# Some Applications Of Topology

## 1.1 Compactness And Linear Algebra

### 1.1.1 Self Adjoint Operators

### 1.1.2 The Right Polar Decomposition

## 1.2 Compactness In Spaces Of Continuous Functions

## 5.3 Weierstrass Approximation Theorem

Therefore, $\phi_k$ has the same shape as the functions above except that the constant $A_k$ will make $\phi_k$ much taller for large $k$ because the area must always equal 1. Now

$$\begin{aligned}\int_{-1}^{1}\left(1-x^2\right)^k dx &= 2\int_0^1\left(1-x^2\right)^k dx \\ &\geq 2\int_0^1 x\left(1-x^2\right)^k dx = \frac{1}{k+1}\end{aligned}$$

and so $A_k \leq k+1$. Therefore, we can conclude that for $1 \geq |x| \geq r > 0$,

$$\phi_k(x) \leq \phi_k(r) \leq (k+1)\left(1-r^2\right)^k$$

and $\lim_{k\to\infty}\phi_k(r) = 0$.

For $\mathbf{x} \in \mathbb{R}^n$, we define $||\mathbf{x}||_\infty \equiv \max\{|x_i|, i = 1,\cdots,n\}$. We leave it as an exercise to verify that $||\cdot||_\infty$ is a norm on $\mathbb{R}^n$ and that if we define $P_r : \mathbb{R}^n \to \mathbb{R}^n$ for $r > 0$ by

$$P_r\mathbf{x} \equiv \begin{cases} \mathbf{x} \text{ if } ||\mathbf{x}||_\infty \leq r \\ \frac{\mathbf{x}}{||\mathbf{x}||_\infty} \text{ if } ||\mathbf{x}||_\infty > r, \end{cases}$$

it follows that $P_r$ is continuous.

With this preparation, we are ready to prove the following lemma which will yield a proof of the Weierstrass theorem.

**Lemma 14** *Let $f : \prod_{i=1}^n [-r,r] \to \mathbb{C}$ be continuous. Then for any $\varepsilon > 0$ there exists a polynomial, $p$ such that*

$$||p - f||_\infty < \varepsilon.$$

**Proof:** For $\mathbf{x} = (x_1,\cdots,x_n) \in \prod_{i=1}^n [-r,r]$ define

$$p_k(\mathbf{x}) \equiv \int_{x_1-1}^{x_1+1}\cdots\int_{x_n-1}^{x_n+1}\overline{f}(\mathbf{y})\prod_{i=1}^n\phi_k(x_i-y_i)\,dy_1\cdots dy_n \tag{5.5}$$

where $\overline{f}$ is the continuous function defined on $\mathbb{R}^n$ by $\overline{f}(\mathbf{y}) \equiv f(P_r(\mathbf{y}))$. Note that in this definition, $\mathbf{y} \in \prod_{i=1}^n [-r-1, r+1]$ because $\mathbf{x} \in \prod_{i=1}^n [-r,r]$. We see this is a polynomial because $\prod_{i=1}^n \phi_k(x_i - y_i)$ is a polynomial in $x_1,\cdots,x_n$ having coefficients which are functions of $\mathbf{y}$. Therefore, the result of the iterated integration yields a polynomial in $x_1,\cdots,x_n$. If you are concerned about the existence of the iterated integral, note that in each iteration the process asks for the integral of a continuous function so there is no problem in writing this. For a more careful discussion of this issue, see Lemma 8.1 on Page 153. Also, since $\int_{x-1}^{x+1}\phi_k(x-y)\,dy = 1$,

$$f(\mathbf{x}) = \int_{x_1-1}^{x_1+1}\cdots\int_{x_n-1}^{x_n+1} f(\mathbf{x})\prod_{i=1}^n\phi_k(x_i-y_i)\,dy_1\cdots dy_n. \tag{5.6}$$

Then we note that if $||\mathbf{z}||_\infty \geq r$, we have

$$\prod_{i=1}^{n} \phi_k (z_i) \leq \phi_k (r)^n \tag{5.7}$$

which converges to zero as $k \to \infty$. Now from 5.5 and 5.6 we find

$$|f(\mathbf{x}) - p(\mathbf{x})| \leq$$

$$\int_{x_1-1}^{x_1+1} \cdots \int_{x_n-1}^{x_n+1} \left|f(\mathbf{x}) - \overline{f}(\mathbf{y})\right| \prod_{i=1}^{n} \phi_k (x_i - y_i)\, dy_1 \cdots dy_n. \tag{5.8}$$

Now since $\prod_{i=1}^n [-r-1, r+1]$ is compact, we can apply Theorem 4.34 on Page 62 and conclude $\overline{f}$ is uniformly continuous on this set and so if $\varepsilon > 0$ is given, there exists a $\delta > 0$ such that if $||\mathbf{x} - \mathbf{y}||_\infty < \delta$, then $\left|\overline{f}(\mathbf{x}) - \overline{f}(\mathbf{y})\right| < \varepsilon/2$. (Note $||\mathbf{x}||_\infty \sqrt{n} \geq |\mathbf{x}|$.) Using 5.7, the expression in 5.8 is dominated by

$$\int_{|x_1-y_1|\geq\delta} \cdots \int_{|x_n-y_n|\geq\delta} \left|f(\mathbf{x}) - \overline{f}(\mathbf{y})\right| \phi_k(\delta)^n\, dy_1 \cdots dy_n +$$

$$\int_{x_1-\delta}^{x_1+\delta} \cdots \int_{x_n-\delta}^{x_n+\delta} (\varepsilon/2) \prod_{i=1}^{n} \phi_k (x_i - y_i)\, dy_1 \cdots dy_n. \tag{5.9}$$

Since $\overline{f}$ is continuous, it is bounded on $\prod_{i=1}^n [-r-1, r+1]$ and so the first integral in 5.9 is dominated by an expression of the form $M\phi_k(\delta)^n$ where $M$ does not depend on $\mathbf{x} \in \prod_{i=1}^n [-r, r]$ while the second is dominated by

$$\int_{x_1-1}^{x_1+1} \cdots \int_{x_n-1}^{x_n+1} \frac{\varepsilon}{2} \prod_{i=1}^{n} \phi_k (x_i - y_i)\, dy_1 \cdots dy_n = \frac{\varepsilon}{2}.$$

Therefore, letting $k$ be large enough, we have shown that $|f(\mathbf{x}) - p_k(\mathbf{x})| < \varepsilon$ for all $\mathbf{x} \in \prod_{i=1}^n [-r, r]$. This proves the lemma.

**Proof of Weierstrass theorem:** Let $R \equiv \prod_{i=1}^n [-r, r] \supseteq K$ and use the Tietze extension theorem (see Problems 8-10) to extend $f$ to a continuous function defined on all of $R$ (or all of $\mathbb{R}^n$), still denoted by $f$. Then apply the above corollary to obtain a polynomial $p$ such that $||p - f||_{\infty,R} < \varepsilon$. Here the notation means $\sup\{|p(\mathbf{x}) - f(\mathbf{x})| : \mathbf{x} \in R\}$. Then since $K \subseteq R$, we have $||p - f||_\infty < \varepsilon$. This proves the Theorem.

## 5.4 Exercises

1. Let $V$ be an open set in $\mathbb{R}^n$. Show there is an increasing sequence of open sets, $\{U_m\}$, such for all $m \in \mathbb{N}$, $\overline{U_m} \subseteq U_{m+1}, \overline{U_m}$ is compact, and $V = \cup_{m=1}^{\infty} U_m$.

