Lecture Notes

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June 6, 2006

Contents

Part I

Preliminary Material

Set Theory

1.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as 1,2,-4, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, $\{\ \}$ specify a set by listing the things which are in the set between the parentheses. For example the set of integers between -1 and 2, including these numbers could be denoted as $\{-1, 0, 1, 2\}$. The notation signifying x is an element of a set S, is written as $x \in S$. Thus, $1 \in \{-1, 0, 1, 2, 3\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.
2. To every set, A, and to every condition $S(x)$ there corresponds a set, B, whose elements are exactly those elements x of A for which $S(x)$ holds.
3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.
4. The Cartesian product of a nonempty family of nonempty sets is nonempty.
5. If A is a set there exists a set, $\mathcal{P}(A)$ such that $\mathcal{P}(A)$ is the set of all subsets of A. This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1, 2, 3\} = \{2, 3, 1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some "condition" which can be used as a test to determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

$$\{x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z}\}.$$

In this notation, the colon is read as "such that" and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way to determine to everyone's satisfaction whether a given judge is an activist. Also, just because something is grammatically correct does not mean it makes any sense. For example consider the following nonsense.

$$S = \{x \in \text{set of dogs} \ : \ \text{it is colder in the mountains than in the winter}\}.$$

So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say "set of sets" or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If $\mathcal{S}$ is such a set whose elements are sets,

$$\cup\{A : A \in \mathcal{S}\} \text{ or } \cup\mathcal{S}$$

signify this union.

Something is in the Cartesian product of a set or "family" of sets if it consists of a single thing taken from each set in the family. Thus $(1, 2, 3) \in \{1, 4, .2\} \times \{1, 2, 7\} \times \{4, 3, 7, 9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathcal{S}$ is a set whose elements are sets,

$$\prod_{A \in \mathcal{S}} A$$

signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathcal{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

A is a subset of B, written $A \subseteq B$, if every element of A is also an element of B. This can also be written as $B \supseteq A$. A is a proper subset of B, written $A \subset B$ or $B \supset A$ if A is a subset of B but A is not equal to $B, A \neq B$. $A \cap B$ denotes the intersection of the two sets, A and B and it means the set of elements of A which are also elements of B. The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, A and B and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of A, denoted as A^C (or more precisely as $X \setminus A$) is a set obtained from using the axiom of specification to write

$$A^C \equiv \{x \in X : x \notin A\}$$

The symbol $\notin$ means: "is not an element of". Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as "all" or "there exists" are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7. Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical even though such usage may be grammatically correct. Quantifiers are used often enough that there are symbols for them. The symbol $\forall$ is read as "for all" or "for every" and the symbol $\exists$ is read as "there exists". Thus $\forall\forall\exists\exists$ could mean for every upside down A there exists a backwards E.

DeMorgan's laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, U. Then

$$\cup \left\{A^C : A \in \mathcal{S}\right\} = \left(\cap \{A : A \in \mathcal{S}\}\right)^C$$

and

$$\cap \left\{A^C : A \in \mathcal{S}\right\} = \left(\cup \{A : A \in \mathcal{S}\}\right)^C .$$

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathcal{S}$ be a set of sets then

$$B \cup \cup \{A : A \in \mathcal{S}\} = \cup \{B \cup A : A \in \mathcal{S}\} .$$

and: Let $\mathcal{S}$ be a set of sets show

$$B \cap \cup \{A : A \in \mathcal{S}\} = \cup \{B \cap A : A \in \mathcal{S}\} .$$

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it S. Then you could consider A the set of all elements of S which are not elements of themselves, this from the axiom of specification. If A is an element of itself, then it fails to qualify for inclusion in A. Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in A so it is an element of itself and so this can't be true either. Thus the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, "Nothing contains everything". Always beware of statements involving quantifiers wherever they occur, even this one.

1.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

Definition 1.1 *Let X and Y be sets.*

$$X \times Y \equiv \{(x, y) : x \in X \text{ and } y \in Y\}$$

A relation is defined to be a subset of $X \times Y$. A function, f, also called a mapping, is a relation which has the property that if (x, y) and (x, y_1) are both elements of the f, then $y = y_1$. The domain of f is defined as

$$D(f) \equiv \{x : (x, y) \in f\},$$

written as $f : D(f) \to Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, x and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, (x, y) which is an element of the function or mapping has an input, x and a unique output, y,denoted as $f(x)$ while the name of the function is f. "mapping" is often a noun meaning function. However, it also is a verb as in "f is mapping A to B ". That which a function is thought of as doing is also referred to using the word "maps" as in: f maps X to Y. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

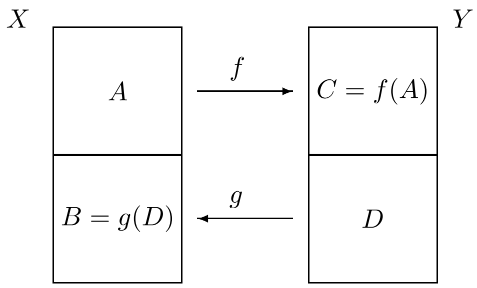
The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

Theorem 1.2 *Let $f : X \to Y$ and $g : Y \to X$ be two functions. Then there exist sets A, B, C, D, such that*

$$A \cup B = X,\ C \cup D = Y,\ A \cap B = \emptyset,\ C \cap D = \emptyset,$$

$$f(A) = C,\ g(D) = B.$$

The following picture illustrates the conclusion of this theorem.



Proof: Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \setminus f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if A, B, C, and D are as described above, A also would have this same property that the empty set has. However, A is probably larger. Therefore, say $A_0 \subseteq X$ satisfies $\mathcal{P}$ if whenever $y \in Y \setminus f(A_0)$, $g(y) \notin A_0$.

$$\mathcal{A} \equiv \{A_0 \subseteq X : A_0 \text{ satisfies } \mathcal{P}\}.$$

Let $A = \cup\mathcal{A}$. If $y \in Y \setminus f(A)$, then for each $A_0 \in \mathcal{A}$, $y \in Y \setminus f(A_0)$ and so $g(y) \notin A_0$. Since $g(y) \notin A_0$ for all $A_0 \in \mathcal{A}$, it follows $g(y) \notin A$. Hence A satisfies $\mathcal{P}$ and is the largest subset of X which does so. Now define

$$C \equiv f(A), \ D \equiv Y \setminus C, \ B \equiv X \setminus A.$$

It only remains to verify that $g(D) = B$.

Suppose $x \in B = X \setminus A$. Then $A \cup \{x\}$ does not satisfy $\mathcal{P}$ and so there exists $y \in Y \setminus f(A \cup \{x\}) \subseteq D$ such that $g(y) \in A \cup \{x\}$. But $y \notin f(A)$ and so since A satisfies $\mathcal{P}$, it follows $g(y) \notin A$. Hence $g(y) = x$ and so $x \in g(D)$ and this proves the theorem.

Theorem 1.3 *(Schroder Bernstein) If $f : X \to Y$ and $g : Y \to X$ are one to one, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: Let A, B, C, D be the sets of Theorem1.2 and define

$$h(x) \equiv \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in B \end{cases}$$

Then h is the desired one to one and onto mapping.

Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 1.4 *Let I be a set and let X_i be a set for each $i \in I$. f is a choice function written as*

$$f \in \prod_{i \in I} X_i$$

if $f(i) \in X_i$ for each $i \in I$.

The axiom of choice says that if $X_i \neq \emptyset$ for each $i \in I$, for I a set, then

$$\prod_{i \in I} X_i \neq \emptyset.$$

Sometimes the two functions, f and g are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 1.5 *If $f : X \to Y$ is onto and $g : Y \to X$ is onto, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: For each $y \in Y$, $f^{-1}(y) \equiv \{x \in X : f(x) = y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_0^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_0^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_0^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then f_0^{-1} is one to one because if $f_0^{-1}(y_1) = f_0^{-1}(y_2)$, then

$$y_1 = f\left(f_0^{-1}(y_1)\right) = f\left(f_0^{-1}(y_2)\right) = y_2.$$

Similarly g_0^{-1} is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h : X \to Y$ which is one to one and onto.

Definition 1.6 *A set S, is finite if there exists a natural number n and a map θ which maps $\{1, \cdots, n\}$ one to one and onto S. S is infinite if it is not finite. A set S, is called countable if there exists a map θ mapping $\mathbb{N}$ one to one and onto S.(When θ maps a set A to a set B, this will be written as $\theta : A \to B$ in the future.) Here $\mathbb{N} \equiv \{1, 2, \cdots\}$, the natural numbers. S is at most countable if there exists a map $\theta : \mathbb{N} \to S$ which is onto.*

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

Theorem 1.7 *If X and Y are both at most countable, then $X \times Y$ is also at most countable. If either X or Y is countable, then $X \times Y$ is also countable.*

Proof: It is given that there exists a mapping $\eta : \mathbb{N} \to X$ which is onto. Define $\eta(i) \equiv x_i$ and consider X as the set $\{x_1, x_2, x_3, \cdots\}$. Similarly, consider Y as the set $\{y_1, y_2, y_3, \cdots\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$\begin{array}{lllll}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots & \leftarrow \text{Those which have } x_1 \text{ in first slot.} \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots & \leftarrow \text{Those which have } x_2 \text{ in first slot.} \\
(x_3, y_1) & (x_3, y_2) & (x_3, y_3) & \cdots & \leftarrow \text{Those which have } x_3 \text{ in first slot.} \cdot \\
\vdots & \vdots & \vdots & & \qquad\qquad \vdots
\end{array}$$

Follow a path through this array as follows.

$$\begin{array}{ccccc}
(x_1, y_1) & \rightarrow & (x_1, y_2) & & (x_1, y_3) \quad \rightarrow \\
 & \swarrow & & \nearrow & \\
(x_2, y_1) & & (x_2, y_2) & & \\
\downarrow & \nearrow & & & \\
(x_3, y_1) & & & &
\end{array}$$

Thus the first element of $X \times Y$ is (x_1, y_1), the second element of $X \times Y$ is (x_1, y_2), the third element of $X \times Y$ is (x_2, y_1) etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that X is countable. Then there exists $\alpha : \mathbb{N} \rightarrow X$ which is one to one and onto. Let $\beta : X \times Y \rightarrow \mathbb{N}$ be defined by $\beta((x,y)) \equiv \alpha^{-1}(x)$. Thus β is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 1.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$. This proves the theorem.

Theorem 1.8 *If X and Y are at most countable, then $X \cup Y$ is at most countable. If either X or Y are countable, then $X \cup Y$ is countable.*

Proof: As in the preceding theorem, $X = \{x_1, x_2, x_3, \cdots\}$ and $Y = \{y_1, y_2, y_3, \cdots\}$. Consider the following array consisting of $X \cup Y$ and path through it.

$$\begin{array}{cccccc} x_1 & \rightarrow & x_2 & & x_3 & \rightarrow \\ & \swarrow & & \nearrow & & \\ y_1 & \rightarrow & y_2 & & & \end{array}$$

Thus the first element of $X \cup Y$ is x_1, the second is x_2 the third is y_1 the fourth is y_2 etc.

Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that X is countable and $\alpha : \mathbb{N} \rightarrow X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$,and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, β maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 1.5 yields the conclusion. This proves the theorem.

1.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 1.9 *Let S be a set. $\sim$ is an equivalence relation on S if it satisfies the following axioms.*

1. *$x \sim x$ for all $x \in S$. (Reflexive)*
2. *If $x \sim y$ then $y \sim x$. (Symmetric)*
3. *If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)*

Definition 1.10 *$[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x.*

With the above definition one can prove the following simple theorem.

Theorem 1.11 *Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.*

1.4 Partially Ordered Sets

Definition 1.12 *Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that*

$$x \leq x \text{ for all } x \in \mathcal{F}.$$

$$\text{If } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 1.13 *(Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

The Riemann Stieltjes Integral

The integral originated in attempts to find areas of various shapes and the ideas involved in finding integrals are much older than the ideas related to finding derivatives. In fact, Archimedes[1] was finding areas of various curved shapes about 250 B.C. using the main ideas of the integral. What is presented here is a generalization of these ideas. The main interest is in the Riemann integral but if it is easy to generalize to the so called Stieltjes integral in which the length of an interval, $[x, y]$ is replaced with an expression of the form $F(y) - F(x)$ where F is an increasing function, then the generalization is given. However, there is much more that can be written about Stieltjes integrals than what is presented here. A good source for this is the book by Apostol, [3].

2.1 Upper And Lower Riemann Stieltjes Sums

The Riemann integral pertains to bounded functions which are defined on a bounded interval. Let $[a, b]$ be a closed interval. A set of points in $[a, b]$, $\{x_0, \cdots, x_n\}$ is a partition if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Such partitions are denoted by P or Q. For f a bounded function defined on $[a, b]$, let

$$M_i(f) \equiv \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$
$$m_i(f) \equiv \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

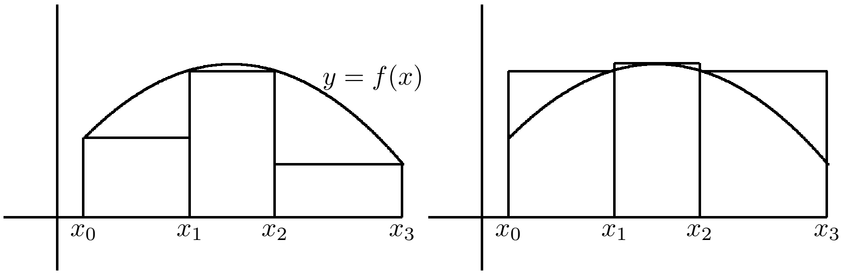
[1] Archimedes 287-212 B.C. found areas of curved regions by stuffing them with simple shapes which he knew the area of and taking a limit. He also made fundamental contributions to physics. The story is told about how he determined that a gold smith had cheated the king by giving him a crown which was not solid gold as had been claimed. He did this by finding the amount of water displaced by the crown and comparing with the amount of water it should have displaced if it had been solid gold.

Definition 2.1 *Let F be an increasing function defined on $[a,b]$ and let $\Delta F_i \equiv F(x_i) - F(x_{i-1})$. Then define upper and lower sums as*

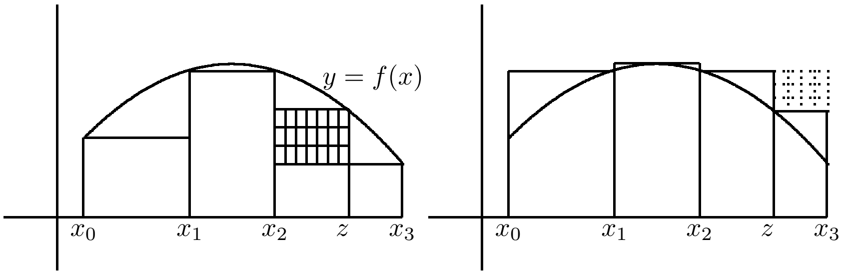
$$U(f,P) \equiv \sum_{i=1}^{n} M_i(f)\,\Delta F_i \text{ and } L(f,P) \equiv \sum_{i=1}^{n} m_i(f)\,\Delta F_i$$

respectively. The numbers, $M_i(f)$ and $m_i(f)$, are well defined real numbers because f is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S = \{f(x) : x \in [x_{i-1}, x_i]\}$ is bounded above and below.

In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition. In these pictures the function, F is given by $F(x) = x$ and these are the ordinary upper and lower sums from calculus.



What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled z has been added in.



Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 2.2 *If $P \subseteq Q$ then*

$$U(f,Q) \leq U(f,P), \text{ and } L(f,P) \leq L(f,Q).$$

Proof: This is verified by adding in one point at a time. Thus let $P = \{x_0, \cdots, x_n\}$ and let $Q = \{x_0, \cdots, x_k, y, x_{k+1}, \cdots, x_n\}$. Thus exactly one point, y, is added between x_k and x_{k+1}. Now the term in the upper sum which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f, P)$ is

$$\sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(x_k)) \tag{2.1}$$

and the term which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f, Q)$ is

$$\sup \{f(x) : x \in [x_k, y]\} (F(y) - F(x_k)) \tag{2.2}$$
$$+ \sup \{f(x) : x \in [y, x_{k+1}]\} (F(x_{k+1}) - F(y)) \tag{2.3}$$
$$\equiv M_1 (F(y) - F(x_k)) + M_2 (F(x_{k+1}) - F(y)) \tag{2.4}$$

All the other terms in the two sums coincide. Now $\sup \{f(x) : x \in [x_k, x_{k+1}]\} \geq \max(M_1, M_2)$ and so the expression in 2.2 is no larger than

$$\begin{gathered}\sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(y)) \\ + \sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(y) - F(x_k))\end{gathered}$$

$$= \sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(x_k)),$$

the term corresponding to the interval, $[x_k, x_{k+1}]$ and $U(f, P)$. This proves the first part of the lemma pertaining to upper sums because if $Q \supseteq P$, one can obtain Q from P by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part about lower sums is similar and is left as an exercise.

Lemma 2.3 *If P and Q are two partitions, then*

$$L(f, P) \leq U(f, Q).$$

Proof: By Lemma 2.2,

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

Definition 2.4

$$\overline{I} \equiv \inf\{U(f, Q) \text{ where } Q \text{ is a partition}\}$$

$$\underline{I} \equiv \sup\{L(f, P) \text{ where } P \text{ is a partition}\}.$$

Note that $\underline{I}$ and $\overline{I}$ are well defined real numbers.

Theorem 2.5 $\underline{I} \leq \overline{I}$.

Proof: From Lemma 2.3,

$$\underline{I} = \sup\{L(f, P) \text{ where } P \text{ is a partition}\} \leq U(f, Q)$$

because $U(f,Q)$ is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since Q is arbitrary,

$$\begin{aligned}\underline{I} &= \sup\{L(f,P) \text{ where } P \text{ is a partition}\} \\ &\leq \inf\{U(f,Q) \text{ where } Q \text{ is a partition}\} \equiv \overline{I}\end{aligned}$$

where the inequality holds because it was just shown that $\underline{I}$ is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 2.6 *A bounded function f is Riemann Stieltjes integrable, written as*

$$f \in R([a,b])$$

if

$$\underline{I} = \overline{I}$$

and in this case,

$$\int_a^b f(x)\, dF \equiv \underline{I} = \overline{I}.$$

When $F(x) = x$, the integral is called the Riemann integral and is written as

$$\int_a^b f(x)\, dx.$$

Thus, in words, the Riemann integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall the following Proposition which comes from the definitions.

Proposition 2.7 *Let S be a nonempty set and suppose* $\sup(S)$ *exists. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ *exists, then for every $\delta > 0$,*

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

This proposition implies the following theorem which is used to determine the question of Riemann Stieltjes integrability.

Theorem 2.8 *A bounded function f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a partition P such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{2.5}$$

Proof: First assume f is Riemann integrable. Then let P and Q be two partitions such that

$$U(f,Q) < \overline{I} + \varepsilon/2,\ L(f,P) > \underline{I} - \varepsilon/2.$$

Then since $\underline{I} = \overline{I}$,

$$U(f,Q\cup P) - L(f,P\cup Q) \leq U(f,Q) - L(f,P) < \overline{I} + \varepsilon/2 - (\underline{I} - \varepsilon/2) = \varepsilon.$$

Now suppose that for all $\varepsilon > 0$ there exists a partition such that 2.5 holds. Then for given ε and partition P corresponding to ε

$$\overline{I} - \underline{I} \leq U(f,P) - L(f,P) \leq \varepsilon.$$

Since ε is arbitrary, this shows $\underline{I} = \overline{I}$ and this proves the theorem.

The condition described in the theorem is called the Riemann criterion .

Not all bounded functions are Riemann integrable. For example, let $F(x) = x$ and

$$f(x) \equiv \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{2.6}$$

Then if $[a,b] = [0,1]$ all upper sums for f equal 1 while all lower sums for f equal 0. Therefore the Riemann criterion is violated for $\varepsilon = 1/2$.

2.2 Exercises

1. Prove the second half of Lemma 2.2 about lower sums.

2. Verify that for f given in 2.6, the lower sums on the interval $[0,1]$ are all equal to zero while the upper sums are all equal to one.

3. Let $f(x) = 1 + x^2$ for $x \in [-1,3]$ and let $P = \left\{-1, -\frac{1}{3}, 0, \frac{1}{2}, 1, 2\right\}$. Find $U(f,P)$ and $L(f,P)$ for $F(x) = x$ and for $F(x) = x^3$.

4. Show that if $f \in R([a,b])$ for $F(x) = x$, there exists a partition, $\{x_0, \cdots, x_n\}$ such that for any $z_k \in [x_k, x_{k+1}]$,

$$\left| \int_a^b f(x)\ dx - \sum_{k=1}^{n} f(z_k)(x_k - x_{k-1}) \right| < \varepsilon$$

   This sum, $\sum_{k=1}^{n} f(z_k)(x_k - x_{k-1})$, is called a Riemann sum and this exercise shows that the Riemann integral can always be approximated by a Riemann sum. For the general Riemann Stieltjes case, does anything change?

5. Let $P = \left\{1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2\right\}$ and $F(x) = x$. Find upper and lower sums for the function, $f(x) = \frac{1}{x}$ using this partition. What does this tell you about $\ln(2)$?

6. If $f \in R([a,b])$ with $F(x) = x$ and f is changed at finitely many points, show the new function is also in $R([a,b])$. Is this still true for the general case where F is only assumed to be an increasing function? Explain.

7. In the case where $F(x) = x$, define a "left sum" as

$$\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

and a "right sum",

$$\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$

Also suppose that all partitions have the property that $x_k - x_{k-1}$ equals a constant, $(b-a)/n$ so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as n gets larger and larger. Show that for f given in 2.6, $\int_0^x f(t)\, dt = 1$ if x is rational and $\int_0^x f(t)\, dt = 0$ if x is irrational. It turns out that the correct answer should always equal zero for that function, regardless of whether x is rational. This is shown when the Lebesgue integral is studied. This illustrates why this method of defining the integral in terms of left and right sums is total nonsense. Show that even though this is the case, it makes no difference if f is continuous.

2.3 Functions Of Riemann Integrable Functions

It is often necessary to consider functions of Riemann integrable functions and a natural question is whether these are Riemann integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

Theorem 2.9 *Let f, g be bounded functions and let $f([a,b]) \subseteq [c_1, d_1]$ and $g([a,b]) \subseteq [c_2, d_2]$. Let $H : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R}$ satisfy,*

$$|H(a_1, b_1) - H(a_2, b_2)| \leq K[|a_1 - a_2| + |b_1 - b_2|]$$

for some constant K. Then if $f, g \in R([a,b])$ it follows that $H \circ (f, g) \in R([a,b])$.

Proof: In the following claim, $M_i(h)$ and $m_i(h)$ have the meanings assigned above with respect to some partition of $[a,b]$ for the function, h.

Claim: The following inequality holds.

$$|M_i(H \circ (f,g)) - m_i(H \circ (f,g))| \leq$$

$$K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|].$$

Proof of the claim: By the above proposition, there exist $x_1, x_2 \in [x_{i-1}, x_i]$ be such that

$$H(f(x_1), g(x_1)) + \eta > M_i(H \circ (f,g)),$$

and

$$H(f(x_2), g(x_2)) - \eta < m_i(H \circ (f,g)).$$

Then

$$|M_i (H \circ (f,g)) - m_i (H \circ (f,g))|$$

$$\begin{aligned}
&< 2\eta + |H (f (x_1), g (x_1)) - H (f (x_2), g (x_2))| \\
&< 2\eta + K [|f (x_1) - f (x_2)| + |g (x_1) - g (x_2)|] \\
&\leq 2\eta + K [|M_i (f) - m_i (f)| + |M_i (g) - m_i (g)|] .
\end{aligned}$$

Since $\eta > 0$ is arbitrary, this proves the claim.

Now continuing with the proof of the theorem, let P be such that

$$\sum_{i=1}^{n} (M_i (f) - m_i (f)) \Delta F_i < \frac{\varepsilon}{2K}, \ \sum_{i=1}^{n} (M_i (g) - m_i (g)) \Delta F_i < \frac{\varepsilon}{2K}.$$

Then from the claim,

$$\sum_{i=1}^{n} (M_i (H \circ (f,g)) - m_i (H \circ (f,g))) \Delta F_i$$

$$< \sum_{i=1}^{n} K [|M_i (f) - m_i (f)| + |M_i (g) - m_i (g)|] \Delta F_i < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows $H \circ (f,g)$ satisfies the Riemann criterion and hence $H \circ (f,g)$ is Riemann integrable as claimed. This proves the theorem.

This theorem implies that if f,g are Riemann Stieltjes integrable, then so is $af + bg, |f|, f^2$, along with infinitely many other such continuous combinations of Riemann Stieltjes integrable functions. For example, to see that $|f|$ is Riemann integrable, let $H (a,b) = |a|$. Clearly this function satisfies the conditions of the above theorem and so $|f| = H (f,f) \in R ([a,b])$ as claimed. The following theorem gives an example of many functions which are Riemann integrable.

Theorem 2.10 *Let $f : [a,b] \to \mathbb{R}$ be either increasing or decreasing on $[a,b]$ and suppose F is continuous. Then $f \in R ([a,b])$.*

Proof: Let $\varepsilon > 0$ be given and let

$$x_i = a + i \left(\frac{b-a}{n} \right), \ i = 0, \cdots, n.$$

Since F is continuous, it follows that it is uniformly continuous. Therefore, if n is large enough, then for all i,

$$F (x_i) - F (x_{i-1}) < \frac{\varepsilon}{f (b) - f (a) + 1}$$

Then since f is increasing,

$$
\begin{aligned}
U(f,P)-L(f,P) &= \sum_{i=1}^{n}\left(f(x_i)-f(x_{i-1})\right)\left(F(x_i)-F(x_{i-1})\right) \\
&\leq \frac{\varepsilon}{f(b)-f(a)+1}\sum_{i=1}^{n}\left(f(x_i)-f(x_{i-1})\right) \\
&= \frac{\varepsilon}{f(b)-f(a)+1}\left(f(b)-f(a)\right)<\varepsilon.
\end{aligned}
$$

Thus the Riemann criterion is satisfied and so the function is Riemann Stieltjes integrable. The proof for decreasing f is similar.

Corollary 2.11 *Let $[a,b]$ be a bounded closed interval and let $\phi:[a,b]\to\mathbb{R}$ be Lipschitz continuous and suppose F is continuous. Then $\phi\in R([a,b])$. Recall that a function, ϕ, is Lipschitz continuous if there is a constant, K, such that for all x,y,*

$$|\phi(x)-\phi(y)|<K|x-y|.$$

Proof: Let $f(x)=x$. Then by Theorem 2.10, f is Riemann Stieltjes integrable. Let $H(a,b)\equiv\phi(a)$. Then by Theorem 2.9 $H\circ(f,f)=\phi\circ f=\phi$ is also Riemann Stieltjes integrable. This proves the corollary.

In fact, it is enough to assume ϕ is continuous, although this is harder. This is the content of the next theorem which is where the difficult theorems about continuity and uniform continuity are used. This is the main result on the existence of the Riemann Stieltjes integral for this book.

Theorem 2.12 *Suppose $f:[a,b]\to\mathbb{R}$ is continuous and F is just an increasing function defined on $[a,b]$. Then $f\in R([a,b])$.*

Proof: Since f is continuous, it follows f is uniformly continuous on $[a,b]$. Therefore, if $\varepsilon>0$ is given, there exists a $\delta>0$ such that if $|x_i-x_{i-1}|<\delta$, then $M_i-m_i<\frac{\varepsilon}{F(b)-F(a)+1}$. Let

$$P\equiv\{x_0,\cdots,x_n\}$$

be a partition with $|x_i-x_{i-1}|<\delta$. Then

$$
\begin{aligned}
U(f,P)-L(f,P) &< \sum_{i=1}^{n}(M_i-m_i)\left(F(x_i)-F(x_{i-1})\right) \\
&< \frac{\varepsilon}{F(b)-F(a)+1}\left(F(b)-F(a)\right)<\varepsilon.
\end{aligned}
$$

By the Riemann criterion, $f\in R([a,b])$. This proves the theorem.

2.4 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.

Lemma 2.13 *Let S be a nonempty set which is bounded above and below. Then if $-S \equiv \{-x : x \in S\}$,*

$$\sup(-S) = -\inf(S) \tag{2.7}$$

and

$$\inf(-S) = -\sup(S). \tag{2.8}$$

Proof: Consider 2.7. Let $x \in S$. Then $-x \leq \sup(-S)$ and so $x \geq -\sup(-S)$. If follows that $-\sup(-S)$ is a lower bound for S and therefore, $-\sup(-S) \leq \inf(S)$. This implies $\sup(-S) \geq -\inf(S)$. Now let $-x \in -S$. Then $x \in S$ and so $x \geq \inf(S)$ which implies $-x \leq -\inf(S)$. Therefore, $-\inf(S)$ is an upper bound for $-S$ and so $-\inf(S) \geq \sup(-S)$. This shows 2.7. Formula 2.8 is similar and is left as an exercise.

In particular, the above lemma implies that for $M_i(f)$ and $m_i(f)$ defined above $M_i(-f) = -m_i(f)$, and $m_i(-f) = -M_i(f)$.

Lemma 2.14 *If $f \in R([a,b])$ then $-f \in R([a,b])$ and*

$$-\int_a^b f(x)\, dF = \int_a^b -f(x)\, dF.$$

Proof: The first part of the conclusion of this lemma follows from Theorem 2.10 since the function $\phi(y) \equiv -y$ is Lipschitz continuous. Now choose P such that

$$\int_a^b -f(x)\, dF - L(-f,P) < \varepsilon.$$

Then since $m_i(-f) = -M_i(f)$,

$$\varepsilon > \int_a^b -f(x)\, dF - \sum_{i=1}^n m_i(-f)\,\Delta F_i = \int_a^b -f(x)\, dF + \sum_{i=1}^n M_i(f)\,\Delta F_i$$

which implies

$$\varepsilon > \int_a^b -f(x)\, dF + \sum_{i=1}^n M_i(f)\,\Delta F_i \geq \int_a^b -f(x)\, dF + \int_a^b f(x)\, dF.$$

Thus, since ε is arbitrary,

$$\int_a^b -f(x)\, dF \leq -\int_a^b f(x)\, dF$$

whenever $f \in R([a,b])$. It follows

$$\int_a^b -f(x)\, dF \leq -\int_a^b f(x)\, dF = -\int_a^b -(-f(x))\, dF \leq \int_a^b -f(x)\, dF$$

and this proves the lemma.

Theorem 2.15 *The integral is linear,*

$$\int_a^b (\alpha f + \beta g)(x)\ dF = \alpha \int_a^b f(x)\ dF + \beta \int_a^b g(x)\ dF.$$

whenever $f, g \in R([a,b])$ *and* $\alpha, \beta \in \mathbb{R}$.

Proof: First note that by Theorem 2.9, $\alpha f + \beta g \in R([a,b])$. To begin with, consider the claim that if $f, g \in R([a,b])$ then

$$\int_a^b (f+g)(x)\ dF = \int_a^b f(x)\ dF + \int_a^b g(x)\ dF. \tag{2.9}$$

Let P_1,Q_1 be such that

$$U(f, Q_1) - L(f, Q_1) < \varepsilon/2,\ U(g, P_1) - L(g, P_1) < \varepsilon/2.$$

Then letting $P \equiv P_1 \cup Q_1$, Lemma 2.2 implies

$$U(f, P) - L(f, P) < \varepsilon/2, \text{ and } U(g, P) - U(g, P) < \varepsilon/2.$$

Next note that

$$m_i(f+g) \geq m_i(f) + m_i(g),\ M_i(f+g) \leq M_i(f) + M_i(g).$$

Therefore,

$$L(g+f, P) \geq L(f, P) + L(g, P),\ U(g+f, P) \leq U(f, P) + U(g, P).$$

For this partition,

$$\begin{aligned}\int_a^b (f+g)(x)\ dF &\in [L(f+g, P), U(f+g, P)] \\ &\subseteq [L(f, P) + L(g, P), U(f, P) + U(g, P)]\end{aligned}$$

and

$$\int_a^b f(x)\ dF + \int_a^b g(x)\ dF \in [L(f, P) + L(g, P), U(f, P) + U(g, P)].$$

Therefore,

$$\left| \int_a^b (f+g)(x)\ dF - \left(\int_a^b f(x)\ dF + \int_a^b g(x)\ dF \right) \right| \leq$$

$$U(f, P) + U(g, P) - (L(f, P) + L(g, P)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves 2.9 since ε is arbitrary.

It remains to show that

$$\alpha \int_a^b f(x)\ dF = \int_a^b \alpha f(x)\ dF.$$

Suppose first that $\alpha \geq 0$. Then

$$\int_a^b \alpha f(x)\ dF \equiv \sup\{L(\alpha f, P) : P \text{ is a partition}\} =$$

$$\alpha \sup\{L(f, P) : P \text{ is a partition}\} \equiv \alpha \int_a^b f(x)\ dF.$$

If $\alpha < 0$, then this and Lemma 2.14 imply

$$\int_a^b \alpha f(x)\ dF = \int_a^b (-\alpha)(-f(x))\ dF$$

$$= (-\alpha) \int_a^b (-f(x))\ dF = \alpha \int_a^b f(x)\ dF.$$

This proves the theorem.

In the next theorem, suppose F is defined on $[a, b] \cup [b, c]$.

Theorem 2.16 *If $f \in R([a, b])$ and $f \in R([b, c])$, then $f \in R([a, c])$ and*

$$\int_a^c f(x)\ dF = \int_a^b f(x)\ dF + \int_b^c f(x)\ dF. \tag{2.10}$$

Proof: Let P_1 be a partition of $[a, b]$ and P_2 be a partition of $[b, c]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon/2,\ i = 1, 2.$$

Let $P \equiv P_1 \cup P_2$. Then P is a partition of $[a, c]$ and

$$U(f, P) - L(f, P)$$
$$= U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{2.11}$$

Thus, $f \in R([a, c])$ by the Riemann criterion and also for this partition,

$$\int_a^b f(x)\ dF + \int_b^c f(x)\ dF \in [L(f, P_1) + L(f, P_2), U(f, P_1) + U(f, P_2)]$$
$$= [L(f, P), U(f, P)]$$

and

$$\int_a^c f(x)\ dF \in [L(f, P), U(f, P)].$$

Hence by 2.11,

$$\left| \int_a^c f(x)\ dF - \left(\int_a^b f(x)\ dF + \int_b^c f(x)\ dF \right) \right| < U(f, P) - L(f, P) < \varepsilon$$

which shows that since ε is arbitrary, 2.10 holds. This proves the theorem.

Corollary 2.17 *Let F be continuous and let $[a, b]$ be a closed and bounded interval and suppose that*

$$a = y_1 < y_2 \cdots < y_l = b$$

and that f is a bounded function defined on $[a, b]$ which has the property that f is either increasing on $[y_j, y_{j+1}]$ or decreasing on $[y_j, y_{j+1}]$ for $j = 1, \cdots, l-1$. Then $f \in R([a, b])$.

Proof: This follows from Theorem 2.16 and Theorem 2.10.

The symbol, $\int_a^b f(x)\, dF$ when $a > b$ has not yet been defined.

Definition 2.18 *Let $[a, b]$ be an interval and let $f \in R([a, b])$. Then*

$$\int_b^a f(x)\, dF \equiv -\int_a^b f(x)\, dF.$$

Note that with this definition,

$$\int_a^a f(x)\, dF = -\int_a^a f(x)\, dF$$

and so

$$\int_a^a f(x)\, dF = 0.$$

Theorem 2.19 *Assuming all the integrals make sense,*

$$\int_a^b f(x)\, dF + \int_b^c f(x)\, dF = \int_a^c f(x)\, dF.$$

Proof: This follows from Theorem 2.16 and Definition 2.18. For example, assume

$$c \in (a, b).$$

Then from Theorem 2.16,

$$\int_a^c f(x)\, dF + \int_c^b f(x)\, dF = \int_a^b f(x)\, dF$$

and so by Definition 2.18,

$$\begin{aligned}\int_a^c f(x)\, dF &= \int_a^b f(x)\, dF - \int_c^b f(x)\, dF \\ &= \int_a^b f(x)\, dF + \int_b^c f(x)\, dF.\end{aligned}$$

The other cases are similar.

The following properties of the integral have either been established or they follow quickly from what has been shown so far.

$$\text{If } f \in R([a,b]) \text{ then if } c \in [a,b],\ f \in R([a,c]), \tag{2.12}$$

$$\int_a^b \alpha\, dF = \alpha\left(F(b) - F(a)\right), \tag{2.13}$$

$$\int_a^b (\alpha f + \beta g)(x)\ dF = \alpha \int_a^b f(x)\ dF + \beta \int_a^b g(x)\ dF, \tag{2.14}$$

$$\int_a^b f(x)\ dF + \int_b^c f(x)\ dF = \int_a^c f(x)\ dF, \tag{2.15}$$

$$\int_a^b f(x)\ dF \geq 0 \text{ if } f(x) \geq 0 \text{ and } a < b, \tag{2.16}$$

$$\left|\int_a^b f(x)\ dF\right| \leq \left|\int_a^b |f(x)|\ dF\right|. \tag{2.17}$$

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

$$|f(x)| - f(x) \geq 0,\ |f(x)| + f(x) \geq 0.$$

Therefore, by 2.16 and 2.14, if $a < b$,

$$\int_a^b |f(x)|\ dF \geq \int_a^b f(x)\ dF$$

and

$$\int_a^b |f(x)|\ dF \geq -\int_a^b f(x)\ dF.$$

Therefore,

$$\int_a^b |f(x)|\ dF \geq \left|\int_a^b f(x)\ dF\right|.$$

If $b < a$ then the above inequality holds with a and b switched. This implies 2.17.

2.5 Fundamental Theorem Of Calculus

In this section $F(x) = x$ so things are specialized to the ordinary Riemann integral. With these properties, it is easy to prove the fundamental theorem of calculus[2].

[2]This theorem is why Newton and Liebnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton's predecessor, Isaac Barrow had made some progress in this direction.

Let $f \in R([a,b])$. Then by 2.12 $f \in R([a,x])$ for each $x \in [a,b]$. The first version of the fundamental theorem of calculus is a statement about the derivative of the function

$$x \to \int_a^x f(t)\, dt.$$

Theorem 2.20 *Let $f \in R([a,b])$ and let*

$$F(x) \equiv \int_a^x f(t)\, dt.$$

Then if f is continuous at $x \in (a,b)$,

$$F'(x) = f(x).$$

Proof: Let $x \in (a,b)$ be a point of continuity of f and let h be small enough that $x+h \in [a,b]$. Then by using 2.15,

$$h^{-1}(F(x+h) - F(x)) = h^{-1} \int_x^{x+h} f(t)\, dt.$$

Also, using 2.13,

$$f(x) = h^{-1} \int_x^{x+h} f(x)\, dt.$$

Therefore, by 2.17,

$$\left|h^{-1}(F(x+h) - F(x)) - f(x)\right| = \left|h^{-1} \int_x^{x+h} (f(t) - f(x))\, dt\right|$$

$$\leq \left|h^{-1} \int_x^{x+h} |f(t) - f(x)|\, dt\right|.$$

Let $\varepsilon > 0$ and let $\delta > 0$ be small enough that if $|t-x| < \delta$, then

$$|f(t) - f(x)| < \varepsilon.$$

Therefore, if $|h| < \delta$, the above inequality and 2.13 shows that

$$\left|h^{-1}(F(x+h) - F(x)) - f(x)\right| \leq |h|^{-1} \varepsilon |h| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\lim_{h\to 0} h^{-1}(F(x+h) - F(x)) = f(x)$$

and this proves the theorem.

Note this gives existence for the initial value problem,

$$F'(x) = f(x),\ F(a) = 0$$

whenever f is Riemann integrable and continuous.[3]

The next theorem is also called the fundamental theorem of calculus.

Theorem 2.21 *Let $f \in R([a,b])$ and suppose there exists an antiderivative for $f, G,$ such that*

$$G'(x) = f(x)$$

for every point of (a,b) and G is continuous on $[a,b]$. Then

$$\int_a^b f(x)\, dx = G(b) - G(a). \tag{2.18}$$

Proof: Let $P = \{x_0, \cdots, x_n\}$ be a partition satisfying

$$U(f,P) - L(f,P) < \varepsilon.$$

Then

$$\begin{aligned} G(b) - G(a) &= G(x_n) - G(x_0) \\ &= \sum_{i=1}^{n} G(x_i) - G(x_{i-1}). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} G(b) - G(a) &= \sum_{i=1}^{n} G'(z_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(z_i)\, \Delta x_i \end{aligned}$$

where z_i is some point in $[x_{i-1}, x_i]$. It follows, since the above sum lies between the upper and lower sums, that

$$G(b) - G(a) \in [L(f,P), U(f,P)],$$

and also

$$\int_a^b f(x)\, dx \in [L(f,P), U(f,P)].$$

Therefore,

$$\left| G(b) - G(a) - \int_a^b f(x)\, dx \right| < U(f,P) - L(f,P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, 2.18 holds. This proves the theorem.

[3]Of course it was proved that if f is continuous on a closed interval, $[a,b]$, then $f \in R([a,b])$ but this is a hard theorem using the difficult result about uniform continuity.

The following notation is often used in this context. Suppose F is an antiderivative of f as just described with F continuous on $[a,b]$ and $F' = f$ on (a,b). Then

$$\int_a^b f(x)\ dx = F(b) - F(a) \equiv F(x)\,|_a^b.$$

Definition 2.22 *Let f be a bounded function defined on a closed interval $[a,b]$ and let $P \equiv \{x_0, \cdots, x_n\}$ be a partition of the interval. Suppose $z_i \in [x_{i-1}, x_i]$ is chosen. Then the sum*

$$\sum_{i=1}^n f(z_i)(x_i - x_{i-1})$$

is known as a Riemann sum. Also,

$$||P|| \equiv \max\{|x_i - x_{i-1}| : i = 1, \cdots, n\}.$$

Proposition 2.23 *Suppose $f \in R([a,b])$. Then there exists a partition, $P \equiv \{x_0, \cdots, x_n\}$ with the property that for any choice of $z_k \in [x_{k-1}, x_k]$,*

$$\left|\int_a^b f(x)\ dx - \sum_{k=1}^n f(z_k)(x_k - x_{k-1})\right| < \varepsilon.$$

Proof: Choose P such that $U(f,P) - L(f,P) < \varepsilon$ and then both $\int_a^b f(x)\ dx$ and $\sum_{k=1}^n f(z_k)(x_k - x_{k-1})$ are contained in $[L(f,P), U(f,P)]$ and so the claimed inequality must hold. This proves the proposition.

It is significant because it gives a way of approximating the integral.

The definition of Riemann integrability given in this chapter is also called Darboux integrability and the integral defined as the unique number which lies between all upper sums and all lower sums which is given in this chapter is called the Darboux integral . The definition of the Riemann integral in terms of Riemann sums is given next.

Definition 2.24 *A bounded function, f defined on $[a,b]$ is said to be Riemann integrable if there exists a number, I with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$P \equiv \{x_0, x_1, \cdots, x_n\}$$

is any partition having $||P|| < \delta$, and $z_i \in [x_{i-1}, x_i]$,

$$\left|I - \sum_{i=1}^n f(z_i)(x_i - x_{i-1})\right| < \varepsilon.$$

The number $\int_a^b f(x)\ dx$ is defined as I.

Thus, there are two definitions of the Riemann integral. It turns out they are equivalent which is the following theorem of of Darboux.

Theorem 2.25 *A bounded function defined on $[a, b]$ is Riemann integrable in the sense of Definition 2.24 if and only if it is integrable in the sense of Darboux. Furthermore the two integrals coincide.*

The proof of this theorem is left for the exercises in Problems 10 - 12. It isn't essential that you understand this theorem so if it does not interest you, leave it out. Note that it implies that given a Riemann integrable function f in either sense, it can be approximated by Riemann sums whenever $||P||$ is sufficiently small. Both versions of the integral are obsolete but entirely adequate for most applications and as a point of departure for a more up to date and satisfactory integral. The reason for using the Darboux approach to the integral is that all the existence theorems are easier to prove in this context.

2.6 Exercises

1. Let $F(x) = \int_{x^2}^{x^3} \frac{t^5+7}{t^7+87t^6+1} \, dt$. Find $F'(x)$.

2. Let $F(x) = \int_2^x \frac{1}{1+t^4} \, dt$. Sketch a graph of F and explain why it looks the way it does.

3. Let a and b be positive numbers and consider the function,

$$F(x) = \int_0^{ax} \frac{1}{a^2 + t^2} \, dt + \int_b^{a/x} \frac{1}{a^2 + t^2} \, dt.$$

   Show that F is a constant.

4. Solve the following initial value problem from ordinary differential equations which is to find a function y such that

$$y'(x) = \frac{x^7 + 1}{x^6 + 97x^5 + 7}, \; y(10) = 5.$$

5. If $F, G \in \int f(x) \, dx$ for all $x \in \mathbb{R}$, show $F(x) = G(x) + C$ for some constant, C. Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion $\int_a^b f(t) \, dt = G(b) - G(a)$ where $G'(x) = f(x)$.

