-5\end{pmatrix}$$

Find $A\begin{pmatrix}1\\-3\\4\end{pmatrix}$

34. 34] Suppose A is a 3×3 matrix and the following information is available.

$$A\begin{pmatrix}-1\\-2\\-2\end{pmatrix}=2\begin{pmatrix}-1\\-2\\-2\end{pmatrix}, A\begin{pmatrix}1\\1\\1\end{pmatrix}=0\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A\begin{pmatrix}-2\\-5\\-4\end{pmatrix}=2\begin{pmatrix}-2\\-5\\-4\end{pmatrix}$$

Find $A\begin{pmatrix}3\\-4\\3\end{pmatrix}$

35. 35] Here is a symmetric matrix.

$$\begin{pmatrix}\frac{13}{6} & -\frac{1}{6} & -\frac{1}{3}\\ -\frac{1}{6} & \frac{13}{6} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{8}{3}\end{pmatrix}$$

Find its largest eigenvalue and an eigenvector associated with this eigenvalue. This eigenvector gives the direction of the principal stretch.

36. 36] Here is a symmetric matrix.

$$\begin{pmatrix}\frac{24}{11} & -\frac{2}{11} & -\frac{6}{11}\\ -\frac{2}{11} & \frac{24}{11} & \frac{6}{11}\\ -\frac{6}{11} & \frac{6}{11} & \frac{40}{11}\end{pmatrix}$$

Find its largest eigenvalue and an eigenvector associated with this eigenvalue. This eigenvector gives the direction of the principal stretch.

37. 37] Here is a symmetric matrix.

$$\begin{pmatrix} \frac{13}{6} & \frac{5}{6} & \frac{2}{3} \\ \frac{5}{6} & \frac{13}{6} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{11}{3} \end{pmatrix}$$

Find its largest eigenvalue and an eigenvector associated with this eigenvalue. This eigenvector gives the direction of the principal stretch.

38. 38] Here is a symmetric matrix.

$$\begin{pmatrix} \frac{25}{6} & -\frac{7}{6} & \frac{5}{3} \\ -\frac{7}{6} & \frac{25}{6} & -\frac{5}{3} \\ \frac{5}{3} & -\frac{5}{3} & \frac{11}{3} \end{pmatrix}$$

Find its largest eigenvalue and an eigenvector associated with this eigenvalue. This eigenvector gives the direction of the principal stretch.

39. 39] Here is a symmetric matrix.

$$\begin{pmatrix} \frac{25}{6} & -\frac{7}{6} & \frac{5}{3} \\ -\frac{7}{6} & \frac{25}{6} & -\frac{5}{3} \\ \frac{5}{3} & -\frac{5}{3} & \frac{11}{3} \end{pmatrix}$$

Find its largest eigenvalue and an eigenvector associated with this eigenvalue. This eigenvector gives the direction of the principal stretch.

40. 40] Here is a Markov (migration matrix) for three locations

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{5}{8} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

The total number of individuals in the migration process is

$$320$$

After a long time, how many are in each location?

41. 41] Here is a Markov (migration matrix) for three locations

$$\begin{pmatrix} \frac{1}{10} & \frac{1}{2} & \frac{2}{9} \\ \frac{2}{5} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{9} \end{pmatrix}$$

The total number of individuals in the migration process is

$$949$$

After a long time, how many are in each location?

42. 42] Here is a Markov (migration matrix) for three locations

$$\begin{pmatrix} \frac{7}{10} & \frac{1}{14} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{10} & \frac{3}{7} & \frac{1}{3} \end{pmatrix}$$

The total number of individuals in the migration process is

$$736$$

After a long time, how many are in each location?

43. 43] The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

$$\begin{pmatrix} & \text{rains} & \text{sunny} & \text{p.c.} \\ \text{rains} & \frac{1}{5} & \frac{1}{5} & \frac{6}{11} \\ \text{sunny} & \frac{1}{5} & \frac{2}{5} & \frac{1}{11} \\ \text{p.c.} & \frac{3}{5} & \frac{2}{5} & \frac{4}{11} \end{pmatrix}$$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

44. 44] The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{1}{3}$ and so forth.

$$\begin{pmatrix} & \text{rains} & \text{sunny} & \text{p.c.} \\ \text{rains} & \frac{1}{10} & \frac{1}{3} & \frac{3}{10} \\ \text{sunny} & \frac{1}{5} & \frac{1}{3} & \frac{2}{5} \\ \text{p.c.} & \frac{7}{10} & \frac{1}{3} & \frac{3}{10} \end{pmatrix}$$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

45. 45] You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE,NE,NW, and SW respectively. Suppose that the following table is observed to take place.

	SE	NE	NW	SW
SE	$\frac{2}{9}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{3}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{9}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{2}{9}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailor starting at NE ends in NW is $1/10$, the probability that a trailor starting at SW ends in NW is $1/5$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailors is 463?

46. 46] The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{1}{5}$ and so forth.

$$\begin{pmatrix} & \text{rains} & \text{sunny} & \text{p.c.} \\ \text{rains} & \frac{1}{10} & \frac{2}{5} & \frac{3}{8} \\ \text{sunny} & \frac{1}{5} & \frac{1}{5} & \frac{3}{8} \\ \text{p.c.} & \frac{7}{10} & \frac{2}{5} & \frac{1}{4} \end{pmatrix}$$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

47. 47] You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE,NE,NW, and SW respectively. Suppose that the following table is observed to take place.

	SE	NE	NW	SW
SE	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailor starting at NE ends in NW is $1/10,$ the probability that a trailor starting at SW ends in NW is $1/5,$ and so forth. Approximately how many will you have in each location after a long time if the total number of trailors is 1030.

48. 48] In the city of Nabal, there are three political persuasions, republicans (R), democrats (D), and neither one (N). The following table shows the transition probabilities between the political parties, the top row being the initial political party and the side row being the political affiliation the following year.

$$\begin{array}{cccc} & \text{R} & \text{D} & \text{N} \\ \text{R} & \frac{1}{10} & \frac{1}{5} & \frac{3}{13} \\ \text{D} & \frac{1}{10} & \frac{1}{5} & \frac{7}{13} \\ \text{N} & \frac{4}{5} & \frac{3}{5} & \frac{3}{13} \end{array}$$

Find the probabilities that a person will be identified with the various political persuasions. Which party will end up being most important?

49. 49] The University of Poohbah offers three degree programs, scouting education (SE), dance appreciation (DA), and engineering (E). It has been determined that the probabilities of transferring from one program to another are as in the following table.

	SE	DA	E
SE	.8	.1	.3
DA	.1	.7	.5
E	.1	.2	.2

where the number indicates the probability of transferring from the top program to the program on the left. Thus the probability of going from DA to E is .2. Find the probability that a student is enrolled in the various programs.

50. 50] Here is a Markov (migration matrix) for three locations

$$\begin{pmatrix} \frac{3}{5} & \frac{1}{8} & \frac{1}{12} \\ \frac{1}{10} & \frac{3}{4} & \frac{7}{12} \\ \frac{3}{10} & \frac{1}{8} & \frac{1}{3} \end{pmatrix}$$

The total number of individuals in the migration process is

$$406$$

After a long time, how many are in each location?

51. 51] You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE,NE,NW, and SW respectively. Suppose that the following table is observed to take place.

	SE	NE	NW	SW
SE	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{10}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailor starting at *NE* ends in *NW* is $1/10$, the probability that a trailor starting at *SW* ends in *NW* is $1/5$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailors is 350?

52. 52] Here is a Markov (migration matrix) for three locations

$$\begin{pmatrix} \frac{2}{5} & \frac{1}{8} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{14} \\ \frac{1}{10} & \frac{3}{8} & \frac{2}{7} \end{pmatrix}$$

The total number of individuals in the migration process is

$$1000$$

After a long time, how many are in each location?

53. 53] Let A be the $n \times n$, $n > 1$, matrix of the linear transformation which comes from the projection $\mathbf{v} \to \text{proj}_{\mathbf{w}}(\mathbf{v})$. Show that A cannot be invertible. Also show that A has an eigenvalue equal to 1 and that for λ an eigenvalue, $|\lambda| \leq 1$.

54. 54] Consider the dynamical system in which A is an $n \times n$ matrix,

$$\mathbf{x}(n+1) = A\mathbf{x}(n),\ \mathbf{x}(0) = \mathbf{x}_0$$

Suppose the eigen pairs are

$$(\mathbf{v}_i, \lambda_i)$$

and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for $\mathbf{R}^n$. If

$$\mathbf{x}_0 = \sum_{i=1}^{n} c_i \mathbf{v}_i,$$

show that the solution to the dynamical system is of the form

$$\mathbf{x}(n) = \sum_{i=1}^{n} c_i \lambda_i^n \mathbf{v}_i$$

55. 55] Consider the recurrance relation $x_{n+2} = x_{n+1} + x_n$ where x_1 and x_0 are given positive numbers. Show that if $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, then it must equal $\frac{1+\sqrt{5}}{2}$.

56. 56] Consider the dynamical system $\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} .8 & .8 \\ -.8 & .8 \end{pmatrix} \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$. Show the eigenvalues and eigenvectors are $0.8 + 0.8i \leftrightarrow \begin{pmatrix} -i \\ 1 \end{pmatrix}, 0.8 - 0.8i \leftrightarrow \begin{pmatrix} i \\ 1 \end{pmatrix}$. Find a formula for the solutiont to the dynamical system for given initial condition $(x_0, y_0)^T$. Show that the magnitude of $(x(n), y(n))^T$ must diverge provided the initial condition is not zero. Next graph the vector field for

$$\begin{pmatrix} .8 & .8 \\ -.8 & .8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that this vector field seems to indicate a conclusion different than what you just obtained. Therefore, in this context of discreet dynamical systems the consideration of such a picture is not all that reliable.