# Abstract Measure And Integration

although all these approximations would likely be too small. Therefore, we define

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih < f]\right)$$

**Lemma 15** *Let* $s(\omega) = \sum_{i=1}^{p} a_i \mathcal{X}_{E_i}(\omega)$ *be a nonnegative simple function. Then*

$$\int s d\mu = \sum_{i=1}^{p} a_i \mu\left(E_i\right). \tag{6.18}$$

*Also, for any nonnegative measurable function,* $f$*,if* $\lambda \geq 0$*, then*

$$\int \lambda f d\mu = \lambda \int f d\mu. \tag{6.19}$$

**Proof:** Without loss of generality, we can assume $0 < a_1 < a_2 < \cdots < a_p$ and that $\mu(E_i) < \infty$. Let $a_0 \equiv 0$. By Problem 15 we can assume $0 < 2h < \min\{a_i - a_{i-1}\}_{i=1}^{p}$. Let $k_i(h)$ be a positive integer defined for each $i = 1, 2, \cdots, p$ by

$$k_i(h) h + r_i = a_i$$

where $0 \leq r_i < h$. Then

$$\sum_{i=1}^{\infty} h\mu\left([ih < s]\right) = \sum_{i=1}^{k_1(h)} h\mu\left([ih < s]\right) + \sum_{i=k_1(h)+1}^{k_2(h)} h\mu\left([ih < s]\right) +$$

$$\cdots + \sum_{i=k_{p-1}(h)+1}^{k_p(h)} h\mu\left([ih < s]\right)$$

$$= hk_1(h) \sum_{i=1}^{p} \mu(E_i) - f_1(h) + \overbrace{h\left(k_2(h) - k_1(h)\right)}^{=a_2-r_2-(a_1-r_1)} \sum_{i=2}^{p} \mu(E_i) - f_2(h) +$$

$$\cdots + \overbrace{h\left(k_p(h) - k_{p-1}(h)\right)}^{=a_p-r_p-(a_{p-1}-r_{p-1})} \mu(E_p) - f_p(h)$$

where $f_j(h)$ either equals 0 or $h\mu(E_j)$, depending on whether $r_j = 0$. Therefore, this equals

$$= \sum_{j=1}^{p} \left(a_j - r_j - \left(a_{j-1} - r_{j-1}\right)\right) \sum_{i=j}^{p} \mu(E_i) - g(h)$$

where $\lim_{h\to 0} g(h) = 0$ and $g(h) \geq 0$. Interchanging the order of the summation, this equals

$$\sum_{i=1}^{p} \mu(E_i) \sum_{j=1}^{i} \left(a_j - r_j - \left(a_{j-1} - r_{j-1}\right)\right) - g(h)$$

$$= \sum_{i=1}^{p} a_i \mu(E_i) - \sum_{i=1}^{p} r_i \mu(E_i) - g(h).$$

Taking the sup yields 6.18 since each $r_i < h$.

To verify (6.19) we note the formula is obvious if $\lambda = 0$ because then $[ih < \lambda f] = \emptyset$ for all $i > 0$. Assume $\lambda > 0$. Then

$$\begin{aligned}
\int \lambda f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < \lambda f]) \\
&= \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih/\lambda < f]) \\
&= \sup_{h>0} \lambda \sum_{i=1}^{\infty} (h/\lambda)\, \mu([i(h/\lambda) < f]) \\
&= \lambda \int f d\mu.
\end{aligned}$$

This proves the lemma.

It follows the conclusion of Lemma 6.32 holds: For $s, t$ nonnegative simple functions, and $a, b$ nonnegative scalars,

$$\int as + bt d\mu = a \int s d\mu + b \int t d\mu \tag{6.20}$$

Now we give a short proof of the monotone convergence theorem based on Lemma 6.38 and this definition of the integral. It is nothing but a computation.

**Theorem 16** *(Monotone Convergence theorem) Let $f \geq 0$ and suppose $\{f_n\}$ is a sequence of nonnegative measurable functions satisfying*

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \text{ for each } \omega.$$

$$\cdots f_n(\omega) \leq f_{n+1}(\omega) \cdots$$

*Then $f$ is measurable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

# The Construction Of Measures

## 8.1 Outer Measures

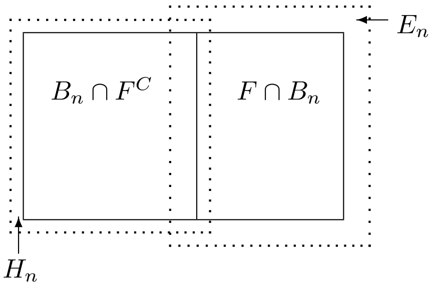
Let

$$E_n \supseteq F \cap B_n\ ,\ \mu(E_n) = \overline{\mu}(F \cap B_n), \tag{8.9}$$

where $E_n \in \mathcal{S}$, and let

$$H_n \supseteq B_n \setminus F = B_n \cap F^C\ ,\ \mu(H_n) = \overline{\mu}(B_n \setminus F), \tag{8.10}$$

where $H_n \in \mathcal{S}$. The following picture may be helpful in visualizing this situation.



Thus $H_n \supseteq B_n \cap F^C$ and so $H_n^C \subseteq B_n^C \cup F$ which implies

$$H_n^C \cap B_n \subseteq F \cap B_n.$$

We have

$$H_n^C \cap B_n \subseteq F \cap B_n \subseteq E_n,\ H_n^C \cap B_n, E_n \in \mathcal{S}. \tag{8.11}$$

**Claim:** If $A, B, D \in \overline{\mathcal{S}}$ and if $A \supseteq B$ with $\overline{\mu}(A \setminus B) = 0$. Then $\overline{\mu}(A \cap D) = \overline{\mu}(B \cap D)$.

**Proof of claim:** This follows from the observation that $(A \cap D) \setminus (B \cap D) \subseteq A \setminus B$.

Now from 8.9 and 8.10 and this claim,

$$\mu\left(E_n \setminus \left(H_n^C \cap B_n\right)\right) = \overline{\mu}\left((F \cap B_n) \setminus \left(H_n^C \cap B_n\right)\right) = \overline{\mu}\left(F \cap B_n \cap \left(B_n^C \cup H_n\right)\right)$$

$$= \overline{\mu}\left(F \cap H_n \cap B_n\right) = \overline{\mu}\left(F \cap \left(B_n \cap F^C\right) \cap B_n\right) = \overline{\mu}\left(\emptyset\right) = 0.$$

Therefore, from the assumption that $(\Omega, \mathcal{S}, \mu)$ is complete and 8.11, $F \cap B_n \in \mathcal{S}$. Therefore,

$$F = \cup_{n=1}^{\infty} F \cap B_n \in \mathcal{S}.$$

This proves the lemma.

## 8.2 Positive Linear Functionals.

One of the most important theorems related to the construction of measures is the Riesz representation theorem. In order to state this theorem, we must give the following definition.

# Lebesgue Measure

Suppose we have shown that for each $r \leq m < n$ (8.5) holds. We have now done it for $m = 1$. Then since the iterated integral makes sense for $f_k$, we know, since the iterated integral makes sense for $f_k$, that

$$x_{k_{m+1}} \rightarrow \int \cdots \int f_k dm\left(x_{k_1}\right) \cdots dm\left(x_{k_m}\right)$$

is Lebesgue measurable and so it follows from (8.5) that

$$x_{k_{m+1}} \rightarrow \int \cdots \int f dm\left(x_{k_1}\right) \cdots dm\left(x_{k_m}\right)$$

is Lebesgue measurable because it is a pointwise limit of measurable functions. Now another application of the monotone convergence theorem implies (8.5) holds for $r \leq m+1$. The last assertion is proved similarly using the dominated convergence theorem instead of the monotone convergence theorem.

Formula (8.6) follows similarly by considering each iterated integral in succession and using the properties of the Lebesgue integral. This proves the lemma.

**Lemma 8** *Let $E$ be a Borel set in $\mathbb{R}^n$ and let $\{k_1, k_2, \cdots, k_n\}$ be distinct integers from $\{1, \cdots, n\}$ and let $Q_p \equiv \prod_{i=1}^n (-p, p]$. Then the iterated integral associated with this list of indices makes sense for the function, $\mathcal{X}_{E \cap Q_p}$.*

**Proof:** If $E$ is an element of $\mathcal{E}$, the algebra of Example 6.7, the conclusion follows readily because $\mathcal{X}_{E \cap Q_p}$ is a sum of functions of the form,

$$\prod_{i=1}^n \mathcal{X}_{(a_i, b_i]}\left(x_{k_i}\right).$$

Therefore, the function of $x_{k_{r+1}}$ of (8.3) with $\mathcal{X}_{E \cap Q_p}$ in place of $f$ is of the form

$$\prod_{i=1}^r \left(b_i - a_i\right) \mathcal{X}_{(a_{r+1}, b_{r+1}]}\left(x_{k_{r+1}}\right) \prod_{i=r+2}^n \mathcal{X}_{(a_i, b_i]}\left(x_{k_i}\right).$$

Let $\mathcal{M}$ be the collection of Borel sets such that (8.3) holds for $f$ replaced with $\mathcal{X}_{E \cap Q_p}$. Using the versions of the dominated convergence and monotone convergence theorems found in Lemma 8.7, we see immediately that $\mathcal{M}$ is a monotone class. In the case of $E_k$ decreasing to $E$, we let the dominating function of this lemma be $\mathcal{X}_{Q_p}$. Since $\mathcal{M}$ is a monotone class, we know from the theorem about monotone classes, Theorem 6.9 on Page 98, that $\mathcal{M} \supseteq \sigma(\mathcal{E})$. However, we know every open set is the countable union of sets of $\mathcal{E}$ by Lemma 8.4 on Page 156 and so $\sigma(\mathcal{E})$ contains the open sets and therefore, $\mathcal{M}$ contains the Borel sets. This proves the lemma.