6. Suppose f is Riemann integrable on $[a, b]$ and continuous. (In fact continuous implies Riemann integrable.) Show there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

   Hint: You might consider the function $F(x) \equiv \int_a^x f(t) \, dt$ and use the mean value theorem for derivatives and the fundamental theorem of calculus.

7. Suppose f and g are continuous functions on $[a,b]$ and that $g(x) \neq 0$ on (a,b). Show there exists $c \in (a,b)$ such that

$$f(c) \int_a^b g(x)\, dx = \int_a^b f(x) g(x)\, dx.$$

Hint: Define $F(x) \equiv \int_a^x f(t) g(t)\, dt$ and let $G(x) \equiv \int_a^x g(t)\, dt$. Then use the Cauchy mean value theorem on these two functions.

8. Consider the function

$$f(x) \equiv \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$

Is f Riemann integrable? Explain why or why not.

9. Prove the second part of Theorem 2.10 about decreasing functions.

10. Suppose f is a bounded function defined on $[a,b]$ and $|f(x)| < M$ for all $x \in [a,b]$. Now let Q be a partition having n points, $\{x_0^*, \cdots, x_n^*\}$ and let P be any other partition. Show that

$$|U(f,P) - L(f,P)| \leq 2Mn\, ||P|| + |U(f,Q) - L(f,Q)|.$$

Hint: Write the sum for $U(f,P) - L(f,P)$ and split this sum into two sums, the sum of terms for which $[x_{i-1}, x_i]$ contains at least one point of Q, and terms for which $[x_{i-1}, x_i]$ does not contain any points of Q. In the latter case, $[x_{i-1}, x_i]$ must be contained in some interval, $\left[x_{k-1}^*, x_k^*\right]$. Therefore, the sum of these terms should be no larger than $|U(f,Q) - L(f,Q)|$.

11. $\uparrow$ If $\varepsilon > 0$ is given and f is a Darboux integrable function defined on $[a,b]$, show there exists $\delta > 0$ such that whenever $||P|| < \delta$, then

$$|U(f,P) - L(f,P)| < \varepsilon.$$

12. $\uparrow$ Prove Theorem 2.25.

Important Linear Algebra

This chapter contains some important linear algebra as distinguished from that which is normally presented in undergraduate courses consisting mainly of uninteresting things you can do with row operations.

The notation, $\mathbb{C}^n$ refers to the collection of ordered lists of n complex numbers. Since every real number is also a complex number, this simply generalizes the usual notion of $\mathbb{R}^n$, the collection of all ordered lists of n real numbers. In order to avoid worrying about whether it is real or complex numbers which are being referred to, the symbol $\mathbb{F}$ will be used. If it is not clear, always pick $\mathbb{C}$.

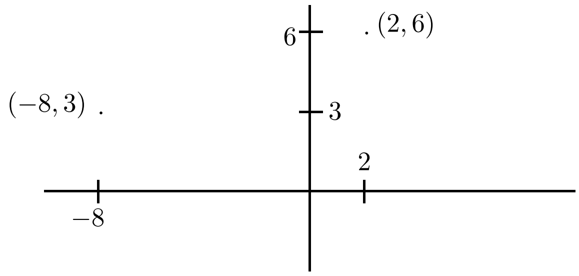
Definition 3.1 *Define* $\mathbb{F}^n \equiv \{(x_1, \cdots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \cdots, n\}$. $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$ *if and only if for all* $j = 1, \cdots, n$, $x_j = y_j$. *When* $(x_1, \cdots, x_n) \in \mathbb{F}^n$, *it is conventional to denote* $(x_1, \cdots, x_n)$ *by the single bold face letter,* $\mathbf{x}$. *The numbers,* x_j *are called the coordinates. The set*

$$\{(0, \cdots, 0, t, 0, \cdots, 0) : t \in \mathbb{F}\}$$

for t *in the* i^{th} *slot is called the* i^{th} *coordinate axis. The point* $\mathbf{0} \equiv (0, \cdots, 0)$ *is called the origin.*

Thus $(1, 2, 4i) \in \mathbb{F}^3$ and $(2, 1, 4i) \in \mathbb{F}^3$ but $(1, 2, 4i) \neq (2, 1, 4i)$ because, even though the same numbers are involved, they don't match up. In particular, the first entries are not equal.

The geometric significance of $\mathbb{R}^n$ for $n \leq 3$ has been encountered already in calculus or in precalculus. Here is a short review. First consider the case when $n = 1$. Then from the definition, $\mathbb{R}^1 = \mathbb{R}$. Recall that $\mathbb{R}$ is identified with the points of a line. Look at the number line again. Observe that this amounts to identifying a point on this line with a real number. In other words a real number determines where you are on this line. Now suppose $n = 2$ and consider two lines which intersect each other at right angles as shown in the following picture.



Notice how you can identify a point shown in the plane with the ordered pair, $(2,6)$. You go to the right a distance of 2 and then up a distance of 6. Similarly, you can identify another point in the plane with the ordered pair $(-8,3)$. Go to the left a distance of 8 and then up a distance of 3. The reason you go to the left is that there is a $-$ sign on the eight. From this reasoning, every ordered pair determines a unique point in the plane. Conversely, taking a point in the plane, you could draw two lines through the point, one vertical and the other horizontal and determine unique points, x_1 on the horizontal line in the above picture and x_2 on the vertical line in the above picture, such that the point of interest is identified with the ordered pair, (x_1,x_2). In short, points in the plane can be identified with ordered pairs similar to the way that points on the real line are identified with real numbers. Now suppose $n=3$. As just explained, the first two coordinates determine a point in a plane. Letting the third component determine how far up or down you go, depending on whether this number is positive or negative, this determines a point in space. Thus, $(1,4,-5)$ would mean to determine the point in the plane that goes with $(1,4)$ and then to go below this plane a distance of 5 to obtain a unique point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

You can't stop here and say that you are only interested in $n\leq 3$. What if you were interested in the motion of two objects? You would need three coordinates to describe where the first object is and you would need another three coordinates to describe where the other object is located. Therefore, you would need to be considering $\mathbb{R}^6$. If the two objects moved around, you would need a time coordinate as well. As another example, consider a hot object which is cooling and suppose you want the temperature of this object. How many coordinates would be needed? You would need one for the temperature, three for the position of the point in the object and one more for the time. Thus you would need to be considering $\mathbb{R}^5$. Many other examples can be given. Sometimes n is very large. This is often the case in applications to business when they are trying to maximize profit subject to constraints. It also occurs in numerical analysis when people try to solve hard problems on a computer.

There are other ways to identify points in space with three numbers but the one presented is the most basic. In this case, the coordinates are known as Cartesian coordinates after Descartes[1] who invented this idea in the first half of the seventeenth century. I will often not bother to draw a distinction between the point in n dimensional space and its Cartesian coordinates.

The geometric significance of $\mathbb{C}^n$ for $n > 1$ is not available because each copy of $\mathbb{C}$ corresponds to the plane or $\mathbb{R}^2$.

3.1 Algebra in $\mathbb{F}^n$

There are two algebraic operations done with elements of $\mathbb{F}^n$. One is addition and the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^n$ the scalars are complex numbers while in the case of $\mathbb{R}^n$ the only allowed scalars are real numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

Definition 3.2 *If $\mathbf{x} \in \mathbb{F}^n$ and $a \in \mathbb{F}$, also called a scalar, then $a\mathbf{x} \in \mathbb{F}^n$ is defined by*

$$a\mathbf{x} = a(x_1, \cdots, x_n) \equiv (ax_1, \cdots, ax_n). \tag{3.1}$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ then $\mathbf{x} + \mathbf{y} \in \mathbb{F}^n$ and is defined by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, \cdots, x_n) + (y_1, \cdots, y_n) \\ &\equiv (x_1 + y_1, \cdots, x_n + y_n) \end{aligned} \tag{3.2}$$

With this definition, the algebraic properties satisfy the conclusions of the following theorem.

Theorem 3.3 *For $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$ and α, β scalars, (real numbers), the following hold.*

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \tag{3.3}$$

the commutative law of addition,

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}), \tag{3.4}$$

the associative law for addition,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \tag{3.5}$$

the existence of an additive identity,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}, \tag{3.6}$$

the existence of an additive inverse, Also

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}, \tag{3.7}$$

[1] René Descartes 1596-1650 is often credited with inventing analytic geometry although it seems the ideas were actually known much earlier. He was interested in many different subjects, physiology, chemistry, and physics being some of them. He also wrote a large book in which he tried to explain the book of Genesis scientifically. Descartes ended up dying in Sweden.

$$(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}, \tag{3.8}$$

$$\alpha\,(\beta\mathbf{v}) = \alpha\beta\,(\mathbf{v}), \tag{3.9}$$

$$1\mathbf{v} = \mathbf{v}. \tag{3.10}$$

In the above $\mathbf{0} = (0, \cdots, 0)$.

You should verify these properties all hold. For example, consider 3.7

$$\begin{aligned}
\alpha\,(\mathbf{v} + \mathbf{w}) &= \alpha\,(v_1 + w_1, \cdots, v_n + w_n) \\
&= (\alpha\,(v_1 + w_1), \cdots, \alpha\,(v_n + w_n)) \\
&= (\alpha v_1 + \alpha w_1, \cdots, \alpha v_n + \alpha w_n) \\
&= (\alpha v_1, \cdots, \alpha v_n) + (\alpha w_1, \cdots, \alpha w_n) \\
&= \alpha\mathbf{v} + \alpha\mathbf{w}.
\end{aligned}$$

As usual subtraction is defined as $\mathbf{x} - \mathbf{y} \equiv \mathbf{x} + (-\mathbf{y})$.

3.2 Subspaces Spans And Bases

Definition 3.4 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ *be vectors in* $\mathbb{F}^n$. *A linear combination is any expression of the form*

$$\sum_{i=1}^{p} c_i \mathbf{x}_i$$

where the c_i *are scalars. The set of all linear combinations of these vectors is called* $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$. *If* $V \subseteq \mathbb{F}^n$, *then* V *is called a subspace if whenever* α, β *are scalars and* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors of* V, *it follows* $\alpha\mathbf{u} + \beta\mathbf{v} \in V$. *That is, it is "closed under the algebraic operations of vector addition and scalar multiplication". A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus* $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ *is called linearly independent if whenever*

$$\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$$

it follows that all the scalars, c_k *equal zero. A set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$, *is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars,* $c_i, i = 1, \cdots, n$, *not all zero such that* $\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$.

Lemma 3.5 *A set of vectors* $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ *is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.*

Proof: Suppose first that $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is linearly independent. If

$$\mathbf{x}_k = \sum_{j \neq k} c_j \mathbf{x}_j,$$

then

$$\mathbf{0} = 1\mathbf{x}_k + \sum_{j \neq k} (-c_j) \mathbf{x}_j,$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ linearly independent? If it is not there exist scalars, c_i, not all zero such that

$$\sum_{i=1}^{p} c_i \mathbf{x}_i = \mathbf{0}.$$

Say $c_k \neq 0$. Then you can solve for $\mathbf{x}_k$ as

$$\mathbf{x}_k = \sum_{j \neq k} (-c_j) / c_k \mathbf{x}_j$$

contrary to assumption. This proves the lemma.

The following is called the exchange theorem.

Theorem 3.6 *(Exchange Theorem) Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be a linearly independent set of vectors such that each $\mathbf{x}_i$ is in span$(\mathbf{y}_1, \cdots, \mathbf{y}_s)$. Then $r \leq s$.*

Proof: Define span$\{\mathbf{y}_1, \cdots, \mathbf{y}_s\} \equiv V$, it follows there exist scalars, $c_1, \cdots, c_s$ such that

$$\mathbf{x}_1 = \sum_{i=1}^{s} c_i \mathbf{y}_i. \tag{3.11}$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_1 = 0$ and so $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would not be linearly independent. Indeed, if $\mathbf{x}_1 = \mathbf{0}$, $1\mathbf{x}_1 + \sum_{i=2}^{r} 0\mathbf{x}_i = \mathbf{x}_1 = \mathbf{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ which equals zero.

Say $c_k \neq 0$. Then solve (3.11) for $\mathbf{y}_k$ and obtain

$$\mathbf{y}_k \in \text{span}\left(\mathbf{x}_1, \overbrace{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s}^{\text{s-1 vectors here}}\right).$$

Define $\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ by

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} \equiv \{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s\}$$

Therefore, span $\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} = V$ because if $\mathbf{v} \in V$, there exist constants $c_1, \cdots, c_s$ such that

$$\mathbf{v} = \sum_{i=1}^{s-1} c_i \mathbf{z}_i + c_s \mathbf{y}_k.$$

Now replace the $\mathbf{y}_k$ in the above with a linear combination of the vectors,

$$\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$$

to obtain $\mathbf{v} \in \text{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$. The vector $\mathbf{y}_k$, in the list $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$, has now been replaced with the vector $\mathbf{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$.

Now suppose that $r > s$ and that span $(\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p) = V$ where the vectors, $\mathbf{z}_1, \cdots, \mathbf{z}_p$ are each taken from the set, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ and $l + p = s$. This has now been done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$ and so $l + 1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\{\mathbf{x}_1, \cdots, \mathbf{x}_l\}$ and since span $\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$, there exist scalars, c_i and d_j such that

$$\mathbf{x}_{l+1} = \sum_{i=1}^{l} c_i \mathbf{x}_i + \sum_{j=1}^{p} d_j \mathbf{z}_j. \tag{3.12}$$

Now not all the d_j can equal zero because if this were so, it would follow that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (3.12) can be solved for one of the $\mathbf{z}_i$, say $\mathbf{z}_k$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_i$ and just as in the above argument, replace that $\mathbf{z}_i$ with $\mathbf{x}_{l+1}$ to obtain

$$\text{span}\left(\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_1, \cdots \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \cdots, \mathbf{z}_p}^{\text{p-1 vectors here}}\right) = V.$$

Continue this way, eventually obtaining

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s) = V.$$

But then $\mathbf{x}_r \in \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s)$ contrary to the assumption that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Definition 3.7 *A finite set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is a basis for* $\mathbb{F}^n$ *if*

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = \mathbb{F}^n$$

and $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is linearly independent.*

Corollary 3.8 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *and* $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ *be two bases*[2] *of* $\mathbb{F}^n$. *Then* $r = s = n$.

[2]This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$\mathbf{e}_i = \overbrace{(0, \cdots, 0, 1, 0 \cdots, 0)}^{1 \text{ is in the } i^{th} \text{ slot}}$$

for $i = 1, 2, \cdots, n$ are a basis for $\mathbb{F}^n$. This proves the corollary.

Lemma 3.9 *Let* $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ *be a set of vectors. Then* $V \equiv \operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_r)$ *is a subspace.*

Proof: Suppose α, β are two scalars and let $\sum_{k=1}^{r} c_k \mathbf{v}_k$ and $\sum_{k=1}^{r} d_k \mathbf{v}_k$ are two elements of V. What about

$$\alpha \sum_{k=1}^{r} c_k \mathbf{v}_k + \beta \sum_{k=1}^{r} d_k \mathbf{v}_k?$$

Is it also in V?

$$\alpha \sum_{k=1}^{r} c_k \mathbf{v}_k + \beta \sum_{k=1}^{r} d_k \mathbf{v}_k = \sum_{k=1}^{r} (\alpha c_k + \beta d_k) \mathbf{v}_k \in V$$

so the answer is yes. This proves the lemma.

Definition 3.10 *A finite set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is a basis for a subspace,* V *of* $\mathbb{F}^n$ *if* $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = V$ *and* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is linearly independent.*

Corollary 3.11 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *and* $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ *be two bases for* V. *Then* $r = s$.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Therefore, this proves the corollary.

Definition 3.12 *Let* V *be a subspace of* $\mathbb{F}^n$. *Then* $\dim(V)$ *read as the dimension of* V *is the number of vectors in a basis.*

Of course you should wonder right now whether an arbitrary subspace even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 3.13 *Suppose* $\mathbf{v} \notin \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$ *and* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ *is linearly independent. Then* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}\}$ *is also linearly independent.*

Proof: Suppose $\sum_{i=1}^{k} c_i \mathbf{u}_i + d\mathbf{v} = \mathbf{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$,

$$\mathbf{v} = -\sum_{i=1}^{k} \left(\frac{c_i}{d}\right) \mathbf{u}_i$$

contrary to assumption. Therefore, $d = 0$. But then $\sum_{i=1}^{k} c_i \mathbf{u}_i = 0$ and the linear independence of $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ implies each $c_i = 0$ also. This proves the lemma.

Theorem 3.14 *Let V be a nonzero subspace of $\mathbb{F}^n$. Then V has a basis.*

Proof: Let $\mathbf{v}_1 \in V$ where $\mathbf{v}_1 \neq 0$. If $\text{span}\{\mathbf{v}_1\} = V$, stop. $\{\mathbf{v}_1\}$ is a basis for V. Otherwise, there exists $\mathbf{v}_2 \in V$ which is not in $\text{span}\{\mathbf{v}_1\}$. By Lemma 3.13 $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$ stop, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq V$, then there exists $\mathbf{v}_3 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 3.15 *Let V be a subspace of $\mathbb{F}^n$ and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a linearly independent set of vectors in V. Then either it is a basis for V or there exist vectors, $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_s$ such that $\{\mathbf{v}_1, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_s\}$ is a basis for V.*

Proof: This follows immediately from the proof of Theorem 3.14. You do exactly the same argument except you start with $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ rather than $\{\mathbf{v}_1\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.

Theorem 3.16 *Let V be a subspace of $\mathbb{F}^n$ and suppose $\text{span}(\mathbf{u}_1 \cdots, \mathbf{u}_p) = V$ where the $\mathbf{u}_i$ are nonzero vectors. Then there exist vectors, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ such that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is a basis for V.*

Proof: Let r be the smallest positive integer with the property that for some set, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$,

$$\text{span}(\mathbf{v}_1 \cdots, \mathbf{v}_r) = V.$$

Then $r \leq p$ and it must be the case that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_k$ would be a linear combination of the others. But then you could delete this vector from $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ and the resulting list of $r-1$ vectors would still span V contrary to the definition of r. This proves the theorem.

3.3 An Application To Matrices

The following is a theorem of major significance.

Theorem 3.17 *Suppose A is an $n \times n$ matrix. Then A is one to one if and only if A is onto. Also, if B is an $n \times n$ matrix and $AB = I$, then it follows $BA = I$.*

Proof: First suppose A is one to one. Consider the vectors, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n\}$ where $\mathbf{e}_k$ is the column vector which is all zeros except for a 1 in the k^{th} position. This set of vectors is linearly independent because if

$$\sum_{k=1}^{n} c_k A\mathbf{e}_k = \mathbf{0},$$

then since A is linear,

$$A\left(\sum_{k=1}^{n} c_k \mathbf{e}_k\right) = \mathbf{0}$$

and since A is one to one, it follows

$$\sum_{k=1}^{n} c_k \mathbf{e}_k = \mathbf{0}^3$$

which implies each $c_k = 0$. Therefore, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n\}$ must be a basis for $\mathbb{F}^n$ because if not there would exist a vector, $\mathbf{y} \notin \text{span}(A\mathbf{e}_1, \cdots, A\mathbf{e}_n)$ and then by Lemma 3.13, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n, \mathbf{y}\}$ would be an independent set of vectors having $n+1$ vectors in it, contrary to the exchange theorem. It follows that for $\mathbf{y} \in \mathbb{F}^n$ there exist constants, c_i such that

$$\mathbf{y} = \sum_{k=1}^{n} c_k A\mathbf{e}_k = A\left(\sum_{k=1}^{n} c_k \mathbf{e}_k\right)$$

showing that, since $\mathbf{y}$ was arbitrary, A is onto.

Next suppose A is onto. This means the span of the columns of A equals $\mathbb{F}^n$. If these columns are not linearly independent, then by Lemma 3.5 on Page 40, one of the columns is a linear combination of the others and so the span of the columns of A equals the span of the $n-1$ other columns. This violates the exchange theorem because $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ would be a linearly independent set of vectors contained in the span of only $n-1$ vectors. Therefore, the columns of A must be independent and this equivalent to saying that $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. This implies A is one to one because if $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ and so $\mathbf{x} - \mathbf{y} = \mathbf{0}$.

Now suppose $AB = I$. Why is $BA = I$? Since $AB = I$ it follows B is one to one since otherwise, there would exist, $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$ and then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0} \neq I\mathbf{x}$. Therefore, from what was just shown, B is also onto. In addition to this, A must be one to one because if $A\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = B\mathbf{x}$ for some $\mathbf{x}$ and then $\mathbf{x} = AB\mathbf{x} = A\mathbf{y} = \mathbf{0}$ showing $\mathbf{y} = \mathbf{0}$. Now from what is given to be so, it follows $(AB)A = A$ and so using the associative law for matrix multiplication,

$$A(BA) - A = A(BA - I) = 0.$$

But this means $(BA - I)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ since otherwise, A would not be one to one. Hence $BA = I$ as claimed. This proves the theorem.

This theorem shows that if an $n \times n$ matrix, B acts like an inverse when multiplied on one side of A it follows that $B = A^{-1}$and it will act like an inverse on both sides of A.

The conclusion of this theorem pertains to square matrices only. For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \tag{3.13}$$

Then

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

3.4 The Mathematical Theory Of Determinants

It is assumed the reader is familiar with matrices. However, the topic of determinants is often neglected in linear algebra books these days. Therefore, I will give a fairly quick and grubby treatment of this topic which includes all the main results. Two books which give a good introduction to determinants are Apostol [3] and Rudin [35]. A recent book which also has a good introduction is Baker [7]

Let $(i_1, \cdots, i_n)$ be an ordered list of numbers from $\{1, \cdots, n\}$. This means the order is important so $(1, 2, 3)$ and $(2, 1, 3)$ are different.

The following Lemma will be essential in the definition of the determinant.

Lemma 3.18 *There exists a unique function,* sgn_n *which maps each list of* n *numbers from* $\{1, \cdots, n\}$ *to one of the three numbers,* $0, 1,$ *or* -1 *which also has the following properties.*

$$\operatorname{sgn}_n (1, \cdots, n) = 1 \tag{3.14}$$

$$\operatorname{sgn}_n (i_1, \cdots, p, \cdots, q, \cdots, i_n) = -\operatorname{sgn}_n (i_1, \cdots, q, \cdots, p, \cdots, i_n) \tag{3.15}$$

In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by -1. *Also, in the case where* $n > 1$ *and* $\{i_1, \cdots, i_n\} = \{1, \cdots, n\}$ *so that every number from* $\{1, \cdots, n\}$ *appears in the ordered list,* $(i_1, \cdots, i_n)$,

$$\operatorname{sgn}_n (i_1, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_n) \equiv$$
$$(-1)^{n-\theta} \operatorname{sgn}_{n-1} (i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_n) \tag{3.16}$$

where $n = i_\theta$ *in the ordered list,* $(i_1, \cdots, i_n)$.

Proof: To begin with, it is necessary to show the existence of such a function. This is clearly true if $n = 1$. Define $\operatorname{sgn}_1 (1) \equiv 1$ and observe that it works. No switching is possible. In the case where $n = 2$, it is also clearly true. Let $\operatorname{sgn}_2 (1, 2) = 1$ and $\operatorname{sgn}_2 (2, 1) = 0$ while $\operatorname{sgn}_2 (2, 2) = \operatorname{sgn}_2 (1, 1) = 0$ and verify it works. Assuming such a function exists for n, sgn_{n+1} will be defined in terms of sgn_n. If there are any repeated numbers in $(i_1, \cdots, i_{n+1})$, $\operatorname{sgn}_{n+1} (i_1, \cdots, i_{n+1}) \equiv 0$. If there are no repeats, then $n + 1$ appears somewhere in the ordered list. Let θ be the position of the number $n + 1$ in the list. Thus, the list is of the form $(i_1, \cdots, i_{\theta-1}, n + 1, i_{\theta+1}, \cdots, i_{n+1})$. From 3.16 it must be that

$$\operatorname{sgn}_{n+1} (i_1, \cdots, i_{\theta-1}, n + 1, i_{\theta+1}, \cdots, i_{n+1}) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n (i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n+1}) .$$

It is necessary to verify this satisfies 3.14 and 3.15 with n replaced with $n+1$. The first of these is obviously true because

$$\operatorname{sgn}_{n+1} (1, \cdots, n, n+1) \equiv (-1)^{n+1-(n+1)} \operatorname{sgn}_n (1, \cdots, n) = 1.$$

If there are repeated numbers in $(i_1, \cdots, i_{n+1})$, then it is obvious 3.15 holds because both sides would equal zero from the above definition. It remains to verify 3.15 in the case where there are no numbers repeated in $(i_1, \cdots, i_{n+1})$. Consider

$$\operatorname{sgn}_{n+1} \left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s}{q}, \cdots, i_{n+1} \right),$$

where the r above the p indicates the number, p is in the r^{th} position and the s above the q indicates that the number, q is in the s^{th} position. Suppose first that $r < \theta < s$. Then

$$\operatorname{sgn}_{n+1} \left(i_1, \cdots, \overset{r}{p}, \cdots, n \overset{\theta}{+} 1, \cdots, \overset{s}{q}, \cdots, i_{n+1} \right) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s-1}{q}, \cdots, i_{n+1} \right)$$

while

$$\operatorname{sgn}_{n+1} \left(i_1, \cdots, \overset{r}{q}, \cdots, n \overset{\theta}{+} 1, \cdots, \overset{s}{p}, \cdots, i_{n+1} \right) =$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s-1}{p}, \cdots, i_{n+1} \right)$$

and so, by induction, a switch of p and q introduces a minus sign in the result. Similarly, if $\theta > s$ or if $\theta < r$ it also follows that 3.15 holds. The interesting case is when $\theta = r$ or $\theta = s$. Consider the case where $\theta = r$ and note the other case is entirely similar.

$$\operatorname{sgn}_{n+1} \left(i_1, \cdots, n \overset{r}{+} 1, \cdots, \overset{s}{q}, \cdots, i_{n+1} \right) =$$

$$(-1)^{n+1-r} \operatorname{sgn}_n \left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1} \right) \tag{3.17}$$

while

$$\operatorname{sgn}_{n+1} \left(i_1, \cdots, \overset{r}{q}, \cdots, n \overset{s}{+} 1, \cdots, i_{n+1} \right) =$$

$$(-1)^{n+1-s} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1} \right). \tag{3.18}$$

By making $s-1-r$ switches, move the q which is in the $s-1^{th}$ position in 3.17 to the r^{th} position in 3.18. By induction, each of these switches introduces a factor of -1 and so

$$\operatorname{sgn}_n \left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1} \right) = (-1)^{s-1-r} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1} \right).$$

Therefore,

$$\operatorname{sgn}_{n+1}\left(i_1,\cdot\cdot\cdot,\overset{r}{n+1},\cdot\cdot\cdot,\overset{s}{q},\cdot\cdot\cdot,i_{n+1}\right)=(-1)^{n+1-r}\operatorname{sgn}_n\left(i_1,\cdot\cdot\cdot,\overset{s-1}{q},\cdot\cdot\cdot,i_{n+1}\right)$$

$$=(-1)^{n+1-r}(-1)^{s-1-r}\operatorname{sgn}_n\left(i_1,\cdot\cdot\cdot,\overset{r}{q},\cdot\cdot\cdot,i_{n+1}\right)$$

$$=(-1)^{n+s}\operatorname{sgn}_n\left(i_1,\cdot\cdot\cdot,\overset{r}{q},\cdot\cdot\cdot,i_{n+1}\right)=(-1)^{2s-1}(-1)^{n+1-s}\operatorname{sgn}_n\left(i_1,\cdot\cdot\cdot,\overset{r}{q},\cdot\cdot\cdot,i_{n+1}\right)$$

$$=-\operatorname{sgn}_{n+1}\left(i_1,\cdot\cdot\cdot,\overset{r}{q},\cdot\cdot\cdot,\overset{s}{n+1},\cdot\cdot\cdot,i_{n+1}\right).$$

This proves the existence of the desired function.

To see this function is unique, note that you can obtain any ordered list of distinct numbers from a sequence of switches. If there exist two functions, f and g both satisfying 3.14 and 3.15, you could start with $f(1,\cdot\cdot\cdot,n)=g(1,\cdot\cdot\cdot,n)$ and applying the same sequence of switches, eventually arrive at $f(i_1,\cdot\cdot\cdot,i_n)=g(i_1,\cdot\cdot\cdot,i_n)$. If any numbers are repeated, then 3.15 gives both functions are equal to zero for that ordered list. This proves the lemma.

In what follows sgn will often be used rather than sgn_n because the context supplies the appropriate n.

Definition 3.19 *Let f be a real valued function which has the set of ordered lists of numbers from $\{1,\cdot\cdot\cdot,n\}$ as its domain. Define*

$$\sum_{(k_1,\cdots,k_n)} f(k_1\cdots k_n)$$

to be the sum of all the $f(k_1\cdots k_n)$ for all possible choices of ordered lists $(k_1,\cdot\cdot\cdot,k_n)$ of numbers of $\{1,\cdot\cdot\cdot,n\}$. For example,

$$\sum_{(k_1,k_2)} f(k_1,k_2)=f(1,2)+f(2,1)+f(1,1)+f(2,2).$$

Definition 3.20 *Let $(a_{ij})=A$ denote an $n\times n$ matrix. The determinant of A, denoted by $\det(A)$ is defined by*

$$\det(A)\equiv\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdot\cdot\cdot,k_n)\,a_{1k_1}\cdots a_{nk_n}$$

where the sum is taken over all ordered lists of numbers from $\{1,\cdot\cdot\cdot,n\}$. Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, $\operatorname{sgn}(k_1,\cdot\cdot\cdot,k_n)=0$ and so that term contributes 0 to the sum.

Let A be an $n\times n$ matrix, $A=(a_{ij})$ and let $(r_1,\cdot\cdot\cdot,r_n)$ denote an ordered list of n numbers from $\{1,\cdot\cdot\cdot,n\}$. Let $A(r_1,\cdot\cdot\cdot,r_n)$ denote the matrix whose k^{th} row is the r_k row of the matrix, A. Thus

$$\det(A(r_1,\cdot\cdot\cdot,r_n))=\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdot\cdot\cdot,k_n)\,a_{r_1k_1}\cdots a_{r_nk_n} \tag{3.19}$$

and

$$A(1,\cdots,n) = A.$$

Proposition 3.21 *Let*

$$(r_1,\cdots,r_n)$$

be an ordered list of numbers from $\{1,\cdots,n\}$. *Then*

$$\operatorname{sgn}(r_1,\cdots,r_n)\det(A)$$

$$= \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{r_1k_1}\cdots a_{r_nk_n} \tag{3.20}$$

$$= \det(A(r_1,\cdots,r_n)). \tag{3.21}$$

Proof: Let $(1,\cdots,n) = (1,\cdots,r,\cdots s,\cdots,n)$ so $r < s$.

$$\det(A(1,\cdots,r,\cdots,s,\cdots,n)) = \tag{3.22}$$

$$\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_r,\cdots,k_s,\cdots,k_n)\, a_{1k_1}\cdots a_{rk_r}\cdots a_{sk_s}\cdots a_{nk_n},$$

and renaming the variables, calling k_s, k_r and k_r, k_s, this equals

$$= \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_s,\cdots,k_r,\cdots,k_n)\, a_{1k_1}\cdots a_{rk_s}\cdots a_{sk_r}\cdots a_{nk_n}$$

$$= \sum_{(k_1,\cdots,k_n)} -\operatorname{sgn}\left(k_1,\cdots, \overbrace{k_r,\cdots,k_s}^{\text{These got switched}}, \cdots,k_n\right) a_{1k_1}\cdots a_{sk_r}\cdots a_{rk_s}\cdots a_{nk_n}$$

$$= -\det(A(1,\cdots,s,\cdots,r,\cdots,n)). \tag{3.23}$$

Consequently,

$$\det(A(1,\cdots,s,\cdots,r,\cdots,n)) =$$
$$-\det(A(1,\cdots,r,\cdots,s,\cdots,n)) = -\det(A)$$

Now letting $A(1,\cdots,s,\cdots,r,\cdots,n)$ play the role of A, and continuing in this way, switching pairs of numbers,

$$\det(A(r_1,\cdots,r_n)) = (-1)^p\det(A)$$

where it took p switches to obtain$(r_1,\cdots,r_n)$ from $(1,\cdots,n)$. By Lemma 3.18, this implies

$$\det(A(r_1,\cdots,r_n)) = (-1)^p\det(A) = \operatorname{sgn}(r_1,\cdots,r_n)\det(A)$$

and proves the proposition in the case when there are no repeated numbers in the ordered list, $(r_1,\cdots,r_n)$. However, if there is a repeat, say the r^{th} row equals the s^{th} row, then the reasoning of 3.22 -3.23 shows that $A(r_1,\cdots,r_n) = 0$ and also $\operatorname{sgn}(r_1,\cdots,r_n) = 0$ so the formula holds in this case also.

Observation 3.22 *There are* $n!$ *ordered lists of distinct numbers from* $\{1, \cdots, n\}$.

To see this, consider n slots placed in order. There are n choices for the first slot. For each of these choices, there are $n-1$ choices for the second. Thus there are $n(n-1)$ ways to fill the first two slots. Then for each of these ways there are $n-2$ choices left for the third slot. Continuing this way, there are $n!$ ordered lists of distinct numbers from $\{1, \cdots, n\}$ as stated in the observation.

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\det(A) = \det(A^T)$.

Corollary 3.23 *The following formula for* $\det(A)$ *is valid.*

$$\det(A) = \frac{1}{n!} \cdot$$

$$\sum_{(r_1, \cdots, r_n)} \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(r_1, \cdots, r_n) \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}. \tag{3.24}$$

And also $\det(A^T) = \det(A)$ *where* A^T *is the transpose of* A. *(Recall that for* $A^T = (a_{ij}^T)$, $a_{ij}^T = a_{ji}$.*)*

Proof: From Proposition 3.21, if the r_i are distinct,

$$\det(A) = \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(r_1, \cdots, r_n) \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}.$$

Summing over all ordered lists, $(r_1, \cdots, r_n)$ where the r_i are distinct, (If the r_i are not distinct, $\operatorname{sgn}(r_1, \cdots, r_n) = 0$ and so there is no contribution to the sum.)

$$n! \det(A) =$$

$$\sum_{(r_1, \cdots, r_n)} \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(r_1, \cdots, r_n) \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}.$$

This proves the corollary since the formula gives the same number for A as it does for A^T.

Corollary 3.24 *If two rows or two columns in an* $n \times n$ *matrix,* A*, are switched, the determinant of the resulting matrix equals* (-1) *times the determinant of the original matrix. If* A *is an* $n \times n$ *matrix in which two rows are equal or two columns are equal then* $\det(A) = 0$. *Suppose the* i^{th} *row of* A *equals* $(xa_1 + yb_1, \cdots, xa_n + yb_n)$. *Then*

$$\det(A) = x \det(A_1) + y \det(A_2)$$

where the i^{th} *row of* A_1 *is* $(a_1, \cdots, a_n)$ *and the* i^{th} *row of* A_2 *is* $(b_1, \cdots, b_n)$, *all other rows of* A_1 *and* A_2 *coinciding with those of* A. *In other words,* $\det$ *is a linear function of each row* A. *The same is true with the word "row" replaced with the word "column".*

Proof: By Proposition 3.21 when two rows are switched, the determinant of the resulting matrix is (-1) times the determinant of the original matrix. By Corollary 3.23 the same holds for columns because the columns of the matrix equal the rows of the transposed matrix. Thus if A_1 is the matrix obtained from A by switching two columns,

$$\det(A) = \det\left(A^T\right) = -\det\left(A_1^T\right) = -\det(A_1).$$

If A has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\det(A) = -\det(A)$ and so $\det(A) = 0$.

It remains to verify the last assertion.

$$\det(A) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots(xa_{k_i}+yb_{k_i})\cdots a_{nk_n}$$

$$= x\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots a_{k_i}\cdots a_{nk_n}$$

$$+y\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots b_{k_i}\cdots a_{nk_n}$$

$$\equiv x\det(A_1) + y\det(A_2).$$

The same is true of columns because $\det\left(A^T\right) = \det(A)$ and the rows of A^T are the columns of A.

Definition 3.25 *A vector,* $\mathbf{w}$*, is a linear combination of the vectors* $\{\mathbf{v}_1,\cdots,\mathbf{v}_r\}$ *if there exists scalars,* $c_1,\cdots c_r$ *such that* $\mathbf{w} = \sum_{k=1}^r c_k\mathbf{v}_k$*. This is the same as saying* $\mathbf{w}\in\operatorname{span}\{\mathbf{v}_1,\cdots,\mathbf{v}_r\}$.

The following corollary is also of great use.

Corollary 3.26 *Suppose A is an $n\times n$ matrix and some column (row) is a linear combination of r other columns (rows). Then* $\det(A)=0$.

Proof: Let $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ be the columns of A and suppose the condition that one column is a linear combination of r of the others is satisfied. Then by using Corollary 3.24 you may rearrange the columns to have the n^{th} column a linear combination of the first r columns. Thus $\mathbf{a}_n = \sum_{k=1}^r c_k\mathbf{a}_k$ and so

$$\det(A) = \det\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \sum_{k=1}^r c_k\mathbf{a}_k \end{pmatrix}.$$

By Corollary 3.24

$$\det(A) = \sum_{k=1}^r c_k\det\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_k \end{pmatrix} = 0.$$

The case for rows follows from the fact that $\det(A) = \det\left(A^T\right)$. This proves the corollary.

Recall the following definition of matrix multiplication.

Definition 3.27 *If A and B are $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, $AB = (c_{ij})$ where*

$$c_{ij} \equiv \sum_{k=1}^{n} a_{ik} b_{kj}.$$

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

Theorem 3.28 *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof: Let c_{ij} be the ij^{th} entry of AB. Then by Proposition 3.21,

$$\det(AB) =$$

$$\begin{aligned}
& \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdot\cdot\cdot,k_n)\, c_{1k_1}\cdots c_{nk_n} \\
= & \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdot\cdot\cdot,k_n) \left(\sum_{r_1} a_{1r_1} b_{r_1k_1}\right)\cdots\left(\sum_{r_n} a_{nr_n} b_{r_nk_n}\right) \\
= & \sum_{(r_1\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdot\cdot\cdot,k_n)\, b_{r_1k_1}\cdots b_{r_nk_n} (a_{1r_1}\cdots a_{nr_n}) \\
= & \sum_{(r_1\cdots,r_n)} \operatorname{sgn}(r_1\cdots r_n)\, a_{1r_1}\cdots a_{nr_n} \det(B) = \det(A)\det(B).
\end{aligned}$$

This proves the theorem.

Lemma 3.29 *Suppose a matrix is of the form*

$$M = \begin{pmatrix} A & * \\ \mathbf{0} & a \end{pmatrix} \tag{3.25}$$

or

$$M = \begin{pmatrix} A & \mathbf{0} \\ * & a \end{pmatrix} \tag{3.26}$$

where a is a number and A is an $(n-1)\times(n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the $\mathbf{0}$ denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then

$$\det(M) = a\det(A).$$

Proof: Denote M by (m_{ij}). Thus in the first case, $m_{nn} = a$ and $m_{ni} = 0$ if $i \neq n$ while in the second case, $m_{nn} = a$ and $m_{in} = 0$ if $i \neq n$. From the definition of the determinant,

$$\det(M) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}_n(k_1,\cdot\cdot\cdot,k_n)\, m_{1k_1}\cdots m_{nk_n}$$

Letting θ denote the position of n in the ordered list, $(k_1, \cdots, k_n)$ then using the earlier conventions used to prove Lemma 3.18, $\det(M)$ equals

$$\sum_{(k_1,\cdots,k_n)} (-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(k_1, \cdots, k_{\theta-1}, \overset{\theta}{k_{\theta+1}}, \cdots, \overset{n-1}{k_n}\right) m_{1k_1} \cdots m_{nk_n}$$

Now suppose 3.26. Then if $k_n \neq n$, the term involving m_{nk_n} in the above expression equals zero. Therefore, the only terms which survive are those for which $\theta = n$ or in other words, those for which $k_n = n$. Therefore, the above expression reduces to

$$a \sum_{(k_1,\cdots,k_{n-1})} \operatorname{sgn}_{n-1}(k_1, \cdots k_{n-1})\, m_{1k_1} \cdots m_{(n-1)k_{n-1}} = a \det(A).$$

To get the assertion in the situation of 3.25 use Corollary 3.23 and 3.26 to write

$$\det(M) = \det\left(M^T\right) = \det\left(\begin{pmatrix} A^T & \mathbf{0} \\ * & a \end{pmatrix}\right) = a \det\left(A^T\right) = a \det(A).$$

This proves the lemma.

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 3.30 *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then a new matrix called the cofactor matrix,* $\operatorname{cof}(A)$ *is defined by* $\operatorname{cof}(A) = (c_{ij})$ *where to obtain c_{ij} delete the i^{th} row and the j^{th} column of A, take the determinant of the $(n-1) \times (n-1)$ matrix which results, (This is called the ij^{th} minor of A.) and then multiply this number by $(-1)^{i+j}$. To make the formulas easier to remember,* $\operatorname{cof}(A)_{ij}$ *will denote the ij^{th} entry of the cofactor matrix.*

The following is the main result. Earlier this was given as a definition and the outrageous totally unjustified assertion was made that the same number would be obtained by expanding the determinant along any row or column. The following theorem proves this assertion.

Theorem 3.31 *Let A be an $n \times n$ matrix where $n \geq 2$. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}. \tag{3.27}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Proof: Let $(a_{i1}, \cdots, a_{in})$ be the i^{th} row of A. Let B_j be the matrix obtained from A by leaving every row the same except the i^{th} row which in B_j equals $(0, \cdots, 0, a_{ij}, 0, \cdots, 0)$. Then by Corollary 3.24,

$$\det(A) = \sum_{j=1}^{n} \det(B_j)$$

Denote by A^{ij} the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of A. Thus $\operatorname{cof}(A)_{ij} \equiv (-1)^{i+j}\det\left(A^{ij}\right)$. At this point, recall that from Proposition 3.21, when two rows or two columns in a matrix, M, are switched, this results in multiplying the determinant of the old matrix by -1 to get the determinant of the new matrix. Therefore, by Lemma 3.29,

$$\begin{aligned}\det(B_j) &= (-1)^{n-j}(-1)^{n-i}\det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij}\end{pmatrix}\right) \\ &= (-1)^{i+j}\det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij}\end{pmatrix}\right) = a_{ij}\operatorname{cof}(A)_{ij}.\end{aligned}$$

Therefore,

$$\det(A) = \sum_{j=1}^{n} a_{ij}\operatorname{cof}(A)_{ij}$$

which is the formula for expanding $\det(A)$ along the i^{th} row. Also,

$$\begin{aligned}\det(A) &= \det\left(A^T\right) = \sum_{j=1}^{n} a_{ij}^T \operatorname{cof}\left(A^T\right)_{ij} \\ &= \sum_{j=1}^{n} a_{ji}\operatorname{cof}(A)_{ji}\end{aligned}$$

which is the formula for expanding $\det(A)$ along the i^{th} column. This proves the theorem.

Note that this gives an easy way to write a formula for the inverse of an $n\times n$ matrix.

Theorem 3.32 *A^{-1} exists if and only if $\det(A)\neq 0$. If $\det(A)\neq 0$, then $A^{-1} = \left(a_{ij}^{-1}\right)$ where*

$$a_{ij}^{-1} = \det(A)^{-1}\operatorname{cof}(A)_{ji}$$

for $\operatorname{cof}(A)_{ij}$ the ij^{th} cofactor of A.

Proof: By Theorem 3.31 and letting $(a_{ir}) = A$, if $\det(A)\neq 0$,

$$\sum_{i=1}^{n} a_{ir}\operatorname{cof}(A)_{ir}\det(A)^{-1} = \det(A)\det(A)^{-1} = 1.$$

Now consider

$$\sum_{i=1}^{n} a_{ir}\operatorname{cof}(A)_{ik}\det(A)^{-1}$$

when $k\neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix, B_k whose determinant equals zero by Corollary 3.24. However, expanding this matrix along the k^{th} column yields

$$0 = \det(B_k)\det(A)^{-1} = \sum_{i=1}^{n} a_{ir}\operatorname{cof}(A)_{ik}\det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.$$

Using the other formula in Theorem 3.31, and similar reasoning,

$$\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

This proves that if $\det(A) \neq 0$, then A^{-1} exists with $A^{-1} = \left(a_{ij}^{-1}\right)$, where

$$a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.$$

Now suppose A^{-1} exists. Then by Theorem 3.28,

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right)$$

so $\det(A) \neq 0$. This proves the theorem.

The next corollary points out that if an $n \times n$ matrix, A has a right or a left inverse, then it has an inverse.

Corollary 3.33 *Let A be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix, B such that $BA = I$. Then A^{-1} exists and $A^{-1} = B$. Also, if there exists C an $n \times n$ matrix such that $AC = I$, then A^{-1} exists and $A^{-1} = C$.*

Proof: Since $BA = I$, Theorem 3.28 implies

$$\det B \det A = 1$$

and so $\det A \neq 0$. Therefore from Theorem 3.32, A^{-1} exists. Therefore,

$$A^{-1} = (BA)A^{-1} = B\left(AA^{-1}\right) = BI = B.$$

The case where $CA = I$ is handled similarly.