57. 57] Let $\mathbf{v}$ be a unit vector and let $A = I - 2\mathbf{v}\mathbf{v}^T$. Show that A has an eigenvalue equal to -1.

58. 58] Let M be an $n \times n$ matrix and suppose $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are n eigenvectors which form a linearly independent set. Form the matrix S by making the columns these vectors. Show that S^{-1} exists and that $S^{-1}MS$ is a **diagonal matrix** (one having zeros everywhere except on the main diagonal) having the eigenvalues of M on the main diagonal. When this can be done the matrix is **diagonalizable**.

59. 59] Show that a matrix, M is diagonalizable if and only if it has a basis of eigenvectors. **Hint**: The first part is done earlier. It only remains to show that if the matrix can be diagonalized by some matrix, S giving $D = S^{-1}MS$ for D a diagonal matrix, then it has a basis of eigenvectors. Try using the columns of the matrix S.

60. 60] Suppose A is an $n \times n$ matrix which is **diagonally dominant**. This means

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

Show that A^{-1} must exist.

61. 61] Let M be an $n \times n$ matrix. Then define the adjoint of M, denoted by M^* to be the transpose of the conjugate of M. For example,

$$\begin{pmatrix} 2 & i \\ 1+i & 3 \end{pmatrix}^* = \begin{pmatrix} 2 & 1-i \\ -i & 3 \end{pmatrix}.$$

A matrix, M, is self adjoint if $M^* = M$. Show the eigenvalues of a self adjoint matrix are all real. If the self adjoint matrix has all real entries, it is called symmetric.

62. 62] Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a + ib$ is a complex eigenvalue having the eigenvector, $\mathbf{x} + i\mathbf{y}$. Here $\mathbf{x}$ and $\mathbf{y}$ are real vectors. Show that then $a - ib$ is also an eigenvalue with the eigenvector, $\mathbf{x} - i\mathbf{y}$. **Hint**: You should remember that the conjugate of a product of complex numbers equals the product of the conjugates. Here $a + ib$ is a complex number whose conjugate equals $a - ib$.

63. 63] Recall an $n \times n$ matrix is said to be symmetric if it has all real entries and if $A = A^T$. Show the eigenvectors and eigenvalues of a real symmetric matrix are real.

64. 64] Recall an $n \times n$ matrix is said to be skew symmetric if it has all real entries and if $A = -A^T$. Show that any nonzero eigenvalues must be of the form ib where $i^2 = -1$. In words, the eigenvalues are either 0 or pure imaginary.

65. 65] A discreet dynamical system is of the form

$$\mathbf{x}(k+1) = A\mathbf{x}(k), \ \mathbf{x}(0) = \mathbf{x}_0$$

where A is an $n \times n$ matrix and $\mathbf{x}(k)$ is a vector in $\mathbf{R}^n$. Show first that

$$\mathbf{x}(k) = A^k \mathbf{x}_0$$

for all $k \geq 1$. If A is nondefective so that it has a basis of eigenvectors, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ where

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j$$

you can write the initial condition $\mathbf{x}_0$ in a unique way as a linear combination of these eigenvectors. Thus

$$\mathbf{x}_0 = \sum_{j=1}^{n} a_j \mathbf{v}_j$$

Now explain why

$$\mathbf{x}(k) = \sum_{j=1}^{n} a_j A^k \mathbf{v}_j = \sum_{j=1}^{n} a_j \lambda_j^k \mathbf{v}_j$$

which gives a formula for $\mathbf{x}(k)$, the solution of the dynamical system.

66. 66] Suppose A is an $n \times n$ matrix and let $\mathbf{v}$ be an eigenvector such that $A\mathbf{v} = \lambda\mathbf{v}$. Also suppose the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Explain why

$$(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I)\mathbf{v} = \mathbf{0}$$

If A is nondefective, give a very easy proof of the Cayley Hamilton theorem based on this. Recall this theorem says A satisfies its characteristic equation,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

67. 67] Suppose an $n \times n$ nondefective matrix A has only 1 and -1 as eigenvalues. Find A^{12}.

68. 68] Suppose the characteristic polynomial of an $n \times n$ matrix A is $1 - \lambda^n$. Find A^{mn} where m is an integer. **Hint**: Note first that A is nondefective. Why?

69. 69] Sometimes sequences come in terms of a recursion formula. An example is the Fibonacci sequence.

$$x_0 = 1 = x_1,\ x_{n+1} = x_n + x_{n-1}$$

Show this can be considered as a discreet dynamical system as follows.

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix},\ \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now find a formula for x_n.

70. 70] Let A be an $n \times n$ matrix having characteristic polynomial

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Show that $a_0 = (-1)^n \det(A)$.

71. 71] Here is a matrix

$$A = \begin{pmatrix} \frac{53}{8} & \frac{25}{8} \\ -\frac{45}{4} & -\frac{21}{4} \end{pmatrix}$$

Find $\lim_{n\to\infty} A^n$.

72. 72] Here is a matrix

$$A = \begin{pmatrix} \frac{19}{9} & \frac{2}{3} \\ -\frac{20}{9} & -\frac{1}{3} \end{pmatrix}$$

Find $\lim_{n\to\infty} A^n$.

73. 73] Find the solution to the initial value problem

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Recall that to do this, you form the matrix exponential e^{At} and then the solution is $e^{At}\mathbf{c}$ where $\mathbf{c}$ is the initial vector.

74. 74] Find the solution to the initial value problem

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -13 & -5 \\ 30 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Recall that to do this, you form the matrix exponential e^{At} and then the solution is $e^{At}\mathbf{c}$ where $\mathbf{c}$ is the initial vector.

75. 75] Find the solution to the initial value problem

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 7 & -2 \\ 12 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Recall that to do this, you form the matrix exponential e^{At} and then the solution is $e^{At}\mathbf{c}$ where $\mathbf{c}$ is the initial vector.

Matrices And Inner Product

1.

2. 76] Here are some matrices. Label according to whether they are symmetric, skew symmetric, or orthogonal. If the matrix is orthogonal, determine whether it is proper or improper.

a. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{pmatrix}$

c. $\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}$

3. 77] Here are some matrices. Label according to whether they are symmetric, skew symmetric, or orthogonal. If the matrix is orthogonal, determine whether it is proper or improper.

a. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{pmatrix}$

c. $\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}$

4. 78] Show that every real matrix may be written as the sum of a skew symmetric and a symmetric matrix in a unique way. **Hint**: If A is an $n \times n$ matrix, show that $B \equiv \frac{1}{2}(A - A^T)$ is skew symmetric.

$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$. There is only one way to do it. If $A = S + W$ where $S = S^T, W = -W^T,$ then you also have $A^T = S - W$ and so $2S = A + A^T$ while $2W = A - A^T$.

5. 79] Let $\mathbf{x}$ be a vector in $\mathbf{R}^n$ and consider the matrix, $I - \frac{2\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}$. Show this matrix is both symmetric and orthogonal.

6. 80] For U an orthogonal matrix, ($U^T U = I$) explain why $||U\mathbf{x}|| = ||\mathbf{x}||$ for any vector, $\mathbf{x}$. Next explain why if U is an $n \times n$ matrix with the property that $||U\mathbf{x}|| = ||\mathbf{x}||$ for all vectors $\mathbf{x}$, then U must be orthogonal. Thus the orthogonal matrices are exactly those which preserve distance.

7. 81] A quadratic form in three variables is an expression of the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. Show that every such quadratic form may be written as

$$\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where A is a symmetric matrix.

8. 82] Given a quadratic form in three variables, $x, y,$ and z, show there exists an orthogonal matrix U and variables x', y', z' such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = U \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

with the property that in terms of the new variables, the quadratic form is

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + \lambda_3 (z')^2$$

where the numbers, $\lambda_1, \lambda_2,$ and λ_3 are the eigenvalues of the matrix, A in the above problem.

9. 83] If A is a symmetric invertible matrix, is it always the case that A^{-1} must be symmetric also? How about A^k for k a positive integer? Explain.

10. 84] If A, B are symmetric matrices, does it follow that AB is also symmetric?

11. 85] Suppose A, B are symmetric and $AB = BA$. Does it follow that AB is symmetric?

12. 86] Here are some matrices. What can you say about the eigenvalues of these matrices just by looking at them?

a. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{pmatrix}$

c. $\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

13. 87] Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} c & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{pmatrix}$. Here b, c are real numbers.

14. 88] Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} c & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$. Here a, b, c are real numbers.

15. 89] Determine which of the following sets of vectors are orthonormal sets. Justify your answer.

a. $\{(1,1),(1,-1)\}$

b. $\left\{\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right),(1,0)\right\}$

c. $\{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}),(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}),(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})\}$

16. 90] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} -\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6} & -\frac{17}{6} \\ -\frac{1}{3} & -\frac{17}{6} & \frac{7}{6} \end{pmatrix}.$$

Hint: Two eigenvalues are $-2, -1$.

17. 91] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} \frac{61}{22} & -\frac{35}{22} & \frac{23}{11} \\ -\frac{35}{22} & -\frac{3}{22} & \frac{7}{11} \\ \frac{23}{11} & \frac{7}{11} & -\frac{18}{11} \end{pmatrix}.$$

Hint: Two eigenvalues are $-3, 0$.

18. 92] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & \frac{1}{2} & -\frac{3}{2} \\ -2 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

Hint: Two eigenvalues are $-3, 3$.

19. 93] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} -\frac{3}{2} & 0 & \frac{7}{2} \\ 0 & -5 & 0 \\ \frac{7}{2} & 0 & -\frac{3}{2} \end{pmatrix}.$$

Hint: Two eigenvalues are $2, -5$.