Now we generalize this lemma.

# Product Measure

# Fourier Series

## 10.1 Pointwise Convergence Of Fourier Series

### 10.2.1 Jordan's Criterion

There is a different condition which implies the Fourier series converges to the mid point of the jump. In order to prove the theorem, there are some interesting lemmas which are needed.

**Lemma 9** *Let $G$ be an increasing function defined on $[a,b]$. Thus $G(x) \leq G(y)$ whenever $x < y$. Then $G(x-) = G(x+) = G(x)$ for every $x$ except for a countable set of exceptions.*

**Proof:** Let $S \equiv \{x \in [a,b] : G(x+) > G(x-)\}$. Then there is a rational number in each interval, $(G(x-), G(x+))$ and also, since $G$ is increasing, these intervals are disjoint. It follows that there are only countably many such intervals. Therefore, $S$ is countable and if $x \notin S$, $G(x+) = G(x-)$ showing that $G$ is continuous on $S^C$ and the claimed equality holds. ■

The next lemma is called the second mean value theorem for integrals.

**Lemma 10** *Let $G$ be an increasing function defined on $[a,b]$ and let $f$ be a continuous function defined on $[a,b]$. Then there exists $t_0 \in [a,b]$ such that*

$$\int_a^b G(s) f(s)\, ds = G(a) \left( \int_a^{t_0} f(s)\, ds \right) + G(b-) \left( \int_{t_0}^b f(s)\, ds \right). \tag{10.1}$$

**Proof:** Letting $h > 0$ define

$$G_h(t) \equiv \frac{1}{h^2} \int_{t-h}^t \int_{s-h}^s G(r)\, dr ds$$

where $G(x) \equiv G(a)$ for all $x < a$. Thus $G_h(a) = G(a)$. Also, from the fundamental theorem of calculus, $G_h'(t) \geq 0$ and is a continuous function of $t$. Also it is clear that $\lim_{h \to 0} G_h(t) = G(t-)$ for all $t \in [a,b]$. Letting $F(t) \equiv \int_a^t f(s)\, ds$,

$$\int_a^b G_h(s) f(s)\, ds = F(t) G_h(t) |_a^b - \int_a^b F(t) G_h'(t)\, dt. \tag{10.2}$$

Now letting $m = \min\{F(t) : t \in [a,b]\}$ and $M = \max\{F(t) : t \in [a,b]\}$, since $G_h'(t) \geq 0$,

$$m \int_a^b G_h'(t)\, dt \leq \int_a^b F(t) G_h'(t)\, dt \leq M \int_a^b G_h'(t)\, dt.$$

Therefore, if $\int_a^b G_h'(t)\, dt \neq 0$,

$$m \leq \frac{\int_a^b F(t) G_h'(t)\, dt}{\int_a^b G_h'(t)\, dt} \leq M$$

and so by the intermediate value theorem from calculus,

$$\left(\int_a^b G_h'(t)\,dt\right) F(t_h) = \int_a^b F(t)\,G_h'(t)\,dt$$

for some $t_h \in [a,b]$. This is true even if $\int_a^b G_h'(t)\,dt = 0$ because in this case, the left side equals 0. Since $G_h' \geq 0$ and is continuous, it must equal 0 and so the right side is also 0. Therefore, substituting for

$$\int_a^b F(t)\,G_h'(t)\,dt$$

in 10.2,

$$\begin{aligned}\int_a^b G_h(s)\,f(s)\,ds &= F(t)\,G_h(t)\,|_a^b - \left[F(t_h)\int_a^b G_h'(t)\,dt\right] \\ &= F(b)\,G_h(b) - F(t_h)\,G_h(b) + F(t_h)\,G_h(a) \\ &= \left(\int_{t_h}^b f(s)\,ds\right) G_h(b) + \left(\int_a^{t_h} f(s)\,ds\right) G(a)\,.\end{aligned}$$

Now selecting a convergent subsequence, still denoted by $h$ which converges to zero, let $t_h \to t_0 \in [a,b]$. Therefore, using the dominated convergence theorem or simply the continuity of $f$ and the above lemma,

$$\begin{aligned}\int_a^b G(s)\,f(s)\,ds &= \int_a^b G(s-)\,f(s)\,ds = \lim_{h\to 0}\int_a^b G_h(s)\,f(s)\,ds \\ &= \lim_{h\to 0}\left(\int_{t_h}^b f(s)\,ds\right) G_h(b) + \left(\int_a^{t_h} f(s)\,ds\right) G(a) \\ &= \left(\int_{t_0}^b f(s)\,ds\right) G(b-) + \left(\int_a^{t_0} f(s)\,ds\right) G(a)\,. \blacksquare\end{aligned}$$

The above lemma will be used in the following lemma from Apostol [**?**].

**Lemma 11** *Let $G$ be increasing. Then for $\delta > 0$,*

$$\lim_{\alpha\to\infty}\int_0^\delta G(y)\,\frac{\sin(\alpha y)}{y}\,dy = \frac{\pi}{2}G(0+)$$

**Proof:** Let $0 < h < \delta$ then $\int_0^\delta G(y)\,\frac{\sin(\alpha y)}{y}\,dy =$

$$\int_0^h (G(y) - G(0+))\,\frac{\sin(\alpha y)}{y}\,dy + G(0+)\int_0^h \frac{\sin(\alpha y)}{y}\,dy + \int_h^\delta G(y)\,\frac{\sin(\alpha y)}{y}\,dy$$

From the mean value theorem above, the first integral equals

$$(G(h-) - G(0+))\int_0^h \frac{\sin(\alpha y)}{y}\,dy$$

This integral converges as $\alpha \to \infty$ to $\int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{2}$. Just change the variable and use the earlier Problem which was about computing this integral. Therefore, if $h$ is chosen small enough, the first term is bounded by $\varepsilon/3$. Fix such an $h$. Then as $\alpha \to \infty$ the second term converges to $\frac{\pi}{2}G(0+)$. The last term converges to 0 by the Riemann Lebesgue lemma. Therefore, fixing $h$ as described,

$$\left|\int_0^\delta G(y)\frac{\sin(\alpha y)}{y}dy - \frac{\pi}{2}G(0+)\right| \leq \left|\int_0^h (G(y) - G(0+))\frac{\sin(\alpha y)}{y}dy\right|$$

$$+\left|G(0+)\int_0^h \frac{\sin(\alpha y)}{y}dy - \frac{\pi}{2}\right| + \left|\int_h^\delta G(y)\frac{\sin(\alpha y)}{y}dy\right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

provided $\alpha$ is large enough. ■

**Definition 1** *Let $f : [a,b] \to \mathbb{C}$ be a function. Then $f$ is of bounded variation if*

$$\sup\left\{\sum_{i=1}^n |f(t_i) - f(t_{i-1})| : a = t_0 < \cdots < t_n = b\right\} \equiv V(f,[a,b]) < \infty$$

*where the sums are taken over all possible lists, $\{a = t_0 < \cdots < t_n = b\}$. The symbol, $V(f,[a,b])$ is known as the total variation on $[a,b]$.*

**Lemma 12** *A real valued function $f$, defined on an interval $[a,b]$ is of bounded variation if and only if there are increasing functions, $H$ and $G$ defined on $[a,b]$ such that $f(t) = H(t) - G(t)$. A complex valued function is of bounded variation if and only if the real and imaginary parts are of bounded variation.*

**Proof:** For $f$ a real valued function of bounded variation, define an increasing function, $H(t) \equiv V(f,[a,t])$ and then note that

$$f(t) = H(t) - \overbrace{[H(t) - f(t)]}^{G(t)}.$$

It is routine to verify that $G(t)$ is increasing. Conversely, if $f(t) = H(t) - G(t)$ where $H$ and $G$ are increasing, the total variation for $H$ is just $H(b) - H(a)$ and the total variation for $G$ is $G(b) - G(a)$. Therefore, the total variation for $f$ is bounded by the sum of these.