The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 3.32 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix A. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, A^{-1} is equal to one over the determinant of A times the adjugate matrix of A.

In case you are solving a system of equations, $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, it follows that if A^{-1} exists,

$$\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{y}$$

thus solving the system. Now in the case that A^{-1} exists, there is a formula for A^{-1} given above. Using this formula,

$$x_i = \sum_{j=1}^{n} a_{ij}^{-1} y_j = \sum_{j=1}^{n} \frac{1}{\det(A)} \operatorname{cof}(A)_{ji} y_j .$$

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{pmatrix} * & \cdots & y_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & y_n & \cdots & * \end{pmatrix},$$

where here the i^{th} column of A is replaced with the column vector, $(y_1 \cdots\cdot, y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

Definition 3.34 *A matrix M, is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form M_{ii} as shown.*

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 3.31.

Corollary 3.35 *Let M be an upper (lower) triangular matrix. Then $\det(M)$ is obtained by taking the product of the entries on the main diagonal.*

Definition 3.36 *A submatrix of a matrix A is the rectangular array of numbers obtained by deleting some rows and columns of A. Let A be an $m \times n$ matrix. The **determinant rank** of the matrix equals r where r is the largest number such that some $r \times r$ submatrix of A has a non zero determinant. The **row rank** is defined to be the dimension of the span of the rows. The **column rank** is defined to be the dimension of the span of the columns.*

Theorem 3.37 *If A has determinant rank, r, then there exist r rows of the matrix such that every other row is a linear combination of these r rows.*

Proof: Suppose the determinant rank of $A = (a_{ij})$ equals r. If rows and columns are interchanged, the determinant rank of the modified matrix is unchanged. Thus rows and columns can be interchanged to produce an $r \times r$ matrix in the upper left corner of the matrix which has non zero determinant. Now consider the $r+1 \times r+1$ matrix, M,

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1p} \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rp} \\ a_{l1} & \cdots & a_{lr} & a_{lp} \end{pmatrix}$$

where C will denote the $r \times r$ matrix in the upper left corner which has non zero determinant. I claim $\det(M) = 0$.

There are two cases to consider in verifying this claim. First, suppose $p > r$. Then the claim follows from the assumption that A has determinant rank r. On the other hand, if $p < r$, then the determinant is zero because there are two identical columns. Expand the determinant along the last column and divide by $\det(C)$ to obtain

$$a_{lp} = -\sum_{i=1}^{r} \frac{\operatorname{cof}(M)_{ip}}{\det(C)} a_{ip}.$$

Now note that $\operatorname{cof}(M)_{ip}$ does not depend on p. Therefore the above sum is of the form

$$a_{lp} = \sum_{i=1}^{r} m_i a_{ip}$$

which shows the l^{th} row is a linear combination of the first r rows of A. Since l is arbitrary, this proves the theorem.

Corollary 3.38 *The determinant rank equals the row rank.*

Proof: From Theorem 3.37, the row rank is no larger than the determinant rank. Could the row rank be smaller than the determinant rank? If so, there exist p rows for $p < r$ such that the span of these p rows equals the row space. But this implies that the $r \times r$ submatrix whose determinant is nonzero also has row rank no larger than p which is impossible if its determinant is to be nonzero because at least one row is a linear combination of the others.

Corollary 3.39 *If A has determinant rank, r, then there exist r columns of the matrix such that every other column is a linear combination of these r columns. Also the column rank equals the determinant rank.*

Proof: This follows from the above by considering A^T. The rows of A^T are the columns of A and the determinant rank of A^T and A are the same. Therefore, from Corollary 3.38, column rank of A = row rank of A^T = determinant rank of A^T = determinant rank of A.

The following theorem is of fundamental importance and ties together many of the ideas presented above.

Theorem 3.40 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. $\det(A) = 0$.

2. *A, A^T are not one to one.*

3. *A is not onto.*

Proof: Suppose $\det(A) = 0$. Then the determinant rank of $A = r < n$. Therefore, there exist r columns such that every other column is a linear combination of these columns by Theorem 3.37. In particular, it follows that for some m, the m^{th} column is a linear combination of all the others. Thus letting $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m & \cdots & \mathbf{a}_n \end{pmatrix}$ where the columns are denoted by $\mathbf{a}_i$, there exists scalars, α_i such that

$$\mathbf{a}_m = \sum_{k \neq m} \alpha_k \mathbf{a}_k.$$

Now consider the column vector, $\mathbf{x} \equiv \begin{pmatrix} \alpha_1 & \cdots & -1 & \cdots & \boldsymbol{\alpha}_n \end{pmatrix}^T$. Then

$$A\mathbf{x} = -\mathbf{a}_m + \sum_{k \neq m} \alpha_k \mathbf{a}_k = \mathbf{0}.$$

Since also $A\mathbf{0} = \mathbf{0}$, it follows A is not one to one. Similarly, A^T is not one to one by the same argument applied to A^T. This verifies that 1.) implies 2.).

Now suppose 2.). Then since A^T is not one to one, it follows there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A^T \mathbf{x} = \mathbf{0}.$$

Taking the transpose of both sides yields

$$\mathbf{x}^T A = \mathbf{0}$$

where the $\mathbf{0}$ is a $1 \times n$ matrix or row vector. Now if $A\mathbf{y} = \mathbf{x}$, then

$$|\mathbf{x}|^2 = \mathbf{x}^T (A\mathbf{y}) = \left(\mathbf{x}^T A\right) \mathbf{y} = \mathbf{0}\mathbf{y} = 0$$

contrary to $\mathbf{x} \neq \mathbf{0}$. Consequently there can be no $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$ and so A is not onto. This shows that 2.) implies 3.).

Finally, suppose 3.). If 1.) does not hold, then $\det(A) \neq 0$ but then from Theorem 3.32 A^{-1} exists and so for every $\mathbf{y} \in \mathbb{F}^n$ there exists a unique $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} = \mathbf{y}$. In fact $\mathbf{x} = A^{-1}\mathbf{y}$. Thus A would be onto contrary to 3.). This shows 3.) implies 1.) and proves the theorem.

Corollary 3.41 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. *$\det(A) \neq 0$.*

2. *A and A^T are one to one.*

3. *A is onto.*

Proof: This follows immediately from the above theorem.

3.5 The Cayley Hamilton Theorem

Definition 3.42 *Let A be an $n \times n$ matrix. The characteristic polynomial is defined as*

$$p_A(t) \equiv \det(tI - A)$$

and the solutions to $p_A(t) = 0$ are called eigenvalues. For A a matrix and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, denote by $p(A)$ the matrix defined by

$$p(A) \equiv A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

The explanation for the last term is that A^0 is interpreted as I, the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $P_A(t) = 0$. It is one of the most important theorems in linear algebra. The following lemma will help with its proof.

Lemma 3.43 *Suppose for all $|\lambda|$ large enough,*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = 0,$$

where the A_i are $n \times n$ matrices. Then each $A_i = 0$.

Proof: Multiply by λ^{-m} to obtain

$$A_0\lambda^{-m} + A_1\lambda^{-m+1} + \cdots + A_{m-1}\lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_m = 0$. With this, multiply by λ to obtain

$$A_0\lambda^{-m+1} + A_1\lambda^{-m+2} + \cdots + A_{m-1} = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by λ and letting $\lambda \to \infty$ to obtain that all the $A_i = 0$. This proves the lemma.

With the lemma, here is a simple corollary.

Corollary 3.44 *Let A_i and B_i be $n \times n$ matrices and suppose*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = B_0 + B_1\lambda + \cdots + B_m\lambda^m$$

for all $|\lambda|$ large enough. Then $A_i = B_i$ for all i. Consequently if λ is replaced by any $n \times n$ matrix, the two sides will be equal. That is, for C any $n \times n$ matrix,

$$A_0 + A_1C + \cdots + A_mC^m = B_0 + B_1C + \cdots + B_mC^m.$$

Proof: Subtract and use the result of the lemma.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

Theorem 3.45 *Let A be an $n \times n$ matrix and let $p(\lambda) \equiv \det(\lambda I - A)$ be the characteristic polynomial. Then $p(A) = 0$.*

Proof: Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I - A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then λ cannot be in the finite list of eigenvalues of A and so for such λ, $(\lambda I - A)^{-1}$ exists.) Therefore, by Theorem 3.32

$$C(\lambda) = p(\lambda)(\lambda I - A)^{-1}.$$

Note that each entry in $C(\lambda)$ is a polynomial in λ having degree no more than $n-1$. Therefore, collecting the terms,

$$C(\lambda) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}$$

for C_j some $n \times n$ matrix. It follows that for all $|\lambda|$ large enough,

$$(A - \lambda I)\left(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}\right) = p(\lambda) I$$

and so Corollary 3.44 may be used. It follows the matrix coefficients corresponding to equal powers of λ are equal on both sides of this equation. Therefore, if λ is replaced with A, the two sides will be equal. Thus

$$0 = (A - A)\left(C_0 + C_1 A + \cdots + C_{n-1}A^{n-1}\right) = p(A) I = p(A).$$

This proves the Cayley Hamilton theorem.

3.6 An Identity Of Cauchy

There is a very interesting identity for determinants due to Cauchy.

Theorem 3.46 *The following identity holds.*

$$\prod_{i,j}(a_i + b_j)\begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix} = \prod_{j<i}(a_i - a_j)(b_i - b_j). \tag{3.28}$$

Proof: What is the exponent of a_2 on the right? It occurs in $(a_2 - a_1)$ and in $(a_m - a_2)$ for $m > 2$. Therefore, there are exactly $n-1$ factors which contain a_2. Therefore, a_2 has an exponent of $n-1$. Similarly, each a_k is raised to the $n-1$ power and the same holds for the b_k as well. Therefore, the right side of 3.28 is of the form

$$ca_1^{n-1}a_2^{n-1}\cdots a_n^{n-1}b_1^{n-1}\cdots b_n^{n-1}$$

where c is some constant. Now consider the left side of 3.28.

This is of the form

$$\frac{1}{n!}\prod_{i,j}(a_i + b_j)\sum_{i_1\cdots i_n, j_1, \cdots j_n}\operatorname{sgn}(i_1\cdots i_n)\operatorname{sgn}(j_1\cdots j_n)\frac{1}{a_{i_1}+b_{j_1}}\frac{1}{a_{i_2}+b_{j_2}}\cdots\frac{1}{a_{i_n}+b_{j_n}}.$$

For a given $i_1 \cdots i_n, j_1, \cdots j_n$, let $S(i_1 \cdots i_n, j_1, \cdots j_n) \equiv \{(i_1, j_1), (i_2, j_2) \cdots, (i_n, j_n)\}$. This equals

$$\frac{1}{n!} \sum_{i_1 \cdots i_n, j_1, \cdots j_n} \operatorname{sgn}(i_1 \cdots i_n) \operatorname{sgn}(j_1 \cdots j_n) \prod_{(i,j) \notin \{(i_1,j_1),(i_2,j_2)\cdots,(i_n,j_n)\}} (a_i + b_j)$$

where you can assume the i_k are all distinct and the j_k are also all distinct because otherwise sgn will produce a 0. Therefore, in $\prod_{(i,j) \notin \{(i_1,j_1),(i_2,j_2)\cdots,(i_n,j_n)\}} (a_i + b_j)$, there are exactly $n-1$ factors which contain a_k for each k and similarly, there are exactly $n-1$ factors which contain b_k for each k. Therefore, the left side of 3.28 is of the form

$$d a_1^{n-1} a_2^{n-1} \cdots a_n^{n-1} b_1^{n-1} \cdots b_n^{n-1}$$

and it remains to verify that $c = d$. Using the properties of determinants, the left side of 3.28 is of the form

$$\prod_{i \neq j} (a_i + b_j) \begin{vmatrix} 1 & \frac{a_1+b_1}{a_1+b_2} & \cdots & \frac{a_1+b_1}{a_1+b_n} \\ \frac{a_2+b_2}{a_2+b_1} & 1 & \cdots & \frac{a_2+b_2}{a_2+b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n+b_n}{a_n+b_1} & \frac{a_n+b_n}{a_n+b_2} & \cdots & 1 \end{vmatrix}$$

Let $a_k \to -b_k$. Then this converges to $\prod_{i \neq j} (-b_i + b_j)$. The right side of 3.28 converges to

$$\prod_{j<i} (-b_i + b_j)(b_i - b_j) = \prod_{i \neq j} (-b_i + b_j).$$

Therefore, $d = c$ and this proves the identity.

3.7 Block Multiplication Of Matrices

Suppose A is a matrix of the form

$$\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rm} \end{pmatrix} \tag{3.29}$$

where A_{ij} is a $s_i \times p_j$ matrix where s_i does not depend on j and p_j does not depend on i. Such a matrix is called a **block matrix**. Let $n = \sum_j p_j$ and $k = \sum_i s_i$ so A is an $k \times n$ matrix. What is $A\mathbf{x}$ where $\mathbf{x} \in \mathbb{F}^n$? From the process of multiplying a matrix times a vector, the following lemma follows.

Lemma 3.47 *Let A be an $m \times n$ block matrix as in 3.29 and let $\mathbf{x} \in \mathbb{F}^n$. Then $A\mathbf{x}$ is of the form*

$$A\mathbf{x} = \begin{pmatrix} \sum_j A_{1j}\mathbf{x}_j \\ \vdots \\ \sum_j A_{rj}\mathbf{x}_j \end{pmatrix}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m)^T$ and $\mathbf{x}_i \in \mathbb{F}^{p_i}$.

Suppose also that B is a $l \times k$ block matrix of the form

$$\begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mp} \end{pmatrix} \tag{3.30}$$

and that for all i, j, it makes sense to multiply $B_{is}A_{sj}$ for all $s \in \{1, \cdots, m\}$. (That is the two matrices are conformable.) and that for each $s, B_{is}A_{sj}$ is the same size so that it makes sense to write $\sum_s B_{is}A_{sj}$.

Theorem 3.48 *Let B be an $l \times k$ block matrix as in 3.30 and let A be a $k \times n$ block matrix as in 3.29 such that B_{is} is conformable with A_{sj} and each product, $B_{is}A_{sj}$ is of the same size so they can be added. Then BA is a $l \times n$ block matrix having rp blocks such that the ij^{th} block is of the form*

$$\sum_s B_{is}A_{sj}. \tag{3.31}$$

Proof: Let B_{is} be a $q_i \times p_s$ matrix and A_{sj} be a $p_s \times r_j$ matrix. Also let $\mathbf{x} \in \mathbb{F}^n$ and let $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m)^T$ and $\mathbf{x}_i \in \mathbb{F}^{r_i}$ so it makes sense to multiply $A_{sj}\mathbf{x}_j$. Then from the associative law of matrix multiplication and Lemma 3.47 applied twice,

$$\begin{aligned} (BA)\mathbf{x} &= B(A\mathbf{x}) \\ &= \begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mp} \end{pmatrix} \begin{pmatrix} \sum_j A_{1j}\mathbf{x}_j \\ \vdots \\ \sum_j A_{rj}\mathbf{x}_j \end{pmatrix} \\ &= \begin{pmatrix} \sum_s \sum_j B_{1s}A_{sj}\mathbf{x}_j \\ \vdots \\ \sum_s \sum_j B_{ms}A_{sj}\mathbf{x}_j \end{pmatrix} = \begin{pmatrix} \sum_j \left(\sum_s B_{1s}A_{sj}\right)\mathbf{x}_j \\ \vdots \\ \sum_j \left(\sum_s B_{ms}A_{sj}\right)\mathbf{x}_j \end{pmatrix}. \end{aligned}$$

By Lemma 3.47, this shows that $(BA)\mathbf{x}$ equals the block matrix whose ij^{th} entry is given by 3.31 times $\mathbf{x}$. Since $\mathbf{x}$ is an arbitrary vector in $\mathbb{F}^n$, this proves the theorem.

The message of this theorem is that you can formally multiply block matrices as though the blocks were numbers. You just have to pay attention to the preservation of order.

This simple idea of block multiplication turns out to be very useful later. For now here is an interesting and significant application. In this theorem, $p_M(t)$ denotes the polynomial, $\det(tI - M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, M.

Theorem 3.49 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then*

$$p_{BA}(t) = t^{n-m}p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n - m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

Since $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ are similar,they have the same characteristic polynomials. Therefore, noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m}\det(tI - AB) = t^{n-m}p_{AB}(t)$. This proves the theorem.

3.8 Shur's Theorem

Every matrix is related to an upper triangular matrix in a particularly significant way. This is Shur's theorem and it is the most important theorem in the spectral theory of matrices.

Lemma 3.50 *Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis for $\mathbb{F}^n$. Then there exists an orthonormal basis for $\mathbb{F}^n$, $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ which has the property that for each $k \leq n$,*

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k).$$

Proof: Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis for $\mathbb{F}^n$. Let $\mathbf{u}_1 \equiv \mathbf{x}_1/|\mathbf{x}_1|$. Thus for $k = 1$, $\text{span}(\mathbf{u}_1) = \text{span}(\mathbf{x}_1)$ and $\{\mathbf{u}_1\}$ is an orthonormal set. Now suppose for some $k < n$, $\mathbf{u}_1, \cdots, \mathbf{u}_k$ have been chosen such that $(\mathbf{u}_j \cdot \mathbf{u}_l) = \delta_{jl}$ and $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$. Then define

$$\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1} - \sum_{j=1}^{k}(\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\mathbf{u}_j}{\left|\mathbf{x}_{k+1} - \sum_{j=1}^{k}(\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\mathbf{u}_j\right|}, \tag{3.32}$$

where the denominator is not equal to zero because the $\mathbf{x}_j$ form a basis and so

$$\mathbf{x}_{k+1} \notin \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$$

Thus by induction,

$$\mathbf{u}_{k+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}).$$

Also, $\mathbf{x}_{k+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1})$ which is seen easily by solving 3.32 for $\mathbf{x}_{k+1}$ and it follows

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
(\mathbf{u}_{k+1} \cdot \mathbf{u}_l) &= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)(\mathbf{u}_j \cdot \mathbf{u}_l)\right) \\
&= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\,\delta_{lj}\right) \\
&= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - (\mathbf{x}_{k+1} \cdot \mathbf{u}_l)\right) = 0.
\end{aligned}$$

The vectors, $\{\mathbf{u}_j\}_{j=1}^n$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process. Recall the following definition.

Definition 3.51 *An $n \times n$ matrix, U, is unitary if $UU^* = I = U^*U$ where U^* is defined to be the transpose of the conjugate of U.*

Theorem 3.52 *Let A be an $n \times n$ matrix. Then there exists a unitary matrix, U such that*

$$U^*AU = T, \tag{3.33}$$

where T is an upper triangular matrix having the eigenvalues of A on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: Let $\mathbf{v}_1$ be a unit eigenvector for A . Then there exists λ_1 such that

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,\ |\mathbf{v}_1| = 1.$$

Extend $\{\mathbf{v}_1\}$ to a basis and then use Lemma 3.50 to obtain $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, an orthonormal basis in $\mathbb{F}^n$. Let U_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then from the above, it follows U_0 is unitary. Then $U_0^* A U_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is an $n-1 \times n-1$ matrix. Repeat the process for the matrix, A_1 above. There exists a unitary matrix $\widetilde{U}_1$ such that $\widetilde{U}_1^* A_1 \widetilde{U}_1$ is of the form

$$\begin{pmatrix} \lambda_2 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{pmatrix}.$$

Now let U_1 be the $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}.$$

This is also a unitary matrix because by block multiplication,

$$\begin{aligned} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}^* \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \widetilde{U}_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \end{aligned}$$

Then using block multiplication, $U_1^* U_0^* A U_0 U_1$ is of the form

$$\begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{pmatrix}$$

where A_2 is an $n-2 \times n-2$ matrix. Continuing in this way, there exists a unitary matrix, U given as the product of the U_i in the above construction such that

$$U^* A U = T$$

where T is some upper triangular matrix. Since the matrix is upper triangular, the characteristic equation is $\prod_{i=1}^{n} (\lambda - \lambda_i)$ where the λ_i are the diagonal entries of T. Therefore, the λ_i are the eigenvalues.

What if A is a real matrix and you only want to consider real unitary matrices?

Theorem 3.53 *Let A be a real $n \times n$ matrix. Then there exists a real unitary matrix, Q and a matrix T of the form*

$$T = \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_r \end{pmatrix} \tag{3.34}$$

where P_i equals either a real 1×1 matrix or P_i equals a real 2×2 matrix having two complex eigenvalues of A such that $Q^T A Q = T$. The matrix, T is called the real Schur form of the matrix A.

Proof: Suppose

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ |\mathbf{v}_1| = 1$$

where λ_1 is real. Then let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be an orthonormal basis of vectors in $\mathbb{R}^n$. Let Q_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is a real $n-1 \times n-1$ matrix. This is just like the proof of Theorem 3.52 up to this point.

Now in case $\lambda_1 = \alpha + i\beta$, it follows since A is real that $\mathbf{v}_1 = \mathbf{z}_1 + i\mathbf{w}_1$ and that $\overline{\mathbf{v}}_1 = \mathbf{z}_1 - i\mathbf{w}_1$ is an eigenvector for the eigenvalue, $\alpha - i\beta$. Here $\mathbf{z}_1$ and $\mathbf{w}_1$ are real vectors. It is clear that $\{\mathbf{z}_1, \mathbf{w}_1\}$ is an independent set of vectors in $\mathbb{R}^n$. Indeed,$\{\mathbf{v}_1, \overline{\mathbf{v}}_1\}$ is an independent set and it follows $\operatorname{span}(\mathbf{v}_1, \overline{\mathbf{v}}_1) = \operatorname{span}(\mathbf{z}_1, \mathbf{w}_1)$. Now using the Gram Schmidt theorem in $\mathbb{R}^n$, there exists $\{\mathbf{u}_1, \mathbf{u}_2\}$, an orthonormal set of real vectors such that $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2) = \operatorname{span}(\mathbf{v}_1, \overline{\mathbf{v}}_1)$. Now let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be an orthonormal basis in $\mathbb{R}^n$ and let Q_0 be a unitary matrix whose i^{th} column is $\mathbf{u}_i$. Then $A\mathbf{u}_j$ are both in $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2)$ for $j = 1, 2$ and so $\mathbf{u}_k^T A\mathbf{u}_j = 0$ whenever $k \geq 3$. It follows that $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} * & * & \cdots & * \\ * & * & & \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is now an $n-2 \times n-2$ matrix. In this case, find $\widetilde{Q}_1$ an $n-2 \times n-2$ matrix to put A_1 in an appropriate form as above and come up with A_2 either an $n-4 \times n-4$ matrix or an $n-3 \times n-3$ matrix. Then the only other difference is to let

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \widetilde{Q}_1 & \\ 0 & 0 & & & \end{pmatrix}$$

thus putting a 2×2 identity matrix in the upper left corner rather than a one. Repeating this process with the above modification for the case of a complex eigenvalue leads eventually to 3.34 where Q is the product of real unitary matrices Q_i above. Finally,

$$\lambda I - T = \begin{pmatrix} \lambda I_1 - P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & \lambda I_r - P_r \end{pmatrix}$$

where I_k is the 2×2 identity matrix in the case that P_k is 2×2 and is the number 1 in the case where P_k is a 1×1 matrix. Now, it follows that $\det(\lambda I - T) = \prod_{k=1}^{r} \det(\lambda I_k - P_k)$. Therefore, λ is an eigenvalue of T if and only if it is an eigenvalue of some P_k. This proves the theorem since the eigenvalues of T are the same as those of A because they have the same characteristic polynomial due to the similarity of A and T.

Definition 3.54 *When a linear transformation, A, mapping a linear space, V to V has a basis of eigenvectors, the linear transformation is called non defective.*

Otherwise it is called defective. An $n \times n$ matrix, A, is called normal if $AA^* = A^*A$. An important class of normal matrices is that of the Hermitian or self adjoint matrices. An $n \times n$ matrix, A is self adjoint or Hermitian if $A = A^*$.

The next lemma is the basis for concluding that every normal matrix is unitarily similar to a diagonal matrix.

Lemma 3.55 *If T is upper triangular and normal, then T is a diagonal matrix.*

Proof: Since T is normal, $T^*T = TT^*$. Writing this in terms of components and using the description of the adjoint as the transpose of the conjugate, yields the following for the ik^{th} entry of $T^*T = TT^*$.

$$\sum_j t_{ij}t_{jk}^* = \sum_j t_{ij}\overline{t_{kj}} = \sum_j t_{ij}^* t_{jk} = \sum_j \overline{t_{ji}} t_{jk}.$$

Now use the fact that T is upper triangular and let $i = k = 1$ to obtain the following from the above.

$$\sum_j |t_{1j}|^2 = \sum_j |t_{j1}|^2 = |t_{11}|^2$$

You see, $t_{j1} = 0$ unless $j = 1$ due to the assumption that T is upper triangular. This shows T is of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

Now do the same thing only this time take $i = k = 2$ and use the result just established. Thus, from the above,

$$\sum_j |t_{2j}|^2 = \sum_j |t_{j2}|^2 = |t_{22}|^2,$$

showing that $t_{2j} = 0$ if $j > 2$ which means T has the form

$$\begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Next let $i = k = 3$ and obtain that T looks like a diagonal matrix in so far as the first 3 rows and columns are concerned. Continuing in this way it follows T is a diagonal matrix.

Theorem 3.56 *Let A be a normal matrix. Then there exists a unitary matrix, U such that U^*AU is a diagonal matrix.*

Proof: From Theorem 3.52 there exists a unitary matrix, U such that U^*AU equals an upper triangular matrix. The theorem is now proved if it is shown that the property of being normal is preserved under unitary similarity transformations. That is, verify that if A is normal and if $B = U^*AU$, then B is also normal. But this is easy.

$$\begin{aligned} B^*B &= U^*A^*UU^*AU = U^*A^*AU \\ &= U^*AA^*U = U^*AUU^*A^*U = BB^*. \end{aligned}$$

Therefore, U^*AU is a normal and upper triangular matrix and by Lemma 3.55 it must be a diagonal matrix. This proves the theorem.

Corollary 3.57 *If A is Hermitian, then all the eigenvalues of A are real and there exists an orthonormal basis of eigenvectors.*

Proof: Since A is normal, there exists unitary, U such that $U^*AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A. Therefore, $D^* = U^*A^*U = U^*AU = D$ showing D is real.

Finally, let

$$U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$$

where the $\mathbf{u}_i$ denote the columns of U and

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The equation, $U^*AU = D$ implies

$$\begin{aligned} AU &= \begin{pmatrix} A\mathbf{u}_1 & A\mathbf{u}_2 & \cdots & A\mathbf{u}_n \end{pmatrix} \\ &= UD = \begin{pmatrix} \lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n \end{pmatrix} \end{aligned}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and since the matrix is unitary, the ij^{th} entry of U^*U equals δ_{ij} and so

$$\delta_{ij} = \overline{\mathbf{u}}_i^T\mathbf{u}_j = \overline{\mathbf{u}_i^T\overline{\mathbf{u}}_j} = \overline{\mathbf{u}_i \cdot \mathbf{u}_j}.$$

This proves the corollary because it shows the vectors $\{\mathbf{u}_i\}$ form an orthonormal basis.

Corollary 3.58 *If A is a real symmetric matrix, then A is Hermitian and there exists a real unitary matrix, U such that $U^TAU = D$ where D is a diagonal matrix.*

Proof: This follows from Theorem 3.53 and Corollary 3.57.

3.9 The Right Polar Decomposition

This is on the right polar decomposition.

Theorem 3.59 *Let F be an $n \times m$ matrix where $m \geq n$. Then there exists an $m \times n$ matrix R and a $n \times n$ matrix U such that*

$$F = RU,\ U = U^*,$$

all eigenvalues of U are non negative,

$$U^2 = F^*F,\ R^*R = I,$$

and $|R\mathbf{x}| = |\mathbf{x}|$.

Proof: $(F^*F)^* = F^*F$ and so F^*F is self adjoint. Also,

$$(F^*Fx, x) = (Fx, Fx) \geq 0.$$

Therefore, all eigenvalues of F^*F must be nonnegative because if $F^*F\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$,

$$0 \leq (Fx, Fx) = (F^*F\mathbf{x}, \mathbf{x}) = (\lambda\mathbf{x}, \mathbf{x}) = \lambda |\mathbf{x}|^2.$$

From linear algebra, there exists Q such that $Q^*Q = I$ and $Q^*F^*FQ = D$ where D is a diagonal matrix of the form

$$\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

where each $\lambda_i \geq 0$. Therefore, you can consider

$$D^{1/2} \equiv \begin{pmatrix} \lambda_1^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^{1/2} \end{pmatrix} \equiv \begin{pmatrix} \mu_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & v \\ \vdots & & \mu_r & & & \vdots \\ \vdots & & & 0 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{3.35}$$

where the μ_i are the positive eigenvalues of $D^{1/2}$.

Let $U \equiv Q^*D^{1/2}Q$. This matrix is the square root of F^*F because

$$\left(Q^*D^{1/2}Q\right)\left(Q^*D^{1/2}Q\right) = Q^*D^{1/2}D^{1/2}Q = Q^*DQ = F^*F$$

It is self adjoint because $\left(Q^*D^{1/2}Q\right)^* = Q^*D^{1/2}Q^{**} = Q^*D^{1/2}Q$.

Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be an orthogonal set of eigenvectors such that $U\mathbf{x}_i = \mu_i \mathbf{x}_i$ and normalize so that

$$\{\mu_1 \mathbf{x}_1, \cdots, \mu_r \mathbf{x}_r\} = \{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$$

is an orthonormal set of vectors. By 3.35 it follows $\operatorname{rank}(U) = r$ and so $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$ is also an orthonormal basis for $U(\mathbb{F}^n)$.

Then $\{F\mathbf{x}_r, \cdots, F\mathbf{x}_r\}$ is also an orthonormal set of vectors in $\mathbb{F}^m$ because

$$(F\mathbf{x}_i, F\mathbf{x}_j) = (F^* F\mathbf{x}_i, \mathbf{x}_j) = \left(U^2 \mathbf{x}_i, \mathbf{x}_j\right) = (U\mathbf{x}_i, U\mathbf{x}_j) = \delta_{ij}.$$

Let $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_n\}$ be an orthonormal basis for $\mathbb{F}^n$ and let

$$\{F\mathbf{x}_r, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_n, \cdots, \mathbf{z}_m\}$$

be an orthonormal basis for $\mathbb{F}^m$. Then a typical vector of $\mathbb{F}^n$ is of the form

$$\sum_{k=1}^{r} a_k U\mathbf{x}_k + \sum_{j=r+1}^{n} b_j \mathbf{y}_j.$$

Define

$$R\left(\sum_{k=1}^{r} a_k U\mathbf{x}_k + \sum_{j=r+1}^{n} b_j \mathbf{y}_j\right) \equiv \sum_{k=1}^{r} a_k F\mathbf{x}_k + \sum_{j=r+1}^{n} b_j \mathbf{z}_j$$

Then since $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_n\}$ and $\{F\mathbf{x}_r, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_n, \cdots, \mathbf{z}_m\}$ are orthonormal,

$$\begin{aligned}
\left|R\left(\sum_{k=1}^{r} a_k U\mathbf{x}_k + \sum_{j=r+1}^{n} b_j \mathbf{y}_j\right)\right|^2 &= \left|\sum_{k=1}^{r} a_k F\mathbf{x}_k + \sum_{j=r+1}^{n} b_j \mathbf{z}_j\right|^2 \\
&= \sum_{k=1}^{r} |a_k|^2 + \sum_{j=r+1}^{n} |b_j|^2 \\
&= \left|\sum_{k=1}^{r} a_k U\mathbf{x}_k + \sum_{j=r+1}^{n} b_j \mathbf{y}_j\right|^2.
\end{aligned}$$

Therefore, R preserves distances.

Letting $\mathbf{x} \in \mathbb{F}^n$,

$$U\mathbf{x} = \sum_{k=1}^{r} a_k U\mathbf{x}_k \tag{3.36}$$

for some unique choice of scalars, a_k because $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$ is a basis for $U(\mathbb{F}^n)$. Therefore,

$$RU\mathbf{x} = R\left(\sum_{k=1}^{r} a_k U\mathbf{x}_k\right) \equiv \sum_{k=1}^{r} a_k F\mathbf{x}_k = F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k\right).$$

Is $F\left(\sum_{k=1}^r a_k\mathbf{x}_k\right) = F(\mathbf{x})$? Using 3.36,

$$\begin{aligned}
F^*F\left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right) &= U^2\left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right) \\
&= \sum_{k=1}^r a_k\mu_k^2\mathbf{x}_k - U(U\mathbf{x}) \\
&= \sum_{k=1}^r a_k\mu_k^2\mathbf{x}_k - U\left(\sum_{k=1}^r a_k U\mathbf{x}_k\right) \\
&= \sum_{k=1}^r a_k\mu_k^2\mathbf{x}_k - U\left(\sum_{k=1}^r a_k\mu_k\mathbf{x}_k\right) = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|F\left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right)\right|^2 &= \left(F\left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right), F\left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right)\right) \\
&= \left(F^*F\left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right), \left(\sum_{k=1}^r a_k\mathbf{x}_k - \mathbf{x}\right)\right) = 0
\end{aligned}$$

and so $F\left(\sum_{k=1}^r a_k\mathbf{x}_k\right) = F(\mathbf{x})$ as hoped. Thus $RU = F$ on $\mathbb{F}^n$.

Since R preserves distances,

$$|\mathbf{x}|^2 + |\mathbf{y}|^2 + 2(\mathbf{x},\mathbf{y}) = |\mathbf{x}+\mathbf{y}|^2 = |R(\mathbf{x}+\mathbf{y})|^2$$

$$= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2(R\mathbf{x},R\mathbf{y}).$$

Therefore,

$$(\mathbf{x},\mathbf{y}) = (R^*R\mathbf{x},\mathbf{y})$$

for all $\mathbf{x},\mathbf{y}$ and so $R^*R = I$ as claimed. This proves the theorem.

3.10 The Space $\mathcal{L}\left(\mathbb{F}^n,\mathbb{F}^m\right)$

Definition 3.60 *The symbol, $\mathcal{L}\left(\mathbb{F}^n,\mathbb{F}^m\right)$ will denote the set of linear transformations mapping $\mathbb{F}^n$ to $\mathbb{F}^m$. Thus $L \in \mathcal{L}\left(\mathbb{F}^n,\mathbb{F}^m\right)$ means that for α,β scalars and $\mathbf{x},\mathbf{y}$ vectors in $\mathbb{F}^n$,*

$$L(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

It is convenient to give a norm for the elements of $\mathcal{L}\left(\mathbb{F}^n,\mathbb{F}^m\right)$. This will allow the consideration of questions such as whether a function having values in this space of linear transformations is continuous.

3.11 The Operator Norm

How do you measure the distance between linear transformations defined on $\mathbb{F}^n$? It turns out there are many ways to do this but I will give the most common one here.

Definition 3.61 *$\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ denotes the space of linear transformations mapping $\mathbb{F}^n$ to $\mathbb{F}^m$. For $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, the* ***operator norm*** *is defined by*

$$||A|| \equiv \max \{|Ax|_{\mathbb{F}^m} : |x|_{\mathbb{F}^n} \leq 1\} < \infty.$$

Theorem 3.62 *Denote by $|\cdot|$ the norm on either $\mathbb{F}^n$ or $\mathbb{F}^m$. Then $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ with this operator norm is a* ***complete normed linear space*** *of dimension nm with*

$$||A\mathbf{x}|| \leq ||A|| \, |\mathbf{x}| \, .$$

Here ***Completeness*** *means that every Cauchy sequence converges.*

Proof: It is necessary to show the norm defined on $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ really is a norm. This means it is necessary to verify

$$||A|| \geq 0 \text{ and equals zero if and only if } A = 0.$$

For α a scalar,

$$||\alpha A|| = |\alpha| \, ||A|| \, ,$$

and for $A, B \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$,

$$||A + B|| \leq ||A|| + ||B||$$

The first two properties are obvious but you should verify them. It remains to verify the norm is well defined and also to verify the triangle inequality above. First if $|\mathbf{x}| \leq 1$, and (A_{ij}) is the matrix of the linear transformation with respect to the usual basis vectors, then

$$\begin{aligned} ||A|| &= \max \left\{ \left(\sum_i |(A\mathbf{x})_i|^2 \right)^{1/2} : |\mathbf{x}| \leq 1 \right\} \\ &= \max \left\{ \left(\sum_i \left| \sum_j A_{ij} x_j \right|^2 \right)^{1/2} : |\mathbf{x}| \leq 1 \right\} \end{aligned}$$

which is a finite number by the extreme value theorem.

It is clear that a basis for $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ consists of linear transformations whose matrices are of the form E_{ij} where E_{ij} consists of the $m \times n$ matrix having all zeros except for a 1 in the ij^{th} position. In effect, this considers $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as $\mathbb{F}^{nm}$. Think of the $m \times n$ matrix as a long vector folded up.

If $\mathbf{x} \neq \mathbf{0}$,

$$|A\mathbf{x}| \frac{1}{|\mathbf{x}|} = \left| A \frac{\mathbf{x}}{|\mathbf{x}|} \right| \leq ||A|| \tag{3.37}$$

It only remains to verify completeness. Suppose then that $\{A_k\}$ is a Cauchy sequence in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Then from 3.37 $\{A_k \mathbf{x}\}$ is a Cauchy sequence for each $\mathbf{x} \in \mathbb{F}^n$. This follows because

$$|A_k \mathbf{x} - A_l \mathbf{x}| \leq ||A_k - A_l|| \, |\mathbf{x}|$$

which converges to 0 as $k, l \to \infty$. Therefore, by completeness of $\mathbb{F}^m$, there exists $A\mathbf{x}$, the name of the thing to which the sequence, $\{A_k \mathbf{x}\}$ converges such that

$$\lim_{k \to \infty} A_k \mathbf{x} = A\mathbf{x}.$$

Then A is linear because

$$\begin{aligned} A(a\mathbf{x} + b\mathbf{y}) &\equiv \lim_{k \to \infty} A_k (a\mathbf{x} + b\mathbf{y}) \\ &= \lim_{k \to \infty} (aA_k \mathbf{x} + bA_k \mathbf{y}) \\ &= a \lim_{k \to \infty} A_k \mathbf{x} + b \lim_{k \to \infty} A_k \mathbf{y} \\ &= aA\mathbf{x} + bA\mathbf{y}. \end{aligned}$$

By the first part of this argument, $||A|| < \infty$ and so $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. This proves the theorem.

Proposition 3.63 *Let $A(\mathbf{x}) \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ for each $\mathbf{x} \in U \subseteq \mathbb{F}^p$. Then letting $(A_{ij}(\mathbf{x}))$ denote the matrix of $A(\mathbf{x})$ with respect to the standard basis, it follows A_{ij} is continuous at $\mathbf{x}$ for each i, j if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then $||A(\mathbf{x}) - A(\mathbf{y})|| < \varepsilon$. That is, A is a continuous function having values in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ at $\mathbf{x}$.*

Proof: Suppose first the second condition holds. Then from the material on linear transformations,

$$\begin{aligned} |A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})| &= |\mathbf{e}_i \cdot (A(\mathbf{x}) - A(\mathbf{y})) \mathbf{e}_j| \\ &\leq |\mathbf{e}_i| \, |(A(\mathbf{x}) - A(\mathbf{y})) \mathbf{e}_j| \\ &\leq ||A(\mathbf{x}) - A(\mathbf{y})||. \end{aligned}$$

Therefore, the second condition implies the first.

Now suppose the first condition holds. That is each A_{ij} is continuous at $\mathbf{x}$. Let $|\mathbf{v}| \leq 1$.

$$\begin{aligned} |(A(\mathbf{x}) - A(\mathbf{y}))(\mathbf{v})| &= \left(\sum_i \left| \sum_j (A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})) v_j \right|^2 \right)^{1/2} \qquad (3.38) \\ &\leq \left(\sum_i \left(\sum_j |A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})| \, |v_j| \right)^2 \right)^{1/2}. \end{aligned}$$

By continuity of each A_{ij}, there exists a $\delta > 0$ such that for each i, j

$$\left|A_{ij}\left(\mathbf{x}\right) - A_{ij}\left(\mathbf{y}\right)\right| < \frac{\varepsilon}{n\sqrt{m}}$$

whenever $|\mathbf{x} - \mathbf{y}| < \delta$. Then from 3.38, if $|\mathbf{x} - \mathbf{y}| < \delta$,

$$\begin{aligned} \left|\left(A\left(\mathbf{x}\right) - A\left(\mathbf{y}\right)\right)\left(\mathbf{v}\right)\right| &< \left(\sum_i \left(\sum_j \frac{\varepsilon}{n\sqrt{m}} |\mathbf{v}|\right)^2\right)^{1/2} \\ &\leq \left(\sum_i \left(\sum_j \frac{\varepsilon}{n\sqrt{m}}\right)^2\right)^{1/2} = \varepsilon \end{aligned}$$

This proves the proposition.

The Frechet Derivative

Let U be an open set in $\mathbb{F}^n$, and let $\mathbf{f} : U \to \mathbb{F}^m$ be a function.

Definition 4.1 *A function* $\mathbf{g}$ *is* $o(\mathbf{v})$ *if*

$$\lim_{|\mathbf{v}| \to 0} \frac{\mathbf{g}(\mathbf{v})}{|\mathbf{v}|} = \mathbf{0} \tag{4.1}$$

A function $\mathbf{f} : U \to \mathbb{F}^m$ *is differentiable at* $\mathbf{x} \in U$ *if there exists a linear transformation* $L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ *such that*

$$\mathbf{f}(\mathbf{x} + \mathbf{v}) = \mathbf{f}(\mathbf{x}) + L\mathbf{v} + o(\mathbf{v})$$

This linear transformation L *is the definition of* $D\mathbf{f}(\mathbf{x})$. *This derivative is often called the Frechet derivative. .*

Usually no harm is occasioned by thinking of this linear transformation as its matrix taken with respect to the usual basis vectors.

The definition 4.1 means that the error,

$$\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}$$

converges to $\mathbf{0}$ faster than $|\mathbf{v}|$. Thus the above definition is equivalent to saying

$$\lim_{|\mathbf{v}| \to 0} \frac{|\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}|}{|\mathbf{v}|} = 0 \tag{4.2}$$

or equivalently,

$$\lim_{\mathbf{y} \to \mathbf{x}} \frac{|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})(\mathbf{y} - \mathbf{x})|}{|\mathbf{y} - \mathbf{x}|} = 0. \tag{4.3}$$

Now it is clear this is just a generalization of the notion of the derivative of a function of one variable because in this more specialized situation,

$$\lim_{|v| \to 0} \frac{|f(x + v) - f(x) - f'(x)v|}{|v|} = 0,$$

due to the definition which says

$$f'(x) = \lim_{v \to 0} \frac{f(x+v) - f(x)}{v}.$$

For functions of n variables, you can't define the derivative as the limit of a difference quotient like you can for a function of one variable because you can't divide by a vector. That is why there is a need for a more general definition.

The term $o(\mathbf{v})$ is notation that is descriptive of the behavior in 4.1 and it is only this behavior that is of interest. Thus, if t and k are constants,

$$o(\mathbf{v}) = o(\mathbf{v}) + o(\mathbf{v}),\ o(t\mathbf{v}) = o(\mathbf{v}),\ ko(\mathbf{v}) = o(\mathbf{v})$$

and other similar observations hold. The sloppiness built in to this notation is useful because it ignores details which are not important. It may help to think of $o(\mathbf{v})$ as an adjective describing what is left over after approximating $\mathbf{f}(\mathbf{x}+\mathbf{v})$ by $\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})\mathbf{v}$.

Theorem 4.2 *The derivative is well defined.*

Proof: First note that for a fixed vector, $\mathbf{v}$, $o(t\mathbf{v}) = o(t)$. Now suppose both L_1 and L_2 work in the above definition. Then let $\mathbf{v}$ be any vector and let t be a real scalar which is chosen small enough that $t\mathbf{v} + \mathbf{x} \in U$. Then

$$\mathbf{f}(\mathbf{x}+t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_1 t\mathbf{v} + o(t\mathbf{v}),\ \mathbf{f}(\mathbf{x}+t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_2 t\mathbf{v} + o(t\mathbf{v}).$$

Therefore, subtracting these two yields $(L_2 - L_1)(t\mathbf{v}) = o(t\mathbf{v}) = o(t)$. Therefore, dividing by t yields $(L_2 - L_1)(\mathbf{v}) = \frac{o(t)}{t}$. Now let $t \to 0$ to conclude that $(L_2 - L_1)(\mathbf{v}) = 0$. Since this is true for all $\mathbf{v}$, it follows $L_2 = L_1$. This proves the theorem.