20. 94] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} \frac{20}{11} & -\frac{6}{11} & \frac{9}{11} \\ -\frac{6}{11} & \frac{15}{11} & -\frac{6}{11} \\ \frac{9}{11} & -\frac{6}{11} & \frac{20}{11} \end{pmatrix}.$$

Hint: One eigenvalue is 1.

21. 95] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hint: Two eigenvalues are $1, 1$.

22. 96] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Hint: One eigenvalue is 3.

23. 97] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A=\begin{pmatrix}3 & 1 & 1\\ 1 & 3 & 1\\ 1 & 1 & 3\end{pmatrix}.$$

Hint: One eigenvalue is 2.

24. 98] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A=\begin{pmatrix}\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{3}{2}\end{pmatrix}.$$

Hint: Two eigenvalues are $2,1$.

25. 99] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A=\begin{pmatrix}\frac{19}{6} & \frac{7}{6} & -\frac{2}{3}\\ \frac{7}{6} & \frac{19}{6} & -\frac{2}{3}\\ -\frac{2}{3} & -\frac{2}{3} & \frac{11}{3}\end{pmatrix}.$$

Hint: Two eigenvalues are $3,2$.

26. 100] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A=\begin{pmatrix}\frac{8}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{5}{3} & \frac{2}{3}\\ -\frac{1}{3} & \frac{2}{3} & \frac{8}{3}\end{pmatrix}.$$

Hint: One eigenvalue is 1.

27. 101] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A=\begin{pmatrix}\frac{13}{6} & \frac{2}{3} & -\frac{7}{6}\\ \frac{2}{3} & \frac{8}{3} & -\frac{2}{3}\\ -\frac{7}{6} & -\frac{2}{3} & \frac{13}{6}\end{pmatrix}.$$

Hint: Two eigenvalues are $1,2$.

28. 102] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A=\begin{pmatrix}\frac{137}{54} & \frac{5}{27} & -\frac{25}{54}\\ \frac{5}{27} & \frac{79}{27} & \frac{5}{27}\\ -\frac{25}{54} & \frac{5}{27} & \frac{137}{54}\end{pmatrix}.$$

Hint: One eigenvalue is 3.

29. 103] Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} \frac{5}{2} & -1 & \frac{1}{2} \\ -1 & 4 & -1 \\ \frac{1}{2} & -1 & \frac{5}{2} \end{pmatrix}.$$

Hint: One eigenvalue is 5.

30. 104] Show that if A is a real symmetric matrix and λ and μ are two different eigenvalues, then if $\mathbf{x}$ is an eigenvector for λ and $\mathbf{y}$ is an eigenvector for μ, then $\mathbf{x} \cdot \mathbf{y} = 0$. Also all eigenvalues are real.

31. 105] Suppose U is an orthogonal $n \times n$ matrix ($UU^T = I$). Explain why $\text{rank}(U) = n$.

32. 106] Show that the eigenvalues and eigenvectors of a real matrix occur in conjugate pairs.

33. 107] If a real matrix, A has all real eigenvalues, does it follow that A must be symmetric. If so, explain why and if not, give an example to the contrary.

34. 108] Suppose A is a 3×3 symmetric matrix and you have found two eigenvectors which form an orthonormal set. Explain why their cross product is also an eigenvector.

35. 109] Show that if $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is an orthonormal set of vectors in $\mathbf{F}^n$, then it is a basis. **Hint**: It was shown earlier that this is a linearly independent set. If you wish, replace $\mathbf{F}^n$ with $\mathbf{R}^n$. Do this version if you do not know the dot product for vectors in $\mathbf{C}^n$.

36. 110] Fill in the missing entries to make the matrix orthogonal.

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \_ & \_ \\ \_ & \frac{\sqrt{6}}{3} & \_ \end{pmatrix}.$$

37. 111] Fill in the missing entries to make the matrix orthogonal.

$$\begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \_ & \_ \\ \_ & 0 & \_ \end{pmatrix}.$$

38. 112] Fill in the missing entries to make the matrix orthogonal.

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & - \\ \frac{2}{3} & 0 & - \\ - & - & \frac{4}{15}\sqrt{5} \end{pmatrix}.$$

39. 113] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

40. 114] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}.$$

Hint: Two eigenvalues are 12 and 18.

41. 115] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

42. 116] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 17 & -7 & -4 \\ -7 & 17 & -4 \\ -4 & -4 & 14 \end{pmatrix}.$$

Hint: Two eigenvalues are 18 and 24.

43. 117] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

44. 118] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}.$$

45. 119] Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

46. 120] Explain why a matrix, A is symmetric if and only if there exists an orthogonal matrix, U such that $A = U^T DU$ for D a diagonal matrix.

47. 121] The proof of a theorem concluded with the following observation. If $-ta + t^2 b \geq 0$ for all $t \in \mathbf{R}$ and $b \geq 0$, then $a = 0$. Why is this so?

48. 122] Using Schur's theorem, show that whenever A is an $n \times n$ matrix, $\det(A)$ equals the product of the eigenvalues of A.

49. 123] In the proof of a theorem, the following argument was used. If $\mathbf{x} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in \mathbf{R}^n$, then $\mathbf{x} = \mathbf{0}$. Why is this so?

50. 124] Suppose A is a 3×2 matrix. Is it possible that A^T is one to one? What does this say about A being onto? Prove your answer.

51. 125] Find the least squares solution to the system

$$\begin{pmatrix} 6y - 3x = 3 \\ x + 3y = 2 \\ x + 4y = 1 \end{pmatrix}$$

52. 126] Find the least squares solution to the system

$$\begin{pmatrix} 3y - 3x = 3 \\ 2x + 3y = 2 \\ x + 6y = 1 \end{pmatrix}$$

53. 127] Find the least squares solution to the system

$$\begin{pmatrix} 2x + y = 3 \\ x + 3y = 1 \\ x + 4y = 1 \end{pmatrix}$$

54. 128] You are doing experiments and have obtained the ordered pairs,

$$(0,1),(1,2),(2,3.5),(3,4)$$

Find m and b such that $y = mx + b$ approximates these four points as well as possible. Now do the same thing for $y = ax^2 + bx + c$, finding a, b, and c to give the best approximation.

55. 129] You are doing experiments and have obtained the ordered pairs,

$$(0,2.0),(1,1.0),(2,2.0),(3,2.0)$$

Find m and b such that $y = mx + b$ approximates these four points as well as possible. Now do the same thing for $y = ax^2 + bx + c$, finding a, b, and c to give the best approximation.

56. 130] You are doing experiments and have obtained the ordered pairs,

$$(0,7.0),(1,-3.0),(2,1.0),(3,1.0)$$

Find m and b such that $y = mx + b$ approximates these four points as well as possible. Now do the same thing for $y = ax^2 + bx + c$, finding a, b, and c to give the best approximation.

57. 131] Suppose you have several ordered triples, (x_i, y_i, z_i). Describe how to find a polynomial,

$$z = a + bx + cy + dxy + ex^2 + fy^2$$

for example giving the best fit to the given ordered triples. Is there any reason you have to use a polynomial? Would similar approaches work for other combinations of functions just as well?

58. 132] The two level surfaces, $2x + 3y - z + w = 0$ and $3x - y + z + 2w = 0$ intersect in a subspace of $\mathbf{R}^4$, find a basis for this subspace. Next find an orthonormal basis for this subspace.

59. 133] Find an orthonormal basis for the span of the vectors $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

60. 134] The set, $V \equiv \{(x,y,z) : 2x + 3y - z = 0\}$ is a subspace of $\mathbf{R}^3$. Find an orthonormal basis for this subspace.

61. 135] Find an orthonormal basis for the span of the vectors $(3,-4,0),(7,-1,0),(1,7,1)$

62. 136] Find an orthonormal basis for the span of the vectors $(3,0,-4),(5,0,10),(-7,1,1)$.

63. 137] Using the Gram Schmidt process, find an orthonormal basis for the span of the vectors, $(1,2,1,0),(2,-1,3,1)$, and $(1,0,0,1)$.

64. 138] Find an orthonormal basis for the span of the vectors $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

65. 139] Find an orthonormal basis for the span of the vectors $(3,0,-4),(11,0,2),(1,1,7)$

66. 140] Let A,B be a $m \times n$ matrices. Define an inner product on the set of $m \times n$ matrices by

$$(A,B)_F \equiv \text{trace}(AB^*).$$

Show this is an inner product satisfying all the inner product axioms. Recall for M an $n \times n$ matrix, $\text{trace}(M) \equiv \sum_{i=1}^n M_{ii}$. The resulting norm, $\|\cdot\|_F$ is called the Frobenius norm and it can be used to measure the distance between two matrices.

67. 141] Let A be an $m \times n$ matrix. Show

$$\|A\|_F^2 \equiv (A,A)_F = \sum_j \sigma_j^2$$

where the σ_j are the singular values of A.

68. 142] Here is a matrix A

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{3}{10}\sqrt{2}\sqrt{10} & -\frac{1}{10}\sqrt{2}\sqrt{10} \\ 0 & \frac{3}{10}\sqrt{2}\sqrt{10} & -\frac{1}{10}\sqrt{2}\sqrt{10} \end{pmatrix}$$

Find a singular value decomposition.