The last claim follows from the observation that

$$|f(t_i) - f(t_{i-1})| \geq \max(|\operatorname{Re} f(t_i) - \operatorname{Re} f(t_{i-1})|, |\operatorname{Im} f(t_i) - \operatorname{Im} f(t_{i-1})|)$$

and

$$|\operatorname{Re} f(t_i) - \operatorname{Re} f(t_{i-1})| + |\operatorname{Im} f(t_i) - \operatorname{Im} f(t_{i-1})| \geq |f(t_i) - f(t_{i-1})|. ■$$

Since a bounded variation function is the difference of increasing functions, this immediately implies the following corollary.

**Corollary 13** *If $f$ is of bounded variation, then*

$$\lim_{\alpha\to\infty}\int_0^{\delta} f(y)\,\frac{\sin(\alpha y)}{y} = \frac{\pi}{2} f(0+)$$

With this corollary, here is the main theorem, the Jordan criterion for pointwise convergence of the Fourier series.

**Theorem 14** *Suppose $f$ is $2\pi$ periodic and is in $L^1(-\pi,\pi)$. Suppose also that for some $\delta > 0$, $f$ is of bounded variation on $[x-\delta, x+\delta]$. Then*

$$\lim_{n\to\infty} S_n f(x) = \frac{f(x+) + f(x-)}{2}. \tag{10.3}$$

**Proof:** First note that from Definition 1, $\lim_{y\to x-}\operatorname{Re} f(y)$ exists because $\operatorname{Re} f$ is the difference of two increasing functions. Similarly this limit will exist for $\operatorname{Im} f$ by the same reasoning, and limits of the form $\lim_{y\to x+}$ will also exist. Then

$$S_n f(x) - \frac{f(x+) + f(x-)}{2} =$$

$$\int_{-\pi}^{\pi} D_n(y)\left(f(x-y) - \frac{f(x+) + f(x-)}{2}\right) dy$$

$$= \int_0^{\pi} D_n(y)\left[(f(x+y) - f(x+)) + (f(x-y) - f(x-))\right] dy.$$

Now the Dirichlet kernel, $D_n(y)$ is a constant multiple of

$$\sin((n+1/2)y)/\sin(y/2)$$

and so the Riemann Lebesgue lemma implies

$$\lim_{n\to\infty}\int_{\delta}^{\pi} D_n(y)\left[(f(x+y) - f(x+)) + (f(x-y) - f(x-))\right] dy = 0.$$

Thus it suffices to show that

$$\lim_{n\to\infty}\int_0^{\delta} D_n(y)\left[(f(x+y) - f(x+)) + (f(x-y) - f(x-))\right] dy = 0. \tag{10.4}$$

Now $y \to (f(x+y) - f(x+)) + (f(x-y) - f(x-)) \equiv h(y)$ is of bounded variation for $y \in [0,\delta]$ and $\lim_{y\to 0+} h(y) = h(0+) = 0$. The above limit equals

$$\lim_{n\to\infty}\int_0^{\delta} \frac{\sin\left(n+\frac{1}{2}\right)y}{y}\,\frac{y}{\sin\left(\frac{y}{2}\right)} h(y)\,dy = \lim_{y\to 0+}\frac{y}{\sin\left(\frac{y}{2}\right)} h(y) = 0$$

by Corollary 13. ■

# The Frechet Derivative

## 11.6 Exercises

1.

where $P \in \mathcal{L}(X, X)$, and $P^2 = P$. Also show $L$ is one to one and onto from $X_1$ to $Y$. **Hint:** Let $\{\mathbf{y}_1 \cdots \mathbf{y}_n\}$ be a basis of $Y$ and let $M\mathbf{x}_i = \mathbf{y}_i$. Then define

$$L\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \text{ where } \mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i.$$

Show $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is a linearly independent set and show you can obtain $\{\mathbf{x}_1, \cdots, \mathbf{x}_n, \cdots, \mathbf{x}_m\}$, a basis for $X$ in which $M\mathbf{x}_j = \mathbf{0}$ for $j > n$. Then let

$$P\mathbf{x} \equiv \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$$

where

$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i.$$

5. ↑ Let $\mathbf{f} : U \subseteq X \to Y$, $\mathbf{f}$ is $C^1$, and $D\mathbf{f}(\mathbf{x})$ is onto for each $\mathbf{x} \in U$. Then show $\mathbf{f}$ maps open subsets of $U$ onto open sets in $Y$. **Hint:** Let $P = LD\mathbf{f}(\mathbf{x})$ as in Problem 4. Argue $L$ maps open sets from $Y$ to open sets of $X_1 \equiv PX$ and $L^{-1}$ maps open sets from $X_1$ to open sets of $Y$. Then $L\mathbf{f}(\mathbf{x} + \mathbf{v}) = L\mathbf{f}(\mathbf{x}) + LD\mathbf{f}(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v})$. Now for $\mathbf{z} \in X_1$, let $\mathbf{h}(\mathbf{z}) = L\mathbf{f}(\mathbf{x} + \mathbf{z}) - L\mathbf{f}(\mathbf{x})$. Then $\mathbf{h}$ is $C^1$ on some small open subset of $X_1$ containing $\mathbf{0}$ and $D\mathbf{h}(\mathbf{0}) = LD\mathbf{f}(\mathbf{x})$ which is seen to be one to one and onto and in $\mathcal{L}(X_1, X_1)$. Therefore, if $r$ is small enough, $\mathbf{h}(B(\mathbf{0}, r))$ equals an open set in $X_1$, $V$. This is by the inverse function theorem. Hence $L(\mathbf{f}(\mathbf{x} + B(\mathbf{0}, r)) - \mathbf{f}(\mathbf{x})) = V$ and so $\mathbf{f}(\mathbf{x} + B(\mathbf{0}, r)) - \mathbf{f}(\mathbf{x}) = L^{-1}(V)$, an open set in $Y$.

6. Suppose $U \subseteq \mathbb{R}^2$ is an open set and $\mathbf{f} : U \to \mathbb{R}^3$ is $C^1$. Suppose $D\mathbf{f}(s_0, t_0)$ has rank two and

$$\mathbf{f}(s_0, t_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

Show that for $(s, t)$ near $(s_0, t_0)$, the points $\mathbf{f}(s, t)$ may be realized in one of the following forms.

$$\{(x, y, \phi(x, y)) : (x, y) \text{ near } (x_0, y_0)\},$$

$$\{(\phi(y, z), y, z) : (y, z) \text{ near } (y_0, z_0)\},$$

or

$$\{(x, \phi(x, z), z, ) : (x, z) \text{ near } (x_0, z_0)\}.$$

This shows that parametrically defined surfaces can be obtained locally in a particularly simple form.

# Change Of Variables For $C^1$Maps

## 12.1 Generalizations

**Proof:** We observe first that $\mathbf{h}(U)$ is measurable because, thanks to Theorem 4.23 on Page 57,

$$U = \cup_{i=1}^{\infty} \overline{U_i}$$

where $\overline{U_i}$ is compact. Therefore, $\mathbf{h}(U) = \cup_{i=1}^{\infty} \mathbf{h}\left(\overline{U_i}\right)$, a Borel set. Now let the open set, $U_+$ be defined as

$$U_+ \equiv \{\mathbf{x} : \det D\mathbf{h}(\mathbf{x}) \neq 0\}.$$

The inverse function theorem applies and we conclude $\mathbf{h}(U_+)$ is an open set. The function,

$$\mathbf{x} \to g(\mathbf{h}(\mathbf{x})) \left|\det D\mathbf{h}(\mathbf{x})\right| \mathcal{X}_{U_+}(\mathbf{x}) = g(\mathbf{h}(\mathbf{x})) \left|\det D\mathbf{h}(\mathbf{x})\right|$$

is measurable from Lemma 12.5. (Note we do not claim that $\mathbf{x} \to g(\mathbf{h}(\mathbf{x}))$ is measurable.) Therefore, if $g \geq 0$ is a Lebesgue measurable function,

$$\int_{\mathbf{h}(U)} g(\mathbf{y})\, dy = \int_{\mathbf{h}(U_+)} g(\mathbf{y})\, dy =$$

$$\int_{U_+} g(\mathbf{h}(\mathbf{x})) \left|\det D\mathbf{h}(\mathbf{x})\right| dx = \int_{U} g(\mathbf{h}(\mathbf{x})) \left|\det D\mathbf{h}(\mathbf{x})\right| dx,$$

the middle equality holding by Theorem 12.8 and the first equality holding by Sard's lemma. This proves the theorem.