Lemma 4.3 *Let $\mathbf{f}$ be differentiable at $\mathbf{x}$. Then $\mathbf{f}$ is continuous at $\mathbf{x}$ and in fact, there exists $K > 0$ such that whenever $|\mathbf{v}|$ is small enough,*

$$|\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})| \le K|\mathbf{v}|$$

Proof: From the definition of the derivative, $\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$. Let $|\mathbf{v}|$ be small enough that $\frac{o(|\mathbf{v}|)}{|\mathbf{v}|} < 1$ so that $|o(\mathbf{v})| \le |\mathbf{v}|$. Then for such $\mathbf{v}$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})| &\le |D\mathbf{f}(\mathbf{x})\mathbf{v}| + |\mathbf{v}| \\ &\le (|D\mathbf{f}(\mathbf{x})| + 1)|\mathbf{v}| \end{aligned}$$

This proves the lemma with $K = |D\mathbf{f}(\mathbf{x})| + 1$.

Theorem 4.4 *(The chain rule) Let U and V be open sets, $U \subseteq \mathbb{F}^n$ and $V \subseteq \mathbb{F}^m$. Suppose $\mathbf{f} : U \to V$ is differentiable at $\mathbf{x} \in U$ and suppose $\mathbf{g} : V \to \mathbb{F}^q$ is differentiable at $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ and*

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = D(\mathbf{g}(\mathbf{f}(\mathbf{x})))\, D(\mathbf{f}(\mathbf{x})).$$

Proof: This follows from a computation. Let $B(\mathbf{x},r) \subseteq U$ and let r also be small enough that for $|\mathbf{v}| \leq r$, it follows that $\mathbf{f}(\mathbf{x}+\mathbf{v}) \in V$. Such an r exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $|\mathbf{v}| < r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) =$$

$$\begin{aligned} & D\mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) + o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ = \; & D\mathbf{g}(\mathbf{f}(\mathbf{x}))[D\mathbf{f}(\mathbf{x})\mathbf{v} + o(\mathbf{v})] + o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ = \; & D(\mathbf{g}(\mathbf{f}(\mathbf{x})))D(\mathbf{f}(\mathbf{x}))\mathbf{v} + o(\mathbf{v}) + o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})). \end{aligned} \tag{4.4}$$

It remains to show $o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) = o(\mathbf{v})$.

By Lemma 4.3, with K given there, letting $\varepsilon > 0$, it follows that for $|\mathbf{v}|$ small enough,

$$|o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}))| \leq (\varepsilon/K)|\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})| \leq (\varepsilon/K)K|\mathbf{v}| = \varepsilon|\mathbf{v}|.$$

Since $\varepsilon > 0$ is arbitrary, this shows $o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) = o(\mathbf{v})$ because whenever $|\mathbf{v}|$ is small enough,

$$\frac{|o(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}))|}{|\mathbf{v}|} \leq \varepsilon.$$

By 4.4, this shows

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) = D(\mathbf{g}(\mathbf{f}(\mathbf{x})))D(\mathbf{f}(\mathbf{x}))\mathbf{v} + o(\mathbf{v})$$

which proves the theorem.

The derivative is a linear transformation. What is the matrix of this linear transformation taken with respect to the usual basis vectors? Let $\mathbf{e}_i$ denote the vector of $\mathbb{F}^n$ which has a one in the i^{th} entry and zeroes elsewhere. Then the matrix of the linear transformation is the matrix whose i^{th} column is $D\mathbf{f}(\mathbf{x})\mathbf{e}_i$. What is this? Let $t \in \mathbb{R}$ such that $|t|$ is sufficiently small.

$$\begin{aligned} \mathbf{f}(\mathbf{x}+t\mathbf{e}_i) - \mathbf{f}(\mathbf{x}) &= D\mathbf{f}(\mathbf{x})t\mathbf{e}_i + \mathbf{o}(t\mathbf{e}_i) \\ &= D\mathbf{f}(\mathbf{x})t\mathbf{e}_i + \mathbf{o}(t). \end{aligned}$$

Then dividing by t and taking a limit,

$$D\mathbf{f}(\mathbf{x})\mathbf{e}_i = \lim_{t\to 0}\frac{\mathbf{f}(\mathbf{x}+t\mathbf{e}_i) - \mathbf{f}(\mathbf{x})}{t} \equiv \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}).$$

Thus the matrix of $D\mathbf{f}(\mathbf{x})$ with respect to the usual basis vectors is the matrix of the form

$$\begin{pmatrix} f_{1,x_1}(\mathbf{x}) & f_{1,x_2}(\mathbf{x}) & \cdots & f_{1,x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ f_{m,x_1}(\mathbf{x}) & f_{m,x_2}(\mathbf{x}) & \cdots & f_{m,x_n}(\mathbf{x}) \end{pmatrix}.$$

As mentioned before, there is no harm in referring to this matrix as $D\mathbf{f}(\mathbf{x})$ but it may also be referred to as $J\mathbf{f}(\mathbf{x})$.

This is summarized in the following theorem.

Theorem 4.5 *Let $\mathbf{f} : \mathbb{F}^n \to \mathbb{F}^m$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_i(\mathbf{x})}{\partial x_j}$ exist and if $J\mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation with respect to the standard basis vectors, then the ij^{th} entry is given by $f_{i,j}$ or $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$.*

What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}.$$

Then from the definition of partial derivatives,

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

and

$$\lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

However f is not even continuous at $(0,0)$ which may be seen by considering the behavior of the function along the line $y = x$ and along the line $x = 0$. By Lemma 4.3 this implies f is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

4.1 C^1 Functions

However, there are theorems which can be used to get differentiability of a function based on existence of the partial derivatives.

Definition 4.6 *When all the partial derivatives exist and are continuous the function is called a C^1 function.*

Because of Proposition 3.63 on Page 73 and Theorem 4.5 which identifies the entries of $J\mathbf{f}$ with the partial derivatives, the following definition is equivalent to the above.

Definition 4.7 *Let $U \subseteq \mathbb{F}^n$ be an open set. Then $\mathbf{f} : U \to \mathbb{F}^m$ is $C^1(U)$ if $\mathbf{f}$ is differentiable and the mapping*

$$\mathbf{x} \to D\mathbf{f}(\mathbf{x}),$$

is continuous as a function from U to $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

The following is an important abstract generalization of the familiar concept of partial derivative.

Definition 4.8 *Let $\mathbf{g} : U \subseteq \mathbb{F}^n \times \mathbb{F}^m \to \mathbb{F}^q$, where U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Denote an element of $\mathbb{F}^n \times \mathbb{F}^m$ by $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{F}^n$ and $\mathbf{y} \in \mathbb{F}^m$. Then the map $\mathbf{x} \to \mathbf{g}(\mathbf{x}, \mathbf{y})$ is a function from the open set in X,*

$$\{\mathbf{x} : (\mathbf{x}, \mathbf{y}) \in U\}$$

to $\mathbb{F}^q$. When this map is differentiable, its derivative is denoted by

$$D_1\mathbf{g}(\mathbf{x}, \mathbf{y}), \text{ or sometimes by } D_{\mathbf{x}}\mathbf{g}(\mathbf{x}, \mathbf{y}).$$

Thus,

$$\mathbf{g}(\mathbf{x} + \mathbf{v}, \mathbf{y}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) = D_1\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{v} + o(\mathbf{v}).$$

A similar definition holds for the symbol $D_{\mathbf{y}}\mathbf{g}$ or $D_2\mathbf{g}$. The special case seen in beginning calculus courses is where $\mathbf{g} : U \to \mathbb{F}^q$ and

$$\mathbf{g}_{x_i}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_i} \equiv \lim_{h \to 0} \frac{\mathbf{g}(\mathbf{x} + h\mathbf{e}_i) - \mathbf{g}(\mathbf{x})}{h}.$$

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem.

Theorem 4.9 *Suppose U is an open subset of $\mathbb{F}^n$ and $\mathbf{f} : U \to \mathbb{F}^m$ has the property that $D\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x}$ in U and that, $\mathbf{x}+t(\mathbf{y} - \mathbf{x}) \in U$ for all $t \in [0,1]$. (The line segment joining the two points lies in U.) Suppose also that for all points on this line segment,*

$$||D\mathbf{f}(\mathbf{x}+t(\mathbf{y} - \mathbf{x}))|| \leq M.$$

Then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| \leq M |\mathbf{y} - \mathbf{x}|.$$

Proof: Let

$$S \equiv \{t \in [0,1] : \text{ for all } s \in [0,t],$$
$$|\mathbf{f}(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| \leq (M + \varepsilon) s |\mathbf{y} - \mathbf{x}|\}.$$

Then $0 \in S$ and by continuity of $\mathbf{f}$, it follows that if $t \equiv \sup S$, then $t \in S$ and if $t < 1$,

$$|\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| = (M + \varepsilon) t |\mathbf{y} - \mathbf{x}|. \tag{4.5}$$

If $t < 1$, then there exists a sequence of positive numbers, $\{h_k\}_{k=1}^{\infty}$ converging to 0 such that

$$|\mathbf{f}(\mathbf{x} + (t + h_k)(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| > (M + \varepsilon)(t + h_k)|\mathbf{y} - \mathbf{x}|$$

which implies that

$$|\mathbf{f}(\mathbf{x} + (t + h_k)(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))|$$
$$+ |\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| > (M + \varepsilon)(t + h_k)|\mathbf{y} - \mathbf{x}|.$$

By 4.5, this inequality implies

$$|\mathbf{f}(\mathbf{x}+(t+h_k)(\mathbf{y}-\mathbf{x}))-\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))|>(M+\varepsilon)h_k|\mathbf{y}-\mathbf{x}|$$

which yields upon dividing by h_k and taking the limit as $h_k\to 0$,

$$|D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})|\geq(M+\varepsilon)|\mathbf{y}-\mathbf{x}|.$$

Now by the definition of the norm of a linear operator,

$$\begin{aligned} M|\mathbf{y}-\mathbf{x}| &\geq ||D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))||\,|\mathbf{y}-\mathbf{x}| \\ &\geq |D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})|\geq(M+\varepsilon)|\mathbf{y}-\mathbf{x}|, \end{aligned}$$

a contradiction. Therefore, $t=1$ and so

$$|\mathbf{f}(\mathbf{x}+(\mathbf{y}-\mathbf{x}))-\mathbf{f}(\mathbf{x})|\leq(M+\varepsilon)|\mathbf{y}-\mathbf{x}|.$$

Since $\varepsilon>0$ is arbitrary, this proves the theorem.

The next theorem proves that if the partial derivatives exist and are continuous, then the function is differentiable.

Theorem 4.10 *Let $\mathbf{g}:U\subseteq\mathbb{F}^n\times\mathbb{F}^m\to\mathbb{F}^q$. Then $\mathbf{g}$ is $C^1(U)$ if and only if $D_1\mathbf{g}$ and $D_2\mathbf{g}$ both exist and are continuous on U. In this case,*

$$D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{v})=D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{v}.$$

Proof: Suppose first that $\mathbf{g}\in C^1(U)$. Then if $(\mathbf{x},\mathbf{y})\in U$,

$$\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})-\mathbf{g}(\mathbf{x},\mathbf{y})=D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{0})+o(\mathbf{u}).$$

Therefore, $D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}=D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{0})$. Then

$$\begin{gathered} |(D_1\mathbf{g}(\mathbf{x},\mathbf{y})-D_1\mathbf{g}(\mathbf{x}',\mathbf{y}'))(\mathbf{u})|= \\ |(D\mathbf{g}(\mathbf{x},\mathbf{y})-D\mathbf{g}(\mathbf{x}',\mathbf{y}'))(\mathbf{u},\mathbf{0})|\leq \\ ||D\mathbf{g}(\mathbf{x},\mathbf{y})-D\mathbf{g}(\mathbf{x}',\mathbf{y}')||\,|(\mathbf{u},\mathbf{0})|. \end{gathered}$$

Therefore,

$$|D_1\mathbf{g}(\mathbf{x},\mathbf{y})-D_1\mathbf{g}(\mathbf{x}',\mathbf{y}')|\leq||D\mathbf{g}(\mathbf{x},\mathbf{y})-D\mathbf{g}(\mathbf{x}',\mathbf{y}')||.$$

A similar argument applies for $D_2\mathbf{g}$ and this proves the continuity of the function, $(\mathbf{x},\mathbf{y})\to D_i\mathbf{g}(\mathbf{x},\mathbf{y})$ for $i=1,2$. The formula follows from

$$\begin{aligned} D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{v}) &= D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{0})+D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{0},\mathbf{v}) \\ &\equiv D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{v}. \end{aligned}$$

Now suppose $D_1\mathbf{g}(\mathbf{x},\mathbf{y})$ and $D_2\mathbf{g}(\mathbf{x},\mathbf{y})$ exist and are continuous.

$$\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y})=\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})$$

$$\begin{aligned}
&+\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y}) \\
&=\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})-\mathbf{g}(\mathbf{x},\mathbf{y})+\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y})+ \\
&[\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})-(\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y}))] \\
&=D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{v}+o(\mathbf{v})+o(\mathbf{u})+ \\
&[\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})-(\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y}))]. \qquad (4.6)
\end{aligned}$$

Let $\mathbf{h}(\mathbf{x},\mathbf{u})\equiv\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})$. Then the expression in [ ] is of the form,

$$\mathbf{h}(\mathbf{x},\mathbf{u})-\mathbf{h}(\mathbf{x},\mathbf{0}).$$

Also

$$D_2\mathbf{h}(\mathbf{x},\mathbf{u})=D_1\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-D_1\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})$$

and so, by continuity of $(\mathbf{x},\mathbf{y})\to D_1\mathbf{g}(\mathbf{x},\mathbf{y})$,

$$||D_2\mathbf{h}(\mathbf{x},\mathbf{u})||<\varepsilon$$

whenever $||(\mathbf{u},\mathbf{v})||$ is small enough. By Theorem 4.9 on Page 79, there exists $\delta>0$ such that if $||(\mathbf{u},\mathbf{v})||<\delta$, the norm of the last term in 4.6 satisfies the inequality,

$$||\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})-(\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x},\mathbf{y}))||<\varepsilon\,||\mathbf{u}||. \qquad (4.7)$$

Therefore, this term is $o((\mathbf{u},\mathbf{v}))$. It follows from 4.7 and 4.6 that

$$\begin{aligned}
&\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})= \\
&\mathbf{g}(\mathbf{x},\mathbf{y})+D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{v}+o(\mathbf{u})+o(\mathbf{v})+o((\mathbf{u},\mathbf{v})) \\
&=\mathbf{g}(\mathbf{x},\mathbf{y})+D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{v}+o((\mathbf{u},\mathbf{v}))
\end{aligned}$$

Showing that $D\mathbf{g}(\mathbf{x},\mathbf{y})$ exists and is given by

$$D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{v})=D_1\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\mathbf{v}.$$

The continuity of $(\mathbf{x},\mathbf{y})\to D\mathbf{g}(\mathbf{x},\mathbf{y})$ follows from the continuity of $(\mathbf{x},\mathbf{y})\to D_i\mathbf{g}(\mathbf{x},\mathbf{y})$. This proves the theorem.

Not surprisingly, it can be generalized to many more factors.

Definition 4.11 *Let $\mathbf{g}:U\subseteq\prod_{i=1}^n\mathbb{F}^{r_i}\to\mathbb{F}^q$, where U is an open set. Then the map $\mathbf{x}_i\to\mathbf{g}(\mathbf{x})$ is a function from the open set in $\mathbb{F}^{r_i}$,*

$$\{\mathbf{x}_i:\mathbf{x}\in U\}$$

to $\mathbb{F}^q$. When this map is differentiable, its derivative is denoted by $D_i\mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v}\in X_i$, let $\theta_i\mathbf{v}\in\prod_{i=1}^n\mathbb{F}^{r_i}$ be the vector $(\mathbf{0},\cdots,\mathbf{v},\cdots,\mathbf{0})$ where the $\mathbf{v}$ is in the i^{th} slot and for $\mathbf{v}\in\prod_{i=1}^n\mathbb{F}^{r_i}$, let $\mathbf{v}_i$ denote the entry in the i^{th} slot of $\mathbf{v}$. Thus by saying $\mathbf{x}_i\to\mathbf{g}(\mathbf{x})$ is differentiable is meant that for $\mathbf{v}\in X_i$ sufficiently small,

$$\mathbf{g}(\mathbf{x}+\theta_i\mathbf{v})-\mathbf{g}(\mathbf{x})=D_i\mathbf{g}(\mathbf{x})\mathbf{v}+o(\mathbf{v}).$$

Here is a generalization of Theorem 4.10.

Theorem 4.12 *Let* $\mathbf{g}, U, \prod_{i=1}^{n} \mathbb{F}^{r_i}$, *be given as in Definition 4.11. Then* $\mathbf{g}$ *is* $C^1(U)$ *if and only if* $D_i\mathbf{g}$ *exists and is continuous on* U *for each* i. *In this case,*

$$D\mathbf{g}(\mathbf{x})(\mathbf{v}) = \sum_k D_k\mathbf{g}(\mathbf{x})\mathbf{v}_k \tag{4.8}$$

Proof: The only if part of the proof is left for you. Suppose then that $D_i\mathbf{g}$ exists and is continuous for each i. Note that $\sum_{j=1}^{k} \theta_j \mathbf{v}_j = (\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{0}, \cdots, \mathbf{0})$. Thus $\sum_{j=1}^{n} \theta_j \mathbf{v}_j = \mathbf{v}$ and define $\sum_{j=1}^{0} \theta_j \mathbf{v}_j \equiv \mathbf{0}$. Therefore,

$$\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^{n} \left[\mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k} \theta_j \mathbf{v}_j \right) - \mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j \right) \right] \tag{4.9}$$

Consider the terms in this sum.

$$\mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k} \theta_j \mathbf{v}_j \right) - \mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j \right) = \mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}_k) - \mathbf{g}(\mathbf{x}) + \tag{4.10}$$

$$\left(\mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k} \theta_j \mathbf{v}_j \right) - \mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}_k) \right) - \left(\mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j \right) - \mathbf{g}(\mathbf{x}) \right) \tag{4.11}$$

and the expression in 4.11 is of the form $\mathbf{h}(\mathbf{v}_k) - \mathbf{h}(\mathbf{0})$ where for small $\mathbf{w} \in \mathbb{F}^{r_k}$,

$$\mathbf{h}(\mathbf{w}) \equiv \mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j + \theta_k \mathbf{w} \right) - \mathbf{g}(\mathbf{x} + \theta_k \mathbf{w}).$$

Therefore,

$$D\mathbf{h}(\mathbf{w}) = D_k\mathbf{g}\left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j + \theta_k \mathbf{w} \right) - D_k\mathbf{g}(\mathbf{x} + \theta_k \mathbf{w})$$

and by continuity, $||D\mathbf{h}(\mathbf{w})|| < \varepsilon$ provided $|\mathbf{v}|$ is small enough. Therefore, by Theorem 4.9, whenever $|\mathbf{v}|$ is small enough, $|\mathbf{h}(\theta_k\mathbf{v}_k) - \mathbf{h}(\mathbf{0})| \leq \varepsilon |\theta_k \mathbf{v}_k| \leq \varepsilon |\mathbf{v}|$ which shows that since ε is arbitrary, the expression in 4.11 is $o(\mathbf{v})$. Now in 4.10 $\mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}_k) - \mathbf{g}(\mathbf{x}) = D_k\mathbf{g}(\mathbf{x})\mathbf{v}_k + o(\mathbf{v}_k) = D_k\mathbf{g}(\mathbf{x})\mathbf{v}_k + o(\mathbf{v})$. Therefore, referring to 4.9,

$$\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^{n} D_k\mathbf{g}(\mathbf{x})\mathbf{v}_k + o(\mathbf{v})$$

which shows $D\mathbf{g}$ exists and equals the formula given in 4.8.

The way this is usually used is in the following corollary, a case of Theorem 4.12 obtained by letting $\mathbb{F}^{r_j} = \mathbb{F}$ in the above theorem.

Corollary 4.13 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^m$ be C^1 in the sense that all the partial derivatives of $\mathbf{f}$ exist and are continuous. Then $\mathbf{f}$ is differentiable and*

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) + \sum_{k=1}^{n} \frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{x}) v_k + \mathbf{o}(\mathbf{v}).$$

4.2 C^k Functions

Recall the notation for partial derivatives in the following definition.

Definition 4.14 *Let $\mathbf{g} : U \to \mathbb{F}^n$. Then*

$$\mathbf{g}_{x_k}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}}{\partial x_k}(\mathbf{x}) \equiv \lim_{h\to 0} \frac{\mathbf{g}(\mathbf{x}+h\mathbf{e}_k) - \mathbf{g}(\mathbf{x})}{h}$$

Higher order partial derivatives are defined in the usual way.

$$\mathbf{g}_{x_k x_l}(\mathbf{x}) \equiv \frac{\partial^2 \mathbf{g}}{\partial x_l \partial x_k}(\mathbf{x})$$

and so forth.

To deal with higher order partial derivatives in a systematic way, here is a useful definition.

Definition 4.15 *$\alpha = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in \mathbb{F}^n$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

and $\mathbf{f}$ a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},\ D^\alpha \mathbf{f}(\mathbf{x}) \equiv \frac{\partial^{|\alpha|}\mathbf{f}(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The following is the definition of what is meant by a C^k function.

Definition 4.16 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^m$. Then for k a nonnegative integer, $\mathbf{f}$ is C^k if for every $|\alpha| \leq k$, $D^\alpha \mathbf{f}$ exists and is continuous.*

4.3 Mixed Partial Derivatives

Under certain conditions the **mixed partial derivatives** will always be equal. This astonishing fact is due to Euler in 1734.

Theorem 4.17 *Suppose $f : U \subseteq \mathbb{F}^2 \to \mathbb{R}$ where U is an open set on which f_x, f_y, f_{xy} and f_{yx} exist. Then if f_{xy} and f_{yx} are continuous at the point $(x, y) \in U$, it follows*

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Proof: Since U is open, there exists $r > 0$ such that $B((x,y), r) \subseteq U$. Now let $|t|, |s| < r/2, t, s$ real numbers and consider

$$\Delta(s,t) \equiv \frac{1}{st}\{\overbrace{f(x+t,y+s) - f(x+t,y)}^{h(t)} - \overbrace{(f(x,y+s) - f(x,y))}^{h(0)}\}. \tag{4.12}$$

Note that $(x+t, y+s) \in U$ because

$$\begin{aligned} |(x+t,y+s) - (x,y)| &= |(t,s)| = \left(t^2+s^2\right)^{1/2} \\ &\leq \left(\frac{r^2}{4} + \frac{r^2}{4}\right)^{1/2} = \frac{r}{\sqrt{2}} < r. \end{aligned}$$

As implied above, $h(t) \equiv f(x+t,y+s) - f(x+t,y)$. Therefore, by the mean value theorem from calculus and the (one variable) chain rule,

$$\begin{aligned} \Delta(s,t) &= \frac{1}{st}(h(t) - h(0)) = \frac{1}{st}h'(\alpha t)\,t \\ &= \frac{1}{s}(f_x(x+\alpha t, y+s) - f_x(x+\alpha t, y)) \end{aligned}$$

for some $\alpha \in (0,1)$. Applying the mean value theorem again,

$$\Delta(s,t) = f_{xy}(x+\alpha t, y+\beta s)$$

where $\alpha, \beta \in (0,1)$.

If the terms $f(x+t,y)$ and $f(x,y+s)$ are interchanged in 4.12, $\Delta(s,t)$ is unchanged and the above argument shows there exist $\gamma, \delta \in (0,1)$ such that

$$\Delta(s,t) = f_{yx}(x+\gamma t, y+\delta s).$$

Letting $(s,t) \to (0,0)$ and using the continuity of f_{xy} and f_{yx} at (x,y),

$$\lim_{(s,t)\to(0,0)} \Delta(s,t) = f_{xy}(x,y) = f_{yx}(x,y).$$

This proves the theorem.

The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 4.18 *Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{R}$ has the property that for two indices, k, l, f_{x_k}, f_{x_l}, $f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

By considering the real and imaginary parts of f in the case where f has values in $\mathbb{F}$ you obtain the following corollary.

Corollary 4.19 *Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{F}$ has the property that for two indices, k, l, f_{x_k}, f_{x_l}, $f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

Finally, by considering the components of $\mathbf{f}$ you get the following generalization.

Corollary 4.20 *Suppose U is an open subset of $\mathbb{F}^n$ and $\mathbf{f} : U \to \mathbb{F}^{m}$ has the property that for two indices, k, l, $\mathbf{f}_{x_k}$, $\mathbf{f}_{x_l}$, $\mathbf{f}_{x_l x_k}$, and $\mathbf{f}_{x_k x_l}$ exist on U and $\mathbf{f}_{x_k x_l}$ and $\mathbf{f}_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $\mathbf{f}_{x_k x_l}(\mathbf{x}) = \mathbf{f}_{x_l x_k}(\mathbf{x})$.*

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [5].

Example 4.21 *Let*

$$f(x,y) = \begin{cases} \frac{xy\left(x^2-y^2\right)}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

From the definition of partial derivatives it follows immediately that $f_x(0,0) = f_y(0,0) = 0$. Using the standard rules of differentiation, for $(x,y) \neq (0,0)$,

$$f_x = y\frac{x^4 - y^4 + 4x^2y^2}{\left(x^2+y^2\right)^2},\ f_y = x\frac{x^4 - y^4 - 4x^2y^2}{\left(x^2+y^2\right)^2}$$

Now

$$\begin{aligned} f_{xy}(0,0) &\equiv \lim_{y\to 0}\frac{f_x(0,y) - f_x(0,0)}{y} \\ &= \lim_{y\to 0}\frac{-y^4}{\left(y^2\right)^2} = -1 \end{aligned}$$

while

$$\begin{aligned} f_{yx}(0,0) &\equiv \lim_{x\to 0}\frac{f_y(x,0) - f_y(0,0)}{x} \\ &= \lim_{x\to 0}\frac{x^4}{\left(x^2\right)^2} = 1 \end{aligned}$$

showing that although the mixed partial derivatives do exist at $(0,0)$, they are not equal there.

4.4 Implicit Function Theorem

The implicit function theorem is one of the greatest theorems in mathematics. There are many versions of this theorem. However, I will give a very simple proof valid in finite dimensional spaces.

Theorem 4.22 *(implicit function theorem) Suppose U is an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Let $\mathbf{f} : U \to \mathbb{R}^n$ be in $C^1(U)$ and suppose*

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0},\ D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n). \tag{4.13}$$

Then there exist positive constants, δ, η, such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \tag{4.14}$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0, \eta))$.

Proof: Let

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} f_1(\mathbf{x}, \mathbf{y}) \\ f_2(\mathbf{x}, \mathbf{y}) \\ \vdots \\ f_n(\mathbf{x}, \mathbf{y}) \end{pmatrix}.$$

Define for $\left(\mathbf{x}^1, \cdots, \mathbf{x}^n\right) \in \overline{B(\mathbf{x}_0, \delta)}^n$ and $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ the following matrix.

$$J\left(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}\right) \equiv \begin{pmatrix} f_{1,x_1}\left(\mathbf{x}^1, \mathbf{y}\right) & \cdots & f_{1,x_n}\left(\mathbf{x}^1, \mathbf{y}\right) \\ \vdots & & \vdots \\ f_{n,x_1}\left(\mathbf{x}^n, \mathbf{y}\right) & \cdots & f_{n,x_n}\left(\mathbf{x}^n, \mathbf{y}\right) \end{pmatrix}.$$

Then by the assumption of continuity of all the partial derivatives, there exists $\delta_0 > 0$ and $\eta_0 > 0$ such that if $\delta < \delta_0$ and $\eta < \eta_0$, it follows that for all $\left(\mathbf{x}^1, \cdots, \mathbf{x}^n\right) \in \overline{B(\mathbf{x}_0, \delta)}^n$ and $\mathbf{y} \in B(\mathbf{y}_0, \eta)$,

$$\det\left(J\left(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}\right)\right) > r > 0. \tag{4.15}$$

and $\overline{B(\mathbf{x}_0, \delta_0)} \times \overline{B(\mathbf{y}_0, \eta_0)} \subseteq U$. Pick $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ and suppose there exist $\mathbf{x}, \mathbf{z} \in \overline{B(\mathbf{x}_0, \delta)}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{z}, \mathbf{y}) = \mathbf{0}$. Consider f_i and let

$$h(t) \equiv f_i(\mathbf{x} + t(\mathbf{z} - \mathbf{x}), \mathbf{y}).$$

Then $h(1) = h(0)$ and so by the mean value theorem, $h'(t_i) = 0$ for some $t_i \in (0, 1)$. Therefore, from the chain rule and for this value of t_i,

$$h'(t_i) = Df_i(\mathbf{x} + t_i(\mathbf{z} - \mathbf{x}), \mathbf{y})(\mathbf{z} - \mathbf{x}) = 0. \tag{4.16}$$

Then denote by $\mathbf{x}^i$ the vector, $\mathbf{x} + t_i(\mathbf{z} - \mathbf{x})$. It follows from 4.16 that

$$J\left(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}\right)(\mathbf{z} - \mathbf{x}) = \mathbf{0}$$

and so from 4.15 $\mathbf{z} - \mathbf{x} = \mathbf{0}$. Now it will be shown that if η is chosen sufficiently small, then for all $\mathbf{y} \in B(\mathbf{y}_0, \eta)$, there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$.

Claim: If η is small enough, then the function, $h_{\mathbf{y}}(\mathbf{x}) \equiv |\mathbf{f}(\mathbf{x}, \mathbf{y})|^2$ achieves its minimum value on $\overline{B(\mathbf{x}_0, \delta)}$ at a point of $B(\mathbf{x}_0, \delta)$.

Proof of claim: Suppose this is not the case. Then there exists a sequence $\eta_k \rightarrow 0$ and for some $\mathbf{y}_k$ having $|\mathbf{y}_k - \mathbf{y}_0| < \eta_k$, the minimum of $h_{\mathbf{y}_k}$ occurs on a point of the boundary of $\overline{B(\mathbf{x}_0, \delta)}$, $\mathbf{x}_k$ such that $|\mathbf{x}_0 - \mathbf{x}_k| = \delta$. Now taking a subsequence, still denoted by k, it can be assumed that $\mathbf{x}_k \to \mathbf{x}$ with $|\mathbf{x} - \mathbf{x}_0| = \delta$ and $\mathbf{y}_k \to \mathbf{y}_0$. Let $\varepsilon > 0$. Then for k large enough, $h_{\mathbf{y}_k}(\mathbf{x}_0) < \varepsilon$ because $\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$. Therefore, from the definition of $\mathbf{x}_k, h_{\mathbf{y}_k}(\mathbf{x}_k) < \varepsilon$. Passing to the limit yields $h_{\mathbf{y}_0}(\mathbf{x}) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $h_{\mathbf{y}_0}(\mathbf{x}) = 0$ which contradicts the first part of the argument in which it was shown that for $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there is at most one point, $\mathbf{x}$ of $\overline{B(\mathbf{x}_0, \delta)}$ where $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Here two have been obtained, $\mathbf{x}_0$ and $\mathbf{x}$. This proves the claim.

Choose $\eta < \eta_0$ and also small enough that the above claim holds and let $\mathbf{x}(\mathbf{y})$ denote a point of $B(\mathbf{x}_0, \delta)$ at which the minimum of $h_{\mathbf{y}}$ on $\overline{B(\mathbf{x}_0, \delta)}$ is achieved. Since $\mathbf{x}(\mathbf{y})$ is an interior point, you can consider $h_{\mathbf{y}}(\mathbf{x}(\mathbf{y}) + t\mathbf{v})$ for $|t|$ small and conclude this function of t has a zero derivative at $t = 0$. Thus

$$Dh_{\mathbf{y}}(\mathbf{x}(\mathbf{y}))\mathbf{v} = 0 = 2\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^T D_1\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})\mathbf{v}$$

for every vector $\mathbf{v}$. But from 4.15 and the fact that $\mathbf{v}$ is arbitrary, it follows $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$. This proves the existence of the function $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ for all $\mathbf{y} \in B(\mathbf{y}_0, \eta)$.

It remains to verify this function is a C^1 function. To do this, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be points of $B(\mathbf{y}_0, \eta)$. Then as before, consider the i^{th} component of $\mathbf{f}$ and consider the same argument using the mean value theorem to write

$$\begin{gathered} 0 = f_i(\mathbf{x}(\mathbf{y}_1), \mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_2) \\ = f_i(\mathbf{x}(\mathbf{y}_1), \mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_1) + f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_2) \\ = D_1 f_i(\mathbf{x}^i, \mathbf{y}_1)(\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)) + D_2 f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}^i)(\mathbf{y}_1 - \mathbf{y}_2). \end{gathered}$$

Therefore,

$$J(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}_1)(\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)) = -M(\mathbf{y}_1 - \mathbf{y}_2) \tag{4.17}$$

where M is the matrix whose i^{th} row is $D_2 f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}^i)$. Then from 4.15 there exists a constant, C independent of the choice of $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ such that

$$\left|\left|J(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y})^{-1}\right|\right| < C$$

whenever $(\mathbf{x}^1, \cdots, \mathbf{x}^n) \in \overline{B(\mathbf{x}_0, \delta)}^n$. By continuity of the partial derivatives of $\mathbf{f}$ it also follows there exists a constant, C_1 such that $||D_2 f_i(\mathbf{x}, \mathbf{y})|| < C_1$ whenever, $(\mathbf{x}, \mathbf{y}) \in \overline{B(\mathbf{x}_0, \delta)} \times B(\mathbf{y}_0, \eta)$. Hence $||M||$ must also be bounded independent of the choice of $\mathbf{y}_1$ and $\mathbf{y}_2$ in $B(\mathbf{y}_0, \eta)$. From 4.17, it follows there exists a constant, C such that for all $\mathbf{y}_1, \mathbf{y}_2$ in $B(\mathbf{y}_0, \eta)$,

$$|\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)| \leq C|\mathbf{y}_1 - \mathbf{y}_2|. \tag{4.18}$$

It follows as in the proof of the chain rule that

$$\mathbf{o}(\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})) = \mathbf{o}(\mathbf{v}). \tag{4.19}$$

Now let $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ and let $|\mathbf{v}|$ be sufficiently small that $\mathbf{y} + \mathbf{v} \in B(\mathbf{y}_0, \eta)$. Then

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y} + \mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \\ &= \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y} + \mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y}) + \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y}) - \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \end{aligned}$$

$$= D_2\mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y})\mathbf{v} + D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})) + \mathbf{o}(|\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})|)$$

$$\begin{aligned} =\ & D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v} + D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})) + \\ & \mathbf{o}(|\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})|) + (D_2\mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y})\mathbf{v} - D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v}) \\ =\ & D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v} + D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})) + \mathbf{o}(\mathbf{v}). \end{aligned}$$

Therefore,

$$\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y}) = -D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v} + \mathbf{o}(\mathbf{v})$$

which shows that $D\mathbf{x}(\mathbf{y}) = -D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$ and $\mathbf{y} \to D\mathbf{x}(\mathbf{y})$ is continuous. This proves the theorem.

In practice, how do you verify the condition, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n)$?

In practice, how do you verify the condition, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n)$?

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} f_1(x_1,\cdots,x_n,y_1,\cdots,y_n) \\ \vdots \\ f_n(x_1,\cdots,x_n,y_1,\cdots,y_n) \end{pmatrix}.$$

The matrix of the linear transformation, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)$ is then

$$\begin{pmatrix} \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \end{pmatrix}$$

and from linear algebra, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n)$ exactly when the above matrix has an inverse. In other words when

$$\det \begin{pmatrix} \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \end{pmatrix} \neq 0$$

at $(\mathbf{x}_0,\mathbf{y}_0)$. The above determinant is important enough that it is given special notation. Letting $\mathbf{z} = \mathbf{f}(\mathbf{x},\mathbf{y})$, the above determinant is often written as

$$\frac{\partial(z_1,\cdots,z_n)}{\partial(x_1,\cdots,x_n)}.$$

Of course you can replace $\mathbb{R}$ with $\mathbb{F}$ in the above by applying the above to the situation in which each $\mathbb{F}$ is replaced with $\mathbb{R}^2$.

Corollary 4.23 *(implicit function theorem) Suppose U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Let $\mathbf{f} : U \to \mathbb{F}^n$ be in $C^1(U)$ and suppose*

$$\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0},\ D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n). \tag{4.20}$$

Then there exist positive constants, δ, η, such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \tag{4.21}$$

Furthermore, the mapping, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0, \eta))$.

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [29] and in [3].

Theorem 4.24 *(inverse function theorem) Let $\mathbf{x}_0 \in U \subseteq \mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^n$. Suppose*

$$\mathbf{f} \text{ is } C^1(U) \ , \quad \text{and} \quad D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \tag{4.22}$$

Then there exist open sets, W, and V such that

$$\mathbf{x}_0 \in W \subseteq U, \tag{4.23}$$

$$\mathbf{f} : W \to V \text{ is one to one and onto}, \tag{4.24}$$

$$\mathbf{f}^{-1} \text{ is } C^1. \tag{4.25}$$

Proof: Apply the implicit function theorem to the function

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{x}) - \mathbf{y}$$

where $\mathbf{y}_0 \equiv \mathbf{f}(\mathbf{x}_0)$. Thus the function $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ defined in that theorem is $\mathbf{f}^{-1}$. Now let

$$W \equiv B(\mathbf{x}_0, \delta) \cap \mathbf{f}^{-1}(B(\mathbf{y}_0, \eta))$$

and

$$V \equiv B(\mathbf{y}_0, \eta).$$

This proves the theorem.

4.5 More Continuous Partial Derivatives

Corollary 4.23 will now be improved slightly. If $\mathbf{f}$ is C^k, it follows that the function which is implicitly defined is also in C^k, not just C^1. Since the inverse function theorem comes as a case of the implicit function theorem, this shows that the inverse function also inherits the property of being C^k.

Theorem 4.25 *(implicit function theorem) Suppose U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Let $\mathbf{f} : U \to \mathbb{F}^n$ be in $C^k(U)$ and suppose*

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}, \ D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \tag{4.26}$$

Then there exist positive constants, δ, η, such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \tag{4.27}$$

Furthermore, the mapping, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is in $C^k(B(\mathbf{y}_0, \eta))$.

Proof: From Corollary 4.23 $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is C^1. It remains to show it is C^k for $k > 1$ assuming that $\mathbf{f}$ is C^k. From 4.27

$$\frac{\partial \mathbf{x}}{\partial y^l} = -D_1(\mathbf{x}, \mathbf{y})^{-1} \frac{\partial \mathbf{f}}{\partial y^l}.$$

Thus the following formula holds for $q = 1$ and $|\alpha| = q$.

$$D^\alpha \mathbf{x}(\mathbf{y}) = \sum_{|\beta| \leq q} M_\beta(\mathbf{x}, \mathbf{y}) D^\beta \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{4.28}$$

where M_β is a matrix whose entries are differentiable functions of $D^\gamma(\mathbf{x})$ for $|\gamma| < q$ and $D^\tau \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q$. This follows easily from the description of $D_1(\mathbf{x}, \mathbf{y})^{-1}$ in terms of the cofactor matrix and the determinant of $D_1(\mathbf{x}, \mathbf{y})$. Suppose 4.28 holds for $|\alpha| = q < k$. Then by induction, this yields $\mathbf{x}$ is C^q. Then

$$\frac{\partial D^\alpha \mathbf{x}(\mathbf{y})}{\partial y^p} = \sum_{|\beta| \leq |\alpha|} \frac{\partial M_\beta(\mathbf{x}, \mathbf{y})}{\partial y^p} D^\beta \mathbf{f}(\mathbf{x}, \mathbf{y}) + M_\beta(\mathbf{x}, \mathbf{y}) \frac{\partial D^\beta \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial y^p}.$$

By the chain rule $\frac{\partial M_\beta(\mathbf{x}, \mathbf{y})}{\partial y^p}$ is a matrix whose entries are differentiable functions of $D^\tau \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q+1$ and $D^\gamma(\mathbf{x})$ for $|\gamma| < q+1$. It follows since y^p was arbitrary that for any $|\alpha| = q + 1$, a formula like 4.28 holds with q being replaced by $q + 1$. By induction, $\mathbf{x}$ is C^k. This proves the theorem.

As a simple corollary this yields an improved version of the inverse function theorem.

Theorem 4.26 *(inverse function theorem) Let $\mathbf{x}_0 \in U \subseteq \mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^n$. Suppose for k a positive integer,*

$$\mathbf{f} \text{ is } C^k(U) \;, \quad \text{and} \quad D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \tag{4.29}$$

Then there exist open sets, W, and V such that

$$\mathbf{x}_0 \in W \subseteq U, \tag{4.30}$$

$$\mathbf{f} : W \to V \text{ is one to one and onto,} \tag{4.31}$$

$$\mathbf{f}^{-1} \text{ is } C^k. \tag{4.32}$$

Part II

Lecture Notes For Math 641 and 642

Metric Spaces And General Topological Spaces

5.1 Metric Space

Definition 5.1 *A metric space is a set, X and a function $d : X \times X \rightarrow [0, \infty)$ which satisfies the following properties.*

$$
\begin{gathered}
d(x, y) = d(y, x) \\
d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ if and only if } x = y \\
d(x, y) \leq d(x, z) + d(z, y).
\end{gathered}
$$

You can check that $\mathbb{R}^n$ and $\mathbb{C}^n$ are metric spaces with $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. However, there are many others. The definitions of open and closed sets are the same for a metric space as they are for $\mathbb{R}^n$.

Definition 5.2 *A set, U in a metric space is open if whenever $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. As before, $B(x, r) \equiv \{y : d(x, y) < r\}$. Closed sets are those whose complements are open. A point p is a limit point of a set, S if for every $r > 0, B(p, r)$ contains infinitely many points of S. A sequence, $\{x_n\}$ converges to a point x if for every $\varepsilon > 0$ there exists N such that if $n \geq N$, then $d(x, x_n) < \varepsilon$. $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that if $m, n \geq N$, then $d(x_n, x_m) < \varepsilon$.*

Lemma 5.3 *In a metric space, X every ball, $B(x, r)$ is open. A set is closed if and only if it contains all its limit points. If p is a limit point of S, then there exists a sequence of distinct points of $S, \{x_n\}$ such that $\lim_{n \to \infty} x_n = p$.*

Proof: Let $z \in B(x, r)$. Let $\delta = r - d(x, z)$. Then if $w \in B(z, \delta)$,

$$d(w, x) \leq d(x, z) + d(z, w) < d(x, z) + r - d(x, z) = r.$$

Therefore, $B(z, \delta) \subseteq B(x, r)$ and this shows $B(x, r)$ is open.

The properties of balls are presented in the following theorem.

Theorem 5.4 *Suppose* (X, d) *is a metric space. Then the sets* $\{B(x, r) : r > 0,\ x \in X\}$ *satisfy*

$$\cup\{B(x, r) : r > 0,\ x \in X\} = X \tag{5.1}$$

If $p \in B(x, r_1) \cap B(z, r_2)$, *there exists* $r > 0$ *such that*

$$B(p, r) \subseteq B(x, r_1) \cap B(z, r_2). \tag{5.2}$$

Proof: Observe that the union of these balls includes the whole space, X so 5.1 is obvious. Consider 5.2. Let $p \in B(x, r_1) \cap B(z, r_2)$. Consider

$$r \equiv \min (r_1 - d(x, p), r_2 - d(z, p))$$

and suppose $y \in B(p, r)$. Then

$$d(y, x) \leq d(y, p) + d(p, x) < r_1 - d(x, p) + d(x, p) = r_1$$

and so $B(p, r) \subseteq B(x, r_1)$. By similar reasoning, $B(p, r) \subseteq B(z, r_2)$. This proves the theorem.

Let K be a closed set. This means $K^C \equiv X \setminus K$ is an open set. Let p be a limit point of K. If $p \in K^C$, then since K^C is open, there exists $B(p, r) \subseteq K^C$. But this contradicts p being a limit point because there are no points of K in this ball. Hence all limit points of K must be in K.

Suppose next that K contains its limit points. Is K^C open? Let $p \in K^C$. Then p is not a limit point of K. Therefore, there exists $B(p, r)$ which contains at most finitely many points of K. Since $p \notin K$, it follows that by making r smaller if necessary, $B(p, r)$ contains no points of K. That is $B(p, r) \subseteq K^C$ showing K^C is open. Therefore, K is closed.

Suppose now that p is a limit point of S. Let $x_1 \in (S \setminus \{p\}) \cap B(p, 1)$. If $x_1, \cdots, x_k$ have been chosen, let

$$r_{k+1} \equiv \min \left\{ d(p, x_i), i = 1, \cdots, k, \frac{1}{k+1} \right\}.$$

Let $x_{k+1} \in (S \setminus \{p\}) \cap B(p, r_{k+1})$. This proves the lemma.

Lemma 5.5 *If* $\{x_n\}$ *is a Cauchy sequence in a metric space,* X *and if some subsequence,* $\{x_{n_k}\}$ *converges to* x, *then* $\{x_n\}$ *converges to* x. *Also if a sequence converges, then it is a Cauchy sequence.*

Proof: Note first that $n_k \geq k$ because in a subsequence, the indices, $n_1, n_2, \cdots$ are strictly increasing. Let $\varepsilon > 0$ be given and let N be such that for $k > N, d(x, x_{n_k}) < \varepsilon/2$ and for $m, n \geq N, d(x_m, x_n) < \varepsilon/2$. Pick $k > n$. Then if $n > N$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, suppose $\lim_{n\to\infty} x_n = x$. Then there exists N such that if $n > N$, then $d(x_n, x) < \varepsilon/2$. it follows that for $m, n > N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the lemma.