69. 143] Here is a matrix A

$$\begin{pmatrix} \frac{1}{3}\sqrt{3}+\frac{5}{2} & \frac{15}{22}\sqrt{11}-\frac{1}{11}\sqrt{3}\sqrt{11}-\frac{1}{33}\sqrt{2}\sqrt{3}\sqrt{11} & \frac{1}{33}\sqrt{2}\sqrt{3}\sqrt{11}-\frac{1}{11}\sqrt{3}\sqrt{11}-\frac{5}{22}\sqrt{2}\sqrt{11} \\ \frac{5}{2}-\frac{1}{3}\sqrt{3} & \frac{1}{11}\sqrt{3}\sqrt{11}+\frac{15}{22}\sqrt{11}+\frac{1}{33}\sqrt{2}\sqrt{3}\sqrt{11} & \frac{1}{11}\sqrt{3}\sqrt{11}-\frac{5}{22}\sqrt{2}\sqrt{11}-\frac{1}{33}\sqrt{2}\sqrt{3}\sqrt{11} \\ \frac{2}{3}\sqrt{3} & \frac{1}{33}\sqrt{2}\sqrt{3}\sqrt{11}-\frac{2}{11}\sqrt{3}\sqrt{11} & \frac{1}{11}\sqrt{3}\sqrt{11}+\frac{2}{33}\sqrt{2}\sqrt{3}\sqrt{11} \end{pmatrix}$$

Find a singular value decomposition. You should probably use some sort of computer algebra system to do this one.

70. 144] Here is a matrix.

$$\begin{pmatrix} \frac{7}{26}\sqrt{10}\sqrt{13} & \frac{3}{26}\sqrt{2}\sqrt{5}\sqrt{13} \\ -\frac{3}{65}\sqrt{10}\sqrt{13} & \frac{21}{65}\sqrt{2}\sqrt{5}\sqrt{13} \end{pmatrix}$$

Find a singular value decomposition.

71. 145] The trace of an $n \times n$ matrix M is defined as $\sum_i M_{ii}$. In other words it is the sum of the entries on the main diagonal. If A, B are $n \times n$ matrices, show $\text{trace}(AB) = \text{trace}(BA)$. Now explain why if $A = S^{-1}BS$ it follows $\text{trace}(A) = \text{trace}(B)$. **Hint**: For the first part, write these in terms of components of the matrices and it just falls out.

72. 146] Using the above problem and Schur's theorem, show that the trace of an $n \times n$ matrix equals the sum of the eigenvalues.

73. 147] If A is a general $n \times n$ matrix having possibly repeated eigenvalues, show there is a sequence $\{A_k\}$ of $n \times n$ matrices having distinct eigenvalues which has the property that the ij^{th} entry of A_k converges to the ij^{th} entry of A for all ij. **Hint**: Use Schur's theorem.

74. 148] Prove the Cayley Hamilton theorem as follows. First suppose A has a basis of eigenvectors $\{\mathbf{v}_k\}_{k=1}^n, A\mathbf{v}_k = \lambda_k \mathbf{v}_k$. Let $p(\lambda)$ be the characteristic polynomial. Show $p(A)\mathbf{v}_k = p(\lambda_k)\mathbf{v}_k = \mathbf{0}$. Then since $\{\mathbf{v}_k\}$ is a basis, it follows $p(A)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ and so $p(A) = 0$. Next in the general case, use the above problem to obtain a sequence $\{A_k\}$ of matrices whose entries converge to the entries of A such that A_k has n distinct eigenvalues. Explain why A_k has a basis of eigenvectors. Therefore, from the first part and for $p_k(\lambda)$ the characteristic polynomial for A_k, it follows $p_k(A_k) = 0$. Now explain why and the sense in which

$$\lim_{k\to\infty} p_k(A_k) = p(A).$$

75. 149] Show that the Moore Penrose inverse A^+ satisfies the following conditions.

$$AA^+A = A,\ \ A^+AA^+ = A^+,\ \ A^+A,\ AA^+ \text{ are Hermitian.}$$

Next show that if A_0 satisfies the above conditions, then it must be the Moore Penrose inverse and that if A is an $n \times n$ invertible matrix, then A^{-1} satisfies the above conditions. Thus the Moore Penrose inverse generalizes the usual notion of inverse but does not contradict it. **Hint**: Let

$$U^*AV = \Sigma \equiv \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and suppose

$$V^+A_0U = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where P is the same size as σ. Now use the conditions to identify $P = \sigma, Q = 0$ etc.

76. 150] Find the least squares solution to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Next suppose ε is so small that all ε^2 terms are ignored by the computer but the terms of order ε are not ignored. Show the least squares equations in this case reduce to

$$\begin{pmatrix} 3 & 3+\varepsilon \\ 3+\varepsilon & 3+2\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+b+c \\ a+b+(1+\varepsilon)c \end{pmatrix}.$$

Find the solution to this and compare the y values of the two solutions. Show that one of these is -2 times the other. This illustrates a problem with the technique for finding least squares solutions presented as the solutions to $aA\mathbf{x} = a\mathbf{y}$. One way of dealing with this problem is to use the QR factorization. This is illustrated in the next problem. It turns out that this helps alleviate some of the round off difficulties of the above.

77. 151] Show that the equations $aA\mathbf{x} = a\mathbf{y}$ can be written as $R^*R\mathbf{x} = R^*Q^*\mathbf{y}$ where R is upper triangular and R^* is lower triangular. Explain how to solve this system efficiently. **Hint**: You first find $R\mathbf{x}$ and then you find $\mathbf{x}$ which will not be hard because R is upper triangular.

Numerical Methods For Linear Systems

1.

2. 152] Solve the system

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

3. 153] Solve the system

$$\begin{pmatrix} 6 & 1 & 1 \\ 1 & 13 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

4. 154] Solve the system

$$\begin{pmatrix} 10 & 1 & 1 \\ 1 & 8 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

5. 155] Solve the system

$$\begin{pmatrix} 12 & 1 & 1 \\ 1 & 8 & 2 \\ 0 & 2 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

6. 156] Solve the system

$$\begin{pmatrix} 13 & 1 & 1 \\ 1 & 13 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

7. 157] If you are considering a system of the form $A\mathbf{x} = \mathbf{b}$ and A^{-1} does not exist, will either the Gauss Seidel or Jacobi methods work? Explain. What does this indicate about using either of these methods for finding eigenvectors for a given eigenvalue?

Numerical Methods For Eigenvalues

1.

2. 158] Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

3. 159] Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

4. 160] Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

5. 161] Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

6. 162] Find the eigenvalues and eigenvectors for the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

7. 163] Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. In this case the exact eigenvalues are $\pm\sqrt{3}, 6$. Compare with the exact answers.

8. 164] Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. The exact eigenvalues are $2, 4+\sqrt{15}, 4-\sqrt{15}$. Compare your numerical results with the exact values. Is it much fun to compute the exact eigenvectors?

9. 165] Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. We don't know the exact eigenvalues in this case. Check your answers by multiplying your numerically computed eigenvectors by the matrix.

10. 166] Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. We don't know the exact eigenvalues in this case. Check your answers by multiplying your numerically computed eigenvectors by the matrix.

11. 167] Consider the matrix $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{pmatrix}$ and the vector $(1,1,1)^T$. Estimate the distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

12. 168] Consider the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & 5 \end{pmatrix}$ and the vector $(1,1,1)^T$. Estimate the distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

13. 169] Using Gerschgorin's theorem, find upper and lower bounds for the eigenvalues of
$A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix}$.

14. 170] The QR algorithm works very well on general matrices. Try the QR algorithm on the following matrix which happens to have some complex eigenvalues.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

Use the QR algorithm to get approximate eigenvalues and then use the shifted inverse power method on one of these to get an approximate eigenvector for one of the complex eigenvalues.

15. 171] Use the QR algorithm to approximate the eigenvalues of the symmetric matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -8 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

16. 172] Try to find the eigenvalues of the matrix $\begin{pmatrix} 3 & 3 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ using the QR algorithm. It has eigenvalues $1, i, -i$. You will see the algorithm won't work well.

Fields

1.

2. 173] Prove the Euclidean algorithm: If m,n are positive integers, then there exist integers $q,r \geq 0$ such that $r < m$ and

$$n = qm + r$$

Hint: You might try considering

$$S \equiv \{n - km : k \in \mathbf{N} \text{ and } n - km < 0\}$$

and picking the smallest integer in S or something like this.

3. 174] The greatest common divisor of two positive integers m,n, denoted as q is a positive number which divides both m and n and if p is any other positive number which divides both $m,n,$ then p divides q. Recall what it means for p to divide q. It means that $q = pk$ for some integer k. Show that the greatest common divisor of m,n is the smallest positive integer in the set S

$$S \equiv \{xm + yn : x,y \in \mathbf{Z} \text{ and } xm + yn > 0\}$$

Two positive integers are called relatively prime if their greatest common divisor is 1.

4. 175] A positive integer larger than 1 is called a prime number if the only positive numbers which divide it are 1 and itself. Thus 2,3,5,7, etc. are prime numbers. If m is a positive integer and p does not divide m where p is a prime number, show that p and m are relatively prime.