## 12.2 Mappings Which Are Not One To One

Next we give a version of this theorem which considers the case where $\mathbf{h}$ is only $C^1$, not necessarily one to one. For

$$U_+ \equiv \{\mathbf{x} \in U : \left|\det D\mathbf{h}(x)\right| > 0\}$$

and $Z$ the set where $\left|\det D\mathbf{h}(\mathbf{x})\right| = 0$, Lemma 12.10 implies $m_n(\mathbf{h}(Z)) = 0$. For $\mathbf{x} \in U_+$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}}$ such that $\mathbf{x} \in B_{\mathbf{x}} \subseteq U_+$, $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\{B_i\}$ be a countable subset of $\{B_{\mathbf{x}}\}_{\mathbf{x} \in U_+}$ such that $U_+ = \cup_{i=1}^{\infty} B_i$. Let $E_1 = B_1$. If $E_1, \cdots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \cup_{i=1}^{k} E_i$. Thus

$$\cup_{i=1}^{\infty} E_i = U_+, \ \ \mathbf{h} \text{ is one to one on } E_i, \ \ E_i \cap E_j = \emptyset,$$

and each $E_i$ is a Borel set contained in the open set $B_i$. Now we define

$$n(\mathbf{y}) = \sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y}).$$

The set, $\mathbf{h}(E_i)$ is Borel measurable because by the inverse function theorem, $\mathbf{h} : B_i \to \mathbf{h}(B_i)$ has a continuous inverse, $\mathbf{g}$ and so $\mathbf{h}(E_i) = \mathbf{g}^{-1}(E_i) = \mathbf{g}^{-1}$ (Borel set). The set, $\mathbf{h}(Z)$ is also a Borel set because $Z$ is a closed set and $Z = \cup_{k=1}^{\infty} Z_k$ where $Z_k = \overline{B(\mathbf{0}, k)} \cap Z$,

# The $L^p$ Spaces

**Example 1** *Let* $U = B(\mathbf{z}, 2r)$

$$\psi(\mathbf{x}) = \begin{cases} \exp\left[\left(\left|\mathbf{x}-\mathbf{z}\right|^2 - r^2\right)^{-1}\right] \text{ if } \left|\mathbf{x}-\mathbf{z}\right| < r, \\ 0 \text{ if } \left|\mathbf{x}-\mathbf{z}\right| \geq r. \end{cases}$$

*Then a little work shows* $\psi \in C_c^\infty(U)$. *We leave this to the reader to verify. The following also is easily obtained.*

**Lemma 22** *Let* $U$ *be any open set. Then* $C_c^\infty(U) \neq \emptyset$.

**Proof:** We pick $\mathbf{z} \in U$ and let $r$ be small enough that $B(\mathbf{z}, 2r) \subseteq U$. Then let $\psi \in C_c^\infty(B(\mathbf{z}, 2r)) \subseteq C_c^\infty(U)$ be the function of the above example.

**Definition 2** *Let* $U = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$. *A sequence* $\{\psi_m\} \subseteq C_c^\infty(U)$ *is called a mollifier (sometimes an approximate identity) if*

$$\psi_m(\mathbf{x}) \geq 0,\ \psi_m(\mathbf{x}) = 0,\ \text{if } |\mathbf{x}| \geq \frac{1}{m},$$

*and* $\int \psi_m(\mathbf{x}) = 1$. *Sometimes we also write* $\{\psi_\varepsilon\}$ *where* $\psi_\varepsilon$ *satisfies the above conditions except we have* $\psi_\varepsilon(\mathbf{x}) = 0$ *if* $|\mathbf{x}| \geq \varepsilon$. *In other words, we let* $\varepsilon$ *take the place of* $1/m$ *and in everything that follows we let* $\varepsilon \to 0$ *instead of* $m \to \infty$.

As before, $\int f(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \to f(\mathbf{x}, \mathbf{y})$ is being integrated. We may also write $dx$ for $dm_n(x)$ in the case of Lebesgue measure.

**Example 2** *Let*

$$\psi \in C_c^\infty(B(0,1)) \ \ (B(0,1) = \{\mathbf{x} : |\mathbf{x}| < 1\})$$

*with* $\psi(\mathbf{x}) \geq 0$ *and* $\int \psi dm = 1$. *Let* $\psi_m(\mathbf{x}) = c_m \psi(m\mathbf{x})$ *where* $c_m$ *is chosen in such a way that* $\int \psi_m dm = 1$. *By the change of variables theorem we see that* $c_m = m^n$.

**Definition 3** *A function,* $f$, *is said to be in* $L^1_{loc}(\mathbb{R}^n, \mu)$ *if* $f$ *is* $\mu$ *measurable and if* $|f|\mathcal{X}_K \in L^1(\mathbb{R}^n, \mu)$ *for every compact set,* $K$. *Here* $\mu$ *is a Radon measure on* $\mathbb{R}^n$. *Usually* $\mu = m_n$, *Lebesgue measure. When this is so, we write* $L^1_{loc}(\mathbb{R}^n)$ *or* $L^p(\mathbb{R}^n)$, *etc. If* $f \in L^1_{loc}(\mathbb{R}^n, \mu)$, *and* $g \in C_c(\mathbb{R}^n)$,

$$f * g(\mathbf{x}) \equiv \int f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) d\mu.$$

The following lemma will be useful in what follows. It says that we can take one of these very unregular functions in $L^1_{loc}(\mathbb{R}^n, \mu)$ and smooth it out by convolving with a mollifier.

**Lemma 23** *Let* $f \in L^1_{loc}(\mathbb{R}^n, \mu)$*, and* $g \in C_c^\infty(\mathbb{R}^n)$*. Then* $f * g$ *is an infinitely differentiable function. Here* $\mu$ *is a Radon measure on* $\mathbb{R}^n$*.*

**Proof:** We look at the difference quotient for calculating a partial derivative of $f * g$.

$$\frac{f * g\left(\mathbf{x} + t\mathbf{e}_j\right) - f * g\left(\mathbf{x}\right)}{t} = \int f(\mathbf{y}) \frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g\left(\mathbf{x} - \mathbf{y}\right)}{t} d\mu\left(y\right).$$

Using the fact that $g \in C_c^\infty\left(\mathbb{R}^n\right)$, we can argue that the quotient,

$$\frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g\left(\mathbf{x} - \mathbf{y}\right)}{t},$$

is uniformly bounded. To see this easily, use Theorem 11.18 on Page 232 to get the existence of a constant, $M$ depending on

$$\max\left\{\left|\left|Dg\left(\mathbf{x}\right)\right|\right| : \mathbf{x} \in \mathbb{R}^n\right\}$$

such that

$$\left|g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g\left(\mathbf{x} - \mathbf{y}\right)\right| \leq M\left|t\right|$$

for any choice of $\mathbf{x}$ and $\mathbf{y}$. Therefore, there exists a dominating function for the integrand of the above integral which is of the form $C\left|f\left(\mathbf{y}\right)\right|\mathcal{X}_K$ where $K$ is a compact set containing the support of $g$. It follows we can take the limit of the difference quotient above inside the integral as $t \to 0$ and write

$$\frac{\partial}{\partial x_j}\left(f * g\right)\left(\mathbf{x}\right) = \int f(\mathbf{y}) \frac{\partial}{\partial x_j} g\left(\mathbf{x} - \mathbf{y}\right) d\mu\left(y\right).$$

Now letting $\frac{\partial}{\partial x_j} g$ play the role of $g$ in the above argument, we can continue taking partial derivatives of all orders. This proves the lemma.