5.2 Compactness In Metric Space

Many existence theorems in analysis depend on some set being compact. Therefore, it is important to be able to identify compact sets. The purpose of this section is to describe compact sets in a metric space.

Definition 5.6 *Let A be a subset of X. A is compact if whenever A is contained in the union of a set of open sets, there exists finitely many of these open sets whose union contains A. (Every open cover admits a finite subcover.) A is "sequentially compact" means every sequence has a convergent subsequence converging to an element of A.*

In a metric space compact is not the same as closed and bounded!

Example 5.7 *Let X be any infinite set and define $d(x,y) = 1$ if $x \neq y$ while $d(x,y) = 0$ if $x = y$.*

You should verify the details that this is a metric space because it satisfies the axioms of a metric. The set X is closed and bounded because its complement is $\emptyset$ which is clearly open because every point of $\emptyset$ is an interior point. (There are none.) Also X is bounded because $X = B(x,2)$. However, X is clearly not compact because $\left\{B\left(x,\frac{1}{2}\right) : x \in X\right\}$ is a collection of open sets whose union contains X but since they are all disjoint and nonempty, there is no finite subset of these whose union contains X. In fact $B\left(x,\frac{1}{2}\right) = \{x\}$.

From this example it is clear something more than closed and bounded is needed. If you are not familiar with the issues just discussed, ignore them and continue.

Definition 5.8 *In any metric space, a set E is totally bounded if for every $\varepsilon > 0$ there exists a finite set of points $\{x_1, \cdots, x_n\}$ such that*

$$E \subseteq \cup_{i=1}^{n} B(x_i, \varepsilon).$$

This finite set of points is called an ε net.

The following proposition tells which sets in a metric space are compact. First here is an interesting lemma.

Lemma 5.9 *Let X be a metric space and suppose D is a countable dense subset of X. In other words, it is being assumed X is a separable metric space. Consider the open sets of the form $B(d,r)$ where r is a positive rational number and $d \in D$. Denote this countable collection of open sets by $\mathcal{B}$. Then every open set is the union of sets of $\mathcal{B}$. Furthermore, if $\mathcal{C}$ is any collection of open sets, there exists a countable subset, $\{U_n\} \subseteq \mathcal{C}$ such that $\cup_n U_n = \cup \mathcal{C}$.*

Proof: Let U be an open set and let $x \in U$. Let $B(x,\delta) \subseteq U$. Then by density of D, there exists $d \in D \cap B(x,\delta/4)$. Now pick $r \in \mathbb{Q} \cap (\delta/4, 3\delta/4)$ and consider $B(d,r)$. Clearly, $B(d,r)$ contains the point x because $r > \delta/4$. Is $B(d,r) \subseteq B(x,\delta)$? if so, this proves the lemma because x was an arbitrary point of U. Suppose $z \in B(d,r)$. Then

$$d(z,x) \leq d(z,d) + d(d,x) < r + \frac{\delta}{4} < \frac{3\delta}{4} + \frac{\delta}{4} = \delta$$

Now let $\mathcal{C}$ be any collection of open sets. Each set in this collection is the union of countably many sets of $\mathcal{B}$. Let $\mathcal{B}'$ denote the sets of $\mathcal{B}$ which are contained in some set of $\mathcal{C}$. Thus $\cup\mathcal{B}' = \cup\mathcal{C}$. Then for each $B \in \mathcal{B}'$, pick $U_B \in \mathcal{C}$ such that $B \subseteq U_B$. Then $\{U_B : B \in \mathcal{B}'\}$ is a countable collection of sets of $\mathcal{C}$ whose union equals $\cup\mathcal{C}$. Therefore, this proves the lemma.

Proposition 5.10 *Let (X,d) be a metric space. Then the following are equivalent.*

$$(X,d) \text{ is compact,} \tag{5.3}$$

$$(X,d) \text{ is sequentially compact,} \tag{5.4}$$

$$(X,d) \text{ is complete and totally bounded.} \tag{5.5}$$

Proof: Suppose 5.3 and let $\{x_k\}$ be a sequence. Suppose $\{x_k\}$ has no convergent subsequence. If this is so, then by Lemma 5.3, $\{x_k\}$ has no limit point and no value of the sequence is repeated more than finitely many times. Thus the set

$$C_n = \cup\{x_k : k \geq n\}$$

is a closed set because it has no limit points and if

$$U_n = C_n^C,$$

then

$$X = \cup_{n=1}^{\infty} U_n$$

but there is no finite subcovering, because no value of the sequence is repeated more than finitely many times. This contradicts compactness of (X,d). This shows 5.3 implies 5.4.

Now suppose 5.4 and let $\{x_n\}$ be a Cauchy sequence. Is $\{x_n\}$ convergent? By sequential compactness $x_{n_k} \to x$ for some subsequence. By Lemma 5.5 it follows that $\{x_n\}$ also converges to x showing that (X,d) is complete. If (X,d) is not totally bounded, then there exists $\varepsilon > 0$ for which there is no ε net. Hence there exists a sequence $\{x_k\}$ with $d(x_k, x_l) \geq \varepsilon$ for all $l \neq k$. By Lemma 5.5 again, this contradicts 5.4 because no subsequence can be a Cauchy sequence and so no subsequence can converge. This shows 5.4 implies 5.5.

Now suppose 5.5. What about 5.4? Let $\{p_n\}$ be a sequence and let $\{x_i^n\}_{i=1}^{m_n}$ be a 2^{-n} net for $n = 1, 2, \cdots$. Let

$$B_n \equiv B\left(x_{i_n}^n, 2^{-n}\right)$$

be such that B_n contains p_k for infinitely many values of k and $B_n \cap B_{n+1} \neq \emptyset$. To do this, suppose B_n contains p_k for infinitely many values of k. Then one of the sets which intersect $B_n, B\left(x_i^{n+1}, 2^{-(n+1)}\right)$ must contain p_k for infinitely many values of k because all these indices of points from $\{p_n\}$ contained in B_n must be accounted for in one of finitely many sets, $B\left(x_i^{n+1}, 2^{-(n+1)}\right)$. Thus there exists a strictly increasing sequence of integers, n_k such that

$$p_{n_k} \in B_k.$$

Then if $k \geq l$,

$$\begin{aligned} d\left(p_{n_k}, p_{n_l}\right) &\leq \sum_{i=l}^{k-1} d\left(p_{n_{i+1}}, p_{n_i}\right) \\ &< \sum_{i=l}^{k-1} 2^{-(i-1)} < 2^{-(l-2)}. \end{aligned}$$

Consequently $\{p_{n_k}\}$ is a Cauchy sequence. Hence it converges because the metric space is complete. This proves 5.4.

Now suppose 5.4 and 5.5 which have now been shown to be equivalent. Let D_n be a n^{-1} net for $n = 1, 2, \cdots$ and let

$$D = \cup_{n=1}^{\infty} D_n.$$

Thus D is a countable dense subset of (X, d).

Now let $\mathcal{C}$ be any set of open sets such that $\cup\mathcal{C} \supseteq X$. By Lemma 5.9, there exists a countable subset of $\mathcal{C}$,

$$\widetilde{\mathcal{C}} = \{U_n\}_{n=1}^{\infty}$$

such that $\cup\widetilde{\mathcal{C}} = \cup\mathcal{C}$. If $\mathcal{C}$ admits no finite subcover, then neither does $\widetilde{\mathcal{C}}$ and there exists $p_n \in X \setminus \cup_{k=1}^n U_k$. Then since X is sequentially compact, there is a subsequence $\{p_{n_k}\}$ such that $\{p_{n_k}\}$ converges. Say

$$p = \lim_{k\to\infty} p_{n_k}.$$

All but finitely many points of $\{p_{n_k}\}$ are in $X \setminus \cup_{k=1}^n U_k$. Therefore $p \in X \setminus \cup_{k=1}^n U_k$ for each n. Hence

$$p \notin \cup_{k=1}^{\infty} U_k$$

contradicting the construction of $\{U_n\}_{n=1}^{\infty}$ which required that $\cup_{n=1}^{\infty} U_n \supseteq X$. Hence X is compact. This proves the proposition.

Consider $\mathbb{R}^n$. In this setting totally bounded and bounded are the same. This will yield a proof of the Heine Borel theorem from advanced calculus.

Lemma 5.11 *A subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded.*

Proof: Let A be totally bounded. Is it bounded? Let $\mathbf{x}_1, \cdots, \mathbf{x}_p$ be a 1 net for A. Now consider the ball $B(\mathbf{0}, r+1)$ where $r > \max(|\mathbf{x}_i| : i = 1, \cdots, p)$. If $\mathbf{z} \in A$, then $\mathbf{z} \in B(\mathbf{x}_j, 1)$ for some j and so by the triangle inequality,

$$|\mathbf{z} - \mathbf{0}| \leq |\mathbf{z} - \mathbf{x}_j| + |\mathbf{x}_j| < 1 + r.$$

Thus $A \subseteq B(\mathbf{0}, r+1)$ and so A is bounded.

Now suppose A is bounded and suppose A is not totally bounded. Then there exists $\varepsilon > 0$ such that there is no ε net for A. Therefore, there exists a sequence of points $\{a_i\}$ with $|a_i - a_j| \geq \varepsilon$ if $i \neq j$. Since A is bounded, there exists $r > 0$ such that

$$A \subseteq [-r, r)^n.$$

($\mathbf{x} \in [-r, r)^n$ means $x_i \in [-r, r)$ for each i.) Now define $\mathcal{S}$ to be all cubes of the form

$$\prod_{k=1}^{n} [a_k, b_k)$$

where

$$a_k = -r + i2^{-p}r, \ b_k = -r + (i+1) 2^{-p}r,$$

for $i \in \{0, 1, \cdots, 2^{p+1} - 1\}$. Thus $\mathcal{S}$ is a collection of $\left(2^{p+1}\right)^n$ non overlapping cubes whose union equals $[-r, r)^n$ and whose diameters are all equal to $2^{-p}r\sqrt{n}$. Now choose p large enough that the diameter of these cubes is less than ε. This yields a contradiction because one of the cubes must contain infinitely many points of $\{a_i\}$. This proves the lemma.

The next theorem is called the Heine Borel theorem and it characterizes the compact sets in $\mathbb{R}^n$.

Theorem 5.12 *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof: Since a set in $\mathbb{R}^n$ is totally bounded if and only if it is bounded, this theorem follows from Proposition 5.10 and the observation that a subset of $\mathbb{R}^n$ is closed if and only if it is complete. This proves the theorem.

5.3 Some Applications Of Compactness

The following corollary is an important existence theorem which depends on compactness.

Corollary 5.13 *Let X be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then $\max\{f(x) : x \in X\}$ and $\min\{f(x) : x \in X\}$ both exist.*

Proof: First it is shown $f(X)$ is compact. Suppose $\mathcal{C}$ is a set of open sets whose union contains $f(X)$. Then since f is continuous $f^{-1}(U)$ is open for all $U \in \mathcal{C}$. Therefore, $\left\{f^{-1}(U) : U \in \mathcal{C}\right\}$ is a collection of open sets whose union contains X. Since X is compact, it follows finitely many of these, $\left\{f^{-1}(U_1), \cdots, f^{-1}(U_p)\right\}$ contains X in their union. Therefore, $f(X) \subseteq \cup_{k=1}^{p} U_k$ showing $f(X)$ is compact as claimed.

Now since $f(X)$ is compact, Theorem 5.12 implies $f(X)$ is closed and bounded. Therefore, it contains its inf and its sup. Thus f achieves both a maximum and a minimum.

Definition 5.14 *Let X, Y be metric spaces and $f : X \to Y$ a function. f is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever x_1 and x_2 are two points of X satisfying $d(x_1, x_2) < \delta$, it follows that $d(f(x_1), f(x_2)) < \varepsilon$.*

A very important theorem is the following.

Theorem 5.15 *Suppose $f : X \to Y$ is continuous and X is compact. Then f is uniformly continuous.*

Proof: Suppose this is not true and that f is continuous but not uniformly continuous. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist points, p_δ and q_δ such that $d(p_\delta, q_\delta) < \delta$ and yet $d(f(p_\delta), f(q_\delta)) \geq \varepsilon$. Let p_n and q_n be the points which go with $\delta = 1/n$. By Proposition 5.10 $\{p_n\}$ has a convergent subsequence, $\{p_{n_k}\}$ converging to a point, $x \in X$. Since $d(p_n, q_n) < \frac{1}{n}$, it follows that $q_{n_k} \to x$ also. Therefore,

$$\varepsilon \leq d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(x)) + d(f(x), f(q_{n_k}))$$

but by continuity of f, both $d(f(p_{n_k}), f(x))$ and $d(f(x), f(q_{n_k}))$ converge to 0 as $k \to \infty$ contradicting the above inequality. This proves the theorem.

Another important property of compact sets in a metric space concerns the finite intersection property.

Definition 5.16 *If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.*

Theorem 5.17 *Suppose $\mathcal{F}$ is a collection of compact sets in a metric space, X which has the finite intersection property. Then there exists a point in their intersection. ($\cap\mathcal{F} \neq \emptyset$).*

Proof: If this were not so, $\cup\left\{F^C : F \in \mathcal{F}\right\} = X$ and so, in particular, picking some $F_0 \in \mathcal{F}$, $\left\{F^C : F \in \mathcal{F}\right\}$ would be an open cover of F_0. Since F_0 is compact, some finite subcover, $F_1^C, \cdot\cdot\cdot, F_m^C$ exists. But then $F_0 \subseteq \cup_{k=1}^m F_k^C$ which means $\cap_{k=0}^\infty F_k = \emptyset$, contrary to the finite intersection property.

Theorem 5.18 *Let X_i be a compact metric space with metric d_i. Then $\prod_{i=1}^m X_i$ is also a compact metric space with respect to the metric, $d(\mathbf{x}, \mathbf{y}) \equiv \max_i (d_i(x_i, y_i))$.*

Proof: This is most easily seen from sequential compactness. Let $\left\{\mathbf{x}^k\right\}_{k=1}^\infty$ be a sequence of points in $\prod_{i=1}^m X_i$. Consider the i^{th} component of $\mathbf{x}^k$, x_i^k. It follows $\left\{x_i^k\right\}$ is a sequence of points in X_i and so it has a convergent subsequence. Compactness of X_1 implies there exists a subsequence of $\mathbf{x}^k$, denoted by $\left\{\mathbf{x}^{k_1}\right\}$ such that

$$\lim_{k_1 \to \infty} x_1^{k_1} \to x_1 \in X_1.$$

Now there exists a further subsequence, denoted by $\left\{\mathbf{x}^{k_2}\right\}$ such that in addition to this, $x_2^{k_2} \to x_2 \in X_2$. After taking m such subsequences, there exists a subsequence, $\left\{\mathbf{x}^l\right\}$ such that $\lim_{l \to \infty} x_i^l = x_i \in X_i$ for each i. Therefore, letting $\mathbf{x} \equiv (x_1, \cdot\cdot\cdot, x_m)$, $\mathbf{x}^l \to \mathbf{x}$ in $\prod_{i=1}^m X_i$. This proves the theorem.

5.4 Ascoli Arzela Theorem

Definition 5.19 *Let (X, d) be a complete metric space. Then it is said to be locally compact if $\overline{B(x, r)}$ is compact for each $r > 0$.*

Thus if you have a locally compact metric space, then if $\{a_n\}$ is a bounded sequence, it must have a convergent subsequence.

Let K be a compact subset of $\mathbb{R}^n$ and consider the continuous functions which have values in a locally compact metric space, (X, d) where d denotes the metric on X. Denote this space as $C(K, X)$.

Definition 5.20 *For $f, g \in C(K, X)$, where K is a compact subset of $\mathbb{R}^n$ and X is a locally compact complete metric space define*

$$\rho_K(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})) : \mathbf{x} \in K\}.$$

Then ρ_K provides a distance which makes $C(K, X)$ into a metric space.

The Ascoli Arzela theorem is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 5.21 *Let $A \subseteq C(K, X)$ for K a compact subset of $\mathbb{R}^n$. Then A is said to be uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x} - \mathbf{y}| < \delta$ and $f \in A$,*

$$d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon.$$

The set, A is said to be uniformly bounded if for some $M < \infty$, and $a \in X$,

$$f(\mathbf{x}) \in B(a, M)$$

for all $f \in A$ and $\mathbf{x} \in K$.

Uniform equicontinuity is like saying that the whole set of functions, A, is uniformly continuous on K uniformly for $f \in A$. The version of the Ascoli Arzela theorem I will present here is the following.

Theorem 5.22 *Suppose K is a nonempty compact subset of $\mathbb{R}^n$ and $A \subseteq C(K, X)$ is uniformly bounded and uniformly equicontinuous. Then if $\{f_k\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, f_{k_l} such that*

$$\lim_{l \to \infty} \rho_K(f_{k_l}, f) = 0.$$

To give a proof of this theorem, I will first prove some lemmas.

Lemma 5.23 *If K is a compact subset of $\mathbb{R}^n$, then there exists $D \equiv \{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq K$ such that D is dense in K. Also, for every $\varepsilon > 0$ there exists a finite set of points, $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\} \subseteq K$, called an ε net such that*

$$\cup_{i=1}^{m} B(\mathbf{x}_i, \varepsilon) \supseteq K.$$

Proof: For $m \in \mathbb{N}$, pick $x_1^m \in K$. If every point of K is within $1/m$ of x_1^m, stop. Otherwise, pick

$$x_2^m \in K \setminus B\left(x_1^m, 1/m\right).$$

If every point of K contained in $B\left(x_1^m, 1/m\right) \cup B\left(x_2^m, 1/m\right)$, stop. Otherwise, pick

$$x_3^m \in K \setminus \left(B\left(x_1^m, 1/m\right) \cup B\left(x_2^m, 1/m\right)\right).$$

If every point of K is contained in $B\left(x_1^m, 1/m\right) \cup B\left(x_2^m, 1/m\right) \cup B\left(x_3^m, 1/m\right)$, stop. Otherwise, pick

$$x_4^m \in K \setminus \left(B\left(x_1^m, 1/m\right) \cup B\left(x_2^m, 1/m\right) \cup B\left(x_3^m, 1/m\right)\right)$$

Continue this way until the process stops, say at $N(m)$. It must stop because if it didn't, there would be a convergent subsequence due to the compactness of K. Ultimately all terms of this convergent subsequence would be closer than $1/m$, violating the manner in which they are chosen. Then $D = \cup_{m=1}^{\infty} \cup_{k=1}^{N(m)} \{x_k^m\}$. This is countable because it is a countable union of countable sets. If $\mathbf{y} \in K$ and $\varepsilon > 0$, then for some m, $2/m < \varepsilon$ and so $B(\mathbf{y}, \varepsilon)$ must contain some point of $\{x_k^m\}$ since otherwise, the process stopped too soon. You could have picked $\mathbf{y}$. This proves the lemma.

Lemma 5.24 *Suppose D is defined above and $\{g_m\}$ is a sequence of functions of A having the property that for every $\mathbf{x}_k \in D$,*

$$\lim_{m\to\infty} g_m(\mathbf{x}_k) \text{ exists.}$$

Then there exists $g \in C(K, X)$ such that

$$\lim_{m\to\infty} \rho(g_m, g) = 0.$$

Proof: Define g first on D.

$$g(\mathbf{x}_k) \equiv \lim_{m\to\infty} g_m(\mathbf{x}_k).$$

Next I show that $\{g_m\}$ converges at every point of K. Let $\mathbf{x} \in K$ and let $\varepsilon > 0$ be given. Choose $\mathbf{x}_k$ such that for all $f \in A$,

$$d\left(f(\mathbf{x}_k), f(\mathbf{x})\right) < \frac{\varepsilon}{3}.$$

I can do this by the equicontinuity. Now if p, q are large enough, say $p, q \geq M$,

$$d\left(g_p(\mathbf{x}_k), g_q(\mathbf{x}_k)\right) < \frac{\varepsilon}{3}.$$

Therefore, for $p, q \geq M$,

$$\begin{aligned} d\left(g_p(\mathbf{x}), g_q(\mathbf{x})\right) &\leq d\left(g_p(\mathbf{x}), g_p(\mathbf{x}_k)\right) + d\left(g_p(\mathbf{x}_k), g_q(\mathbf{x}_k)\right) + d\left(g_q(\mathbf{x}_k), g_q(\mathbf{x})\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It follows that $\{g_m(\mathbf{x})\}$ is a Cauchy sequence having values X. Therefore, it converges. Let $g(\mathbf{x})$ be the name of the thing it converges to.

Let $\varepsilon > 0$ be given and pick $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ and $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $d(f(\mathbf{x}), f(\mathbf{y})) < \frac{\varepsilon}{3}$ for all $f \in A$. Now let $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ be a δ net for K as in Lemma 5.23. Since there are only finitely many points in this δ net, it follows that there exists N such that for all $p, q \geq N$,

$$d(g_q(\mathbf{x}_i), g_p(\mathbf{x}_i)) < \frac{\varepsilon}{3}$$

for all $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\} \cdot$ Therefore, for arbitrary $\mathbf{x} \in K$, pick $\mathbf{x}_i \in \{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ such that $|\mathbf{x}_i - \mathbf{x}| < \delta$. Then

$$\begin{aligned} d(g_q(\mathbf{x}), g_p(\mathbf{x})) &\leq d(g_q(\mathbf{x}), g_q(\mathbf{x}_i)) + d(g_q(\mathbf{x}_i), g_p(\mathbf{x}_i)) + d(g_p(\mathbf{x}_i), g_p(\mathbf{x})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since N does not depend on the choice of $\mathbf{x}$, it follows this sequence $\{g_m\}$ is uniformly Cauchy. That is, for every $\varepsilon > 0$, there exists N such that if $p, q \geq N$, then

$$\rho(g_p, g_q) < \varepsilon.$$

Next, I need to verify that the function, g is a continuous function. Let N be large enough that whenever $p, q \geq N$, the above holds. Then for all $\mathbf{x} \in K$,

$$d(g(\mathbf{x}), g_p(\mathbf{x})) \leq \frac{\varepsilon}{3} \tag{5.6}$$

whenever $p \geq N$. This follows from observing that for $p, q \geq N$,

$$d(g_q(\mathbf{x}), g_p(\mathbf{x})) < \frac{\varepsilon}{3}$$

and then taking the limit as $q \to \infty$ to obtain 5.6. In passing to the limit, you can use the following simple claim.

Claim: In a metric space, if $a_n \to a$, then $d(a_n, b) \to d(a, b)$.

Proof of the claim: You note that by the triangle inequality, $d(a_n, b) - d(a, b) \leq d(a_n, a)$ and $d(a, b) - d(a_n, b) \leq d(a_n, a)$ and so

$$|d(a_n, b) - d(a, b)| \leq d(a_n, a).$$

Now let p satisfy 5.6 for all $\mathbf{x}$ whenever $p > N$. Also pick $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then

$$d(g_p(\mathbf{x}), g_p(\mathbf{y})) < \frac{\varepsilon}{3}.$$

Then if $|\mathbf{x} - \mathbf{y}| < \delta$,

$$\begin{aligned} d(g(\mathbf{x}), g(\mathbf{y})) &\leq d(g(\mathbf{x}), g_p(\mathbf{x})) + d(g_p(\mathbf{x}), g_p(\mathbf{y})) + d(g_p(\mathbf{y}), g(\mathbf{y})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, this shows that g is continuous.

It only remains to verify that $\rho(g, g_k) \to 0$. But this follows from 5.6. This proves the lemma.

With these lemmas, it is time to prove Theorem 5.22.

Proof of Theorem 5.22: Let $D = \{\mathbf{x}_k\}$ be the countable dense set of K gauranteed by Lemma 5.23 and let $\{(1,1), (1,2), (1,3), (1,4), (1,5), \cdots\}$ be a subsequence of $\mathbb{N}$ such that

$$\lim_{k\to\infty} f_{(1,k)}(\mathbf{x}_1) \text{ exists.}$$

This is where the local compactness of X is being used. Now let

$$\{(2,1), (2,2), (2,3), (2,4), (2,5), \cdots\}$$

be a subsequence of $\{(1,1), (1,2), (1,3), (1,4), (1,5), \cdots\}$ which has the property that

$$\lim_{k\to\infty} f_{(2,k)}(\mathbf{x}_2) \text{ exists.}$$

Thus it is also the case that

$$f_{(2,k)}(\mathbf{x}_1) \text{ converges to } \lim_{k\to\infty} f_{(1,k)}(\mathbf{x}_1).$$

because every subsequence of a convergent sequence converges to the same thing as the convergent sequence. Continue this way and consider the array

$$\begin{array}{c} f_{(1,1)}, f_{(1,2)}, f_{(1,3)}, f_{(1,4)}, \cdots \text{ converges at } \mathbf{x}_1 \\ f_{(2,1)}, f_{(2,2)}, f_{(2,3)}, f_{(2,4)} \cdots \text{ converges at } \mathbf{x}_1 \text{ and } \mathbf{x}_2 \\ f_{(3,1)}, f_{(3,2)}, f_{(3,3)}, f_{(3,4)} \cdots \text{ converges at } \mathbf{x}_1, \mathbf{x}_2, \text{ and } \mathbf{x}_3 \\ \vdots \end{array}$$

Now let $g_k \equiv f_{(k,k)}$. Thus g_k is ultimately a subsequence of $\{f_{(m,k)}\}$ whenever $k > m$ and therefore, $\{g_k\}$ converges at each point of D. By Lemma 5.24 it follows there exists $g \in C(K)$ such that

$$\lim_{k\to\infty} \rho(g, g_k) = 0.$$

This proves the Ascoli Arzela theorem.

Actually there is an if and only if version of it but the most useful case is what is presented here. The process used to get the subsequence in the proof is called the Cantor diagonalization procedure.

5.5 General Topological Spaces

It turns out that metric spaces are not sufficiently general for some applications. This section is a brief introduction to general topology. In making this generalization, the properties of balls which are the conclusion of Theorem 5.4 on Page 94 are stated as axioms for a subset of the power set of a given set which will be known as a basis for the topology. More can be found in [28] and the references listed there.

Definition 5.25 *Let X be a nonempty set and suppose $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for a topology if it satisfies the following axioms.*

1.) Whenever $p \in A \cap B$ for $A, B \in \mathcal{B}$, it follows there exists $C \in \mathcal{B}$ such that $p \in C \subseteq A \cap B$.

2.) $\cup \mathcal{B} = X$.

Then a subset, U, of X is an open set if for every point, $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Thus the open sets are exactly those which can be obtained as a union of sets of $\mathcal{B}$. Denote these subsets of X by the symbol τ and refer to τ as the topology or the set of open sets.

Note that this is simply the analog of saying a set is open exactly when every point is an interior point.

Proposition 5.26 *Let X be a set and let $\mathcal{B}$ be a basis for a topology as defined above and let τ be the set of open sets determined by $\mathcal{B}$. Then*

$$\emptyset \in \tau, \ X \in \tau, \tag{5.7}$$

$$\textit{If}\, \mathcal{C} \subseteq \tau, \ \textit{then} \ \cup \mathcal{C} \in \tau \tag{5.8}$$

$$\textit{If}\ A, B \in \tau, \ \textit{then}\ A \cap B \in \tau. \tag{5.9}$$

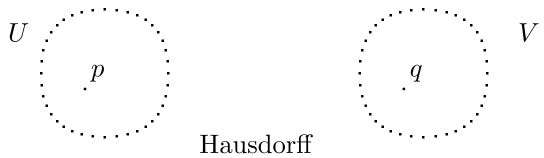
Proof: If $p \in \emptyset$ then there exists $B \in \mathcal{B}$ such that $p \in B \subseteq \emptyset$ because there are no points in $\emptyset$. Therefore, $\emptyset \in \tau$. Now if $p \in X$, then by part 2.) of Definition 5.25 $p \in B \subseteq X$ for some $B \in \mathcal{B}$ and so $X \in \tau$.

If $\mathcal{C} \subseteq \tau$, and if $p \in \cup \mathcal{C}$, then there exists a set, $B \in \mathcal{C}$ such that $p \in B$. However, B is itself a union of sets from $\mathcal{B}$ and so there exists $C \in \mathcal{B}$ such that $p \in C \subseteq B \subseteq \cup \mathcal{C}$. This verifies 5.8.

Finally, if $A, B \in \tau$ and $p \in A \cap B$, then since A and B are themselves unions of sets of $\mathcal{B}$, it follows there exists $A_1, B_1 \in \mathcal{B}$ such that $A_1 \subseteq A, B_1 \subseteq B$, and $p \in A_1 \cap B_1$. Therefore, by 1.) of Definition 5.25 there exists $C \in \mathcal{B}$ such that $p \in C \subseteq A_1 \cap B_1 \subseteq A \cap B$, showing that $A \cap B \in \tau$ as claimed. Of course if $A \cap B = \emptyset$, then $A \cap B \in \tau$. This proves the proposition.

Definition 5.27 *A set X together with such a collection of its subsets satisfying 5.7-5.9 is called a topological space. τ is called the topology or set of open sets of X.*

Definition 5.28 *A topological space is said to be Hausdorff if whenever p and q are distinct points of X, there exist disjoint open sets U, V such that $p \in U$, $q \in V$. In other words points can be separated with open sets.*



Hausdorff

Definition 5.29 *A subset of a topological space is said to be closed if its complement is open. Let p be a point of X and let $E \subseteq X$. Then p is said to be a limit point of E if every open set containing p contains a point of E distinct from p.*

Note that if the topological space is Hausdorff, then this definition is equivalent to requiring that every open set containing p contains infinitely many points from E. Why?

Theorem 5.30 *A subset, E, of X is closed if and only if it contains all its limit points.*

Proof: Suppose first that E is closed and let x be a limit point of E. Is $x \in E$? If $x \notin E$, then E^C is an open set containing x which contains no points of E, a contradiction. Thus $x \in E$.

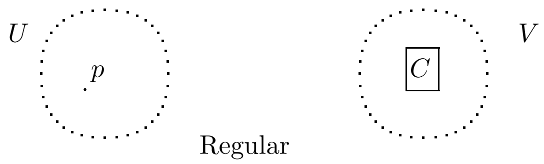
Now suppose E contains all its limit points. Is the complement of E open? If $x \in E^C$, then x is not a limit point of E because E has all its limit points and so there exists an open set, U containing x such that U contains no point of E other than x. Since $x \notin E$, it follows that $x \in U \subseteq E^C$ which implies E^C is an open set because this shows E^C is the union of open sets.

Theorem 5.31 *If (X, τ) is a Hausdorff space and if $p \in X$, then $\{p\}$ is a closed set.*

Proof: If $x \neq p$, there exist open sets U and V such that $x \in U, p \in V$ and $U \cap V = \emptyset$. Therefore, $\{p\}^C$ is an open set so $\{p\}$ is closed.

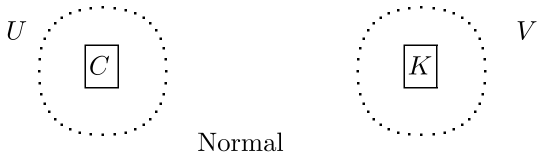
Note that the Hausdorff axiom was stronger than needed in order to draw the conclusion of the last theorem. In fact it would have been enough to assume that if $x \neq y$, then there exists an open set containing x which does not intersect y.

Definition 5.32 *A topological space (X, τ) is said to be regular if whenever C is a closed set and p is a point not in C, there exist disjoint open sets U and V such that $p \in U$, $C \subseteq V$. Thus a closed set can be separated from a point not in the closed set by two disjoint open sets.*



Regular

Definition 5.33 *The topological space, (X, τ) is said to be normal if whenever C and K are disjoint closed sets, there exist disjoint open sets U and V such that $C \subseteq U$, $K \subseteq V$. Thus any two disjoint closed sets can be separated with open sets.*



Normal

Definition 5.34 *Let E be a subset of X. $\overline{E}$ is defined to be the smallest closed set containing E.*

Lemma 5.35 *The above definition is well defined.*

Proof: Let $\mathcal{C}$ denote all the closed sets which contain E. Then $\mathcal{C}$ is nonempty because $X \in \mathcal{C}$.

$$(\cap \{A : A \in \mathcal{C}\})^C = \cup \left\{A^C : A \in \mathcal{C}\right\},$$

an open set which shows that $\cap\mathcal{C}$ is a closed set and is the smallest closed set which contains E.

Theorem 5.36 $\overline{E} = E \cup \{$*limit points of* $E\}$.

Proof: Let $x \in \overline{E}$ and suppose that $x \notin E$. If x is not a limit point either, then there exists an open set, U,containing x which does not intersect E. But then U^C is a closed set which contains E which does not contain x, contrary to the definition that $\overline{E}$ is the intersection of all closed sets containing E. Therefore, x must be a limit point of E after all.

Now $E \subseteq \overline{E}$ so suppose x is a limit point of E. Is $x \in \overline{E}$? If H is a closed set containing E, which does not contain x, then H^C is an open set containing x which contains no points of E other than x negating the assumption that x is a limit point of E.

The following is the definition of continuity in terms of general topological spaces. It is really just a generalization of the ε - δ definition of continuity given in calculus.

Definition 5.37 *Let (X, τ) and (Y, η) be two topological spaces and let $f : X \to Y$. f is continuous at $x \in X$ if whenever V is an open set of Y containing $f(x)$, there exists an open set $U \in \tau$ such that $x \in U$ and $f(U) \subseteq V$. f is continuous if $f^{-1}(V) \in \tau$ whenever $V \in \eta$.*

You should prove the following.

Proposition 5.38 *In the situation of Definition 5.37 f is continuous if and only if f is continuous at every point of X.*

Definition 5.39 *Let (X_i, τ_i) be topological spaces. $\prod_{i=1}^n X_i$ is the Cartesian product. Define a product topology as follows. Let $\mathcal{B} = \prod_{i=1}^n A_i$ where $A_i \in \tau_i$. Then $\mathcal{B}$ is a basis for the product topology.*

Theorem 5.40 *The set $\mathcal{B}$ of Definition 5.39 is a basis for a topology.*

Proof: Suppose $\mathbf{x} \in \prod_{i=1}^{n} A_i \cap \prod_{i=1}^{n} B_i$ where A_i and B_i are open sets. Say

$$\mathbf{x} = (x_1, \cdots, x_n).$$

Then $x_i \in A_i \cap B_i$ for each i. Therefore, $\mathbf{x} \in \prod_{i=1}^{n} A_i \cap B_i \in \mathcal{B}$ and $\prod_{i=1}^{n} A_i \cap B_i \subseteq \prod_{i=1}^{n} A_i$.

The definition of compactness is also considered for a general topological space. This is given next.

Definition 5.41 *A subset, E, of a topological space (X, τ) is said to be compact if whenever $\mathcal{C} \subseteq \tau$ and $E \subseteq \cup\mathcal{C}$, there exists a finite subset of $\mathcal{C}, \{U_1 \cdots U_n\}$, such that $E \subseteq \cup_{i=1}^{n} U_i$. (Every open covering admits a finite subcovering.) E is precompact if $\overline{E}$ is compact. A topological space is called locally compact if it has a basis $\mathcal{B}$, with the property that $\overline{B}$ is compact for each $B \in \mathcal{B}$.*

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space.

Definition 5.42 *Suppose (X, τ) is a locally compact Hausdorff space. Then let $\widetilde{X} \equiv X \cup \{\infty\}$ where ∞ is just the name of some point which is not in X which is called the point at infinity. A basis for the topology $\widetilde{\tau}$ for $\widetilde{X}$ is*

$$\tau \cup \left\{K^C \text{ where } K \text{ is a compact subset of } X\right\}.$$

The complement is taken with respect to $\widetilde{X}$ and so the open sets, K^C are basic open sets which contain ∞.

The reason this is called a compactification is contained in the next lemma.

Lemma 5.43 *If (X, τ) is a locally compact Hausdorff space, then $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a compact Hausdorff space.*

Proof: Since (X, τ) is a locally compact Hausdorff space, it follows $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and ∞. Since (X, τ) is locally compact, there exists an open set of τ, U having compact closure which contains p. Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, p and ∞ respectively. Now let $\mathcal{C}$ be an open cover of $\widetilde{X}$ with sets from $\widetilde{\tau}$. Then ∞ must be in some set, U_∞ from $\mathcal{C}$, which must contain a set of the form K^C where K is a compact subset of X. Then there exist sets from $\mathcal{C}$, $U_1, \cdots, U_r$ which cover K. Therefore, a finite subcover of $\widetilde{X}$ is $U_1, \cdots, U_r, U_\infty$.

In general topological spaces there may be no concept of "bounded". Even if there is, closed and bounded is not necessarily the same as compactness. However, in any Hausdorff space every compact set must be a closed set.

Theorem 5.44 *If (X, τ) is a Hausdorff space, then every compact subset must also be a closed set.*

Proof: Suppose $p \notin K$. For each $x \in X$, there exist open sets, U_x and V_x such that

$$x \in U_x,\ p \in V_x,$$

and

$$U_x \cap V_x = \emptyset.$$

If K is assumed to be compact, there are finitely many of these sets, $U_{x_1}, \cdot \cdot \cdot, U_{x_m}$ which cover K. Then let $V \equiv \cap_{i=1}^m V_{x_i}$. It follows that V is an open set containing p which has empty intersection with each of the U_{x_i}. Consequently, V contains no points of K and is therefore not a limit point of K. This proves the theorem.

Definition 5.45 *If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.*

Theorem 5.46 *Let $\mathcal{K}$ be a set whose elements are compact subsets of a Hausdorff topological space, (X, τ). Suppose $\mathcal{K}$ has the finite intersection property. Then $\emptyset \neq \cap \mathcal{K}$.*

Proof: Suppose to the contrary that $\emptyset = \cap \mathcal{K}$. Then consider

$$\mathcal{C} \equiv \left\{ K^C : K \in \mathcal{K} \right\}.$$

It follows $\mathcal{C}$ is an open cover of K_0 where K_0 is any particular element of $\mathcal{K}$. But then there are finitely many $K \in \mathcal{K}$, $K_1, \cdot \cdot \cdot, K_r$ such that $K_0 \subseteq \cup_{i=1}^r K_i^C$ implying that $\cap_{i=0}^r K_i = \emptyset$, contradicting the finite intersection property.

Lemma 5.47 *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ. Then K is compact if and only if every open cover of basic open sets admits a finite subcover.*

Proof: Suppose first that X is compact. Then if $\mathcal{C}$ is an open cover consisting of basic open sets, it follows it admits a finite subcover because these are open sets in $\mathcal{C}$.

Next suppose that every basic open cover admits a finite subcover and let $\mathcal{C}$ be an open cover of X. Then define $\widetilde{\mathcal{C}}$ to be the collection of basic open sets which are contained in some set of $\mathcal{C}$. It follows $\widetilde{\mathcal{C}}$ is a basic open cover of X and so it admits a finite subcover, $\{U_1, \cdot \cdot \cdot, U_p\}$. Now each U_i is contained in an open set of $\mathcal{C}$. Let O_i be a set of $\mathcal{C}$ which contains U_i. Then $\{O_1, \cdot \cdot \cdot, O_p\}$ is an open cover of X. This proves the lemma.

In fact, much more can be said than Lemma 5.47. However, this is all which I will present here.

5.6 Connected Sets

Stated informally, connected sets are those which are in one piece. More precisely,

Definition 5.48 *A set, S in a general topological space is separated if there exist sets, A, B such that*

$$S = A \cup B,\ A, B \neq \emptyset,\ \text{and}\ \overline{A} \cap B = \overline{B} \cap A = \emptyset.$$

In this case, the sets A and B are said to separate S. A set is connected if it is not separated.

One of the most important theorems about connected sets is the following.

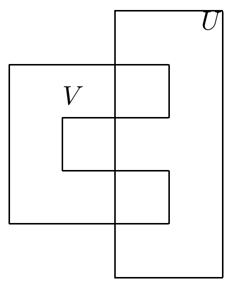
Theorem 5.49 *Suppose U and V are connected sets having nonempty intersection. Then $U \cup V$ is also connected.*

Proof: Suppose $U \cup V = A \cup B$ where $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. Consider the sets, $A \cap U$ and $B \cup U$. Since

$$\overline{(A \cap U)} \cap (B \cap U) = (A \cap U) \cap \left(\overline{B \cap U}\right) = \emptyset,$$

It follows one of these sets must be empty since otherwise, U would be separated. It follows that U is contained in either A or B. Similarly, V must be contained in either A or B. Since U and V have nonempty intersection, it follows that both V and U are contained in one of the sets, A, B. Therefore, the other must be empty and this shows $U \cup V$ cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.



Theorem 5.50 *Let $f : X \to Y$ be continuous where X and Y are topological spaces and X is connected. Then $f(X)$ is also connected.*

Proof: To do this you show $f(X)$ is not separated. Suppose to the contrary that $f(X) = A \cup B$ where A and B separate $f(X)$. Then consider the sets, $f^{-1}(A)$ and $f^{-1}(B)$. If $z \in f^{-1}(B)$, then $f(z) \in B$ and so $f(z)$ is not a limit point of

A. Therefore, there exists an open set, U containing $f(z)$ such that $U \cap A = \emptyset$. But then, the continuity of f implies that $f^{-1}(U)$ is an open set containing z such that $f^{-1}(U) \cap f^{-1}(A) = \emptyset$. Therefore, $f^{-1}(B)$ contains no limit points of $f^{-1}(A)$. Similar reasoning implies $f^{-1}(A)$ contains no limit points of $f^{-1}(B)$. It follows that X is separated by $f^{-1}(A)$ and $f^{-1}(B)$, contradicting the assumption that X was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 5.51 *Let S be a set and let $p \in S$. Denote by C_p the union of all connected subsets of S which contain p. This is called the connected component determined by p.*

Theorem 5.52 *Let C_p be a connected component of a set S in a general topological space. Then C_p is a connected set and if $C_p \cap C_q \neq \emptyset$, then $C_p = C_q$.*

Proof: Let $\mathcal{C}$ denote the connected subsets of S which contain p. If $C_p = A \cup B$ where

$$\overline{A} \cap B = \overline{B} \cap A = \emptyset,$$

then p is in one of A or B. Suppose without loss of generality $p \in A$. Then every set of $\mathcal{C}$ must also be contained in A also since otherwise, as in Theorem 5.49, the set would be separated. But this implies B is empty. Therefore, C_p is connected. From this, and Theorem 5.49, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, I is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 5.53 *A set, C in $\mathbb{R}$ is connected if and only if C is an interval.*

Proof: Let C be connected. If C consists of a single point, p, there is nothing to prove. The interval is just $[p, p]$. Suppose $p < q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$. If

$$x \in (p, q) \setminus C$$

let $C \cap (-\infty, x) \equiv A$, and $C \cap (x, \infty) \equiv B$. Then $C = A \cup B$ and the sets, A and B separate C contrary to the assumption that C is connected.

Conversely, let I be an interval. Suppose I is separated by A and B. Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x < y$. Now define the set,

$$S \equiv \{t \in [x, y] : [x, t] \subseteq A\}$$

and let l be the least upper bound of S. Then $l \in \overline{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \overline{B}$, then for some $\delta > 0$,

$$(l, l + \delta) \cap B = \emptyset$$

contradicting the definition of l as an upper bound for S. Therefore, $l \in \overline{B}$ which implies $l \notin A$ after all, a contradiction. It follows I must be connected.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.

Theorem 5.54 *Let U be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $U = \cup_{i=1}^{\infty} (a_i, b_i)$.*

Proof: Let $p \in U$ and let $z \in C_p$, the connected component determined by p. Since U is open, there exists, $\delta > 0$ such that $(z - \delta, z + \delta) \subseteq U$. It follows from Theorem 5.49 that

$$(z - \delta, z + \delta) \subseteq C_p.$$

This shows C_p is open. By Theorem 5.53, this shows C_p is an open interval, (a, b) where $a, b \in [-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\{(a_i, b_i)\}_{i=1}^{\infty}$ the set of these connected components. This proves the theorem.

Definition 5.55 *A topological space, E is arcwise connected if for any two points, $p, q \in E$, there exists a closed interval, $[a, b]$ and a continuous function, $\gamma : [a, b] \to E$ such that $\gamma(a) = p$ and $\gamma(b) = q$. E is locally connected if it has a basis of connected open sets. E is locally arcwise connected if it has a basis of arcwise connected open sets.*

An example of an arcwise connected topological space would be the any subset of $\mathbb{R}^n$ which is the continuous image of an interval. Locally connected is not the same as connected. A well known example is the following.

$$\left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0, 1] \right\} \cup \{(0, y) : y \in [-1, 1]\} \tag{5.10}$$

You can verify that this set of points considered as a metric space with the metric from $\mathbb{R}^2$ is not locally connected or arcwise connected but is connected.