5. 176] There are lots of fields. This will give an example of a finite field. Let $\mathbf{Z}$ denote the set of integers. Thus $\mathbf{Z} = \{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$. Also let p be a prime number. We will say that two integers, a,b are equivalent and write $a \sim b$ if $a - b$ is divisible by p. Thus they are equivalent if $a - b = px$ for some integer x. First show that $a \sim a$. Next show that if $a \sim b$ then $b \sim a$. Finally show that if $a \sim b$ and $b \sim c$ then $a \sim c$. For a an integer, denote by $[a]$ the set of all integers which is equivalent to $a,$ the equivalence class of a. Show first that is suffices to consider only $[a]$ for $a = 0,1,2,\cdots,p-1$ and that for $0 \leq a < b \leq p-1, [a] \neq [b]$. That is, $[a] = [r]$ where $r \in \{0,1,2,\cdots,p-1\}$. Thus there are exactly p of these equivalence classes. **Hint**: Recall the Euclidean algorithm. For $a > 0,\ a = mp + r$ where $r < p.$ Next define the following operations.

$$[a] + [b] \equiv [a+b]$$
$$[a][b] \equiv [ab]$$

Show these operations are well defined. That is, if $[a] = [a']$ and $[b] = [b'],$ then $[a] + [b] = [a'] + [b']$ with a similar conclusion holding for multiplication. Thus for addition you need to verify $[a+b] = [a'+b']$ and for multiplication you need to verify $[ab] = [a'b']$. For example, if $p = 5$ you have $[3] = [8]$ and $[2] = [7]$. Is $[2 \times 3] = [8 \times 7]$? Is $[2+3] = [8+7]$? Clearly so in this example because when you subtract, the result is divisible by 5. So why is this so in general? Now verify that $\{[0],[1],\cdots,[p-1]\}$ with these operations is a Field. This is called the integers modulo a prime and is written $\mathbf{Z}_p$. Since there are infinitely many primes $p,$ it follows there are infinitely many of these finite fields. **Hint**: Most of the axioms are easy once you have shown the operations are well defined. The only two which are tricky are the ones which give the existence of the additive inverse and the multiplicative inverse. Of these, the first is not hard. $-[x] = [-x]$. Since p is prime, there exist integers x,y such that $1 = px + ky$ and so $1 - ky = px$

which says $1 \sim ky$ and so $[1] = [ky]$. Now you finish the argument. What is the multiplicative identity in this collection of equivalence classes?

Abstract Vector Spaces

1.

2. 177] Let $M = \{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbf{R}^4 : |u_1| \leq 4\}$. Is M a subspace? Explain.

3. 178] Let $M = \{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbf{R}^4 : \sin(u_1) = 1\}$. Is M a subspace? Explain.

4. 179] If you have 5 vectors in $\mathbf{F}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbf{F}^5$? Explain.

5. 180] If you have 6 vectors in $\mathbf{F}^5$, is it possible they are linearly independent? Explain.

6. 181] Show in any vector space, $\mathbf{0}$ is unique.

7. 182] ↑In any vector space, show that if $\mathbf{x} + \mathbf{y} = \mathbf{0}$, then $\mathbf{y} = -\mathbf{x}$. In other words, if it acts like the additive inverse, it is the additive inverse.

8. 183] ↑Show that in any vector space, $0\mathbf{x} = \mathbf{0}$. That is, the scalar 0 times the vector $\mathbf{x}$ gives the vector $\mathbf{0}$.

9. 184] ↑Show that in any vector space, $(-1)\mathbf{x} = -\mathbf{x}$.

10. 185] Let X be a vector space and suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is a set of vectors from X. Show that $\mathbf{0}$ is in $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k)$.

11. 186] Let X consist of the real valued functions which are defined on an interval $[a, b]$. For $f, g \in X$, $f + g$ is the name of the function which satisfies $(f + g)(x) = f(x) + g(x)$ and for α a real number, $(\alpha f)(x) \equiv \alpha(f(x))$. Show this is a vector space with field of scalars equal to $\mathbf{R}$. Also explain why it cannot possibly be finite dimensional.

12. 187] Let S be a nonempty set and let V denote the set of all functions which are defined on S and have values in W a vector space having field of scalars $\mathbf{F}$. Also define vector addition according to the usual rule, $(f + g)(s) \equiv f(s) + g(s)$ and scalar multiplication by $(\alpha f)(s) \equiv \alpha f(s)$. Show that V is a vector space with field of scalars $\mathbf{F}$.

13. 188] Verify that any field $\mathbf{F}$ is a vector space with field of scalars $\mathbf{F}$. However, show that $\mathbf{R}$ is a vector space with field of scalars $\mathbf{Q}$.

14. 189] Let $\mathbf{F}$ be a field and consider functions defined on $\{1, 2, \cdots, n\}$ having values in $\mathbf{F}$. Explain how, if V is the set of all such functions, V can be considered as $\mathbf{F}^n$.

15. 190] Let V be the set of all functions defined on $\mathbf{N} \equiv \{1, 2, \cdots\}$ having values in a field $\mathbf{F}$ such that vector addition and scalar multiplication are defined by $(\mathbf{f}+\mathbf{g})(s) \equiv \mathbf{f}(s)+\mathbf{g}(s)$ and $(\alpha\mathbf{f})(s) \equiv \alpha\mathbf{f}(s)$ respectively, for $\mathbf{f}, \mathbf{g} \in V$ and $\alpha \in \mathbf{F}$. Explain how this is a vector space and show that for $\mathbf{e}_i$ given by

$$\mathbf{e}_i(k) \equiv \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases},$$

the vectors $\{\mathbf{e}_k\}_{k=1}^{\infty}$ are linearly independent.

16. 191] Suppose you have smooth functions $\{y_1, y_2, \cdots, y_n\}$ (all derivatives exist) defined on an interval $[a, b]$. Then the Wronskian of these functions is the determinant

$$W(y_1, \cdots, y_n)(x) = \det\begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

Show that if $W(y_1, \cdots, y_n)(x) \neq 0$ for some x, then the functions are linearly independent.

17. 192] Give an example of two functions, y_1, y_2 defined on $[-1, 1]$ such that $W(y_1, y_2)(x) = 0$ for all $x \in [-1, 1]$ and yet $\{y_1, y_2\}$ is linearly independent. **Hint**: Prove that x^2 and $x|x|$ work.

18. 193] Let the vectors be polynomials of degree no more than 3. Show that with the usual definitions of scalar multiplication and addition wherein, for $p(x)$ a polynomial, $(\alpha p)(x) = \alpha p(x)$ and for p, q polynomials $(p+q)(x) \equiv p(x)+q(x)$, this is a vector space.

19. 194] In the previous problem show that a basis for the vector space is $\{1, x, x^2, x^3\}$.

20. 195] Let V be the polynomials of degree no more than 3. Determine which of the following are bases for this vector space.

a. $\{x+1, x^3+x^2+2x, x^2+x, x^3+x^2+x\}$

b. $\{x^3+1, x^2+x, 2x^3+x^2, 2x^3-x^2-3x+1\}$

21. 196] Let V be the space of polynomials of degree three or less. Consider a collection of polynomials in this vector space which are of the form

$$\left\{a_i x^3 + b_i x^2 + c_i x + d_i,\ i = 1, 2, 3, 4\right\}$$

Show that this collection of polynomials is linearly independent on an interval $[a, b]$ if and only if

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

is an invertible matrix.

22. 197] Let the field of scalars be $\mathbf{Q}$, the rational numbers and let the vectors be of the form $a + b\sqrt{2}$ where a, b are rational numbers. Show that this collection of vectors is a vector space with field of scalars $\mathbf{Q}$ and give a basis for this vector space.

23. 198] Suppose V is a finite dimensional vector space. Based on the exchange theorem above, it was shown that any two bases have the same number of vectors in them. Give a different proof of this fact using the earlier material in the book. **Hint**: Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_m\}$ are two bases with $m < n$. Then define

$$\phi : \mathbf{F}^n \to V, \psi : \mathbf{F}^m \to V$$

by

$$\phi(\mathbf{a}) \equiv \sum_{k=1}^{n} a_k x_k, \ \ \psi(\mathbf{b}) \equiv \sum_{j=1}^{m} b_j y_j$$

Consider the linear transformation, $\psi^{-1} \circ \phi$. Argue it is a one to one and onto mapping from $\mathbf{F}^n$ to $\mathbf{F}^m$. Now consider a matrix of this linear transformation and its row reduced echelon form.

24. 199] This and the following problems will present most of a differential equations course. To begin with, consider the scalar initial value problem

$$y' = ay, \ \ y(t_0) = y_0$$

When a is real, show the unique solution to this problem is $y = y_0 e^{a(t-t_0)}$. Next suppose

$$y' = (a + ib)y, \ \ y(t_0) = y_0$$

where $y(t) = u(t) + iv(t)$. Show there exists a unique solution and it is

$$y(t) = y_0 e^{a(t-t_0)}(\cos b(t - t_0) + i \sin b(t - t_0))$$
$$\equiv e^{(a+ib)(t-t_0)} y_0.$$

Next show that for a real or complex there exists a unique solution to the initial value problem

$$y' = ay + f, \ \ y(t_0) = y_0$$

and it is given by

$$y(t) = e^{a(t-t_0)} y_0 + e^{at} \int_{t_0}^{t} e^{-as} f(s) ds.$$

25. 200] Now consider A an $n \times n$ matrix. By Schur's theorem there exists unitary Q such that

$$Q^{-1} A Q = T$$

where T is upper triangular. Now consider the first order initial value problem

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{x}_0.$$

Show there exists a unique solution to this first order system. **Hint**: Let $\mathbf{y} = Q^{-1}\mathbf{x}$ and so the system becomes

$$\mathbf{y}' = T\mathbf{y},\ \mathbf{y}(t_0) = Q^{-1}\mathbf{x}_0 \qquad *$$

Now letting $\mathbf{y} = (y_1, \cdots, y_n)^T$, the bottom equation becomes

$$y_n' = t_{nn}y_n,\ y_n(t_0) = (Q^{-1}\mathbf{x}_0)_n.$$

Then use the solution you get in this to get the solution to the initial value problem which occurs one level up, namely

$$y_{n-1}' = t_{(n-1)(n-1)}y_{n-1} + t_{(n-1)n}y_n,\ y_{n-1}(t_0) = (Q^{-1}\mathbf{x}_0)_{n-1}$$

Continue doing this to obtain a unique solution to *.