**Theorem 24** *Let* $K$ *be a compact subset of an open set,* $U$*. Then there exists a function,* $h \in C_c^\infty(U)$*, such that* $h(\mathbf{x}) = 1$ *for all* $\mathbf{x} \in K$ *and* $h(\mathbf{x}) \in [0, 1]$ *for all* $\mathbf{x}$*.*

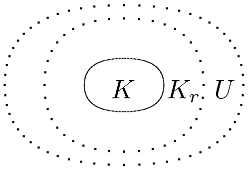
**Proof:** Let $r > 0$ be small enough that $K + B(\mathbf{0}, 3r) \subseteq U$. The symbol, $K + B(\mathbf{0}, 3r)$ means

$$\left\{\mathbf{k} + \mathbf{x} : \mathbf{k} \in K \text{ and } \mathbf{x} \in B\left(\mathbf{0}, 3r\right)\right\}.$$

Thus this is simply a way to write

$$\cup\left\{B\left(\mathbf{k}, 3r\right) : \mathbf{k} \in K\right\}.$$

Think of it as fattening up the set, $K$. Let $K_r = K + B(0, r)$. A picture of what is happening follows.



Consider $\mathcal{X}_{K_r} * \psi_m$where $\psi_m$is a mollifier. Let $m$ be so large that $\frac{1}{m} < r$. Then from the definition of what we mean by a convolution, and using that $\psi_m$ has support in $B\left(\mathbf{0}, \frac{1}{m}\right)$, we see that $\mathcal{X}_{K_r} * \psi_m = 1$ on $K$ and that its support is in $K + B(\mathbf{0}, 3r)$. Now using Lemma 13.27 we see that $\mathcal{X}_{K_r} * \psi_m$ is also infinitely differentiable. Therefore, we let $h = \mathcal{X}_{K_r} * \psi_m$.

Although we will not use the following corollary till later, it follows as an easy consequence of the above theorem and is useful. Therefore, we state it here.

**Corollary 25** *Let $K$ be a compact set in $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $K$. Then there exist functions, $\psi_k \in C_c^{\infty}(U_i)$ such that $\psi_i \prec U_i$ and*

$$\sum_{i=1}^{\infty} \psi_i(\mathbf{x}) = 1.$$

*If $K_1$ is a compact subset of $U_1$ we may also take $\psi_1$ such that $\psi_1(\mathbf{x}) = 1$ for all $\mathbf{x} \in K_1$.*

**Proof:** This follows from a repeat of the proof of Theorem 7.12 on Page 140, replacing the lemma used in that proof with Theorem 13.28.

**Theorem 26** *For each $p \geq 1$, $C_c^{\infty}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Here the measure is Lebesgue measure.*

**Proof:** Let $f \in L^p(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given. Choose $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_p < \frac{\varepsilon}{2}$. This can be done by using Theorem 13.15. Now let

$$g_m(\mathbf{x}) = g * \psi_m(\mathbf{x}) \equiv \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) \, dm_n(y) = \int g(\mathbf{y}) \psi_m(\mathbf{x} - \mathbf{y}) \, dm_n(y)$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 13.27 that $g_m \in C_c^{\infty}(\mathbb{R}^n)$. It vanishes if $\mathbf{x} \notin \text{spt}(g) + B(0, \frac{1}{m})$.

$$\begin{aligned}
||g - g_m||_p &= \left( \int |g(\mathbf{x}) - \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\
&\leq \left( \int ( \int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y}) dm_n(\mathbf{y}))^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\
&\leq \int \left( \int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \\
&= \int_{B(0, \frac{1}{m})} ||g - g_{\mathbf{y}}||_p \psi_m(\mathbf{y}) dm_n(\mathbf{y}) < \frac{\varepsilon}{2}
\end{aligned}$$

whenever $m$ is large enough. This follows from Corollary 13.20. Theorem 13.11 was used to obtain the third inequality. There is no measurability problem because the function

$$(\mathbf{x}, \mathbf{y}) \rightarrow |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y})$$

is continuous. Thus when $m$ is large enough,

$$||f - g_m||_p \leq ||f - g||_p + ||g - g_m||_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the theorem.

This is a very remarkable result. Functions in $L^p(\mathbb{R}^n)$ don't need to be continuous anywhere and yet we can get arbitrarily close to any such function in the sense of the $L^p$ norm with a function which is infinitely differentiable having compact support.

Another thing should probably be mentioned. If you have had a course in complex analysis, you may be wondering whether these infinitely differentiable functions having compact support have anything to do with analytic functions which also have infinitely many derivatives. The answer is no! Recall that if an analytic function has a limit point in the set of zeros then it is identically equal to zero. Thus these functions in $C_c^\infty(\mathbb{R}^n)$ are not analytic. What we are dealing with here is a strictly real analysis phenomenon and has absolutely nothing to do with the theory of functions of a complex variable. For those who have no idea what is being discussed here, ignore the above comments and continue.

## 13.5 Exercises

1. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set
$$E - E = \{x - y : x \in E, y \in E\}.$$
Show that $E - E$ contains an interval. **Hint:** Let
$$f(x) = \int \mathcal{X}_E(t) \mathcal{X}_E(x + t) dt.$$

Note $f$ is continuous at 0 and $f(0) > 0$ and use continuity of translation in $L^p$.

1.

22.

If $q, r, s \geq 1$ this says that

$$||f||_q \leq ||f||_r^{\theta} ||f||_s^{1-\theta}.$$

Using this, show that

$$\ln\left(||f||_q\right) \leq \theta \ln\left(||f||_r\right) + (1-\theta)\ln\left(||f||_s\right).$$

**Hint:**

$$\int |f|^q d\mu = \int |f|^{q\theta} |f|^{q(1-\theta)} d\mu.$$

Now note that $1 = \frac{\theta q}{r} + \frac{q(1-\theta)}{s}$ and use Holder's inequality.

23. Suppose $f$ is a function in $L^1(\mathbb{R})$ and $f$ is infinitely differentiable. Can we conclude $f' \in L^1(\mathbb{R})$? **Hint:** What if $\phi \in C_c^{\infty}(0,1)$ and $f(x) = \phi(2^n(x-n))$ for $x \in (n, n+1)$, $f(x) = 0$ if $x < 0$?

24. Can you show the conclusion of Theorem 26 holds for $L^p(\mathbb{R}^n, \mu)$ where $\mu$ is only a Radon measure? **Hint:** Use the argument of Theorem 13.15 with the conclusion of Theorem 24.

## 13.6 Chapter Notes

This chapter includes all the standard material on the $L^p$ spaces. See most books on measure and integration for similar presentations. The theorems about density of smooth functions are useful in studying function spaces such as the Sobolev spaces. The technique of using a mollifier to smooth out a function is used extensively in books on this subject. See Adams [1] for example. There are generalizations of $L^p$ spaces known as Orlitz spaces in which the function $t^p$ is replaced by more general increasing functions. The book by Adams has an introduction to these spaces.

# Fourier Transforms

## 14.1 Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$

in simply assuming these functions are Borel measurable. Therefore, by Fubini's theorem, Corollary 8.12 on Page 162,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(\mathbf{x}-\mathbf{y})\,g(\mathbf{y})\right|dydx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(\mathbf{x}-\mathbf{y})\,g(\mathbf{y})\right|dxdy \\ &= ||f||_1\,||g||_1 < \infty.\end{aligned}$$

It follows that for a.e. $\mathbf{x}$, we have $\int_{\mathbb{R}^n}\left|f(\mathbf{x}-\mathbf{y})\,g(\mathbf{y})\right|dy < \infty$ and for each of these values of $\mathbf{x}$, it follows that $\int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\,g(\mathbf{y})\,dy$ exists and equals a function of $\mathbf{x}$ which is in $L^1(\mathbb{R}^n)$, $f*g(\mathbf{x})$. Now

$$\begin{aligned}F(f*g)(\mathbf{t}) &\equiv (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f*g(\mathbf{x})\,dx \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\,g(\mathbf{y})\,dydx \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{y}}g(\mathbf{y})\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot(\mathbf{x}-\mathbf{y})}f(\mathbf{x}-\mathbf{y})\,dxdy \\ &= (2\pi)^{n/2}\,Ff(\mathbf{t})\,Fg(\mathbf{t}).\end{aligned}$$

There are many other considerations involving Fourier transforms of functions in $L^1(\mathbb{R}^n)$ and we refer to the exercises for a few others.