Proposition 5.56 *If a topological space is arcwise connected, then it is connected.*

Proof: Let X be an arcwise connected space and suppose it is separated. Then $X = A \cup B$ where A, B are two separated sets. Pick $p \in A$ and $q \in B$. Since X is given to be arcwise connected, there must exist a continuous function $\gamma : [a, b] \to X$ such that $\gamma(a) = p$ and $\gamma(b) = q$. But then we would have $\gamma([a, b]) = (\gamma([a, b]) \cap A) \cup (\gamma([a, b]) \cap B)$ and the two sets, $\gamma([a, b]) \cap A$ and $\gamma([a, b]) \cap B$ are separated thus showing that $\gamma([a, b])$ is separated and contradicting Theorem 5.53 and Theorem 5.50. It follows that X must be connected as claimed.

Theorem 5.57 *Let U be an open subset of a locally arcwise connected topological space, X. Then U is arcwise connected if and only if U if connected. Also the connected components of an open set in such a space are open sets, hence arcwise connected.*

Proof: By Proposition 5.56 it is only necessary to verify that if U is connected and open in the context of this theorem, then U is arcwise connected. Pick $p \in U$.

Say $x \in U$ satisfies $\mathcal{P}$ if there exists a continuous function, $\gamma : [a,b] \to U$ such that $\gamma(a) = p$ and $\gamma(b) = x$.

$$A \equiv \{x \in U \text{ such that } x \text{ satisfies } \mathcal{P}.\}$$

If $x \in A$, there exists, according to the assumption that X is locally arcwise connected, an open set, V, containing x and contained in U which is arcwise connected. Thus letting $y \in V$, there exist intervals, $[a,b]$ and $[c,d]$ and continuous functions having values in U, γ, η such that $\gamma(a) = p, \gamma(b) = x, \eta(c) = x$, and $\eta(d) = y$. Then let $\gamma_1 : [a, b+d-c] \to U$ be defined as

$$\gamma_1(t) \equiv \begin{cases} \gamma(t) & \text{if } t \in [a,b] \\ \eta(t) & \text{if } t \in [b, b+d-c] \end{cases}$$

Then it is clear that γ_1 is a continuous function mapping p to y and showing that $V \subseteq A$. Therefore, A is open. $A \neq \emptyset$ because there is an open set, V containing p which is contained in U and is arcwise connected.

Now consider $B \equiv U \setminus A$. This is also open. If B is not open, there exists a point $z \in B$ such that every open set containing z is not contained in B. Therefore, letting V be one of the basic open sets chosen such that $z \in V \subseteq U$, there exist points of A contained in V. But then, a repeat of the above argument shows $z \in A$ also. Hence B is open and so if $B \neq \emptyset$, then $U = B \cup A$ and so U is separated by the two sets, B and A contradicting the assumption that U is connected.

It remains to verify the connected components are open. Let $z \in C_p$ where C_p is the connected component determined by p. Then picking V an arcwise connected open set which contains z and is contained in U, $C_p \cup V$ is connected and contained in U and so it must also be contained in C_p. This proves the theorem.

As an application, consider the following corollary.

Corollary 5.58 *Let $f : \Omega \to \mathbb{Z}$ be continuous where Ω is a connected open set. Then f must be a constant.*

Proof: Suppose not. Then it achieves two different values, k and $l \neq k$. Then $\Omega = f^{-1}(l) \cup f^{-1}(\{m \in \mathbb{Z} : m \neq l\})$ and these are disjoint nonempty open sets which separate Ω. To see they are open, note

$$f^{-1}(\{m \in \mathbb{Z} : m \neq l\}) = f^{-1}\left(\cup_{m \neq l}\left(m - \frac{1}{6}, n + \frac{1}{6}\right)\right)$$

which is the inverse image of an open set.

5.7 Exercises

1. Let V be an open set in $\mathbb{R}^n$. Show there is an increasing sequence of open sets, $\{U_m\}$, such for all $m \in \mathbb{N}$, $\overline{U_m} \subseteq U_{m+1}, \overline{U_m}$ is compact, and $V = \cup_{m=1}^{\infty} U_m$.

2. Completeness of $\mathbb{R}$ is an axiom. Using this, show $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces with respect to the distance given by the usual norm.

3. Let X be a metric space. Can we conclude $\overline{B(x,r)} = \{y : d(x,y) \leq r\}$? **Hint:** Try letting X consist of an infinite set and let $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$.

4. The usual version of completeness in $\mathbb{R}$ involves the assertion that a nonempty set which is bounded above has a least upper bound. Show this is equivalent to saying that every Cauchy sequence converges.

5. If (X,d) is a metric space, prove that whenever K, H are disjoint non empty closed sets, there exists $f : X \to [0,1]$ such that f is continuous, $f(K) = \{0\}$, and $f(H) = \{1\}$.

6. Consider $\mathbb{R}$ with the usual metric, $d(x,y) = |x-y|$ and the metric,
$$\rho(x,y) = |\arctan x - \arctan y|$$
Thus we have two metric spaces here although they involve the same sets of points. Show the identity map is continuous and has a continuous inverse. Show that $\mathbb{R}$ with the metric, ρ is not complete while $\mathbb{R}$ with the usual metric is complete. The first part of this problem shows the two metric spaces are homeomorphic. (That is what it is called when there is a one to one onto continuous map having continuous inverse between two topological spaces.) Thus completeness is not a topological property although it will likely be referred to as such.

7. If M is a separable metric space and $T \subseteq M$, then T is also a separable metric space with the same metric.

8. Prove the Heine Borel theorem as follows. First show $[a,b]$ is compact in $\mathbb{R}$. Next show that $\prod_{i=1}^{n} [a_i, b_i]$ is compact. Use this to verify that compact sets are exactly those which are closed and bounded.

9. Give an example of a metric space in which closed and bounded subsets are not necessarily compact. **Hint:** Let X be any infinite set and let $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. Show this is a metric space. What about $\overline{B(x,2)}$?

10. If $f : [a,b] \to \mathbb{R}$ is continuous, show that f is Riemann integrable.**Hint:** Use the theorem that a function which is continuous on a compact set is uniformly continuous.

11. Give an example of a set, $X \subseteq \mathbb{R}^2$ which is connected but not arcwise connected. Recall arcwise connected means for every two points, $p, q \in X$ there exists a continuous function $f : [0,1] \to X$ such that $f(0) = p, f(1) = q$.

12. Let (X,d) be a metric space where d is a bounded metric. Let $\mathcal{C}$ denote the collection of closed subsets of X. For $A, B \in \mathcal{C}$, define
$$\rho(A,B) \equiv \inf\{\delta > 0 : A_\delta \supseteq B \text{ and } B_\delta \supseteq A\}$$

where for a set S,

$$S_\delta \equiv \{x : \text{dist}(x, S) \equiv \inf\{d(x, s) : s \in S\} \leq \delta\}.$$

Show S_δ is a closed set containing S. Also show that ρ is a metric on $\mathcal{C}$. This is called the Hausdorff metric.

13. Using 12, suppose (X, d) is a compact metric space. Show $(\mathcal{C}, \rho)$ is a complete metric space. **Hint:** Show first that if $W_n \downarrow W$ where W_n is closed, then $\rho(W_n, W) \to 0$. Now let $\{A_n\}$ be a Cauchy sequence in $\mathcal{C}$. Then if $\varepsilon > 0$ there exists N such that when $m, n \geq N$, then $\rho(A_n, A_m) < \varepsilon$. Therefore, for each $n \geq N$,
$$(A_n)_\varepsilon \supseteq \overline{\cup_{k=n}^\infty A_k}.$$
Let $A \equiv \cap_{n=1}^\infty \overline{\cup_{k=n}^\infty A_k}$. By the first part, there exists $N_1 > N$ such that for $n \geq N_1$,
$$\rho\left(\overline{\cup_{k=n}^\infty A_k}, A\right) < \varepsilon, \text{ and } (A_n)_\varepsilon \supseteq \overline{\cup_{k=n}^\infty A_k}.$$
Therefore, for such n, $A_\varepsilon \supseteq W_n \supseteq A_n$ and $(W_n)_\varepsilon \supseteq (A_n)_\varepsilon \supseteq A$ because
$$(A_n)_\varepsilon \supseteq \overline{\cup_{k=n}^\infty A_k} \supseteq A.$$

14. In the situation of the last two problems, let X be a compact metric space. Show $(\mathcal{C}, \rho)$ is compact. **Hint:** Let $\mathcal{D}_n$ be a 2^{-n} net for X. Let $\mathcal{K}_n$ denote finite unions of sets of the form $\overline{B(p, 2^{-n})}$ where $p \in \mathcal{D}_n$. Show $\mathcal{K}_n$ is a $2^{-(n-1)}$ net for $(\mathcal{C}, \rho)$.

15. Suppose U is an open connected subset of $\mathbb{R}^n$ and $f : U \to \mathbb{N}$ is continuous. That is f has values only in $\mathbb{N}$. Also $\mathbb{N}$ is a metric space with respect to the usual metric on $\mathbb{R}$. Show that f must actually be constant.

Approximation Theorems

6.1 The Bernstein Polynomials

To begin with I will give a famous theorem due to Weierstrass which shows that every continuous function can be uniformly approximated by polynomials on an interval. The proof I will give is not the one Weierstrass used. That proof is found in [35] and also in [28].

The following estimate will be the basis for the Weierstrass approximation theorem. It is actually a statement about the variance of a binomial random variable.

Lemma 6.1 *The following estimate holds for $x \in [0,1]$.*

$$\sum_{k=0}^{m} \binom{m}{k} (k - mx)^2 x^k (1-x)^{m-k} \leq \frac{1}{4} m$$

Proof: By the Binomial theorem,

$$\sum_{k=0}^{m} \binom{m}{k} \left(e^t x\right)^k (1-x)^{m-k} = \left(1 - x + e^t x\right)^m . \tag{6.1}$$

Differentiating both sides with respect to t and then evaluating at $t = 0$ yields

$$\sum_{k=0}^{m} \binom{m}{k} k x^k (1-x)^{m-k} = mx.$$

Now doing two derivatives of 6.1 with respect to t yields

$$\begin{aligned} \textstyle\sum_{k=0}^{m} \binom{m}{k} k^2 \left(e^t x\right)^k (1-x)^{m-k} = m (m-1) \left(1 - x + e^t x\right)^{m-2} e^{2t} x^2 \\ + m \left(1 - x + e^t x\right)^{m-1} x e^t . \end{aligned}$$

Evaluating this at $t = 0$,

$$\sum_{k=0}^{m} \binom{m}{k} k^2 (x)^k (1-x)^{m-k} = m (m-1) x^2 + mx.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(k-mx)^2 x^k(1-x)^{m-k} &= m(m-1)x^2+mx-2m^2x^2+m^2x^2\\ &= m\left(x-x^2\right)\leq\frac{1}{4}m.\end{aligned}$$

This proves the lemma.

Definition 6.2 *Let $f\in C([0,1])$. Then the following polynomials are known as the Bernstein polynomials.*

$$p_n(x)\equiv\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}.$$

Theorem 6.3 *Let $f\in C([0,1])$ and let p_n be given in Definition 6.2. Then*

$$\lim_{n\to\infty}||f-p_n||_\infty=0.$$

Proof: Since f is continuous on the compact $[0,1]$, it follows f is uniformly continuous there and so if $\varepsilon>0$ is given, there exists $\delta>0$ such that if

$$|y-x|\leq\delta,$$

then

$$|f(x)-f(y)|<\varepsilon/2.$$

By the Binomial theorem,

$$f(x)=\sum_{k=0}^{n}\binom{n}{k}f(x)x^k(1-x)^{n-k}$$

and so

$$|p_n(x)-f(x)|\leq\sum_{k=0}^{n}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f(x)\right|x^k(1-x)^{n-k}$$

$$\begin{aligned}&\leq\sum_{|k/n-x|>\delta}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f(x)\right|x^k(1-x)^{n-k}+\\ &\quad\sum_{|k/n-x|\leq\delta}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f(x)\right|x^k(1-x)^{n-k}\\ &<\varepsilon/2+2||f||_\infty\sum_{(k-nx)^2>n^2\delta^2}\binom{n}{k}x^k(1-x)^{n-k}\\ &\leq\frac{2||f||_\infty}{n^2\delta^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^k(1-x)^{n-k}+\varepsilon/2.\end{aligned}$$

By the lemma,

$$\leq \frac{4\,||f||_\infty}{\delta^2 n} + \varepsilon/2 < \varepsilon$$

whenever n is large enough. This proves the theorem.

The next corollary is called the Weierstrass approximation theorem.

Corollary 6.4 *The polynomials are dense in $C([a,b])$.*

Proof: Let $f \in C([a,b])$ and let $h : [0,1] \to [a,b]$ be linear and onto. Then $f \circ h$ is a continuous function defined on $[0,1]$ and so there exists a poynomial, p_n such that

$$|f(h(t)) - p_n(t)| < \varepsilon$$

for all $t \in [0,1]$. Therefore for all $x \in [a,b]$,

$$\left|f(x) - p_n\left(h^{-1}(x)\right)\right| < \varepsilon.$$

Since h is linear $p_n \circ h^{-1}$ is a polynomial. This proves the theorem.

The next result is the key to the profound generalization of the Weierstrass theorem due to Stone in which an interval will be replaced by a compact or locally compact set and polynomials will be replaced with elements of an algebra satisfying certain axioms.

Corollary 6.5 *On the interval $[-M,M]$, there exist polynomials p_n such that*

$$p_n(0) = 0$$

and

$$\lim_{n\to\infty} ||p_n - |\cdot|||_\infty = 0.$$

Proof: Let $\tilde{p}_n \to |\cdot|$ uniformly and let

$$p_n \equiv \tilde{p}_n - \tilde{p}_n(0).$$

This proves the corollary.

6.2 Stone Weierstrass Theorem

6.2.1 The Case Of Compact Sets

There is a profound generalization of the Weierstrass approximation theorem due to Stone.

Definition 6.6 *$\mathcal{A}$ is an algebra of functions if $\mathcal{A}$ is a vector space and if whenever $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$.*

To begin with assume that the field of scalars is $\mathbb{R}$. This will be generalized later.

Definition 6.7 *An algebra of functions, $\mathcal{A}$ defined on A, annihilates no point of A if for all $x \in A$, there exists $g \in \mathcal{A}$ such that $g(x) \neq 0$. The algebra separates points if whenever $x_1 \neq x_2$, then there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$.*

The following generalization is known as the Stone Weierstrass approximation theorem.

Theorem 6.8 *Let A be a compact topological space and let $\mathcal{A} \subseteq C(A;\mathbb{R})$ be an algebra of functions which separates points and annihilates no point. Then $\mathcal{A}$ is dense in $C(A;\mathbb{R})$.*

Proof: First here is a lemma.

Lemma 6.9 *Let c_1 and c_2 be two real numbers and let $x_1 \neq x_2$ be two points of A. Then there exists a function $f_{x_1 x_2}$ such that*

$$f_{x_1 x_2}(x_1) = c_1,\ f_{x_1 x_2}(x_2) = c_2.$$

Proof of the lemma: Let $g \in \mathcal{A}$ satisfy

$$g(x_1) \neq g(x_2).$$

Such a g exists because the algebra separates points. Since the algebra annihilates no point, there exist functions h and k such that

$$h(x_1) \neq 0,\ k(x_2) \neq 0.$$

Then let

$$u \equiv gh - g(x_2)\,h,\ v \equiv gk - g(x_1)\,k.$$

It follows that $u(x_1) \neq 0$ and $u(x_2) = 0$ while $v(x_2) \neq 0$ and $v(x_1) = 0$. Let

$$f_{x_1 x_2} \equiv \frac{c_1 u}{u(x_1)} + \frac{c_2 v}{v(x_2)}.$$

This proves the lemma. Now continue the proof of Theorem 6.8.

First note that $\overline{\mathcal{A}}$ satisfies the same axioms as $\mathcal{A}$ but in addition to these axioms, $\overline{\mathcal{A}}$ is closed. The closure of $\mathcal{A}$ is taken with respect to the usual norm on $C(A)$,

$$||f||_\infty \equiv \max\{|f(x)| : x \in A\}.$$

Suppose $f \in \overline{\mathcal{A}}$ and suppose M is large enough that

$$||f||_\infty < M.$$

Using Corollary 6.5, there exists $\{p_n\}$, a sequence of polynomials such that

$$||p_n - |\cdot|||_\infty \to 0,\ p_n(0) = 0.$$

It follows that $p_n \circ f \in \overline{\mathcal{A}}$ and so $|f| \in \overline{\mathcal{A}}$ whenever $f \in \overline{\mathcal{A}}$. Also note that

$$\max(f,g) = \frac{|f-g| + (f+g)}{2}$$

$$\min(f,g) = \frac{(f+g) - |f-g|}{2}.$$

Therefore, this shows that if $f, g \in \overline{\mathcal{A}}$ then

$$\max(f,g)\,,\ \min(f,g) \in \overline{\mathcal{A}}.$$

By induction, if $f_i, i = 1, 2, \cdots, m$ are in $\overline{\mathcal{A}}$ then

$$\max(f_i, i = 1, 2, \cdots, m)\,,\ \ \min(f_i, i = 1, 2, \cdots, m) \in \overline{\mathcal{A}}.$$

Now let $h \in C(A;\mathbb{R})$ and let $x \in A$. Use Lemma 6.9 to obtain f_{xy}, a function of $\overline{\mathcal{A}}$ which agrees with h at x and y. Letting $\varepsilon > 0$, there exists an open set $U(y)$ containing y such that

$$f_{xy}(z) > h(z) - \varepsilon \ \text{ if } z \in U(y).$$

Since A is compact, let $U(y_1), \cdots, U(y_l)$ cover A. Let

$$f_x \equiv \max(f_{xy_1}, f_{xy_2}, \cdots, f_{xy_l}).$$

Then $f_x \in \overline{\mathcal{A}}$ and

$$f_x(z) > h(z) - \varepsilon$$

for all $z \in A$ and $f_x(x) = h(x)$. This implies that for each $x \in A$ there exists an open set $V(x)$ containing x such that for $z \in V(x)$,

$$f_x(z) < h(z) + \varepsilon.$$

Let $V(x_1), \cdots, V(x_m)$ cover A and let

$$f \equiv \min(f_{x_1}, \cdots, f_{x_m}).$$

Therefore,

$$f(z) < h(z) + \varepsilon$$

for all $z \in A$ and since $f_x(z) > h(z) - \varepsilon$ for all $z \in A$, it follows

$$f(z) > h(z) - \varepsilon$$

also and so

$$|f(z) - h(z)| < \varepsilon$$

for all z. Since ε is arbitrary, this shows $h \in \overline{\mathcal{A}}$ and proves $\overline{\mathcal{A}} = C(A;\mathbb{R})$. This proves the theorem.

6.2.2 The Case Of Locally Compact Sets

Definition 6.10 *Let (X, τ) be a locally compact Hausdorff space. $C_0(X)$ denotes the space of real or complex valued continuous functions defined on X with the property that if $f \in C_0(X)$, then for each $\varepsilon > 0$ there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \notin K$. Define*

$$||f||_\infty = \sup \{|f(x)| : x \in X\}.$$

Lemma 6.11 *For (X, τ) a locally compact Hausdorff space with the above norm, $C_0(X)$ is a complete space.*

Proof: Let $\left(\widetilde{X}, \widetilde{\tau}\right)$ be the one point compactification described in Lemma 5.43.

$$D \equiv \left\{ f \in C\left(\widetilde{X}\right) : f(\infty) = 0 \right\}.$$

Then D is a closed subspace of $C\left(\widetilde{X}\right)$. For $f \in C_0(X)$,

$$\widetilde{f}(x) \equiv \begin{cases} f(x) \text{ if } x \in X \\ 0 \text{ if } x = \infty \end{cases}$$

and let $\theta : C_0(X) \to D$ be given by $\theta f = \widetilde{f}$. Then θ is one to one and onto and also satisfies $||f||_\infty = ||\theta f||_\infty$. Now D is complete because it is a closed subspace of a complete space and so $C_0(X)$ with $||\cdot||_\infty$ is also complete. This proves the lemma.

The above refers to functions which have values in $\mathbb{C}$ but the same proof works for functions which have values in any complete normed linear space.

In the case where the functions in $C_0(X)$ all have real values, I will denote the resulting space by $C_0(X; \mathbb{R})$ with similar meanings in other cases.

With this lemma, the generalization of the Stone Weierstrass theorem to locally compact sets is as follows.

Theorem 6.12 *Let $\mathcal{A}$ be an algebra of functions in $C_0(X; \mathbb{R})$ where (X, τ) is a locally compact Hausdorff space which separates the points and annihilates no point. Then $\mathcal{A}$ is dense in $C_0(X; \mathbb{R})$.*

Proof: Let $\left(\widetilde{X}, \widetilde{\tau}\right)$ be the one point compactification as described in Lemma 5.43. Let $\widetilde{\mathcal{A}}$ denote all finite linear combinations of the form

$$\left\{ \sum_{i=1}^{n} c_i \widetilde{f}_i + c_0 : f \in \mathcal{A}, \; c_i \in \mathbb{R} \right\}$$

where for $f \in C_0(X; \mathbb{R})$,

$$\widetilde{f}(x) \equiv \begin{cases} f(x) \text{ if } x \in X \\ 0 \text{ if } x = \infty \end{cases}.$$

Then $\widetilde{\mathcal{A}}$ is obviously an algebra of functions in $C\left(\widetilde{X};\mathbb{R}\right)$. It separates points because this is true of $\mathcal{A}$. Similarly, it annihilates no point because of the inclusion of c_0 an arbitrary element of $\mathbb{R}$ in the definition above. Therefore from Theorem 6.8, $\widetilde{\mathcal{A}}$ is dense in $C\left(\widetilde{X};\mathbb{R}\right)$. Letting $f \in C_0(X;\mathbb{R})$, it follows $\widetilde{f} \in C\left(\widetilde{X};\mathbb{R}\right)$ and so there exists a sequence $\{h_n\} \subseteq \widetilde{\mathcal{A}}$ such that h_n converges uniformly to $\widetilde{f}$. Now h_n is of the form $\sum_{i=1}^{n} c_i^n \widetilde{f_i^n} + c_0^n$ and since $\widetilde{f}(\infty) = 0$, you can take each $c_0^n = 0$ and so this has shown the existence of a sequence of functions in $\mathcal{A}$ such that it converges uniformly to f. This proves the theorem.

6.2.3 The Case Of Complex Valued Functions

What about the general case where $C_0(X)$ consists of complex valued functions and the field of scalars is $\mathbb{C}$ rather than $\mathbb{R}$? The following is the version of the Stone Weierstrass theorem which applies to this case. You have to assume that for $f \in \mathcal{A}$ it follows $\overline{f} \in \mathcal{A}$. Such an algebra is called self adjoint.

Theorem 6.13 *Suppose $\mathcal{A}$ is an algebra of functions in $C_0(X)$, where X is a locally compact Hausdorff space, which separates the points, annihilates no point, and has the property that if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C_0(X)$.*

Proof: Let $\operatorname{Re}\mathcal{A} \equiv \{\operatorname{Re} f : f \in \mathcal{A}\}$, $\operatorname{Im}\mathcal{A} \equiv \{\operatorname{Im} f : f \in \mathcal{A}\}$. First I will show that $\mathcal{A} = \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A} = \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A}$. Let $f \in \mathcal{A}$. Then

$$f = \frac{1}{2}\left(f + \overline{f}\right) + \frac{1}{2}\left(f - \overline{f}\right) = \operatorname{Re} f + i\operatorname{Im} f \in \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A}$$

and so $\mathcal{A} \subseteq \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A}$. Also

$$f = \frac{1}{2i}\left(if + i\overline{f}\right) - \frac{i}{2}\left(if + \overline{(if)}\right) = \operatorname{Im}(if) + i\operatorname{Re}(if) \in \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A}$$

This proves one half of the desired equality. Now suppose $h \in \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A}$. Then $h = \operatorname{Re} g_1 + i\operatorname{Im} g_2$ where $g_i \in \mathcal{A}$. Then since $\operatorname{Re} g_1 = \frac{1}{2}\left(g_1 + \overline{g_1}\right)$, it follows $\operatorname{Re} g_1 \in \mathcal{A}$. Similarly $\operatorname{Im} g_2 \in \mathcal{A}$. Therefore, $h \in \mathcal{A}$. The case where $h \in \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A}$ is similar. This establishes the desired equality.

Now $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ are both real algebras. I will show this now. First consider $\operatorname{Im}\mathcal{A}$. It is obvious this is a real vector space. It only remains to verify that the product of two functions in $\operatorname{Im}\mathcal{A}$ is in $\operatorname{Im}\mathcal{A}$. Note that from the first part, $\operatorname{Re}\mathcal{A}, \operatorname{Im}\mathcal{A}$ are both subsets of $\mathcal{A}$ because, for example, if $u \in \operatorname{Im}\mathcal{A}$ then $u + 0 \in \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A} = \mathcal{A}$. Therefore, if $v, w \in \operatorname{Im}\mathcal{A}$, both iv and w are in $\mathcal{A}$ and so $\operatorname{Im}(ivw) = vw$ and $ivw \in \mathcal{A}$. Similarly, $\operatorname{Re}\mathcal{A}$ is an algebra.

Both $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ must separate the points. Here is why: If $x_1 \neq x_2$, then there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$. If $\operatorname{Im} f(x_1) \neq \operatorname{Im} f(x_2)$, this shows there is a function in $\operatorname{Im}\mathcal{A}$, $\operatorname{Im} f$ which separates these two points. If $\operatorname{Im} f$ fails to separate the two points, then $\operatorname{Re} f$ must separate the points and so you could consider $\operatorname{Im}(if)$ to get a function in $\operatorname{Im}\mathcal{A}$ which separates these points. This shows $\operatorname{Im}\mathcal{A}$ separates the points. Similarly $\operatorname{Re}\mathcal{A}$ separates the points.

Neither $\operatorname{Re}\mathcal{A}$ nor $\operatorname{Im}\mathcal{A}$ annihilate any point. This is easy to see because if x is a point there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. Thus either $\operatorname{Re} f(x) \neq 0$ or $\operatorname{Im} f(x) \neq 0$. If $\operatorname{Im} f(x) \neq 0$, this shows this point is not annihilated by $\operatorname{Im}\mathcal{A}$. If $\operatorname{Im} f(x) = 0$, consider $\operatorname{Im}(if)(x) = \operatorname{Re} f(x) \neq 0$. Similarly, $\operatorname{Re}\mathcal{A}$ does not annihilate any point.

It follows from Theorem 6.12 that $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ are dense in the real valued functions of $C_0(X)$. Let $f \in C_0(X)$. Then there exists $\{h_n\} \subseteq \operatorname{Re}\mathcal{A}$ and $\{g_n\} \subseteq \operatorname{Im}\mathcal{A}$ such that $h_n \to \operatorname{Re} f$ uniformly and $g_n \to \operatorname{Im} f$ uniformly. Therefore, $h_n + ig_n \in \mathcal{A}$ and it converges to f uniformly. This proves the theorem.

6.3 Exercises

1. Let X be a finite dimensional normed linear space, real or complex. Show that X is separable. **Hint:** Let $\{v_i\}_{i=1}^n$ be a basis and define a map from $\mathbb{F}^n$ to X, θ, as follows. $\theta\left(\sum_{k=1}^n x_k \mathbf{e}_k\right) \equiv \sum_{k=1}^n x_k v_k$. Show θ is continuous and has a continuous inverse. Now let D be a countable dense set in $\mathbb{F}^n$ and consider $\theta(D)$.

2. Let $B(X;\mathbb{R}^n)$ be the space of functions $\mathbf{f}$, mapping X to $\mathbb{R}^n$ such that
$$\sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\} < \infty.$$
Show $B(X;\mathbb{R}^n)$ is a complete normed linear space if we define
$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}.$$

3. Let $\alpha \in (0,1]$. We define, for X a compact subset of $\mathbb{R}^p$,
$$C^\alpha(X;\mathbb{R}^n) \equiv \{\mathbf{f} \in C(X;\mathbb{R}^n) : \rho_\alpha(\mathbf{f}) + ||\mathbf{f}|| \equiv ||\mathbf{f}||_\alpha < \infty\}$$
where
$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}$$
and
$$\rho_\alpha(\mathbf{f}) \equiv \sup\{\frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha} : \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x} \neq \mathbf{y}\}.$$
Show that $\left(C^\alpha(X;\mathbb{R}^n), ||\cdot||_\alpha\right)$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha > 1$?

4. Let $\{\mathbf{f}_n\}_{n=1}^\infty \subseteq C^\alpha(X;\mathbb{R}^n)$ where X is a compact subset of $\mathbb{R}^p$ and suppose
$$||\mathbf{f}_n||_\alpha \leq M$$
for all n. Show there exists a subsequence, n_k, such that $\mathbf{f}_{n_k}$ converges in $C(X;\mathbb{R}^n)$. We say the given sequence is precompact when this happens. (This also shows the embedding of $C^\alpha(X;\mathbb{R}^n)$ into $C(X;\mathbb{R}^n)$ is a compact embedding.) **Hint:** You might want to use the Ascoli Arzela theorem.

5. Let $\mathbf{f}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous and bounded and let $\mathbf{x}_0\in\mathbb{R}^n$. If

$$\mathbf{x}:[0,T]\to\mathbb{R}^n$$

and $h>0$, let

$$\tau_h\mathbf{x}(s)\equiv\begin{cases}\mathbf{x}_0 \text{ if } s\leq h,\\ \mathbf{x}(s-h), \text{ if } s>h.\end{cases}$$

For $t\in[0,T]$, let

$$\mathbf{x}_h(t)=\mathbf{x}_0+\int_0^t\mathbf{f}(s,\tau_h\mathbf{x}_h(s))\,ds.$$

Show using the Ascoli Arzela theorem that there exists a sequence $h\to 0$ such that

$$\mathbf{x}_h\to\mathbf{x}$$

in $C([0,T];\mathbb{R}^n)$. Next argue

$$\mathbf{x}(t)=\mathbf{x}_0+\int_0^t\mathbf{f}(s,\mathbf{x}(s))\,ds$$

and conclude the following theorem. If $\mathbf{f}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous and bounded, and if $\mathbf{x}_0\in\mathbb{R}^n$ is given, there exists a solution to the following initial value problem.

$$\begin{aligned}\mathbf{x}' &= \mathbf{f}(t,\mathbf{x}),\quad t\in[0,T]\\ \mathbf{x}(0) &= \mathbf{x}_0.\end{aligned}$$

This is the Peano existence theorem for ordinary differential equations.

6. Let H and K be disjoint closed sets in a metric space, (X,d), and let

$$g(x)\equiv\frac{2}{3}h(x)-\frac{1}{3}$$

where

$$h(x)\equiv\frac{\operatorname{dist}(x,H)}{\operatorname{dist}(x,H)+\operatorname{dist}(x,K)}.$$

Show $g(x)\in\left[-\frac{1}{3},\frac{1}{3}\right]$ for all $x\in X$, g is continuous, and g equals $\frac{-1}{3}$ on H while g equals $\frac{1}{3}$ on K.

7. Suppose M is a closed set in X where X is the metric space of problem 6 and suppose $f:M\to[-1,1]$ is continuous. Show there exists $g:X\to[-1,1]$ such that g is continuous and $g=f$ on M. **Hint:** Show there exists

$$g_1\in C(X),\; g_1(x)\in\left[\frac{-1}{3},\frac{1}{3}\right],$$

and $|f(x) - g_1(x)| \leq \frac{2}{3}$ for all $x \in H$. To do this, consider the disjoint closed sets

$$H \equiv f^{-1}\left(\left[-1, \frac{-1}{3}\right]\right), \; K \equiv f^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$$

and use Urysohn's lemma or something to obtain a continuous function g_1 defined on X such that $g_1(H) = -1/3, g_1(K) = 1/3$ and g_1 has values in $[-1/3, 1/3]$. When this has been done, let

$$\frac{3}{2}(f(x) - g_1(x))$$

play the role of f and let g_2 be like g_1. Obtain

$$\left|f(x) - \sum_{i=1}^{n}\left(\frac{2}{3}\right)^{i-1} g_i(x)\right| \leq \left(\frac{2}{3}\right)^n$$

and consider

$$g(x) \equiv \sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1} g_i(x).$$

8. $\uparrow$ Let M be a closed set in a metric space (X, d) and suppose $f \in C(M)$. Show there exists $g \in C(X)$ such that $g(x) = f(x)$ for all $x \in M$ and if $f(M) \subseteq [a, b]$, then $g(X) \subseteq [a, b]$. This is a version of the Tietze extension theorem.

9. This problem gives an outline of the way Weierstrass originally proved the theorem. Choose a_n such that $\int_{-1}^{1}(1 - x^2)^n a_n dx = 1$. Show $a_n < \frac{n+1}{2}$ or something like this. Now show that for $\delta \in (0, 1)$,

$$\lim_{n \to \infty}\left(\int_{\delta}^{1}(1 - x^2)^n a_n + \int_{-1}^{-\delta}(1 - x^2)^n dx\right) = 0.$$

Next for f a continuous function defined on $\mathbb{R}$, define the polynomial, $p_n(x)$ by

$$p_n(x) \equiv \int_{x-1}^{x+1}\left(1 - (x - t)^2\right)^n f(t)\,dt = \int_{-1}^{1} f(x - t)(1 - t^2)^n dt.$$

Then show $\lim_{n \to \infty}||p_n - f||_{\infty} = 0.$ where $||f||_{\infty} = \max\{|f(x)| : x \in [-1, 1]\}$.

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f \geq 0$ on $[-1, 1]$ with $f(-1) = f(1) = 0$ and $f(x) < 0$ for all $x \notin [-1, 1]$. Can you use a modification of the proof of the Weierstrass approximation theorem given in Problem 9 to show that for all $\varepsilon > 0$ there exists a polynomial, p, such that $|p(x) - f(x)| < \varepsilon$ for $x \in [-1, 1]$ and $p(x) \leq 0$ for all $x \notin [-1, 1]$? **Hint:**Let $f_{\varepsilon}(x) = f(x) - \frac{\varepsilon}{2}$. Thus there exists δ such that $1 > \delta > 0$ and $f_{\varepsilon} < 0$ on $(-1, -1 + \delta)$ and $(1 - \delta, 1)$. Now consider $\phi_k(x) = a_k\left(\delta^2 - \left(\frac{x}{\delta}\right)^2\right)^k$ and try something similar to the proof given for the Weierstrass approximation theorem above.

Abstract Measure And Integration

7.1 σ Algebras

This chapter is on the basics of measure theory and integration. A measure is a real valued mapping from some subset of the power set of a given set which has values in $[0, \infty]$. Many apparently different things can be considered as measures and also there is an integral defined. By discussing this in terms of axioms and in a very abstract setting, many different topics can be considered in terms of one general theory. For example, it will turn out that sums are included as an integral of this sort. So is the usual integral as well as things which are often thought of as being in between sums and integrals.

Let Ω be a set and let $\mathcal{F}$ be a collection of subsets of Ω satisfying

$$\emptyset \in \mathcal{F},\ \Omega \in \mathcal{F}, \tag{7.1}$$

$$E \in \mathcal{F} \text{ implies } E^C \equiv \Omega \setminus E \in \mathcal{F},$$

$$\text{If } \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{F}, \text{ then } \cup_{n=1}^{\infty} E_n \in \mathcal{F}. \tag{7.2}$$

Definition 7.1 *A collection of subsets of a set, Ω, satisfying Formulas 7.1-7.2 is called a σ algebra.*

As an example, let Ω be any set and let $\mathcal{F} = \mathcal{P}(\Omega)$, the set of all subsets of Ω (power set). This obviously satisfies Formulas 7.1-7.2.

Lemma 7.2 *Let $\mathcal{C}$ be a set whose elements are σ algebras of subsets of Ω. Then $\cap\mathcal{C}$ is a σ algebra also.*

Be sure to verify this lemma. It follows immediately from the above definitions but it is important for you to check the details.

Example 7.3 *Let τ denote the collection of all open sets in $\mathbb{R}^n$ and let $\sigma(\tau) \equiv$ intersection of all σ algebras that contain τ. $\sigma(\tau)$ is called the σ algebra of Borel sets . In general, for a collection of sets, Σ, $\sigma(\Sigma)$ is the smallest σ algebra which contains Σ.*

This is a very important σ algebra and it will be referred to frequently as the Borel sets. Attempts to describe a typical Borel set are more trouble than they are worth and it is not easy to do so. Rather, one uses the definition just given in the example. Note, however, that all countable intersections of open sets and countable unions of closed sets are Borel sets. Such sets are called G_δ and F_σ respectively.

Definition 7.4 *Let $\mathcal{F}$ be a σ algebra of sets of Ω and let $\mu : \mathcal{F} \to [0, \infty]$. μ is called a measure if*

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \tag{7.3}$$

whenever the E_i are disjoint sets of $\mathcal{F}$. The triple, $(\Omega, \mathcal{F}, \mu)$ is called a measure space and the elements of $\mathcal{F}$ are called the measurable sets. $(\Omega, \mathcal{F}, \mu)$ is a finite measure space when $\mu(\Omega) < \infty$.

The following theorem is the basis for most of what is done in the theory of measure and integration. It is a very simple result which follows directly from the above definition.

Theorem 7.5 *Let $\{E_m\}_{m=1}^{\infty}$ be a sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$. Then if $\cdots E_n \subseteq E_{n+1} \subseteq E_{n+2} \subseteq \cdots$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_n) \tag{7.4}$$

and if $\cdots E_n \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \cdots$ and $\mu(E_1) < \infty$, then

$$\mu(\cap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_n). \tag{7.5}$$

Stated more succinctly, $E_k \uparrow E$ implies $\mu(E_k) \uparrow \mu(E)$ and $E_k \downarrow E$ with $\mu(E_1) < \infty$ implies $\mu(E_k) \downarrow \mu(E)$.

Proof: First note that $\cap_{i=1}^{\infty} E_i = (\cup_{i=1}^{\infty} E_i^C)^C \in \mathcal{F}$ so $\cap_{i=1}^{\infty} E_i$ is measurable. Also note that for A and B sets of $\mathcal{F}$, $A \setminus B \equiv \left(A^C \cup B\right)^C \in \mathcal{F}$. To show 7.4, note that 7.4 is obviously true if $\mu(E_k) = \infty$ for any k. Therefore, assume $\mu(E_k) < \infty$ for all k. Thus

$$\mu(E_{k+1} \setminus E_k) + \mu(E_k) = \mu(E_{k+1})$$

and so

$$\mu(E_{k+1} \setminus E_k) = \mu(E_{k+1}) - \mu(E_k).$$

Also,

$$\bigcup_{k=1}^{\infty} E_k = E_1 \cup \bigcup_{k=1}^{\infty} (E_{k+1} \setminus E_k)$$

and the sets in the above union are disjoint. Hence by 7.3,

$$\mu(\cup_{i=1}^{\infty} E_i) = \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} \setminus E_k) = \mu(E_1)$$

$$+\sum_{k=1}^{\infty}\mu(E_{k+1})-\mu(E_k)$$

$$=\mu(E_1)+\lim_{n\to\infty}\sum_{k=1}^{n}\mu(E_{k+1})-\mu(E_k)=\lim_{n\to\infty}\mu(E_{n+1}).$$

This shows part 7.4.

To verify 7.5,

$$\mu(E_1)=\mu(\cap_{i=1}^{\infty}E_i)+\mu(E_1\setminus\cap_{i=1}^{\infty}E_i)$$

since $\mu(E_1)<\infty$, it follows $\mu(\cap_{i=1}^{\infty}E_i)<\infty$. Also, $E_1\setminus\cap_{i=1}^{n}E_i\uparrow E_1\setminus\cap_{i=1}^{\infty}E_i$ and so by 7.4,

$$\mu(E_1)-\mu(\cap_{i=1}^{\infty}E_i)=\mu(E_1\setminus\cap_{i=1}^{\infty}E_i)=\lim_{n\to\infty}\mu(E_1\setminus\cap_{i=1}^{n}E_i)$$

$$=\mu(E_1)-\lim_{n\to\infty}\mu(\cap_{i=1}^{n}E_i)=\mu(E_1)-\lim_{n\to\infty}\mu(E_n),$$

Hence, subtracting $\mu(E_1)$ from both sides,

$$\lim_{n\to\infty}\mu(E_n)=\mu(\cap_{i=1}^{\infty}E_i).$$

This proves the theorem.

It is convenient to allow functions to take the value $+\infty$. You should think of $+\infty$, usually referred to as ∞ as something out at the right end of the real line and its only importance is the notion of sequences converging to it. $x_n\to\infty$ exactly when for all $l\in\mathbb{R}$, there exists N such that if $n\geq N$, then

$$x_n>l.$$

This is what it means for a sequence to converge to ∞. Don't think of ∞ as a number. It is just a convenient symbol which allows the consideration of some limit operations more simply. Similar considerations apply to $-\infty$ but this value is not of very great interest. In fact the set of most interest is the complex numbers or some vector space. Therefore, this topic is not considered.

Lemma 7.6 *Let $f:\Omega\to(-\infty,\infty]$ where $\mathcal{F}$ is a σ algebra of subsets of Ω. Then the following are equivalent.*

$$f^{-1}((d,\infty])\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}((-\infty,d))\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}([d,\infty])\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}((-\infty,d])\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}((a,b))\in\mathcal{F}\text{ for all }a<b,-\infty<a<b<\infty.$$

Proof: First note that the first and the third are equivalent. To see this, observe

$$f^{-1}([d,\infty]) = \cap_{n=1}^{\infty} f^{-1}((d-1/n,\infty]),$$

and so if the first condition holds, then so does the third.

$$f^{-1}((d,\infty]) = \cup_{n=1}^{\infty} f^{-1}([d+1/n,\infty]),$$

and so if the third condition holds, so does the first.

Similarly, the second and fourth conditions are equivalent. Now

$$f^{-1}((-\infty,d]) = (f^{-1}((d,\infty]))^C$$

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for $-\infty < a < b < \infty$,

$$f^{-1}((a,b)) = f^{-1}((-\infty,b)) \cap f^{-1}((a,\infty]) \in \mathcal{F}.$$

Finally, if the last condition holds,

$$f^{-1}([d,\infty]) = \left(\cup_{k=1}^{\infty} f^{-1}\left((-k+d,d)\right)\right)^C \in \mathcal{F}$$

and so the third condition holds. Therefore, all five conditions are equivalent. This proves the lemma.

This lemma allows for the following definition of a measurable function having values in $(-\infty,\infty]$.

Definition 7.7 *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and let $f:\Omega\to(-\infty,\infty]$. Then f is said to be measurable if any of the equivalent conditions of Lemma 7.6 hold. When the σ algebra, $\mathcal{F}$ equals the Borel σ algebra, $\mathcal{B}$, the function is called Borel measurable. More generally, if $f:\Omega\to X$ where X is a topological space, f is said to be measurable if $f^{-1}(U)\in\mathcal{F}$ whenever U is open.*

You should verify this last condition is verified in the special cases considered above.

Theorem 7.8 *Let f_n and f be functions mapping Ω to $(-\infty,\infty]$ where $\mathcal{F}$ is a σ algebra of measurable sets of Ω. Then if f_n is measurable, and $f(\omega)=\lim_{n\to\infty} f_n(\omega)$, it follows that f is also measurable. (Pointwise limits of measurable functions are measurable.)*

Proof: First is is shown $f^{-1}((a,b)) \in \mathcal{F}$. Let $V_m \equiv \left(a+\frac{1}{m}, b-\frac{1}{m}\right)$ and $\overline{V}_m = \left[a+\frac{1}{m}, b-\frac{1}{m}\right]$. Then for all m, $V_m \subseteq (a,b)$ and

$$(a,b) = \cup_{m=1}^{\infty} V_m = \cup_{m=1}^{\infty} \overline{V}_m.$$

Note that $V_m \neq \emptyset$ for all m large enough. Since f is the pointwise limit of f_n,

$$f^{-1}(V_m) \subseteq \{\omega : f_k(\omega)\in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\overline{V}_m).$$

You should note that the expression in the middle is of the form

$$\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m).$$

Therefore,

$$f^{-1}((a,b)) = \cup_{m=1}^{\infty} f^{-1}(V_m) \subseteq \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m)$$
$$\subseteq \cup_{m=1}^{\infty} f^{-1}(\overline{V}_m) = f^{-1}((a,b)).$$

It follows $f^{-1}((a,b)) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. This shows f is measurable.

Theorem 7.9 *Let $\mathcal{B}$ consist of open cubes of the form*

$$Q_{\mathbf{x}} \equiv \prod_{i=1}^{n} (x_i - \delta, x_i + \delta)$$

where δ is a positive rational number and $\mathbf{x} \in \mathbb{Q}^n$. Then every open set in $\mathbb{R}^n$ can be written as a countable union of open cubes from $\mathcal{B}$. Furthermore, $\mathcal{B}$ is a countable set.