26. 201] Now suppose $\Phi(t)$ is an $n \times n$ matrix of the form

$$\Phi(t) = \begin{pmatrix} \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \qquad \spadesuit$$

where

$$\mathbf{x}_k'(t) = A\mathbf{x}_k(t).$$

Explain why

$$\Phi'(t) = A\Phi(t)$$

if and only if $\Phi(t)$ is given in the form of ♠. Also explain why if $\mathbf{c} \in \mathbf{F}^n$,

$$\mathbf{y}(t) \equiv \Phi(t)\mathbf{c}$$

solves the equation

$$\mathbf{y}'(t) = A\mathbf{y}(t).$$

27. 202] In the above problem, consider the question whether all solutions to

$$\mathbf{x}' = A\mathbf{x} \qquad \clubsuit$$

are obtained in the form $\Phi(t)\mathbf{c}$ for some choice of $\mathbf{c} \in \mathbf{F}^n$. In other words, is the general solution to this equation $\Phi(t)\mathbf{c}$ for $\mathbf{c} \in \mathbf{F}^n$? Prove the following theorem using linear algebra.

Theorem Suppose $\Phi(t)$ is an $n \times n$ matrix which satisfies

$$\Phi'(t) = A\Phi(t).$$

Then the general solution to ♣ is $\Phi(t)\mathbf{c}$ if and only if $\Phi(t)^{-1}$ exists for some t. Furthermore, if $\Phi'(t) = A\Phi(t)$, then either $\Phi(t)^{-1}$ exists for all t or $\Phi(t)^{-1}$ never exists for any t.

(det($\Phi(t)$)) is called the Wronskian and this theorem is sometimes called the Wronskian alternative.)

28. 203] Let $\Phi'(t) = A\Phi(t)$. Then $\Phi(t)$ is called a fundamental matrix if $\Phi(t)^{-1}$ exists for all t. Show there exists a unique solution to the equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \mathbf{x}(t_0) = \mathbf{x}_0 \qquad *$$

and it is given by the formula

$$\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}\mathbf{f}(s)ds$$

Now these few problems have done virtually everything of significance in an entire undergraduate differential equations course, illustrating the superiority of linear algebra. The above formula is called the variation of constants formula.

29. 204] Show there exists a special Φ such that $\Phi'(t) = A\Phi(t),\ \Phi(0) = I,$ and suppose $\Phi(t)^{-1}$ exists for all t. Show using uniqueness that

$$\Phi(-t) = \Phi(t)^{-1}$$

and that for all $t, s \in \mathbf{R}$

$$\Phi(t+s) = \Phi(t)\Phi(s)$$

Explain why with this special $\Phi,$ the solution to the differential equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \ \mathbf{x}(t_0) = \mathbf{x}_0$$

can be written as

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t-s)\mathbf{f}(s)ds.$$

Hint: Let $\Phi(t)$ be such that the j^{th} column is $\mathbf{x}_j(t)$ where

$$\mathbf{x}_j' = A\mathbf{x}_j,\ \ \mathbf{x}_j(0) = \mathbf{e}_j.$$

Use uniqueness as required.

30. 205] *Using the Lindemann Weierstrass theorem show that if σ is an algebraic number $\sin\sigma, \cos\sigma, \ln\sigma,$ and e are all transcendental. **Hint**: Observe, that

$$ee^{-1} + (-1)e^0 = 0,\ 1e^{\ln(\sigma)} + (-1)\sigma e^0 = 0,$$

$$\frac{1}{2i}e^{i\sigma} - \frac{1}{2i}e^{-i\sigma} + (-1)\sin(\sigma)e^0 = 0.$$

Inner Product Spaces

1.

2. 206] Let V be the continuous complex valued functions defined on a finite closed interval I. Define an inner product as follows.

$$\langle f, g\rangle \equiv \int_I f(x)\overline{g(x)}p(x)dx$$

where $p(x)$ some function which is strictly positive on the closed interval I. It is understood in writing this that

$$\int_I f(x) + ig(x)dx \equiv \int_I f(x)dx + i\int_I g(x)dx$$

Then with this convention, the usual calculus theorems hold about evaluating integrals using the fundamental theorem of calculus and so forth. You simply apply these theorems to the real and imaginary parts of a complex valued function. Show V is an inner product space with the given inner product.

3. 207] Let V be the polynomials of degree at most n which are defined on a closed interval I and let $\{x_0, x_1, \cdots, x_n\}$ be $n+1$ distinct points in I. Then define

$$\langle f, g\rangle \equiv \sum_{k=0}^{n} f(x_k)\overline{g(x_k)}$$

Show that V is an inner product space with respect to the given inner product.

4. 208] Let V be any complex vector space and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis. **Decree** that

$$\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \delta_{ij}.$$

Then define

$$\left\langle \sum_{j=1}^{n} c_j\mathbf{v}_j, \sum_{k=1}^{n} d_k\mathbf{v}_k \right\rangle \equiv \sum_{j,k} c_j\overline{d_k}\langle \mathbf{v}_j, \mathbf{v}_k\rangle = \sum_{k=1}^{n} c_k\overline{d_k}$$

This makes the complex vector space into an inner product space.

5. 209] Let V consist of sequences $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$, $a_k \in \mathbf{C}$, with the property that

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty$$

and the inner product is then defined as

$$\langle \mathbf{a}, \mathbf{b}\rangle \equiv \sum_{k=1}^{\infty} a_k\overline{b_k}$$

6. 210] In each of the examples of the above problems, write the Cauchy Schwarz inequality.

7. 211] Verify that in an inner product space,

$$|\alpha \mathbf{z}| = |\alpha||\mathbf{z}|$$

and that $|\mathbf{z}| \geq 0$ and equals zero if and only if $\mathbf{z} = \mathbf{0}$.

8. 212] Consider the Cauchy Schwarz inequality. Show that it still holds under the assumptions $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}, \langle (a\mathbf{u} + b\mathbf{v}), \mathbf{z} \rangle = a\langle \mathbf{u}, \mathbf{z} \rangle + b\langle \mathbf{v}, \mathbf{z} \rangle,$ and $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. Thus it is not necessary to say that $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ only if $\mathbf{u} = \mathbf{0}$. It is enough to simply state that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.

9. 213] Consider the integers modulo a prime, $\mathbf{Z}_p$. This is a field of scalars. Now let the vector space be $(\mathbf{Z}_p)^n$ where $n \geq p$. Define now

$$\langle \mathbf{z}, \mathbf{w} \rangle \equiv \sum_{i=1}^{n} z_i w_i$$

Does this satisfy the axioms of an inner product? Does the Cauchy Schwarz inequality hold for this $\langle \quad \rangle$? Does the Cauchy Schwarz inequality even make any sense?

10. 214] If you only know that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ along with the other axioms of the inner product and if you define $|\mathbf{z}|$ the same way, which properties of a norm are lost?

11. 215] In an inner product space, an open ball is the set

$$B(\mathbf{x}, r) \equiv \{\mathbf{y} : |\mathbf{y} - \mathbf{x}| < r\}.$$

If $\mathbf{z} \in B(\mathbf{x}, r)$, show there exists $\delta > 0$ such that $B(\mathbf{z}, \delta) \subseteq B(\mathbf{x}, r)$. In words, this says that an open ball is open. **Hint**: This depends on the triangle inequality.

12. 216] Let V be the real inner product space consisting of continuous functions defined on $[-1, 1]$ with the inner product given by

$$\int_{-1}^{1} f(x)g(x)dx$$

Show that $\{1, x, x^2\}$ are linearly independent and find an orthonormal basis for the span of these vectors.

13. 217] A **regular Sturm Liouville problem** involves the differential equation for an unknown function of x which is denoted here by y,

$$(p(x)y')' + (\lambda q(x) + r(x))y = 0, \ \ x \in [a, b]$$

and it is assumed that $p(t), q(t) > 0$ for any t along with boundary conditions,

$$C_1 y(a) + C_2 y'(a) = 0$$
$$C_3 y(b) + C_4 y'(b) = 0$$

where

$$C_1^2 + C_2^2 > 0, \text{ and } C_3^2 + C_4^2 > 0.$$

There is an immense theory connected to these important problems. The constant λ is called an eigenvalue. Show that if y is a solution to the above problem corresponding to $\lambda = \lambda_1$ and if z is a solution corresponding to $\lambda = \lambda_2 \neq \lambda_1$, then

$$\int_a^b q(x)y(x)z(x)dx = 0.$$

Hint: Let y go with λ and z go with μ.

$$z(p(x)y')' + (\lambda q(x) + r(x))yz = 0$$

$$y(p(x)z')' + (\mu q(x) + r(x))zy = 0$$

Now subtract and then integrate and use the boundary conditions, in particular the fact that $C_1^2 + C_2^2 > 0,$ and $C_3^2 + C_4^2 > 0.$

14. 218] Using the above problem or standard techniques of calculus, show that

$$\left\{ \frac{\sqrt{2}}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}$$

are orthonormal with respect to the inner product

$$\langle f, g \rangle = \int_0^{\pi} f(x)g(x)dx$$

Hint: If you want to use the above problem, show that $\sin(nx)$ is a solution to the boundary value problem

$$y'' + n^2 y = 0,\ y(0) = y(\pi) = 0$$

15. 219] Find $S_5 f(x)$ where $f(x) = x$ on $[-\pi, \pi]$. Then graph both $S_5 f(x)$ and $f(x)$ if you have access to a system which will do a good job of it.

16. 220] Find $S_5 f(x)$ where $f(x) = |x|$ on $[-\pi, \pi]$. Then graph both $S_5 f(x)$ and $f(x)$ if you have access to a system which will do a good job of it.

17. 221] Find $S_5 f(x)$ where $f(x) = x^2$ on $[-\pi, \pi]$. Then graph both $S_5 f(x)$ and $f(x)$ if you have access to a system which will do a good job of it.

18. 222] Let V be the set of real polynomials defined on $[0, 1]$ which have degree at most 2. Make this into a real inner product space by defining

$$\langle f, g \rangle \equiv f(0)g(0) + f(1/2)g(1/2) + f(1)g(1)$$

Find an orthonormal basis and explain why this is an inner product.