## 14.2 The Schwartz Class

Recall Fourier transform of a function in $L^1(\mathbb{R}^n)$ is given by

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}f(\mathbf{x})dx.$$

It turns out that you can give a meaning to the Fourier transform of just about anything, including any function in $L^p$. However, the above straight forward definition is doomed to failure because the integral may not even be defined for such a function. For example, if $f \in L^2(\mathbb{R}^n)$, the above integral might not even exist. In defining what is meant by the Fourier Transform of more general functions, it is convenient to use a special class of functions known as the Schwartz class which is a subset of $L^p(\mathbb{R}^n)$ for all $p \geq 1$. We will define the Fourier transform for more general classes of functions in terms of what it does to Fourier transforms of functions in the Schwartz class.

The functions in the Schwartz class are infinitely differentiable and they vanish very rapidly as $|\mathbf{x}| \to \infty$ along with all their partial derivatives. To describe precisely what we mean by this, we need to present some notation.

**Definition 4** *$\alpha = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For $\alpha$ a multi-index, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in \mathbb{R}^n$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

# Banach Spaces

## 15.1 Baire Category Theorem

## 15.2 Applications Of Baire's Theorem

## 15.3 Hahn Banach Theorem

## 15.4 Exercises

Finally, to show the assertion about the norm of $x^*$, use what was just shown applied to the James map from $X'$ to $X'''$ still referred to as $J$.

$$||x^*|| = \sup\{|x^*(x)| : ||x|| \leq 1\} = \sup\{|J(x)(x^*)| : ||Jx|| \leq 1\}$$

$$\leq \sup\{|x^{**}(x^*)| : ||x^{**}|| \leq 1\} = \sup\{|J(x^*)(x^{**})| : ||x^{**}|| \leq 1\}$$

$$\equiv ||Jx^*|| = ||x^*||.$$

This proves the theorem.

**Definition 5** *When $J$ maps $X$ onto $X''$, we say that $X$ is Reflexive.*

Later we will show the $L^p$ spaces are reflexive whenever $p > 1$.

## 15.4 Exercises

1. Is $\mathbb{N}$ a $G_\delta$ set? What about $\mathbb{Q}$? What about a countable dense subset of a complete metric space?

2. ↑ Let $f : \mathbb{R} \to \mathbb{C}$ be a function. We define the oscillation of a function in $B(x,r)$ by $\omega_r f(x) = \sup\{|f(z) - f(y)| : y, z \in B(x,r)\}$. We define the oscillation of the function at the point, $x$ by $\omega f(x) = \lim_{r\to 0} \omega_r f(x)$. Show $f$ is continuous at $x$ if and only if $\omega f(x) = 0$. Then show the set of points where $f$ is continuous is a $G_\delta$ set (try $U_n = \{x : \omega f(x) < \frac{1}{n}\}$). Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? **Hint:** Suppose $D$ is any countable set, $D = \{d_i\}_{i=1}^\infty$, and define the function, $f_n(x)$ to equal zero for every $x \notin \{d_1, \cdots, d_n\}$ and $2^{-n}$ for $x$ in this finite set. Then consider $g(x) \equiv \sum_{n=1}^\infty f_n(x)$. Show that this series converges uniformly.

3. Let $f \in C([0,1])$ and suppose $f'(x)$ exists. Show there exists a constant, $K$, such that $|f(x) - f(y)| \leq K|x - y|$ for all $y \in [0,1]$. Let $U_n = \{f \in C([0,1])$ such that for each $x \in [0,1]$ there exists $y \in [0,1]$ such that $|f(x) - f(y)| > n|x - y|\}$. Show that $U_n$ is open and dense in $C([0,1])$ where for $f \in C([0,1])$,

$$||f|| \equiv \sup\{|f(x)| : x \in [0,1]\}.$$

   Show that $\cap_n U_n$ is a dense $G_\delta$ set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.

4. ↑Suppose $f(x) = \sum_{k=1}^\infty u_k(x)$ where the convergence is uniform and each $u_k$ is a polynomial. Is it reasonable to conclude that $f'(x) = \sum_{k=1}^\infty u_k'(x)$? The answer is no. Use Problem 3 and the Weierstrass approximation theorem do show this.

5. Let $X$ be a normed linear space. We say $A \subseteq X$ is "weakly bounded" if for each $x^* \in X'$, $\sup\{|x^*(x)| : x \in A\} < \infty$, while $A$ is bounded if $\sup\{||x|| : x \in A\} < \infty$. Show $A$ is weakly bounded if and only if it is bounded.

6. Let $X$ and $Y$ be two Banach spaces. Define the norm
$$|||(x, y)||| \equiv ||x||_X + ||y||_Y .$$
Show this is a norm on $X \times Y$ which is equivalent to the norm given in the chapter for $X \times Y$. Can you do the same for the norm defined for $p > 1$ by
$$|(x, y)| \equiv (||x||_X^p + ||y||_Y^p)^{1/p}?$$

7. Review the Dirichlet kernel for the Fourier series showing the $n^{th}$ partial sum of the Fourier series of a function is of the form $S_n f(x) = \int_{-\pi}^{\pi} D_n(x - y) f(y)\, dy$ where
$$D_n(t) = \frac{\sin((n + \frac{1}{2})t)}{2\pi \sin(\frac{t}{2})}.$$
If you have trouble, review the chapter on Fourier series.

8. ↑ Let $Y = \{f$ such that $f$ is continuous, defined on $\mathbb{R}$, and $2\pi$ periodic$\}$. Define $||f||_Y = \sup\{|f(x)| : x \in [-\pi, \pi]\}$. Show that $(Y, ||\ ||_Y)$ is a Banach space. Let $x \in \mathbb{R}$ and define $L_n(f) = S_n f(x)$. Show $L_n \in Y'$ but $\lim_{n\to\infty} ||L_n|| = \infty$. Show that for each $x \in \mathbb{R}$, there exists a dense $G_\delta$ subset of $Y$ such that for $f$ in this set, $|S_n f(x)|$ is unbounded. Finally, show there is a dense $G_\delta$ subset of $Y$ having the property that $|S_n f(x)|$ is unbounded on the rational numbers. **Hint:** To do the first part, let $f(y)$ approximate $\text{sgn}(D_n(x - y))$. Here $\text{sgn}\, r = 1$ if $r > 0, -1$ if $r < 0$ and 0 if $r = 0$. This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense $G_\delta$ sets must also be a dense $G_\delta$ set.

9. Let $\Lambda_n f = \int_0^{\pi} \sin\left(\left(n + \frac{1}{2}\right) y\right) f(y)\, dy$ for $f \in L^1(0, \pi)$. Show that
$$\sup\{||\Lambda_n|| : n = 1, 2, \cdots\} < \infty$$
using the Riemann Lebesgue lemma. Also show it is obvious $||\Lambda_n|| \leq 1$.

10. ↑Now recall Problem 4 on Page 91 about the Holder spaces. Suppose we let $X$ be the Holder functions which are periodic of period $2\pi$. Define $L_n f(x) = S_n f(x)$ where $L_n : X \to Y$ for $Y$ given in Problem 8 . Show $||L_n||$ is bounded independent of $n$. Conclude that $L_n f \to f$ in $Y$ for all $f \in X$. In other words, for the Holder continuous and $2\pi$ periodic functions, the Fourier series converges to the function uniformly. **Hint:** $L_n f(x)$ is given by
$$L_n f(x) = \int_{-\pi}^{\pi} D_n(y) f(x - y)\, dy$$

# Integrals And Derivatives

## 20.1 The Vitali Covering Theorem

The Vitali covering theorem is concerned with the situation in which a set is contained in the union of balls. You can imagine that it might be very hard to get disjoint balls from this collection of balls which would cover the given set. However, it is possible to get disjoint balls from this collection of balls which have the property that if each ball is enlarged appropriately, the resulting enlarged balls do cover the set. When this result is established, we use it to prove another form of this theorem in which the disjoint balls do not cover the set but they only miss a set of measure zero. We have in mind that the balls involved are taken with respect to the usual Euclidean norm in $\mathbb{R}^n$ but this is not necessary, they could be balls taken with respect to any norm on $\mathbb{R}^n$ and the same theorems would all hold. In particular, the balls could be little cubes if we used the norm given by $||\mathbf{x}|| = \max\{|x_i| : 1 = 1, \cdots, n\}$.

For different versions of this theorem or of the proof, see Evans and Gariepy [15], Stein [44], or Rudin [41].