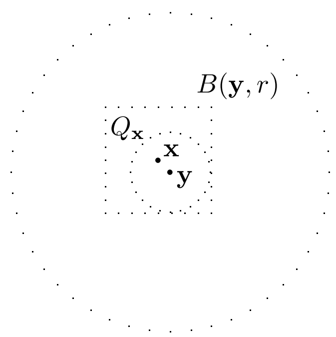
Proof: Let U be an open set and let $\mathbf{y} \in U$. Since U is open, $B(\mathbf{y}, r) \subseteq U$ for some $r > 0$ and it can be assumed $r/\sqrt{n} \in \mathbb{Q}$. Let

$$\mathbf{x} \in B\left(\mathbf{y}, \frac{r}{10\sqrt{n}}\right) \cap \mathbb{Q}^n$$

and consider the cube, $Q_{\mathbf{x}} \in \mathcal{B}$ defined by

$$Q_{\mathbf{x}} \equiv \prod_{i=1}^{n} (x_i - \delta, x_i + \delta)$$

where $\delta = r/4\sqrt{n}$. The following picture is roughly illustrative of what is taking place.



Then the diameter of $Q_{\mathbf{x}}$ equals

$$\left(n\left(\frac{r}{2\sqrt{n}}\right)^2\right)^{1/2}=\frac{r}{2}$$

and so, if $\mathbf{z}\in Q_{\mathbf{x}}$, then

$$\begin{aligned}|\mathbf{z}-\mathbf{y}| &\leq |\mathbf{z}-\mathbf{x}|+|\mathbf{x}-\mathbf{y}| \\ &< \frac{r}{2}+\frac{r}{2}=r.\end{aligned}$$

Consequently, $Q_{\mathbf{x}}\subseteq U$. Now also,

$$\left(\sum_{i=1}^n (x_i-y_i)^2\right)^{1/2}<\frac{r}{10\sqrt{n}}$$

and so it follows that for each i,

$$|x_i-y_i|<\frac{r}{4\sqrt{n}}$$

since otherwise the above inequality would not hold. Therefore, $\mathbf{y}\in Q_{\mathbf{x}}\subseteq U$. Now let $\mathcal{B}_U$ denote those sets of $\mathcal{B}$ which are contained in U. Then $\cup\mathcal{B}_U=U$.

To see $\mathcal{B}$ is countable, note there are countably many choices for $\mathbf{x}$ and countably many choices for δ. This proves the theorem.

Recall that $g:\mathbb{R}^n\to\mathbb{R}$ is continuous means g^{-1} (open set) $=$ an open set. In particular $g^{-1}((a,b))$ must be an open set.

Theorem 7.10 *Let $f_i:\Omega\to\mathbb{R}$ for $i=1,\cdot\cdot\cdot,n$ be measurable functions and let $g:\mathbb{R}^n\to\mathbb{R}$ be continuous where $\mathbf{f}\equiv(f_1\cdots f_n)^T$. Then $g\circ\mathbf{f}$ is a measurable function from Ω to $\mathbb{R}$.*

Proof: First it is shown

$$(g\circ\mathbf{f})^{-1}((a,b))\in\mathcal{F}.$$

Now $(g\circ\mathbf{f})^{-1}((a,b))=\mathbf{f}^{-1}\left(g^{-1}((a,b))\right)$ and since g is continuous, it follows that $g^{-1}((a,b))$ is an open set which is denoted as U for convenience. Now by Theorem 7.9 above, it follows there are countably many open cubes, $\{Q_k\}$ such that

$$U=\cup_{k=1}^{\infty}Q_k$$

where each Q_k is a cube of the form

$$Q_k=\prod_{i=1}^n(x_i-\delta,x_i+\delta).$$

Now

$$\mathbf{f}^{-1}\left(\prod_{i=1}^{n}(x_i-\delta,x_i+\delta)\right)=\cap_{i=1}^{n}f_i^{-1}\left((x_i-\delta,x_i+\delta)\right)\in\mathcal{F}$$

and so

$$\begin{aligned}(g\circ\mathbf{f})^{-1}\left((a,b)\right) &= \mathbf{f}^{-1}\left(g^{-1}\left((a,b)\right)\right)=\mathbf{f}^{-1}\left(U\right)\\ &= \mathbf{f}^{-1}\left(\cup_{k=1}^{\infty}Q_k\right)=\cup_{k=1}^{\infty}\mathbf{f}^{-1}\left(Q_k\right)\in\mathcal{F}.\end{aligned}$$

This proves the theorem.

Corollary 7.11 *Sums, products, and linear combinations of measurable functions are measurable.*

Proof: To see the product of two measurable functions is measurable, let $g(x,y)=xy$, a continuous function defined on $\mathbb{R}^2$. Thus if you have two measurable functions, f_1 and f_2 defined on Ω,

$$g\circ(f_1,f_2)(\omega)=f_1(\omega)f_2(\omega)$$

and so $\omega\rightarrow f_1(\omega)f_2(\omega)$ is measurable. Similarly you can show the sum of two measurable functions is measurable by considering $g(x,y)=x+y$ and you can show a linear combination of two measurable functions is measurable by considering $g(x,y)=ax+by$. More than two functions can also be considered as well.

The message of this corollary is that starting with measurable real valued functions you can combine them in pretty much any way you want and you end up with a measurable function.

Here is some notation which will be used whenever convenient.

Definition 7.12 *Let* $f:\Omega\rightarrow[-\infty,\infty]$. *Define*

$$[\alpha<f]\equiv\{\omega\in\Omega:f(\omega)>\alpha\}\equiv f^{-1}\left((\alpha,\infty]\right)$$

with obvious modifications for the symbols $[\alpha\leq f],[\alpha\geq f],[\alpha\geq f\geq\beta]$*, etc.*

Definition 7.13 *For a set* E*,*

$$\mathcal{X}_E(\omega)=\begin{cases}1 \text{ if } \omega\in E,\\ 0 \text{ if } \omega\notin E.\end{cases}$$

This is called the characteristic function of E. Sometimes this is called the indicator function which I think is better terminology since the term characteristic function has another meaning. Note that this "indicates" whether a point, ω is contained in E. It is exactly when the function has the value 1.

Theorem 7.14 *(Egoroff) Let* $(\Omega,\mathcal{F},\mu)$ *be a finite measure space,*

$$(\mu(\Omega)<\infty)$$

and let f_n, f be complex valued functions such that $\operatorname{Re} f_n, \operatorname{Im} f_n$ *are all measurable and*

$$\lim_{n\to\infty} f_n(\omega) = f(\omega)$$

for all $\omega \notin E$ where $\mu(E) = 0$. Then for every $\varepsilon > 0$, there exists a set,

$$F \supseteq E,\ \mu(F) < \varepsilon,$$

such that f_n converges uniformly to f on F^C.

Proof: First suppose $E = \emptyset$ so that convergence is pointwise everywhere. It follows then that $\operatorname{Re} f$ and $\operatorname{Im} f$ are pointwise limits of measurable functions and are therefore measurable. Let $E_{km} = \{\omega \in \Omega : |f_n(\omega) - f(\omega)| \geq 1/m$ for some $n > k\}$. Note that

$$|f_n(\omega) - f(\omega)| = \sqrt{(\operatorname{Re} f_n(\omega) - \operatorname{Re} f(\omega))^2 + (\operatorname{Im} f_n(\omega) - \operatorname{Im} f(\omega))^2}$$

and so, By Theorem 7.10,

$$\left[|f_n - f| \geq \frac{1}{m}\right]$$

is measurable. Hence E_{km} is measurable because

$$E_{km} = \cup_{n=k+1}^{\infty} \left[|f_n - f| \geq \frac{1}{m}\right].$$

For fixed $m, \cap_{k=1}^{\infty} E_{km} = \emptyset$ because f_n converges to f. Therefore, if $\omega \in \Omega$ there exists k such that if $n > k$, $|f_n(\omega) - f(\omega)| < \frac{1}{m}$ which means $\omega \notin E_{km}$. Note also that

$$E_{km} \supseteq E_{(k+1)m}.$$

Since $\mu(E_{1m}) < \infty$, Theorem 7.5 on Page 126 implies

$$0 = \mu(\cap_{k=1}^{\infty} E_{km}) = \lim_{k\to\infty} \mu(E_{km}).$$

Let $k(m)$ be chosen such that $\mu(E_{k(m)m}) < \varepsilon 2^{-m}$ and let

$$F = \bigcup_{m=1}^{\infty} E_{k(m)m}.$$

Then $\mu(F) < \varepsilon$ because

$$\mu(F) \leq \sum_{m=1}^{\infty} \mu\left(E_{k(m)m}\right) < \sum_{m=1}^{\infty} \varepsilon 2^{-m} = \varepsilon$$

Now let $\eta > 0$ be given and pick m_0 such that $m_0^{-1} < \eta$. If $\omega \in F^C$, then

$$\omega \in \bigcap_{m=1}^{\infty} E_{k(m)m}^C.$$

Hence $\omega \in E_{k(m_0)m_0}^C$ so

$$|f_n(\omega) - f(\omega)| < 1/m_0 < \eta$$

for all $n > k(m_0)$. This holds for all $\omega \in F^C$and so f_n converges uniformly to f on F^C.

Now if $E \neq \emptyset$, consider $\{\mathcal{X}_{E^C} f_n\}_{n=1}^\infty$. Each $\mathcal{X}_{E^C} f_n$ has real and imaginary parts measurable and the sequence converges pointwise to $\mathcal{X}_E f$ everywhere. Therefore, from the first part, there exists a set of measure less than ε, F such that on $F^C, \{\mathcal{X}_{E^C} f_n\}$ converges uniformly to $\mathcal{X}_{E^C} f$. Therefore, on $(E \cup F)^C$, $\{f_n\}$ converges uniformly to f. This proves the theorem.

Finally here is a comment about notation.

Definition 7.15 *Something happens for μ a.e. ω said as μ almost everywhere, if there exists a set E with $\mu(E) = 0$ and the thing takes place for all $\omega \notin E$. Thus $f(\omega) = g(\omega)$ a.e. if $f(\omega) = g(\omega)$ for all $\omega \notin E$ where $\mu(E) = 0$. A measure space, $(\Omega, \mathcal{F}, \mu)$ is σ finite if there exist measurable sets, Ω_n such that $\mu(\Omega_n) < \infty$ and $\Omega = \cup_{n=1}^\infty \Omega_n$.*

7.2 The Abstract Lebesgue Integral

7.2.1 Preliminary Observations

This section is on the Lebesgue integral and the major convergence theorems which are the reason for studying it. In all that follows μ will be a measure defined on a σ algebra $\mathcal{F}$ of subsets of Ω. $0 \cdot \infty = 0$ is always defined to equal zero. This is a meaningless expression and so it can be defined arbitrarily but a little thought will soon demonstrate that this is the right definition in the context of measure theory. To see this, consider the zero function defined on $\mathbb{R}$. What should the integral of this function equal? Obviously, by an analogy with the Riemann integral, it should equal zero. Formally, it is zero times the length of the set or infinity. This is why this convention will be used.

Lemma 7.16 *Let $f(a,b) \in [-\infty, \infty]$ for $a \in A$ and $b \in B$ where A, B are sets. Then*

$$\sup_{a \in A} \sup_{b \in B} f(a,b) = \sup_{b \in B} \sup_{a \in A} f(a,b).$$

Proof: Note that for all a, b, $f(a,b) \leq \sup_{b \in B} \sup_{a \in A} f(a,b)$ and therefore, for all a,

$$\sup_{b \in B} f(a,b) \leq \sup_{b \in B} \sup_{a \in A} f(a,b).$$

Therefore,

$$\sup_{a \in A} \sup_{b \in B} f(a,b) \leq \sup_{b \in B} \sup_{a \in A} f(a,b).$$

Repeating the same argument interchanging a and b, gives the conclusion of the lemma.

Lemma 7.17 *If* $\{A_n\}$ *is an increasing sequence in* $[-\infty, \infty]$*, then* $\sup\{A_n\} = \lim_{n\to\infty} A_n$.

The following lemma is useful also and this is a good place to put it. First $\{b_j\}_{j=1}^{\infty}$ is an enumeration of the a_{ij} if

$$\cup_{j=1}^{\infty}\{b_j\} = \cup_{i,j}\{a_{ij}\}.$$

In other words, the countable set, $\{a_{ij}\}_{i,j=1}^{\infty}$ is listed as $b_1, b_2, \cdots$.

Lemma 7.18 *Let* $a_{ij} \geq 0$*. Then* $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$*. Also if* $\{b_j\}_{j=1}^{\infty}$ *is any enumeration of the* a_{ij}*, then* $\sum_{j=1}^{\infty} b_j = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$.

Proof: First note there is no trouble in defining these sums because the a_{ij} are all nonnegative. If a sum diverges, it only diverges to ∞ and so ∞ is written as the answer.

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \geq \sup_n \sum_{j=1}^{\infty}\sum_{i=1}^{n} a_{ij} = \sup_n \lim_{m\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}$$

$$= \sup_n \lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sup_n \sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}. \tag{7.6}$$

Interchanging the i and j in the above argument the first part of the lemma is proved.

Finally, note that for all p,

$$\sum_{j=1}^{p} b_j \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

and so $\sum_{j=1}^{\infty} b_j \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$. Now let $m, n > 1$ be given. Then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{p} b_j$$

where p is chosen large enough that $\{b_1, \cdots, b_p\} \supseteq \{a_{ij} : i \leq m \text{ and } j \leq n\}$. Therefore, since such a p exists for any choice of m, n,it follows that for any m, n,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{\infty} b_j.$$

Therefore, taking the limit as $n \to \infty$,

$$\sum_{i=1}^{m}\sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j$$

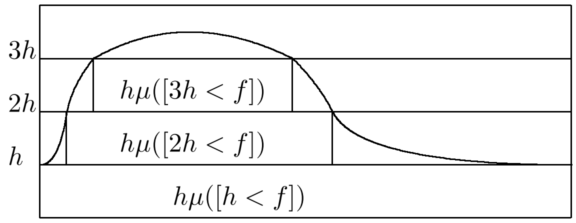
and finally, taking the limit as $m \to \infty$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j$$

proving the lemma.

7.2.2 Definition Of The Lebesgue Integral For Nonnegative Measurable Functions

The following picture illustrates the idea used to define the Lebesgue integral to be like the area under a curve.



You can see that by following the procedure illustrated in the picture and letting h get smaller, you would expect to obtain better approximations to the area under the curve[1] although all these approximations would likely be too small. Therefore, define

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih < f]\right)$$

Lemma 7.19 *The following inequality holds.*

$$\sum_{i=1}^{\infty} h\mu\left([ih < f]\right) \leq \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right).$$

Also, it suffices to consider only h smaller than a given positive number in the above definition of the integral.

Proof:
Let $N \in \mathbb{N}$.

$$\sum_{i=1}^{2N} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) = \sum_{i=1}^{2N} \frac{h}{2}\mu\left([ih < 2f]\right)$$

$$= \sum_{i=1}^{N} \frac{h}{2}\mu\left([(2i-1)\,h < 2f]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([(2i)\,h < 2f]\right)$$

$$= \sum_{i=1}^{N} \frac{h}{2}\mu\left(\left[\frac{(2i-1)}{2}h < f\right]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right)$$

[1]Note the difference between this picture and the one usually drawn in calculus courses where the little rectangles are upright rather than on their sides. This illustrates a fundamental philosophical difference between the Riemann and the Lebesgue integrals. With the Riemann integral intervals are measured. With the Lebesgue integral, it is inverse images of intervals which are measured.

$$\geq \sum_{i=1}^{N} \frac{h}{2} \mu ([ih < f]) + \sum_{i=1}^{N} \frac{h}{2} \mu ([ih < f]) = \sum_{i=1}^{N} h\mu ([ih < f]) .$$

Now letting $N \to \infty$ yields the claim of the lemma.

To verify the last claim, suppose $M < \int f d\mu$ and let $\delta > 0$ be given. Then there exists $h > 0$ such that

$$M < \sum_{i=1}^{\infty} h\mu ([ih < f]) \leq \int f d\mu.$$

By the first part of this lemma,

$$M < \sum_{i=1}^{\infty} \frac{h}{2} \mu \left(\left[i\frac{h}{2} < f \right] \right) \leq \int f d\mu$$

and continuing to apply the first part,

$$M < \sum_{i=1}^{\infty} \frac{h}{2^n} \mu \left(\left[i\frac{h}{2^n} < f \right] \right) \leq \int f d\mu.$$

Choose n large enough that $h/2^n < \delta$. It follows $M < \sup_{\delta > h > 0} \sum_{i=1}^{\infty} h\mu ([ih < f]) \leq \int f d\mu$. Since M is arbitrary, this proves the last claim.

7.2.3 The Lebesgue Integral For Nonnegative Simple Functions

Definition 7.20 *A function, s, is called simple if it is a measurable real valued function and has only finitely many values. These values will never be $\pm\infty$. Thus a simple function is one which may be written in the form*

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i} (\omega)$$

where the sets, E_i are disjoint and measurable. s takes the value c_i at E_i.

Note that by taking the union of some of the E_i in the above definition, you can assume that the numbers, c_i are the distinct values of s. Simple functions are important because it will turn out to be very easy to take their integrals as shown in the following lemma.

Lemma 7.21 *Let $s(\omega) = \sum_{i=1}^{p} a_i \mathcal{X}_{E_i} (\omega)$ be a nonnegative simple function with the a_i the distinct non zero values of s. Then*

$$\int s d\mu = \sum_{i=1}^{p} a_i \mu (E_i) . \tag{7.7}$$

Also, for any nonnegative measurable function, f, if $\lambda \geq 0$, then

$$\int \lambda f d\mu = \lambda \int f d\mu. \tag{7.8}$$

Proof: Consider 7.7 first. Without loss of generality, you can assume $0 < a_1 < a_2 < \cdots < a_p$ and that $\mu(E_i) < \infty$. Let $\varepsilon > 0$ be given and let

$$\delta_1 \sum_{i=1}^{p} \mu(E_i) < \varepsilon.$$

Pick $\delta < \delta_1$ such that for $h < \delta$ it is also true that

$$h < \frac{1}{2} \min(a_1, a_2 - a_1, a_3 - a_2, \cdots, a_n - a_{n-1}).$$

Then for $0 < h < \delta$

$$\begin{aligned} \sum_{k=1}^{\infty} h\mu([s > kh]) &= \sum_{k=1}^{\infty} h \sum_{i=k}^{\infty} \mu([ih < s \leq (i+1)h]) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} h\mu([ih < s \leq (i+1)h]) \\ &= \sum_{i=1}^{\infty} ih\mu([ih < s \leq (i+1)h]). \end{aligned} \tag{7.9}$$

Because of the choice of h there exist positive integers, i_k such that $i_1 < i_2 < \cdots, < i_p$ and

$$i_1 h < a_1 \leq (i_1 + 1) h < \cdots < i_2 h < a_2 < (i_2 + 1) h < \cdots < i_p h < a_p \leq (i_p + 1) h$$

Then in the sum of 7.9 the only terms which are nonzero are those for which $i \in \{i_1, i_2 \cdots, i_p\}$. From the above, you see that

$$\mu([i_k h < s \leq (i_k + 1) h]) = \mu(E_k).$$

Therefore,

$$\sum_{k=1}^{\infty} h\mu([s > kh]) = \sum_{k=1}^{p} i_k h\mu(E_k).$$

It follows that for all h this small,

$$\begin{aligned} 0 &< \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{\infty} h\mu([s > kh]) \\ &= \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{p} i_k h\mu(E_k) \leq h \sum_{k=1}^{p} \mu(E_k) < \varepsilon. \end{aligned}$$

Taking the inf for h this small and using Lemma 7.19,

$$0 \leq \sum_{k=1}^{p} a_k \mu(E_k) - \sup_{\delta > h > 0} \sum_{k=1}^{\infty} h\mu([s > kh]) = \sum_{k=1}^{p} a_k \mu(E_k) - \int s d\mu \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the first part.

To verify 7.8 Note the formula is obvious if $\lambda = 0$ because then $[ih < \lambda f] = \emptyset$ for all $i > 0$. Assume $\lambda > 0$. Then

$$\begin{aligned}
\int \lambda f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < \lambda f]) \\
&= \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih/\lambda < f]) \\
&= \sup_{h>0} \lambda \sum_{i=1}^{\infty} (h/\lambda)\, \mu([i(h/\lambda) < f]) \\
&= \lambda \int f d\mu.
\end{aligned}$$

This proves the lemma.

Lemma 7.22 *Let the nonnegative simple function, s be defined as*

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega)$$

where the c_i are not necessarily distinct but the E_i are disjoint. It follows that

$$\int s = \sum_{i=1}^{n} c_i \mu(E_i).$$

Proof: Let the values of s be $\{a_1, \cdots, a_m\}$. Therefore, since the E_i are disjoint, each a_i equal to one of the c_j. Let $A_i \equiv \cup\{E_j : c_j = a_i\}$. Then from Lemma 7.21 it follows that

$$\begin{aligned}
\int s &= \sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} a_i \sum_{\{j:c_j=a_i\}} \mu(E_j) \\
&= \sum_{i=1}^{m} \sum_{\{j:c_j=a_i\}} c_j \mu(E_j) = \sum_{i=1}^{n} c_i \mu(E_i).
\end{aligned}$$

This proves the lemma.

Note that $\int s$ could equal $+\infty$ if $\mu(A_k) = \infty$ and $a_k > 0$ for some k, but $\int s$ is well defined because $s \geq 0$. Recall that $0 \cdot \infty = 0$.

Lemma 7.23 *If $a, b \geq 0$ and if s and t are nonnegative simple functions, then*

$$\int as + bt = a \int s + b \int t.$$

Proof: Let

$$s(\omega) = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{A_i}(\omega),\ t(\omega) = \sum_{i=1}^{m} \beta_j \mathcal{X}_{B_j}(\omega)$$

where α_i are the distinct values of s and the β_j are the distinct values of t. Clearly $as + bt$ is a nonnegative simple function because it is measurable and has finitely many values. Also,

$$(as + bt)(\omega) = \sum_{j=1}^{m} \sum_{i=1}^{n} (a\alpha_i + b\beta_j) \mathcal{X}_{A_i \cap B_j}(\omega)$$

where the sets $A_i \cap B_j$ are disjoint. By Lemma 7.22,

$$\begin{aligned} \int as + bt &= \sum_{j=1}^{m} \sum_{i=1}^{n} (a\alpha_i + b\beta_j)\mu(A_i \cap B_j) \\ &= a \sum_{i=1}^{n} \alpha_i \mu(A_i) + b \sum_{j=1}^{m} \beta_j \mu(B_j) \\ &= a \int s + b \int t. \end{aligned}$$

This proves the lemma.

7.2.4 Simple Functions And Measurable Functions

There is a fundamental theorem about the relationship of simple functions to measurable functions given in the next theorem.

Theorem 7.24 *Let $f \geq 0$ be measurable. Then there exists a sequence of simple functions $\{s_n\}$ satisfying*

$$0 \leq s_n(\omega) \tag{7.10}$$

$$\cdots s_n(\omega) \leq s_{n+1}(\omega) \cdots$$

$$f(\omega) = \lim_{n \to \infty} s_n(\omega) \text{ for all } \omega \in \Omega. \tag{7.11}$$

If f is bounded the convergence is actually uniform.

Proof: Letting $I \equiv \{\omega : f(\omega) = \infty\}$, define

$$t_n(\omega) = \sum_{k=0}^{2^n} \frac{k}{n} \mathcal{X}_{[k/n \leq f < (k+1)/n]}(\omega) + n\mathcal{X}_I(\omega).$$

Then $t_n(\omega) \leq f(\omega)$ for all ω and $\lim_{n\to\infty} t_n(\omega) = f(\omega)$ for all ω. This is because $t_n(\omega) = n$ for $\omega \in I$ and if $f(\omega) \in [0, \frac{2^n+1}{n})$, then

$$0 \leq f(\omega) - t_n(\omega) \leq \frac{1}{n}. \tag{7.12}$$

Thus whenever $\omega \notin I$, the above inequality will hold for all n large enough. Let

$$s_1 = t_1,\ s_2 = \max(t_1, t_2),\ s_3 = \max(t_1, t_2, t_3), \cdots.$$

Then the sequence $\{s_n\}$ satisfies 7.10-7.11.

To verify the last claim, note that in this case the term $n\mathcal{X}_I(\omega)$ is not present. Therefore, for all n large enough, 7.12 holds for all ω. Thus the convergence is uniform. This proves the theorem.

7.2.5 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral.

Theorem 7.25 *(Monotone Convergence theorem) Let f have values in $[0,\infty]$ and suppose $\{f_n\}$ is a sequence of nonnegative measurable functions having values in $[0,\infty]$ and satisfying*

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \text{ for each } \omega.$$

$$\cdots f_n(\omega) \leq f_{n+1}(\omega) \cdots$$

Then f is measurable and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Proof: From Lemmas 7.16 and 7.17,

$$\begin{aligned}
\int f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < f]) \\
&= \sup_{h>0} \sup_{k} \sum_{i=1}^{k} h\mu([ih < f]) \\
&= \sup_{h>0} \sup_{k} \sup_{m} \sum_{i=1}^{k} h\mu([ih < f_m]) \\
&= \sup_{m} \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < f_m]) \\
&\equiv \sup_{m} \int f_m d\mu \\
&= \lim_{m\to\infty} \int f_m d\mu.
\end{aligned}$$

The third equality follows from the observation that

$$\lim_{m\to\infty} \mu([ih < f_m]) = \mu([ih < f])$$

which follows from Theorem 7.5 since the sets, $[ih < f_m]$ are increasing in m and their union equals $[ih < f]$. This proves the theorem.

To illustrate what goes wrong without the Lebesgue integral, consider the following example.

Example 7.26 *Let $\{r_n\}$ denote the rational numbers in $[0,1]$ and let*

$$f_n(t) \equiv \begin{cases} 1 \text{ if } t \notin \{r_1, \cdots, r_n\} \\ 0 \text{ otherwise} \end{cases}$$

Then $f_n(t) \uparrow f(t)$ where f is the function which is one on the rationals and zero on the irrationals. Each f_n is Riemann integrable (why?) but f is not Riemann integrable. Therefore, you can't write $\int f dx = \lim_{n\to\infty} \int f_n dx$.

A meta-mathematical observation related to this type of example is this. If you can choose your functions, you don't need the Lebesgue integral. The Riemann integral is just fine. It is when you can't choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books.

7.2.6 Other Definitions

To review and summarize the above, if $f \geq 0$ is measurable,

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([f > ih]) \tag{7.13}$$

another way to get the same thing for $\int f d\mu$ is to take an increasing sequence of nonnegative simple functions, $\{s_n\}$ with $s_n(\omega) \to f(\omega)$ and then by monotone convergence theorem,

$$\int f d\mu = \lim_{n\to\infty} \int s_n$$

where if $s_n(\omega) = \sum_{j=1}^{m} c_i \mathcal{X}_{E_i}(\omega)$,

$$\int s_n d\mu = \sum_{i=1}^{m} c_i m(E_i).$$

Similarly this also shows that for such nonnegative measurable function,

$$\int f d\mu = \sup\left\{\int s : 0 \leq s \leq f,\ s \text{ simple}\right\}$$

which is the usual way of defining the Lebesgue integral for nonnegative simple functions in most books. I have done it differently because this approach led to such an easy proof of the Monotone convergence theorem. Here is an equivalent definition of the integral. The fact it is well defined has been discussed above.

Definition 7.27 *For s a nonnegative simple function, $s(\omega) = \sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega)$, $\int s = \sum_{k=1}^{n} c_k \mu(E_k)$. For f a nonnegative measurable function,*

$$\int f d\mu = \sup \left\{ \int s : 0 \leq s \leq f, \ s \text{ simple} \right\}.$$

7.2.7 Fatou's Lemma

Sometimes the limit of a sequence does not exist. There are two more general notions known as $\limsup$ and $\liminf$ which do always exist in some sense. These notions are dependent on the following lemma.

Lemma 7.28 *Let $\{a_n\}$ be an increasing (decreasing) sequence in $[-\infty, \infty]$. Then $\lim_{n\to\infty} a_n$ exists.*

Proof: Suppose first $\{a_n\}$ is increasing. Recall this means $a_n \leq a_{n+1}$ for all n. If the sequence is bounded above, then it has a least upper bound and so $a_n \to a$ where a is its least upper bound. If the sequence is not bounded above, then for every $l \in \mathbb{R}$, it follows l is not an upper bound and so eventually, $a_n > l$. But this is what is meant by $a_n \to \infty$. The situation for decreasing sequences is completely similar.

Now take any sequence, $\{a_n\} \subseteq [-\infty, \infty]$ and consider the sequence $\{A_n\}$ where $A_n \equiv \inf \{a_k : k \geq n\}$. Then as n increases, the set of numbers whose inf is being taken is getting smaller. Therefore, A_n is an increasing sequence and so it must converge. Similarly, if $B_n \equiv \sup \{a_k : k \geq n\}$, it follows B_n is decreasing and so $\{B_n\}$ also must converge. With this preparation, the following definition can be given.

Definition 7.29 *Let $\{a_n\}$ be a sequence of points in $[-\infty, \infty]$. Then define*

$$\liminf_{n\to\infty} a_n \equiv \lim_{n\to\infty} \inf \{a_k : k \geq n\}$$

and

$$\limsup_{n\to\infty} a_n \equiv \lim_{n\to\infty} \sup \{a_k : k \geq n\}$$

In the case of functions having values in $[-\infty, \infty]$,

$$\left(\liminf_{n\to\infty} f_n\right)(\omega) \equiv \liminf_{n\to\infty} (f_n(\omega)).$$

A similar definition applies to $\limsup_{n\to\infty} f_n$.

Lemma 7.30 *Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$. Then $\lim_{n\to\infty} a_n$ exists if and only if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and in this case, the limit equals the common value of these two numbers.

Proof: Suppose first $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, $a_n \in (a-\varepsilon, a+\varepsilon)$ for all n large enough, say $n \geq N$. Therefore, both $\inf\{a_k : k \geq n\}$ and $\sup\{a_k : k \geq n\}$ are contained in $[a-\varepsilon, a+\varepsilon]$ whenever $n \geq N$. It follows $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both in $[a-\varepsilon, a+\varepsilon]$, showing

$$\left|\liminf_{n\to\infty} a_n - \limsup_{n\to\infty} a_n\right| < 2\varepsilon.$$

Since ε is arbitrary, the two must be equal and they both must equal a. Next suppose $\lim_{n\to\infty} a_n = \infty$. Then if $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$l \leq a_n$$

and therefore, for such n,

$$l \leq \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

and this shows, since l is arbitrary that

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty.$$

The case for $-\infty$ is similar.

Conversely, suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$. Suppose first that $a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, there exists N such that if $n \geq N$,

$$\sup\{a_k : k \geq n\} - \inf\{a_k : k \geq n\} < \varepsilon$$

therefore, if $k, m > N$, and $a_k > a_m$,

$$|a_k - a_m| = a_k - a_m \leq \sup\{a_k : k \geq n\} - \inf\{a_k : k \geq n\} < \varepsilon$$

showing that $\{a_n\}$ is a Cauchy sequence. Therefore, it converges to $a \in \mathbb{R}$, and as in the first part, the $\liminf$ and $\limsup$ both equal a. If $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$, then given $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$\inf_{n>N} a_n > l.$$

Therefore, $\lim_{n\to\infty} a_n = \infty$. The case for $-\infty$ is similar. This proves the lemma.

The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.

Theorem 7.31 *(Fatou's lemma) Let f_n be a nonnegative measurable function with values in $[0,\infty]$. Let $g(\omega) = \liminf_{n\to\infty} f_n(\omega)$. Then g is measurable and*

$$\int g d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

In other words,

$$\int \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

Proof: Let $g_n(\omega) = \inf\{f_k(\omega) : k \geq n\}$. Then

$$g_n^{-1}([a,\infty]) = \cap_{k=n}^{\infty} f_k^{-1}([a,\infty]) \in \mathcal{F}.$$

Thus g_n is measurable by Lemma 7.6 on Page 127. Also $g(\omega) = \lim_{n\to\infty} g_n(\omega)$ so g is measurable because it is the pointwise limit of measurable functions. Now the functions g_n form an increasing sequence of nonnegative measurable functions so the monotone convergence theorem applies. This yields

$$\int g d\mu = \lim_{n\to\infty} \int g_n d\mu \leq \lim \inf_{n\to\infty} \int f_n d\mu.$$

The last inequality holding because

$$\int g_n d\mu \leq \int f_n d\mu.$$

(Note that it is not known whether $\lim_{n\to\infty} \int f_n d\mu$ exists.) This proves the Theorem.

7.2.8 The Righteous Algebraic Desires Of The Lebesgue Integral

The monotone convergence theorem shows the integral wants to be linear. This is the essential content of the next theorem.

Theorem 7.32 *Let f, g be nonnegative measurable functions and let a, b be nonnegative numbers. Then*

$$\int (af + bg)\, d\mu = a \int f d\mu + b \int g d\mu. \tag{7.14}$$

Proof: By Theorem 7.24 on Page 139 there exist sequences of nonnegative simple functions, $s_n \to f$ and $t_n \to g$. Then by the monotone convergence theorem and Lemma 7.23,

$$\begin{aligned}
\int (af + bg)\, d\mu &= \lim_{n\to\infty} \int a s_n + b t_n d\mu \\
&= \lim_{n\to\infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\
&= a \int f d\mu + b \int g d\mu.
\end{aligned}$$

As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f + (-g)$ and so you can't very well expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. This is discussed next.

7.3 The Space L^1

The functions considered here have values in $\mathbb{C}$, a vector space.

Definition 7.33 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose $f : \Omega \to \mathbb{C}$. Then f is said to be measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real valued functions.*

Definition 7.34 *A complex simple function will be a function which is of the form*

$$s(\omega) = \sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega)$$

where $c_k \in \mathbb{C}$ and $\mu(E_k) < \infty$. For s a complex simple function as above, define

$$I(s) \equiv \sum_{k=1}^{n} c_k \mu(E_k).$$

Lemma 7.35 *The definition, 7.34 is well defined. Furthermore, I is linear on the vector space of complex simple functions. Also the triangle inequality holds,*

$$|I(s)| \leq I(|s|).$$

Proof: Suppose $\sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega) = 0$. Does it follow that $\sum_k c_k \mu(E_k) = 0$? The supposition implies

$$\sum_{k=1}^{n} \operatorname{Re} c_k \mathcal{X}_{E_k}(\omega) = 0, \ \sum_{k=1}^{n} \operatorname{Im} c_k \mathcal{X}_{E_k}(\omega) = 0. \tag{7.15}$$

Choose λ large and positive so that $\lambda + \operatorname{Re} c_k \geq 0$. Then adding $\sum_k \lambda \mathcal{X}_{E_k}$ to both sides of the first equation above,

$$\sum_{k=1}^{n} (\lambda + \operatorname{Re} c_k) \mathcal{X}_{E_k}(\omega) = \sum_{k=1}^{n} \lambda \mathcal{X}_{E_k}$$

and by Lemma 7.23 on Page 138, it follows upon taking $\int$ of both sides that

$$\sum_{k=1}^{n} (\lambda + \operatorname{Re} c_k) \mu(E_k) = \sum_{k=1}^{n} \lambda \mu(E_k)$$

which implies $\sum_{k=1}^{n} \operatorname{Re} c_k \mu(E_k) = 0$. Similarly, $\sum_{k=1}^{n} \operatorname{Im} c_k \mu(E_k) = 0$ and so $\sum_{k=1}^{n} c_k \mu(E_k) = 0$. Thus if

$$\sum_j c_j \mathcal{X}_{E_j} = \sum_k d_k \mathcal{X}_{F_k}$$

then $\sum_j c_j \mathcal{X}_{E_j} + \sum_k (-d_k) \mathcal{X}_{F_k} = 0$ and so the result just established verifies $\sum_j c_j \mu(E_j) - \sum_k d_k \mu(F_k) = 0$ which proves I is well defined.

That I is linear is now obvious. It only remains to verify the triangle inequality. Let s be a simple function,

$$s = \sum_j c_j \mathcal{X}_{E_j}$$

Then pick $\theta \in \mathbb{C}$ such that $\theta I(s) = |I(s)|$ and $|\theta| = 1$. Then from the triangle inequality for sums of complex numbers,

$$\begin{aligned} |I(s)| &= \theta I(s) = I(\theta s) = \sum_j \theta c_j \mu(E_j) \\ &= \left| \sum_j \theta c_j \mu(E_j) \right| \leq \sum_j |\theta c_j| \mu(E_j) = I(|s|) . \end{aligned}$$

This proves the lemma.

With this lemma, the following is the definition of $L^1(\Omega)$.

Definition 7.36 *$f \in L^1(\Omega)$ means there exists a sequence of complex simple functions, $\{s_n\}$ such that*

$$\begin{gathered} s_n(\omega) \to f(\omega) \text{ for all } \omega \in \Omega \\ \lim_{m,n\to\infty} I(|s_n - s_m|) = \lim_{n,m\to\infty} \int |s_n - s_m| \, d\mu = 0 \end{gathered} \tag{7.16}$$

Then

$$I(f) \equiv \lim_{n\to\infty} I(s_n) . \tag{7.17}$$

Lemma 7.37 *Definition 7.36 is well defined.*

Proof: There are several things which need to be verified. First suppose 7.16. Then by Lemma 7.35

$$|I(s_n) - I(s_m)| = |I(s_n - s_m)| \leq I(|s_n - s_m|)$$

and for m, n large enough this last is given to be small so $\{I(s_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so it converges. This verifies the limit in 7.17 at least exists. It remains to consider another sequence $\{t_n\}$ having the same properties as $\{s_n\}$ and verifying $I(f)$ determined by this other sequence is the same. By Lemma 7.35 and Fatou's lemma, Theorem 7.31 on Page 143,

$$\begin{aligned} |I(s_n) - I(t_n)| &\leq I(|s_n - t_n|) = \int |s_n - t_n| \, d\mu \\ &\leq \int |s_n - f| + |f - t_n| \, d\mu \\ &\leq \liminf_{k\to\infty} \int |s_n - s_k| \, d\mu + \liminf_{k\to\infty} \int |t_n - t_k| \, d\mu < \varepsilon \end{aligned}$$

whenever n is large enough. Since ε is arbitrary, this shows the limit from using the t_n is the same as the limit from using s_n. This proves the lemma.

What if f has values in $[0,\infty)$? Earlier $\int f d\mu$ was defined for such functions and now $I(f)$ has been defined. Are they the same? If so, I can be regarded as an extension of $\int d\mu$ to a larger class of functions.

Lemma 7.38 *Suppose f has values in $[0,\infty)$ and $f \in L^1(\Omega)$. Then f is measurable and*

$$I(f) = \int f d\mu.$$

Proof: Since f is the pointwise limit of a sequence of complex simple functions, $\{s_n\}$ having the properties described in Definition 7.36, it follows $f(\omega) = \lim_{n\to\infty} \operatorname{Re} s_n(\omega)$ and so f is measurable. Also

$$\int \left|(\operatorname{Re} s_n)^+ - (\operatorname{Re} s_m)^+\right| d\mu \leq \int |\operatorname{Re} s_n - \operatorname{Re} s_m| \, d\mu \leq \int |s_n - s_m| \, d\mu$$

where $x^+ \equiv \frac{1}{2}(|x| + x)$, the positive part of the real number, x. [2]Thus there is no loss of generality in assuming $\{s_n\}$ is a sequence of complex simple functions having values in $[0,\infty)$. Then since for such complex simple functions, $I(s) = \int s d\mu$,

$$\left|I(f) - \int f d\mu\right| \leq |I(f) - I(s_n)| + \left|\int s_n d\mu - \int f d\mu\right| < \varepsilon + \int |s_n - f| \, d\mu$$

whenever n is large enough. But by Fatou's lemma, Theorem 7.31 on Page 143, the last term is no larger than

$$\liminf_{k\to\infty} \int |s_n - s_k| \, d\mu < \varepsilon$$

whenever n is large enough. Since ε is arbitrary, this shows $I(f) = \int f d\mu$ as claimed.

As explained above, I can be regarded as an extension of $\int d\mu$ so from now on, the usual symbol, $\int d\mu$ will be used. It is now easy to verify $\int d\mu$ is linear on $L^1(\Omega)$.

Theorem 7.39 *$\int d\mu$ is linear on $L^1(\Omega)$ and $L^1(\Omega)$ is a complex vector space. If $f \in L^1(\Omega)$, then $\operatorname{Re} f, \operatorname{Im} f$, and $|f|$ are all in $L^1(\Omega)$. Furthermore, for $f \in L^1(\Omega)$,*

$$\int f d\mu = \int (\operatorname{Re} f)^+ d\mu - \int (\operatorname{Re} f)^- d\mu + i\left(\int (\operatorname{Im} f)^+ d\mu - \int (\operatorname{Im} f)^- d\mu\right)$$

Also the triangle inequality holds,

$$\left|\int f d\mu\right| \leq \int |f| \, d\mu$$

[2]The negative part of the real number x is defined to be $x^- \equiv \frac{1}{2}(|x| - x)$. Thus $|x| = x^+ + x^-$ and $x = x^+ - x^-$. .

Proof: First it is necessary to verify that $L^1(\Omega)$ is really a vector space because it makes no sense to speak of linear maps without having these maps defined on a vector space. Let f, g be in $L^1(\Omega)$ and let $a, b \in \mathbb{C}$. Then let $\{s_n\}$ and $\{t_n\}$ be sequences of complex simple functions associated with f and g respectively as described in Definition 7.36. Consider $\{as_n + bt_n\}$, another sequence of complex simple functions. Then $as_n(\omega) + bt_n(\omega) \to af(\omega) + bg(\omega)$ for each ω. Also, from Lemma 7.35

$$\int |as_n + bt_n - (as_m + bt_m)| \, d\mu \leq |a| \int |s_n - s_m| \, d\mu + |b| \int |t_n - t_m| \, d\mu$$

and the sum of the two terms on the right converge to zero as $m, n \to \infty$. Thus $af + bg \in L^1(\Omega)$. Also

$$\begin{aligned}
\int (af + bg) \, d\mu &= \lim_{n\to\infty} \int (as_n + bt_n) \, d\mu \\
&= \lim_{n\to\infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\
&= a \lim_{n\to\infty} \int s_n d\mu + b \lim_{n\to\infty} \int t_n d\mu \\
&= a \int f d\mu + b \int g d\mu.
\end{aligned}$$

If $\{s_n\}$ is a sequence of complex simple functions described in Definition 7.36 corresponding to f, then $\{|s_n|\}$ is a sequence of complex simple functions satisfying the conditions of Definition 7.36 corresponding to $|f|$. This is because $|s_n(\omega)| \to |f(\omega)|$ and

$$\int ||s_n| - |s_m|| \, d\mu \leq \int |s_m - s_n| \, d\mu$$

with this last expression converging to 0 as $m, n \to \infty$. Thus $|f| \in L^1(\Omega)$. Also, by similar reasoning, $\{\operatorname{Re} s_n\}$ and $\{\operatorname{Im} s_n\}$ correspond to $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively in the manner described by Definition 7.36 showing that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $L^1(\Omega)$. Now $(\operatorname{Re} f)^+ = \frac{1}{2}(|\operatorname{Re} f| + \operatorname{Re} f)$ and $(\operatorname{Re} f)^- = \frac{1}{2}(|\operatorname{Re} f| - \operatorname{Re} f)$ so both of these functions are in $L^1(\Omega)$. Similar formulas establish that $(\operatorname{Im} f)^+$ and $(\operatorname{Im} f)^-$ are in $L^1(\Omega)$.

The formula follows from the observation that

$$f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i\left((\operatorname{Im} f)^+ - (\operatorname{Im} f)^-\right)$$

and the fact shown first that $\int d\mu$ is linear.

To verify the triangle inequality, let $\{s_n\}$ be complex simple functions for f as in Definition 7.36. Then

$$\left| \int f d\mu \right| = \lim_{n\to\infty} \left| \int s_n d\mu \right| \leq \lim_{n\to\infty} \int |s_n| \, d\mu = \int |f| \, d\mu.$$

This proves the theorem.

Now here is an equivalent description of $L^1(\Omega)$ which is the version which will be used more often than not.

Corollary 7.40 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{C}$. Then $f \in L^1(\Omega)$ if and only if f is measurable and $\int |f| \, d\mu < \infty$.*

Proof: Suppose $f \in L^1(\Omega)$. Then from Definition 7.36, it follows both real and imaginary parts of f are measurable. Just take real and imaginary parts of s_n and observe the real and imaginary parts of f are limits of the real and imaginary parts of s_n respectively. By Theorem 7.39 this shows the only if part.

The more interesting part is the if part. Suppose then that f is measurable and $\int |f| \, d\mu < \infty$. Suppose first that f has values in $[0, \infty)$. It is necessary to obtain the sequence of complex simple functions. By Theorem 7.24, there exists a sequence of nonnegative simple functions, $\{s_n\}$ such that $s_n(\omega) \uparrow f(\omega)$. Then by the monotone convergence theorem,

$$\lim_{n\to\infty} \int (2f - (f - s_n)) \, d\mu = \int 2f d\mu$$

and so

$$\lim_{n\to\infty} \int (f - s_n) \, d\mu = 0.$$

Letting m be large enough, it follows $\int (f - s_m) \, d\mu < \varepsilon$ and so if $n > m$

$$\int |s_m - s_n| \, d\mu \le \int |f - s_m| \, d\mu < \varepsilon.$$

Therefore, $f \in L^1(\Omega)$ because $\{s_n\}$ is a suitable sequence.