19. 223] Consider $\mathbf{R}^n$ with the following definition.

$$\langle \mathbf{x}, \mathbf{y} \rangle \equiv \sum_{i=1}^{n} x_i y_i i$$

Does this define an inner product? If so, explain why and state the Cauchy Schwarz inequality in terms of sums.

20. 224] For f a piecewise continuous function,

$$S_n f(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx}\left(\int_{-\pi}^{\pi} f(y)e^{-iky}dy\right).$$

Show this can be written in the form

$$S_n f(x) = \int_{-\pi}^{\pi} f(y)D_n(x-y)dy$$

where

$$D_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikt}$$

This is called the Dirichlet kernel. Show that

$$D_n(t) = \frac{1}{2\pi}\frac{\sin(n+(1/2))t}{\sin(t/2)}$$

For V the vector space of piecewise continuous functions, define $S_n : V \to V$ by

$$S_n f(x) = \int_{-\pi}^{\pi} f(y)D_n(x-y)dy.$$

Show that S_n is a linear transformation. (In fact, $S_n f$ is not just piecewise continuous but infinitely differentiable. Why?) Explain why $\int_{-\pi}^{\pi} D_n(t)dt = 1$. **Hint**: To obtain the formula, do the following.

$$e^{i(t/2)}D_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{i(k+(1/2))t}$$

$$e^{i(-t/2)}D_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{i(k-(1/2))t}$$

Change the variable of summation in the bottom sum and then subtract and solve for $D_n(t)$.

$$S_n f(x) = \frac{1}{\sqrt{2\pi}}\sum_{k=-n}^{n} e^{ikx}\left(\int_{-\pi}^{\pi} f(y)e^{-iky}dy\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\sum_{k=-n}^{n} e^{ik(x-y)}dy.$$

Let $D_n(t)$ be as described. Then the above equals

$$\int_{-\pi}^{\pi} D_n(x-y)f(y)dy.$$

21. 225] Let V be an inner product space and let U be a finite dimensional subspace with an orthonormal basis $\{\mathbf{u}_i\}_{i=1}^{n}$. If $\mathbf{y} \in V$, show

$$|\mathbf{y}|^2 \geq \sum_{k=1}^{n} |\langle \mathbf{y}, \mathbf{u}_k\rangle|^2$$

Now suppose that $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is an orthonormal set of vectors of V. Explain why

$$\lim_{k\to\infty}\langle \mathbf{y}, \mathbf{u}_k\rangle = 0.$$

When applied to functions, this is a special case of the Riemann Lebesgue lemma.

22. 226] Let f be any piecewise continuous function which is bounded on $[-\pi, \pi]$. Show, using the above problem, that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = 0$$

23. 227] *Let f be a function which is defined on $(-\pi,\pi]$. The 2π periodic extension is given by the formula $f(x+2\pi) = f(x)$. In the rest of this problem, f will refer to this 2π periodic extension. Assume that f is piecewise continuous, bounded, and also that the following limits exist

$$\lim_{y\to 0+}\frac{f(x+y)-f(x+)}{y},\ \lim_{y\to 0+}\frac{f(x-y)-f(x+)}{y}$$

Here it is assumed that

$$f(x+) \equiv \lim_{h\to 0+} f(x+h),\ f(x-) \equiv \lim_{h\to 0+} f(x-h)$$

both exist at every point. The above conditions rule out functions where the slope taken from either side becomes infinite. Justify the following assertions and eventually conclude that under these very reasonable conditions

$$\lim_{n\to\infty} S_n f(x) = (f(x+)+f(x-))/2$$

the mid point of the jump. In words, the Fourier series converges to the midpoint of the jump of the function.

$$S_n f(x) = \int_{-\pi}^{\pi} f(x-y)D_n(y)dy$$

$$\left|S_n f(x) - \frac{f(x+)+f(x-)}{2}\right| = \left|\int_{-\pi}^{\pi}\left(f(x-y) - \frac{f(x+)+f(x-)}{2}\right)D_n(y)dy\right|$$

$$= \left|\int_0^{\pi} f(x-y)D_n(y)dy + \int_0^{\pi} f(x+y)D_n(y)dy\right.$$

$$\left. - \int_0^{\pi}(f(x+)+f(x-))D_n(y)dy\right|$$

$$\leq \left|\int_0^{\pi}(f(x-y)-f(x-))D_n(y)dy\right| + \left|\int_0^{\pi}(f(x+y)-f(x+))D_n(y)dy\right|$$

Now apply some trig. identities and use the result of the above problems to conclude that both of these terms must converge to 0.

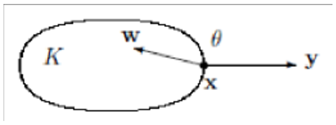
24. 228] Using the Fourier series obtained above for $f(x) = x$ and the result that the Fourier series converges to the midpoint of the jump obtained above to find an interesting formula by examining where the Fourier series converges when $x = \pi/2$. Of course you can get many other interesting formulas in the same way. **Hint**: You should get

$$S_n f(x) = \sum_{k=1}^{n}\frac{2(-1)^{k+1}}{k}\sin(kx)$$

25. 229] Let V be an inner product space and let K be a convex subset of V. This means that if $\mathbf{x},\mathbf{z} \in K$, then the line segment $\mathbf{x} + t(\mathbf{z}-\mathbf{x}) = (1-t)\mathbf{x} + t\mathbf{z}$ is contained in K for all $t \in [0,1]$. Note that every subspace is a convex set. Let $\mathbf{y} \in V$ and let $\mathbf{x} \in K$. Show that $\mathbf{x}$ is the closest point to $\mathbf{y}$ out of all points in K if and only if for all $\mathbf{w} \in K$,

$$\mathrm{Re}\langle \mathbf{y}-\mathbf{x}, \mathbf{w}-\mathbf{x}\rangle \leq 0.$$

In $\mathbf{R}^n$, a picture of the above situation where $\mathbf{x}$ is the closest point to $\mathbf{y}$ is as follows.



The condition of the above **variational inequality** is that the angle θ shown in the picture is larger than 90 degrees. Recall the geometric description of the dot product in terms of an included angle.

26. 230] Show that in any inner product space the parallelogram identity holds.

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Next show that in a real inner product space, the polarization identity holds.

$$\langle \mathbf{x}, \mathbf{y}\rangle = \frac{1}{4}(|\mathbf{x}+\mathbf{y}|^2 - |\mathbf{x}-\mathbf{y}|^2).$$

27. 231] *This problem is for those who know about Cauchy sequences and completeness of $\mathbf{F}^p$ for $\mathbf{F}$ either $\mathbf{R}$ or $\mathbf{C}$, and about closed sets. Suppose K is a closed nonempty convex subset of a finite dimensional subspace U of an inner product space V. Let $\mathbf{y} \in V$. Then show there exists a unique point $\mathbf{x} \in K$ which is closest to $\mathbf{y}$. **Hint**: Let

$$\lambda = \inf\{|\mathbf{y}-\mathbf{z}| : \mathbf{z} \in K\}$$

Let $\{\mathbf{x}_n\}$ be a minimizing sequence,

$$|\mathbf{y}-\mathbf{x}_n| \to \lambda.$$

Use the parallelogram identity in the above problem to show that $\{\mathbf{x}_n\}$ is a Cauchy sequence. Now let $\{\mathbf{u}_k\}_{k=1}^p$ be an orthonormal basis for U. Say

$$\mathbf{x}_n = \sum_{k=1}^p c_k^n \mathbf{u}_k$$

Verify that for $\mathbf{c}^n \equiv (c_1^n, \cdots, c_p^n) \in \mathbf{F}^p$

$$|\mathbf{x}_n - \mathbf{x}_m| = |\mathbf{c}^n - \mathbf{c}^m|_{\mathbf{F}^p}.$$

Now use completeness of $\mathbf{F}^p$ and the assumption that K is closed to get the existence of $\mathbf{x} \in K$ such that $|\mathbf{x}-\mathbf{y}| = \lambda$.

28. 232] *Let K be a closed nonempty convex subset of a finite dimensional subspace U of a real inner product space V. (It is true for complex ones also.) For $\mathbf{x} \in V$, denote by $P\mathbf{x}$ the unique closest point to $\mathbf{x}$ in K. Verify that P is Lipschitz continuous with Lipschitz constant 1,

$$|P\mathbf{x} - P\mathbf{y}| \leq |\mathbf{x}-\mathbf{y}|.$$

Hint: Use the above problem which involves characterizing the closest point with a variational inequality.

29. 233] * This problem is for people who know about compactness. It is an analysis problem. If you have only had the usual undergraduate calculus course, don't waste your time with this problem. Suppose V is a finite dimensional normed linear space. Recall this means that there exists a norm $\|\cdot\|$ defined on V as described above,

$$\|\mathbf{v}\| \geq 0 \text{ equals } 0 \text{ if and only if } \mathbf{v} = \mathbf{0}$$

$$\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|,\ \|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|.$$

Let $|\cdot|$ denote the norm which is defined by

$$|\mathbf{v}|^2 \equiv \sum_{j=1}^{n} |v_j|^2 \text{ where } \mathbf{v} = \sum_{j=1}^{n} v_j \mathbf{u}_j$$

$\{\mathbf{u}_j\}$ a basis, the norm which comes from the inner product by decree. Show $|\cdot|$ and $\|\cdot\|$ are equivalent. That is, there exist constants $\delta, \Delta > 0$ such that for all $\mathbf{x} \in V$,

$$\delta|\mathbf{x}| \leq \|\mathbf{x}\| \leq \Delta|\mathbf{x}|.$$

Next explain why every two norms on a finite dimensional vector space must be equivalent in the above sense.

Linear Transformations

1.