**Lemma 1** *Let $\mathcal{F}$ be a countable collection of balls satisfying*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

*and let $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that*

$$\text{If } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \tag{20.1}$$

$$\text{If } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \tag{20.2}$$

$$\mathcal{G} \text{ is maximal with respect to Formulas 20.1 and 20.2.} \tag{20.3}$$

**Proof:** If no ball of $\mathcal{F}$ has radius larger than $k$, let $\mathcal{G} = \emptyset$. Assume therefore, that some balls have radius larger than $k$. Let $\mathcal{F} \equiv \{B_i\}_{i=1}^{\infty}$. Now let $B_{n_1}$ be the first ball in the list which has radius greater than $k$. If every ball having radius larger than $k$ intersects this one, then stop. We have found our maximal set. Otherwise, let $B_{n_2}$ be the next ball having radius larger than $k$ which is disjoint from $B_{n_1}$. Continue this way obtaining $\{B_{n_i}\}_{i=1}^{\infty}$, a finite or infinite sequence of disjoint balls having radius larger than $k$. Then let $\mathcal{G} \equiv \{B_{n_i}\}$. To see that $\mathcal{G}$ is maximal with respect to 20.1 and 20.2, suppose $B \in \mathcal{F}$, $B$ has radius larger than $k$, and $\mathcal{G} \cup \{B\}$ satisfies 20.1 and 20.2. Then at some point in the process, $B$ would have been chosen because it would be the ball of radius larger than $k$ which has the smallest index. Therefore, $B \in \mathcal{G}$ and this shows $\mathcal{G}$ is maximal with respect to 20.1 and 20.2.

For the next lemma, if we have an open ball, $B = B(\mathbf{x}, r)$, we denote by $\widetilde{B}$ the open ball, $B(\mathbf{x}, 4r)$.

**Lemma 2** *Let $\mathcal{F}$ be a collection of open balls, and let*

$$A \equiv \cup \{B : B \in \mathcal{F}\}.$$

*Suppose*

$$\infty > M \equiv \sup \{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

*Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and*

$$A \subseteq \cup\{\widetilde{B} : B \in \mathcal{G}\}.$$

**Proof:** Without loss of generality we can assume $\mathcal{F}$ is countable. This is because of Corollary 4.11 on Page 54 which says that since $\mathbb{R}^n$ is a separable metric space, it has the Lindelöf property. Thus there is a countable subset of $\mathcal{F}$, $\mathcal{F}'$ such that $\cup\mathcal{F}' = A$. We could then work with $\mathcal{F}'$ instead of $\mathcal{F}$. Therefore, we assume at the outset $\mathcal{F}$ is countable. By Lemma 1, there exists $\mathcal{G}_1 \subseteq \mathcal{F}$ which satisfies 20.1, 20.2, and 20.3 with $k = \frac{2M}{3}$.

Suppose $\mathcal{G}_1, \cdots, \mathcal{G}_{m-1}$ have been chosen for $m \geq 2$. Let

$$\mathcal{F}_m = \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \overbrace{\cup\{\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{m-1}\}}^{\text{union of the balls in these } \mathcal{G}_j} \}$$

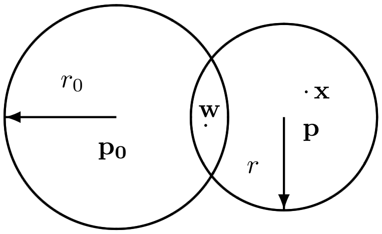
and using Lemma 1, let $\mathcal{G}_m$ be a maximal collection of disjoint balls from $\mathcal{F}_m$ with the property that each ball has radius larger than $\left(\frac{2}{3}\right)^m M$. Let $\mathcal{G} \equiv \cup_{k=1}^{\infty} \mathcal{G}_k$. Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$. Choose $m$ such that

$$\left(\frac{2}{3}\right)^m M < r \leq \left(\frac{2}{3}\right)^{m-1} M$$

Then $B(\mathbf{p}, r)$ must have nonempty intersection with some ball from $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ because if it didn't, then $\mathcal{G}_m$ would fail to be maximal. Denote by $B(\mathbf{p}_0, r_0)$ a ball in $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ which has nonempty intersection with $B(\mathbf{p}, r)$. Thus

$$r_0 > \left(\frac{2}{3}\right)^m M.$$

Consider the picture, in which $\mathbf{w} \in B(\mathbf{p}_0, r_0) \cap B(\mathbf{p}, r)$.



Then

$$\begin{aligned} |\mathbf{x}-\mathbf{p}_0| &\leq |\mathbf{x}-\mathbf{p}|+|\mathbf{p}-\mathbf{w}|+\overbrace{|\mathbf{w}-\mathbf{p}_0|}^{<r_0} \\ &< r+r+r_0 \leq 2\overbrace{\left(\frac{2}{3}\right)^{m-1} M}^{<\frac{3}{2}r_0}+r_0 \\ &< 2\left(\frac{3}{2}\right)r_0+r_0=4r_0. \end{aligned}$$

This proves the lemma since it shows $B(\mathbf{p},r)\subseteq B(\mathbf{p}_0,4r_0)$.

With this Lemma we can present the version of the Vitali covering theorem in which the balls do not have to be open. A ball centered at $\mathbf{x}$ of radius $r$ will denote something which contains the open ball, $B(\mathbf{x},r)$ and is contained in the closed ball, $\overline{B(\mathbf{x},r)}$. Thus the balls could be open or they could contain some but not all of their boundary points.

**Definition 6** *Let $B$ be a ball centered at $\mathbf{x}$ having radius $r$. We denote by $\widehat{B}$ the open ball, $B(\mathbf{x},5r)$.*

**Theorem 3** *(Vitali) Let $\mathcal{F}$ be a collection of balls, and let*

$$A\equiv\cup\{B:B\in\mathcal{F}\}.$$

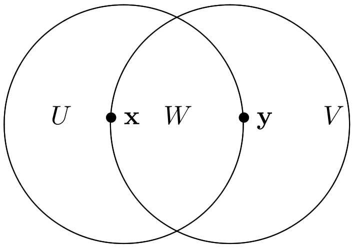
*Suppose*

$$\infty>M\equiv\sup\{r:B(\mathbf{p},r)\in\mathcal{F}\}>0.$$

*Then there exists $\mathcal{G}\subseteq\mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and*

$$A\subseteq\cup\{\widehat{B}:B\in\mathcal{G}\}.$$

## 20.4 Rademacher's Theorem



It is clear there exists a constant, $C$, depending only on $n$ such that

$$\frac{m(W)}{m(U)} = \frac{m(W)}{m(V)} = \frac{1}{C}.$$

In fact we could get this constant by letting $r = 1$ and using the change of variables formula to argue that the same number would be obtained for arbitrary $r$. Then from (20.25),

$$\begin{aligned}
|u(\mathbf{x}) - u(\mathbf{y})| &= \fint_W |u(\mathbf{x}) - u(\mathbf{y})| \, dz \\
&\leq \fint_W |u(\mathbf{x}) - u(\mathbf{z})| \, dz + \fint_W |u(\mathbf{z}) - u(\mathbf{y})| \, dz \\
&= \frac{C}{m(U)} \left[ \int_W |u(\mathbf{x}) - u(\mathbf{z})| \, dz + \int_W |u(\mathbf{z}) - u(\mathbf{y})| \, dz \right] \\
&\leq C \left[ \fint_U |u(\mathbf{x}) - u(\mathbf{z})| \, dz + \fint_V |u(\mathbf{y}) - u(\mathbf{z})| \, dz \right]
\end{aligned}$$

$$\leq C \overbrace{\left[ \int_U |\nabla u(\mathbf{z})| |\mathbf{z} - \mathbf{x}|^{1-n} dz + \int_V |\nabla u(\mathbf{z})| |\mathbf{z} - \mathbf{y}|^{1-n} dz \right]}^{\text{from (20.25 )}}. \tag{20.26}$$

Consider the first of these two integrals. Using Holder's inequality, polar coordinates, and adjusting $C$, this is no smaller than

$$\leq C \left( \int_{B(\mathbf{x},r)} |\nabla u(\mathbf{z})|^p dz \right)^{1/p} \left( \int_{B(\mathbf{x},r)} (|\mathbf{z} - \mathbf{x}|^{1-n})^{p/(p-1)} dz \right)^{(p-1)/p}$$