2. 234] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $-\frac{1}{4}\pi$.

3. 235] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{3}{4}\pi$.

4. 236] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{4}{3}\pi$.

5. 237] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{4}\pi$. and then reflects through the x axis.

6. 238] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $5\pi/12$. **Hint**: Note that $5\pi/12 = 2\pi/3 - \pi/4$.

7. 239] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{3}\pi$. and then reflects through the y axis.

8. 240] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{1}{6}\pi$.

9. 241] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\frac{7}{6}\pi$.

10. 242] Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/12$. **Hint**: Note that $\pi/12 = \pi/3 - \pi/4$.

11. 243] Let V be a finite dimensional inner product space, the field of scalars equal to either $\mathbf{R}$ or $\mathbf{C}$. Verify that f given by $f\mathbf{v} \equiv \langle \mathbf{v}, \mathbf{z} \rangle$ is in $\mathrm{L}(V, \mathbf{F})$, the space of linear transformations from V to the field $\mathbf{F}$. Next suppose f is an arbitrary element of $\mathrm{L}(V, \mathbf{F})$. Show the following.

- **a**. If $f = 0$, the zero mapping, then $f\mathbf{v} = \langle \mathbf{v}, \mathbf{0} \rangle$ for all $\mathbf{v} \in V$.
- **b**. If $f \neq 0$ then there exists $\mathbf{z} \neq \mathbf{0}$ satisfying $\langle \mathbf{u}, \mathbf{z} \rangle = 0$ for all $\mathbf{u} \in \ker(f)$.
- **c**. Explain why $f(\mathbf{y})\mathbf{z} - f(\mathbf{z})\mathbf{y} \in \ker(f)$.
- **d**. Use the above to show that there exists $\mathbf{w}$ such that $f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w} \rangle$ for all $\mathbf{y} \in V$.
- **e**. Show there is at most one such $\mathbf{w}$.

You have now proved the Riesz representation theorem which states that every $f \in \mathrm{L}(V, \mathbf{F})$ is of the form

$$f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w} \rangle$$

for a unique $\mathbf{w} \in V$.

12. 244] ↑Let $A \in \mathrm{L}(V, W)$ where V, W are two finite dimensional inner product spaces, both having field of scalars equal to $\mathbf{F}$ which is either $\mathbf{R}$ or $\mathbf{C}$. Let $f \in \mathrm{L}(V, \mathbf{F})$ be given by

$$f(\mathbf{y}) \equiv \langle A\mathbf{y}, \mathbf{z} \rangle$$

where $\langle \quad \rangle$ now refers to the inner product in W. Use the above problem to verify that there exists a unique $\mathbf{w} \in V$ such that $f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w} \rangle$, the inner product here being the one on V. Let $A^*\mathbf{z} \equiv \mathbf{w}$. Show that $A^* \in \mathrm{L}(W, V)$ and by construction,

$$\langle A\mathbf{y},\mathbf{z}\rangle = \langle \mathbf{y},A^{*}\mathbf{z}\rangle.$$

In the case that $V = \mathbf{F}^n$ and $W = \mathbf{F}^m$ and A consists of multiplication on the left by an $m \times n$ matrix, give a description of A^*.

13. 245] Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = (D^2 + 2D + 1)f.$ Find $\ker(A)$.

14. 246] Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = (D^2 + 5D + 4)f.$ Find $\ker(A)$. Note that you could first find $\ker(D + 4)$ where D is the differentiation operator and then consider $\ker(D + 1)(D + 4) = \ker(A)$ and consider Sylvester's theorem.

15. 247] Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = (D^2 + 6D + 8)f.$ Find $\ker(A)$. Note that you could first find $\ker(D + 2)$ where D is the differentiation operator and then consider $\ker(D + 2)(D + 4) = \ker(A)$ and consider Sylvester's theorem.

16. 248] Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = (D^2 + 3D - 4)f.$ Find $\ker(A)$. Note that you could first find $\ker(D - 1)$ where D is the differentiation operator and then consider $\ker(D - 1)(D + 4) = \ker(A)$ and consider Sylvester's theorem.

17. 249] Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = (D^2 - 7D + 12)f.$ Find $\ker(A)$. Note that you could first find $\ker(D - 4)$ where D is the differentiation operator and then consider $\ker(D - 4)(D - 3) = \ker(A)$ and consider Sylvester's theorem.

18. 250] Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = (D^2 + 4D + 4)f.$ Find $\ker(A)$.

19. 251] Suppose $A\mathbf{x} = \mathbf{b}$ has a solution where A is a linear transformation. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution.

20. 252] Verify the linear transformation determined by the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

maps $\mathbf{R}^3$ onto $\mathbf{R}^2$ but the linear transformation determined by this matrix is not one to one.

21. 253] Let L be the linear transformation taking polynomials of degree at most three to polynomials of degree at most three given by

$$D^2 + 4D + 3$$

where D is the differentiation operator. Find the matrix of this linear transformation relative to the basis $\{1, x, x^2, x^3\}$. Also find the matrices with of $D + a$ and $D + b$ and multiply these. You should get the same thing. Why?

22. 254] Let L be the linear transformation taking polynomials of degree at most three to polynomials of degree at most three given by

$$D^2 - 16$$

where D is the differentiation operator. Find the matrix of this linear transformation relative to the basis $\{1, x, x^2, x^3\}$. Also find the matrices with of $D - 4$ and $D + 4$ and multiply these. You should get the same thing. Why?

23. 255] Let L be the linear transformation taking polynomials of degree at most three to polynomials of degree at most three given by

$$D^2 + 2D + 1$$

where D is the differentiation operator. Find the matrix of this linear transformation relative to the basis $\{1, x, x^2, x^3\}$. Find the matrix directly and then find the matrix with respect to the differential operator $D + 1$ and multiply this matrix by itself. You should get the same thing. Why?

24. 256] Show that if $L \in \mathrm{L}(V, W)$ (linear transformation) where V and W are vector spaces, then if $L\mathbf{y}_p = \mathbf{f}$ for some $\mathbf{y}_p \in V$, then the general solution of $L\mathbf{y} = \mathbf{f}$ is of the form

$$\ker(L) + \mathbf{y}_p.$$

25. 257] Let $L \in \mathrm{L}(V, W)$ where V, W are vector spaces, finite or infinite dimensional, and define $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x} - \mathbf{y} \in \ker(L)$. Show that $\sim$ is an equivalence relation. Next define addition and scalar multiplication on the space of equivalence classes as follows.

$$[\mathbf{x}] + [\mathbf{y}] \equiv [\mathbf{x} + \mathbf{y}]$$
$$\alpha[\mathbf{x}] = [\alpha\mathbf{x}]$$

Show that these are well defined definitions and that the set of equivalence classes is a vector space with respect to these operations. The zero is $[\ker L]$. Denote the resulting vector space by $V/\ker(L)$. Now suppose L is onto W. Define a mapping $A : V/\ker(K) \to W$ as follows.

$$A[\mathbf{x}] \equiv L\mathbf{x}$$

Show that A is well defined, linear, one to one and onto.

26. 258] If V is a finite dimensional vector space and $L \in \mathrm{L}(V, V)$, show that the minimal polynomial for L equals the minimal polynomial of A where A is the $n \times n$ matrix of L with respect to some basis.

27. 259] Let A be an $n \times n$ matrix. Describe a fairly simple method based on row operations for computing the minimal polynomial of A. Recall, that this is a monic polynomial $p(\lambda)$ such that $p(A) = 0$ and it has smallest degree of all such monic polynomials. **Hint**: Consider $I, A^2, \cdots$. Regard each as a vector in $\mathbf{F}^{n^2}$ and consider taking the row reduced echelon form or something like this. You might also use the Cayley Hamilton theorem to note that you can stop the above sequence at A^n.

28. 260] Let A be an $n \times n$ matrix which is non defective. That is, there exists a basis of eigenvectors. Show that if $p(\lambda)$ is the minimal polynomial, then $p(\lambda)$ has no repeated roots. **Hint**: First show that the minimal polynomial of A is the same as the minimal polynomial of the diagonal matrix

$$D = \begin{pmatrix} D(\lambda_1) & & \\ & \ddots & \\ & & D(\lambda_r) \end{pmatrix}$$

Where $D(\lambda)$ is a diagonal matrix having λ down the main diagonal and in the above, the λ_i are distinct. Show that the minimal polynomial is $\prod_{i=1}^{r}(\lambda - \lambda_i)$.

29. 261] Show that if A is an $n \times n$ matrix and the minimal polynomial has no repeated roots, then A is non defective and there exists a basis of eigenvectors. Thus, from the above problem, a matrix may be diagonalized if and only if its minimal polynomial has no repeated roots. It turns out this condition is something which is relatively easy to determine. (One simply checks whether the polynomial and its derivative are relatively prime meaning that the only polynomials which divide both are constant polynomials. There are systematic ways to check this.)

30. 262] Recall the linearization of the Lotka Volterra equations used to model the interaction between predators and prey. It was shown earlier that if x, y are the deviations from the equilibrium point, then

$$\begin{aligned} x' &= -bxy - b\frac{c}{d}y \\ y' &= dxy + \frac{a}{b}dx \end{aligned}$$

If one is interested only in small values of $x, y,$ that is, in behavior near the equilibrium point, these are approximately equal to the system

$$\begin{aligned} x' &= -b\frac{c}{d}y \\ y' &= \frac{a}{b}dx \end{aligned}$$

Written more simply, for $\alpha, \beta > 0,$

$$\begin{aligned} x' &= -\alpha y \\ y' &= \beta x \end{aligned}$$

Find the solution to the initial value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where x_0, y_0 are both real. Also have a computer graph the vector fields $\left(-xy - y, xy + x\right)$ and $(-y, x)$ which result from setting all the parameters equal to 1.