The general case follows from considering positive and negative parts of real and imaginary parts of f. These are each measurable and nonnegative and their integral is finite so each is in $L^1(\Omega)$ by what was just shown. Thus

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i \left(\operatorname{Im} f^+ - \operatorname{Im} f^- \right)$$

and so $f \in L^1(\Omega)$. This proves the corollary.

Theorem 7.41 *(Dominated Convergence theorem) Let $f_n \in L^1(\Omega)$ and suppose*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega),$$

and there exists a measurable function g, with values in $[0, \infty]$,[3] such that

$$|f_n(\omega)| \le g(\omega) \text{ and } \int g(\omega) d\mu < \infty.$$

Then $f \in L^1(\Omega)$ and

$$0 = \lim_{n\to\infty} \int |f_n - f| \, d\mu = \lim_{n\to\infty} \left| \int f d\mu - \int f_n d\mu \right|$$

[3]Note that, since g is allowed to have the value ∞, it is not known that $g \in L^1(\Omega)$.

Proof: f is measurable by Theorem 7.8. Since $|f| \leq g$, it follows that

$$f \in L^1(\Omega) \text{ and } |f - f_n| \leq 2g.$$

By Fatou's lemma (Theorem 7.31),

$$\begin{aligned} \int 2g d\mu &\leq \liminf_{n\to\infty} \int 2g - |f - f_n| d\mu \\ &= \int 2g d\mu - \limsup_{n\to\infty} \int |f - f_n| d\mu. \end{aligned}$$

Subtracting $\int 2g d\mu$,

$$0 \leq -\limsup_{n\to\infty} \int |f - f_n| d\mu.$$

Hence

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} (\int |f - f_n| d\mu) \\ &\geq \liminf_{n\to\infty} |\int |f - f_n| d\mu| \geq \left| \int f d\mu - \int f_n d\mu \right| \geq 0. \end{aligned}$$

This proves the theorem by Lemma 7.30 on Page 142 because the lim sup and lim inf are equal.

Corollary 7.42 *Suppose $f_n \in L^1(\Omega)$ and $f(\omega) = \lim_{n\to\infty} f_n(\omega)$. Suppose also there exist measurable functions, g_n, g with values in $[0, \infty]$ such that $\lim_{n\to\infty} \int g_n d\mu = \int g d\mu$, $g_n(\omega) \to g(\omega)$ μ a.e. and both $\int g_n d\mu$ and $\int g d\mu$ are finite. Also suppose $|f_n(\omega)| \leq g_n(\omega)$. Then*

$$\lim_{n\to\infty} \int |f - f_n| d\mu = 0.$$

Proof: It is just like the above. This time $g + g_n - |f - f_n| \geq 0$ and so by Fatou's lemma,

$$\int 2g d\mu - \limsup_{n\to\infty} \int |f - f_n| d\mu =$$

$$\begin{aligned} &\liminf_{n\to\infty} \int (g_n + g) - \limsup_{n\to\infty} \int |f - f_n| d\mu \\ =\ &\liminf_{n\to\infty} \int ((g_n + g) - |f - f_n|) d\mu \geq \int 2g d\mu \end{aligned}$$

and so $-\limsup_{n\to\infty} \int |f - f_n| d\mu \geq 0$.

Definition 7.43 *Let E be a measurable subset of Ω.*

$$\int_E f d\mu \equiv \int f \mathcal{X}_E d\mu.$$

If $L^1(E)$ is written, the σ algebra is defined as

$$\{E \cap A : A \in \mathcal{F}\}$$

and the measure is μ restricted to this smaller σ algebra. Clearly, if $f \in L^1(\Omega)$, then

$$f\mathcal{X}_E \in L^1(E)$$

and if $f \in L^1(E)$, then letting $\tilde{f}$ be the 0 extension of f off of E, it follows $\tilde{f} \in L^1(\Omega)$.

7.4 Vitali Convergence Theorem

The Vitali convergence theorem is a convergence theorem which in the case of a finite measure space is superior to the dominated convergence theorem.

Definition 7.44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^1(\Omega)$. $\mathfrak{S}$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathfrak{S}$*

$$|\int_E f d\mu| < \varepsilon \text{ whenever } \mu(E) < \delta.$$

Lemma 7.45 *If $\mathfrak{S}$ is uniformly integrable, then $|\mathfrak{S}| \equiv \{|f| : f \in \mathfrak{S}\}$ is uniformly integrable. Also $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.*

Proof: Let $\varepsilon > 0$ be given and suppose $\mathfrak{S}$ is uniformly integrable. First suppose the functions are real valued. Let δ be such that if $\mu(E) < \delta$, then

$$\left|\int_E f d\mu\right| < \frac{\varepsilon}{2}$$

for all $f \in \mathfrak{S}$. Let $\mu(E) < \delta$. Then if $f \in \mathfrak{S}$,

$$\begin{aligned}
\int_E |f|\, d\mu &\leq \int_{E\cap[f\leq 0]} (-f)\, d\mu + \int_{E\cap[f>0]} f d\mu \\
&= \left|\int_{E\cap[f\leq 0]} f d\mu\right| + \left|\int_{E\cap[f>0]} f d\mu\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

In general, if $\mathfrak{S}$ is a uniformly integrable set of complex valued functions, the inequalities,

$$\left|\int_E \operatorname{Re} f d\mu\right| \leq \left|\int_E f d\mu\right|, \left|\int_E \operatorname{Im} f d\mu\right| \leq \left|\int_E f d\mu\right|,$$

imply $\operatorname{Re}\mathfrak{S} \equiv \{\operatorname{Re} f : f \in \mathfrak{S}\}$ and $\operatorname{Im}\mathfrak{S} \equiv \{\operatorname{Im} f : f \in \mathfrak{S}\}$ are also uniformly integrable. Therefore, applying the above result for real valued functions to these sets of functions, it follows $|\mathfrak{S}|$ is uniformly integrable also.

For the last part, is suffices to verify a single function in $L^1(\Omega)$ is uniformly integrable. To do so, note that from the dominated convergence theorem,

$$\lim_{R\to\infty}\int_{[|f|>R]}|f|\,d\mu = 0.$$

Let $\varepsilon > 0$ be given and choose R large enough that $\int_{[|f|>R]}|f|\,d\mu < \frac{\varepsilon}{2}$. Now let $\mu(E) < \frac{\varepsilon}{2R}$. Then

$$\begin{aligned}\int_E |f|\,d\mu &= \int_{E\cap[|f|\le R]}|f|\,d\mu + \int_{E\cap[|f|>R]}|f|\,d\mu \\ &< R\mu(E) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This proves the lemma.

The following theorem is Vitali's convergence theorem.

Theorem 7.46 *Let $\{f_n\}$ be a uniformly integrable set of complex valued functions, $\mu(\Omega) < \infty$, and $f_n(x) \to f(x)$ a.e. where f is a measurable complex valued function. Then $f \in L^1(\Omega)$ and*

$$\lim_{n\to\infty}\int_\Omega |f_n - f|d\mu = 0. \tag{7.18}$$

Proof: First it will be shown that $f \in L^1(\Omega)$. By uniform integrability, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then

$$\int_E |f_n|\,d\mu < 1$$

for all n. By Egoroff's theorem, there exists a set, E of measure less than δ such that on E^C, $\{f_n\}$ converges uniformly. Therefore, for p large enough, and $n > p$,

$$\int_{E^C}|f_p - f_n|\,d\mu < 1$$

which implies

$$\int_{E^C}|f_n|\,d\mu < 1 + \int_\Omega |f_p|\,d\mu.$$

Then since there are only finitely many functions, f_n with $n \le p$, there exists a constant, M_1 such that for all n,

$$\int_{E^C}|f_n|\,d\mu < M_1.$$

But also,

$$\begin{aligned}\int_\Omega |f_m|\,d\mu &= \int_{E^C}|f_m|\,d\mu + \int_E |f_m| \\ &\le M_1 + 1 \equiv M.\end{aligned}$$

Therefore, by Fatou's lemma,

$$\int_\Omega |f|\, d\mu \le \lim \inf_{n\to\infty} \int |f_n|\, d\mu \le M,$$

showing that $f \in L^1$ as hoped.

Now $\mathfrak{S}\cup\{f\}$ is uniformly integrable so there exists $\delta_1 > 0$ such that if $\mu(E) < \delta_1$, then $\int_E |g|\, d\mu < \varepsilon/3$ for all $g \in \mathfrak{S} \cup \{f\}$. By Egoroff's theorem, there exists a set, F with $\mu(F) < \delta_1$ such that f_n converges uniformly to f on F^C. Therefore, there exists N such that if $n > N$, then

$$\int_{F^C} |f - f_n|\, d\mu < \frac{\varepsilon}{3}.$$

It follows that for $n > N$,

$$\begin{aligned} \int_\Omega |f - f_n|\, d\mu &\le \int_{F^C} |f - f_n|\, d\mu + \int_F |f|\, d\mu + \int_F |f_n|\, d\mu \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which verifies 7.18.

7.5 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \cdots\}$. Let $\mathcal{F} = \mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$, and let $\mu(S) =$ number of elements in S. Thus $\mu(\{1\}) = 1 = \mu(\{2\})$, $\mu(\{1, 2\}) = 2$, etc. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. It is called counting measure. What functions are measurable in this case?

2. For a measurable nonnegative function, f, the integral was defined as

$$\sup_{\delta > h > 0} \sum_{i=1}^{\infty} h\mu([f > ih])$$

Show this is the same as

$$\int_0^\infty \mu([f > t])\, dt$$

where this integral is just the improper Riemann integral defined by

$$\int_0^\infty \mu([f > t])\, dt = \lim_{R\to\infty} \int_0^R \mu([f > t])\, dt.$$

3. Using the Problem 2, show that for s a nonnegative simple function, $s(\omega) = \sum_{i=1}^n c_i \mathcal{X}_{E_i}(\omega)$ where $0 < c_1 < c_2 \cdots < c_n$ and the sets, E_k are disjoint,

$$\int s d\mu = \sum_{i=1}^n c_i \mu(E_i).$$

Give a really easy proof of this.

4. Let Ω be any uncountable set and let $\mathcal{F} = \{A \subseteq \Omega : \text{either } A \text{ or } A^C \text{ is countable}\}$. Let $\mu(A) = 1$ if A is uncountable and $\mu(A) = 0$ if A is countable. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. This is a well known bad example.

5. Let $\mathcal{F}$ be a σ algebra of subsets of Ω and suppose $\mathcal{F}$ has infinitely many elements. Show that $\mathcal{F}$ is uncountable. **Hint:** You might try to show there exists a countable sequence of disjoint sets of $\mathcal{F}$, $\{A_i\}$. It might be easiest to verify this by contradiction if it doesn't exist rather than a direct construction however, I have seen this done several ways. Once this has been done, you can define a map, θ, from $\mathcal{P}(\mathbb{N})$ into $\mathcal{F}$ which is one to one by $\theta(S) = \cup_{i\in S} A_i$. Then argue $\mathcal{P}(\mathbb{N})$ is uncountable and so $\mathcal{F}$ is also uncountable.

6. An algebra $\mathcal{A}$ of subsets of Ω is a subset of the power set such that Ω is in the algebra and for $A, B \in \mathcal{A}$, $A \setminus B$ and $A \cup B$ are both in $\mathcal{A}$. Let $\mathcal{C} \equiv \{E_i\}_{i=1}^\infty$ be a countable collection of sets and let $\Omega_1 \equiv \cup_{i=1}^\infty E_i$. Show there exists an algebra of sets, $\mathcal{A}$, such that $\mathcal{A} \supseteq \mathcal{C}$ and $\mathcal{A}$ is countable. Note the difference between this problem and Problem 5. **Hint:** Let $\mathcal{C}_1$ denote all finite unions of sets of $\mathcal{C}$ and Ω_1. Thus $\mathcal{C}_1$ is countable. Now let $\mathcal{B}_1$ denote all complements with respect to Ω_1 of sets of $\mathcal{C}_1$. Let $\mathcal{C}_2$ denote all finite unions of sets of $\mathcal{B}_1 \cup \mathcal{C}_1$. Continue in this way, obtaining an increasing sequence $\mathcal{C}_n$, each of which is countable. Let
$$\mathcal{A} \equiv \cup_{i=1}^\infty \mathcal{C}_i.$$

7. We say g is Borel measurable if whenever U is open, $g^{-1}(U)$ is a Borel set. Let $f : \Omega \to X$ and let $g : X \to Y$ where X is a topological space and Y equals $\mathbb{C}, \mathbb{R}$, or $(-\infty, \infty]$ and $\mathcal{F}$ is a σ algebra of sets of Ω. Suppose f is measurable and g is Borel measurable. Show $g \circ f$ is measurable.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]$ by
$$\overline{\mu}(A) = \inf\{\mu(B) : B \supseteq A,\ B \in \mathcal{F}\}.$$
Show $\overline{\mu}$ satisfies
$$\begin{aligned} \overline{\mu}(\emptyset) &= 0, \text{ if } A \subseteq B,\ \overline{\mu}(A) \leq \overline{\mu}(B), \\ \overline{\mu}(\cup_{i=1}^\infty A_i) &\leq \sum_{i=1}^\infty \overline{\mu}(A_i),\ \mu(A) = \overline{\mu}(A) \text{ if } A \in \mathcal{F}. \end{aligned}$$
If $\overline{\mu}$ satisfies these conditions, it is called an outer measure. This shows every measure determines an outer measure on the power set. Outer measures are discussed more later.

9. Let $\{E_i\}$ be a sequence of measurable sets with the property that
$$\sum_{i=1}^\infty \mu(E_i) < \infty.$$

Let $S = \{\omega \in \Omega$ such that $\omega \in E_i$ for infinitely many values of $i\}$. Show $\mu(S) = 0$ and S is measurable. This is part of the Borel Cantelli lemma. **Hint:** Write S in terms of intersections and unions. Something is in S means that for every n there exists $k > n$ such that it is in E_k. Remember the tail of a convergent series is small.

10. $\uparrow$ Let $\{f_n\}, f$ be measurable functions with values in $\mathbb{C}$. $\{f_n\}$ converges in measure if

$$\lim_{n\to\infty} \mu(x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon) = 0$$

for each fixed $\varepsilon > 0$. Prove the theorem of F. Riesz. If f_n converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges to f a.e. **Hint:** Choose n_1 such that

$$\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.$$

Choose $n_2 > n_1$ such that

$$\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,$$

$n_3 > n_2$ such that

$$\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,$$

etc. Now consider what it means for $f_{n_k}(x)$ to fail to converge to $f(x)$. Then use Problem 9.

11. Let $\Omega = \mathbb{N} = \{1, 2, \cdots\}$ and $\mu(S) =$ number of elements in S. If

$$f : \Omega \to \mathbb{C}$$

what is meant by $\int f d\mu$? Which functions are in $L^1(\Omega)$? Which functions are measurable? See Problem 1.

12. For the measure space of Problem 11, give an example of a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function f, such that inequality is obtained in Fatou's lemma.

13. Suppose (Ω, μ) is a finite measure space ($\mu(\Omega) < \infty$) and $\mathfrak{S} \subseteq L^1(\Omega)$. Show $\mathfrak{S}$ is uniformly integrable and bounded in $L^1(\Omega)$ if there exists an increasing function h which satisfies

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty, \ \sup\left\{\int_\Omega h(|f|)\, d\mu : f \in \mathfrak{S}\right\} < \infty.$$

$\mathfrak{S}$ is bounded if there is some number, M such that

$$\int |f|\, d\mu \leq M$$

for all $f \in \mathfrak{S}$.

14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to (-\infty, \infty]$ are measurable. Prove the sets

$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$

are measurable. **Hint:** The easy way to do this is to write

$$\{\omega : f(\omega) < g(\omega)\} = \cup_{r \in \mathbb{Q}} [f < r] \cap [g > r] .$$

Note that $l(x, y) = x - y$ is not continuous on $(-\infty, \infty]$ so the obvious idea doesn't work.

15. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let

$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$

Show S is measurable. **Hint:** You might try to exhibit the set where f_n converges in terms of countable unions and intersections using the definition of a Cauchy sequence.

16. Suppose $u_n(t)$ is a differentiable function for $t \in (a, b)$ and suppose that for $t \in (a, b)$,

$$|u_n(t)|,\ |u_n'(t)| < K_n$$

where $\sum_{n=1}^{\infty} K_n < \infty$. Show

$$(\sum_{n=1}^{\infty} u_n(t))' = \sum_{n=1}^{\infty} u_n'(t).$$

Hint: This is an exercise in the use of the dominated convergence theorem and the mean value theorem from calculus.

17. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions defined on a measure space, $(\Omega, \mathcal{S}, \mu)$. Show that

$$\int \sum_{k=1}^{\infty} f_k d\mu = \sum_{k=1}^{\infty} \int f_k d\mu.$$

Hint: Use the monotone convergence theorem along with the fact the integral is linear.

The Construction Of Measures

8.1 Outer Measures

What are some examples of measure spaces? In this chapter, a general procedure is discussed called the method of outer measures. It is due to Caratheodory (1918). This approach shows how to obtain measure spaces starting with an outer measure. This will then be used to construct measures determined by positive linear functionals.

Definition 8.1 *Let Ω be a nonempty set and let $\mu : \mathcal{P}(\Omega) \to [0,\infty]$ satisfy*

$$\mu(\emptyset) = 0,$$

$$\textit{If } A \subseteq B, \ \textit{then } \mu(A) \leq \mu(B),$$

$$\mu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Such a function is called an outer measure. For $E \subseteq \Omega$, E is μ measurable if for all $S \subseteq \Omega$,

$$\mu(S) = \mu(S \setminus E) + \mu(S \cap E). \tag{8.1}$$

To help in remembering 8.1, think of a measurable set, E, as a process which divides a given set into two pieces, the part in E and the part not in E as in 8.1. In the Bible, there are four incidents recorded in which a process of division resulted in more stuff than was originally present.[1] Measurable sets are exactly

[1] 1 Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^3$ into five disjoint pieces and assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.

those for which no such miracle occurs. You might think of the measurable sets as the nonmiraculous sets. The idea is to show that they form a σ algebra on which the outer measure, μ is a measure.

First here is a definition and a lemma.

Definition 8.2 *$(\mu\lfloor S)(A) \equiv \mu(S \cap A)$ for all $A \subseteq \Omega$. Thus $\mu\lfloor S$ is the name of a new outer measure, called μ restricted to S.*

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.

Lemma 8.3 *If A is μ measurable, then A is $\mu\lfloor S$ measurable.*

Proof: Suppose A is μ measurable. It is desired to to show that for all $T \subseteq \Omega$,

$$(\mu\lfloor S)(T) = (\mu\lfloor S)(T \cap A) + (\mu\lfloor S)(T \setminus A).$$

Thus it is desired to show

$$\mu(S \cap T) = \mu(T \cap A \cap S) + \mu(T \cap S \cap A^C). \tag{8.2}$$

But 8.2 holds because A is μ measurable. Apply Definition 8.1 to $S \cap T$ instead of S.

If A is $\mu\lfloor S$ measurable, it does not follow that A is μ measurable. Indeed, if you believe in the existence of non measurable sets, you could let $A = S$ for such a μ non measurable set and verify that S is $\mu\lfloor S$ measurable.

The next theorem is the main result on outer measures. It is a very general result which applies whenever one has an outer measure on the power set of any set. This theorem will be referred to as Caratheodory's procedure in the rest of the book.

Theorem 8.4 *The collection of μ measurable sets, $\mathcal{S}$, forms a σ algebra and*

$$\textit{If } F_i \in \mathcal{S},\ F_i \cap F_j = \emptyset,\ \textit{then } \mu(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i). \tag{8.3}$$

If $\cdots F_n \subseteq F_{n+1} \subseteq \cdots$, then if $F = \cup_{n=1}^{\infty} F_n$ and $F_n \in \mathcal{S}$, it follows that

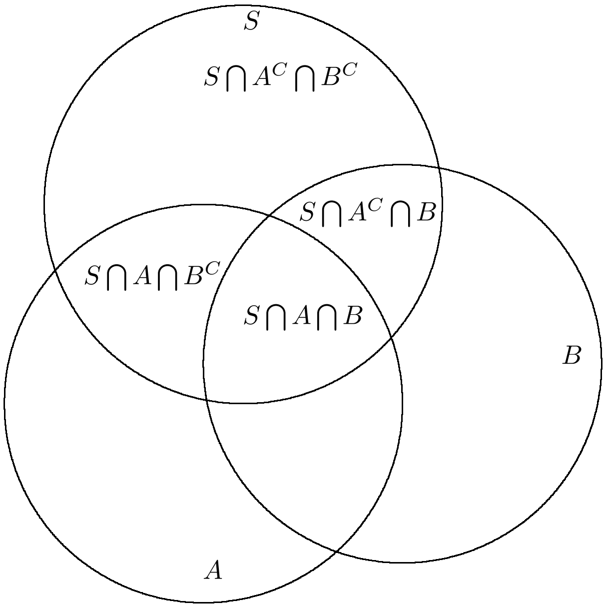
$$\mu(F) = \lim_{n\to\infty} \mu(F_n). \tag{8.4}$$

If $\cdots F_n \supseteq F_{n+1} \supseteq \cdots$, and if $F = \cap_{n=1}^{\infty} F_n$ for $F_n \in \mathcal{S}$ then if $\mu(F_1) < \infty$,

$$\mu(F) = \lim_{n\to\infty} \mu(F_n). \tag{8.5}$$

Also, $(\mathcal{S}, \mu)$ is complete. By this it is meant that if $F \in \mathcal{S}$ and if $E \subseteq \Omega$ with $\mu(E \setminus F) + \mu(F \setminus E) = 0$, then $E \in \mathcal{S}$.

Proof: First note that $\emptyset$ and Ω are obviously in $\mathcal{S}$. Now suppose $A, B \in \mathcal{S}$. I will show $A \setminus B \equiv A \cap B^C$ is in $\mathcal{S}$. To do so, consider the following picture.



Since μ is subadditive,

$$\mu(S) \leq \mu\left(S \cap A \cap B^C\right) + \mu(A \cap B \cap S) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right).$$

Now using $A, B \in \mathcal{S}$,

$$\begin{aligned}\mu(S) &\leq \mu\left(S \cap A \cap B^C\right) + \mu(S \cap A \cap B) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right) \\ &= \mu(S \cap A) + \mu\left(S \cap A^C\right) = \mu(S)\end{aligned}$$

It follows equality holds in the above. Now observe using the picture if you like that

$$(A \cap B \cap S) \cup \left(S \cap B \cap A^C\right) \cup \left(S \cap A^C \cap B^C\right) = S \setminus (A \setminus B)$$

and therefore,

$$\begin{aligned}\mu(S) &= \mu\left(S \cap A \cap B^C\right) + \mu(A \cap B \cap S) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right) \\ &\geq \mu(S \cap (A \setminus B)) + \mu(S \setminus (A \setminus B)).\end{aligned}$$

Therefore, since S is arbitrary, this shows $A \setminus B \in \mathcal{S}$.

Since $\Omega \in \mathcal{S}$, this shows that $A \in \mathcal{S}$ if and only if $A^C \in \mathcal{S}$. Now if $A, B \in \mathcal{S}$, $A \cup B = (A^C \cap B^C)^C = (A^C \setminus B)^C \in \mathcal{S}$. By induction, if $A_1, \cdots, A_n \in \mathcal{S}$, then so is $\cup_{i=1}^n A_i$. If $A, B \in \mathcal{S}$, with $A \cap B = \emptyset$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \setminus A) = \mu(A) + \mu(B).$$

By induction, if $A_i \cap A_j = \emptyset$ and $A_i \in \mathcal{S}$, $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$.

Now let $A = \cup_{i=1}^\infty A_i$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\sum_{i=1}^\infty \mu(A_i) \geq \mu(A) \geq \mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$

Since this holds for all n, you can take the limit as $n \to \infty$ and conclude,

$$\sum_{i=1}^\infty \mu(A_i) = \mu(A)$$

which establishes 8.3. Part 8.4 follows from part 8.3 just as in the proof of Theorem 7.5 on Page 126. That is, letting $F_0 \equiv \emptyset$, use part 8.3 to write

$$\begin{aligned}
\mu(F) &= \mu\left(\cup_{k=1}^\infty (F_k \setminus F_{k-1})\right) = \sum_{k=1}^\infty \mu(F_k \setminus F_{k-1}) \\
&= \lim_{n\to\infty} \sum_{k=1}^n \left(\mu(F_k) - \mu(F_{k-1})\right) = \lim_{n\to\infty} \mu(F_n).
\end{aligned}$$

In order to establish 8.5, let the F_n be as given there. Then, since $(F_1 \setminus F_n)$ increases to $(F_1 \setminus F)$, 8.4 implies

$$\lim_{n\to\infty} \left(\mu(F_1) - \mu(F_n)\right) = \mu(F_1 \setminus F).$$

Now $\mu(F_1 \setminus F) + \mu(F) \geq \mu(F_1)$ and so $\mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$. Hence

$$\lim_{n\to\infty} \left(\mu(F_1) - \mu(F_n)\right) = \mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$$

which implies

$$\lim_{n\to\infty} \mu(F_n) \leq \mu(F).$$

But since $F \subseteq F_n$,

$$\mu(F) \leq \lim_{n\to\infty} \mu(F_n)$$

and this establishes 8.5. Note that it was assumed $\mu(F_1) < \infty$ because $\mu(F_1)$ was subtracted from both sides.

It remains to show $\mathcal{S}$ is closed under countable unions. Recall that if $A \in \mathcal{S}$, then $A^C \in \mathcal{S}$ and $\mathcal{S}$ is closed under finite unions. Let $A_i \in \mathcal{S}$, $A = \cup_{i=1}^\infty A_i$, $B_n = \cup_{i=1}^n A_i$. Then

$$\begin{aligned}
\mu(S) &= \mu(S \cap B_n) + \mu(S \setminus B_n) \qquad (8.6)\\
&= (\mu\lfloor S)(B_n) + (\mu\lfloor S)(B_n^C).
\end{aligned}$$

By Lemma 8.3 B_n is $(\mu\lfloor S)$ measurable and so is B_n^C. I want to show $\mu(S) \geq \mu(S \setminus A) + \mu(S \cap A)$. If $\mu(S) = \infty$, there is nothing to prove. Assume $\mu(S) < \infty$.

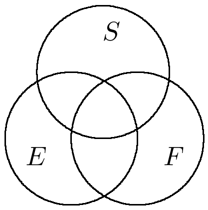
Then apply Parts 8.5 and 8.4 to the outer measure, $\mu\lfloor S$ in 8.6 and let $n \to \infty$. Thus

$$B_n \uparrow A, \;\; B_n^C \downarrow A^C$$

and this yields $\mu(S) = (\mu\lfloor S)(A) + (\mu\lfloor S)(A^C) = \mu(S \cap A) + \mu(S \setminus A)$.

Therefore $A \in \mathcal{S}$ and this proves Parts 8.3, 8.4, and 8.5. It remains to prove the last assertion about the measure being complete.

Let $F \in \mathcal{S}$ and let $\mu(E \setminus F) + \mu(F \setminus E) = 0$. Consider the following picture.



Then referring to this picture and using $F \in \mathcal{S}$,

$$\begin{aligned}
\mu(S) &\leq \mu(S \cap E) + \mu(S \setminus E) \\
&\leq \mu(S \cap E \cap F) + \mu((S \cap E) \setminus F) + \mu(S \setminus F) + \mu(F \setminus E) \\
&\leq \mu(S \cap F) + \mu(E \setminus F) + \mu(S \setminus F) + \mu(F \setminus E) \\
&= \mu(S \cap F) + \mu(S \setminus F) = \mu(S)
\end{aligned}$$

Hence $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$ and so $E \in \mathcal{S}$. This shows that $(\mathcal{S}, \mu)$ is complete and proves the theorem.

Completeness usually occurs in the following form. $E \subseteq F \in \mathcal{S}$ and $\mu(F) = 0$. Then $E \in \mathcal{S}$.

Where do outer measures come from? One way to obtain an outer measure is to start with a measure μ, defined on a σ algebra of sets, $\mathcal{S}$, and use the following definition of the outer measure induced by the measure.

Definition 8.5 *Let μ be a measure defined on a σ algebra of sets, $\mathcal{S} \subseteq \mathcal{P}(\Omega)$. Then the outer measure induced by μ, denoted by $\overline{\mu}$ is defined on $\mathcal{P}(\Omega)$ as*

$$\overline{\mu}(E) = \inf\{\mu(F) \; : F \in \mathcal{S} \text{ and } F \supseteq E\}.$$

A measure space, $(\mathcal{S}, \Omega, \mu)$ is σ finite if there exist measurable sets, Ω_i with $\mu(\Omega_i) < \infty$ and $\Omega = \cup_{i=1}^{\infty} \Omega_i$.

You should prove the following lemma.

Lemma 8.6 *If $(\mathcal{S}, \Omega, \mu)$ is σ finite then there exist disjoint measurable sets, $\{B_n\}$ such that $\mu(B_n) < \infty$ and $\cup_{n=1}^{\infty} B_n = \Omega$.*

The following lemma deals with the outer measure generated by a measure which is σ finite. It says that if the given measure is σ finite and complete then no new measurable sets are gained by going to the induced outer measure and then considering the measurable sets in the sense of Caratheodory.

Lemma 8.7 *Let $(\Omega, \mathcal{S}, \mu)$ be any measure space and let $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]$ be the outer measure induced by μ. Then $\overline{\mu}$ is an outer measure as claimed and if $\overline{\mathcal{S}}$ is the set of $\overline{\mu}$ measurable sets in the sense of Caratheodory, then $\overline{\mathcal{S}} \supseteq \mathcal{S}$ and $\overline{\mu} = \mu$ on $\mathcal{S}$. Furthermore, if μ is σ finite and $(\Omega, \mathcal{S}, \mu)$ is complete, then $\overline{\mathcal{S}} = \mathcal{S}$.*

Proof: It is easy to see that $\overline{\mu}$ is an outer measure. Let $E \in \mathcal{S}$. The plan is to show $E \in \overline{\mathcal{S}}$ and $\overline{\mu}(E) = \mu(E)$. To show this, let $S \subseteq \Omega$ and then show

$$\overline{\mu}(S) \geq \overline{\mu}(S \cap E) + \overline{\mu}(S \setminus E). \tag{8.7}$$

This will verify that $E \in \overline{\mathcal{S}}$. If $\overline{\mu}(S) = \infty$, there is nothing to prove, so assume $\overline{\mu}(S) < \infty$. Thus there exists $T \in \mathcal{S}$, $T \supseteq S$, and

$$\begin{aligned}
\overline{\mu}(S) &> \mu(T) - \varepsilon = \mu(T \cap E) + \mu(T \setminus E) - \varepsilon \\
&\geq \overline{\mu}(T \cap E) + \overline{\mu}(T \setminus E) - \varepsilon \\
&\geq \overline{\mu}(S \cap E) + \overline{\mu}(S \setminus E) - \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves 8.7 and verifies $\mathcal{S} \subseteq \overline{\mathcal{S}}$. Now if $E \in \mathcal{S}$ and $V \supseteq E$ with $V \in \mathcal{S}$, $\mu(E) \leq \mu(V)$. Hence, taking inf, $\mu(E) \leq \overline{\mu}(E)$. But also $\mu(E) \geq \overline{\mu}(E)$ since $E \in \mathcal{S}$ and $E \supseteq E$. Hence

$$\overline{\mu}(E) \leq \mu(E) \leq \overline{\mu}(E).$$

Next consider the claim about not getting any new sets from the outer measure in the case the measure space is σ finite and complete.

Claim 1: If $E, D \in \mathcal{S}$, and $\mu(E \setminus D) = 0$, then if $D \subseteq F \subseteq E$, it follows $F \in \mathcal{S}$.

Proof of claim 1:

$$F \setminus D \subseteq E \setminus D \in \mathcal{S},$$

and $E \setminus D$ is a set of measure zero. Therefore, since $(\Omega, \mathcal{S}, \mu)$ is complete, $F \setminus D \in \mathcal{S}$ and so

$$F = D \cup (F \setminus D) \in \mathcal{S}.$$

Claim 2: Suppose $F \in \overline{\mathcal{S}}$ and $\overline{\mu}(F) < \infty$. Then $F \in \mathcal{S}$.

Proof of the claim 2: From the definition of $\overline{\mu}$, it follows there exists $E \in \mathcal{S}$ such that $E \supseteq F$ and $\mu(E) = \overline{\mu}(F)$. Therefore,

$$\overline{\mu}(E) = \overline{\mu}(E \setminus F) + \overline{\mu}(F)$$

so

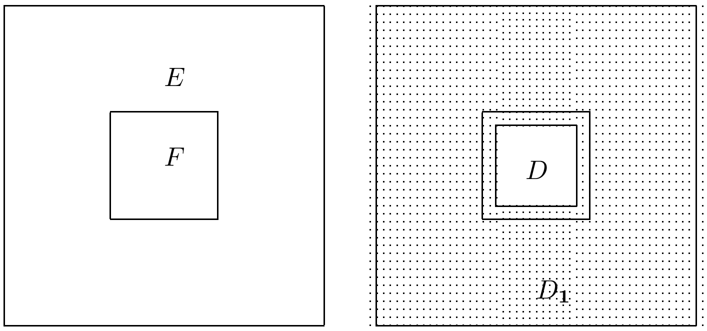
$$\overline{\mu}(E \setminus F) = 0. \tag{8.8}$$

Similarly, there exists $D_1 \in \mathcal{S}$ such that

$$D_1 \subseteq E, \ D_1 \supseteq (E \setminus F), \ \mu(D_1) = \overline{\mu}(E \setminus F).$$

and

$$\overline{\mu}(D_1 \setminus (E \setminus F)) = 0. \tag{8.9}$$



Now let $D = E \setminus D_1$. It follows $D \subseteq F$ because if $x \in D$, then $x \in E$ but $x \notin (E \setminus F)$ and so $x \in F$. Also $F \setminus D = D_1 \setminus (E \setminus F)$ because both sides equal $D_1 \cap F \setminus E$.

Then from 8.8 and 8.9,

$$\begin{aligned} \mu (E \setminus D) &\leq \overline{\mu} (E \setminus F) + \overline{\mu} (F \setminus D) \\ &= \overline{\mu} (E \setminus F) + \overline{\mu} (D_1 \setminus (E \setminus F)) = 0. \end{aligned}$$

By Claim 1, it follows $F \in \mathcal{S}$. This proves Claim 2.

Now since $(\Omega, \mathcal{S},\mu)$ is σ finite, there are sets of $\mathcal{S}$, $\{B_n\}_{n=1}^{\infty}$ such that $\mu (B_n) < \infty, \cup_n B_n = \Omega$. Then $F \cap B_n \in \mathcal{S}$ by Claim 2. Therefore, $F = \cup_{n=1}^{\infty} F \cap B_n \in \mathcal{S}$ and so $\mathcal{S} = \overline{\mathcal{S}}$. This proves the lemma.

8.2 Regular measures

Usually Ω is not just a set. It is also a topological space. It is very important to consider how the measure is related to this topology.

Definition 8.8 *Let μ be a measure on a σ algebra $\mathcal{S}$, of subsets of Ω, where (Ω, τ) is a topological space. μ is a Borel measure if $\mathcal{S}$ contains all Borel sets. μ is called outer regular if μ is Borel and for all $E \in \mathcal{S}$,*

$$\mu(E) = \inf\{\mu(V) : V \text{ is open and } V \supseteq E\}.$$

μ is called inner regular if μ is Borel and

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, \text{ and } K \text{ is compact}\}.$$

If the measure is both outer and inner regular, it is called regular.

It will be assumed in what follows that (Ω, τ) is a locally compact Hausdorff space. This means it is Hausdorff: If $p, q \in \Omega$ such that $p \neq q$, there exist open sets, U_p and U_q containing p and q respectively such that $U_p \cap U_q = \emptyset$ and Locally compact: There exists a basis of open sets for the topology, $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $\overline{U}$ is compact. Recall $\mathcal{B}$ is a basis for the topology if $\cup\mathcal{B} = \Omega$ and if every open set in τ is the union of sets of $\mathcal{B}$. Also recall a Hausdorff space is normal if whenever H and C are two closed sets, there exist disjoint open sets, U_H and U_C containing H and C respectively. A regular space is one which has the property that if p is a point not in H, a closed set, then there exist disjoint open sets, U_p and U_H containing p and H respectively.

8.3 Urysohn's lemma

Urysohn's lemma which characterizes normal spaces is a very important result which is useful in general topology and in the construction of measures. Because it is somewhat technical a proof is given for the part which is needed.

Theorem 8.9 *(Urysohn) Let (X, τ) be normal and let $H \subseteq U$ where H is closed and U is open. Then there exists $g : X \to [0, 1]$ such that g is continuous, $g(x) = 1$ on H and $g(x) = 0$ if $x \notin U$.*

Proof: Let $D \equiv \{r_n\}_{n=1}^{\infty}$ be the rational numbers in $(0, 1)$. Choose V_{r_1} an open set such that

$$H \subseteq V_{r_1} \subseteq \overline{V}_{r_1} \subseteq U.$$

This can be done by applying the assumption that X is normal to the disjoint closed sets, H and U^C, to obtain open sets V and W with

$$H \subseteq V,\ U^C \subseteq W, \text{and } V \cap W = \emptyset.$$

Then

$$H \subseteq V \subseteq \overline{V},\ \overline{V} \cap U^C = \emptyset$$

and so let $V_{r_1} = V$.

Suppose $V_{r_1}, \cdots, V_{r_k}$ have been chosen and list the rational numbers $r_1, \cdots, r_k$ in order,

$$r_{l_1} < r_{l_2} < \cdots < r_{l_k} \text{ for } \{l_1, \cdots, l_k\} = \{1, \cdots, k\}.$$

If $r_{k+1} > r_{l_k}$ then letting $p = r_{l_k}$, let $V_{r_{k+1}}$ satisfy

$$\overline{V}_p \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq U.$$

If $r_{k+1} \in (r_{l_i}, r_{l_{i+1}})$, let $p = r_{l_i}$ and let $q = r_{l_{i+1}}$. Then let $V_{r_{k+1}}$ satisfy

$$\overline{V}_p \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq V_q.$$

If $r_{k+1} < r_{l_1}$, let $p = r_{l_1}$ and let $V_{r_{k+1}}$ satisfy

$$H \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq V_p.$$

Thus there exist open sets V_r for each $r \in \mathbb{Q} \cap (0,1)$ with the property that if $r < s$,

$$H \subseteq V_r \subseteq \overline{V}_r \subseteq V_s \subseteq \overline{V}_s \subseteq U.$$

Now let

$$f(x) = \inf\{t \in D : x \in V_t\}, \ f(x) \equiv 1 \text{ if } x \notin \bigcup_{t \in D} V_t.$$

I claim f is continuous.

$$f^{-1}([0,a)) = \cup\{V_t : t < a, t \in D\},$$

an open set.

Next consider $x \in f^{-1}([0,a])$ so $f(x) \leq a$. If $t > a$, then $x \in V_t$ because if not, then

$$\inf\{t \in D : x \in V_t\} > a.$$

Thus

$$f^{-1}([0,a]) = \cap\{V_t : t > a\} = \cap\{\overline{V}_t : t > a\}$$

which is a closed set. If $a = 1$, $f^{-1}([0,1]) = f^{-1}([0,a]) = X$. Therefore,

$$f^{-1}((a,1]) = X \setminus f^{-1}([0,a]) = \text{open set}.$$

It follows f is continuous. Clearly $f(x) = 0$ on H. If $x \in U^C$, then $x \notin V_t$ for any $t \in D$ so $f(x) = 1$ on U^C. Let $g(x) = 1 - f(x)$. This proves the theorem.

In any metric space there is a much easier proof of the conclusion of Urysohn's lemma which applies.

Lemma 8.10 *Let S be a nonempty subset of a metric space, (X,d). Define*

$$f(x) \equiv \operatorname{dist}(x,S) \equiv \inf\{d(x,y) : y \in S\}.$$

Then f is continuous.

Proof: Consider $|f(x) - f(x_1)|$and suppose without loss of generality that $f(x_1) \geq f(x)$. Then choose $y \in S$ such that $f(x) + \varepsilon > d(x,y)$. Then

$$\begin{aligned}
|f(x_1) - f(x)| &= f(x_1) - f(x) \leq f(x_1) - d(x,y) + \varepsilon \\
&\leq d(x_1,y) - d(x,y) + \varepsilon \\
&\leq d(x,x_1) + d(x,y) - d(x,y) + \varepsilon \\
&= d(x_1,x) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that $|f(x_1) - f(x)| \leq d(x_1,x)$ and this proves the lemma.

Theorem 8.11 *(Urysohn's lemma for metric space) Let H be a closed subset of an open set, U in a metric space, (X,d). Then there exists a continuous function, $g : X \to [0,1]$ such that $g(x) = 1$ for all $x \in H$ and $g(x) = 0$ for all $x \notin U$.*

Proof: If $x \notin C$, a closed set, then $\text{dist}(x, C) > 0$ because if not, there would exist a sequence of points of C converging to x and it would follow that $x \in C$. Therefore, $\text{dist}(x, H) + \text{dist}\left(x, U^C\right) > 0$ for all $x \in X$. Now define a continuous function, g as

$$g(x) \equiv \frac{\text{dist}\left(x, U^C\right)}{\text{dist}(x, H) + \text{dist}\left(x, U^C\right)}.$$

It is easy to see this verifies the conclusions of the theorem and this proves the theorem.

Theorem 8.12 *Every compact Hausdorff space is normal.*

Proof: First it is shown that X, is regular. Let H be a closed set and let $p \notin H$. Then for each $h \in H$, there exists an open set U_h containing p and an open set V_h containing h such that $U_h \cap V_h = \emptyset$. Since H must be compact, it follows there are finitely many of the sets V_h, $V_{h_1} \cdots V_{h_n}$ such that $H \subseteq \cup_{i=1}^n V_{h_i}$. Then letting $U = \cap_{i=1}^n U_{h_i}$ and $V = \cup_{i=1}^n V_{h_i}$, it follows that $p \in U$, $H \in V$ and $U \cap V = \emptyset$. Thus X is regular as claimed.

Next let K and H be disjoint nonempty closed sets.Using regularity of X, for every $k \in K$, there exists an open set U_k containing k and an open set V_k containing H such that these two open sets have empty intersection. Thus $H \cap \overline{U}_k = \emptyset$. Finitely many of the U_k, $U_{k_1}, \cdots, U_{k_p}$ cover K and so $\cup_{i=1}^p \overline{U}_{k_i}$ is a closed set which has empty intersection with H. Therefore, $K \subseteq \cup_{i=1}^p U_{k_i}$ and $H \subseteq \left(\cup_{i=1}^p \overline{U}_{k_i}\right)^C$. This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space discussed earlier. However, it is reviewed here for the sake of convenience or in case you have not read the earlier treatment.

Definition 8.13 *Suppose (X, τ) is a locally compact Hausdorff space. Then let $\widetilde{X} \equiv X \cup \{\infty\}$ where ∞ is just the name of some point which is not in X which is called the point at infinity. A basis for the topology $\widetilde{\tau}$ for $\widetilde{X}$ is*

$$\tau \cup \left\{K^C \text{ where } K \text{ is a compact subset of } X\right\}.$$

The complement is taken with respect to $\widetilde{X}$ and so the open sets, K^C are basic open sets which contain ∞.

The reason this is called a compactification is contained in the next lemma.

Lemma 8.14 *If (X, τ) is a locally compact Hausdorff space, then $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a compact Hausdorff space.*

Proof: Since (X, τ) is a locally compact Hausdorff space, it follows $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and ∞. Since (X, τ) is locally compact, there exists an open set of τ, U

having compact closure which contains p. Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, p and ∞ respectively. Now let $\mathcal{C}$ be an open cover of $\widetilde{X}$ with sets from $\widetilde{\tau}$. Then ∞ must be in some set, U_∞ from $\mathcal{C}$, which must contain a set of the form K^C where K is a compact subset of X. Then there exist sets from $\mathcal{C}$, $U_1, \cdots, U_r$ which cover K. Therefore, a finite subcover of $\widetilde{X}$ is $U_1, \cdots, U_r, U_\infty$.

Theorem 8.15 *Let X be a locally compact Hausdorff space, and let K be a compact subset of the open set V. Then there exists a continuous function, $f : X \to [0,1]$, such that f equals 1 on K and $\overline{\{x : f(x) \neq 0\}} \equiv \operatorname{spt}(f)$ is a compact subset of V.*

Proof: Let $\widetilde{X}$ be the space just described. Then K and V are respectively closed and open in $\widetilde{\tau}$. By Theorem 8.12 there exist open sets in $\widetilde{\tau}$, U, and W such that $K \subseteq U, \infty \in V^C \subseteq W$, and $U \cap W = U \cap (W \setminus \{\infty\}) = \emptyset$. Thus $W \setminus \{\infty\}$ is an open set in the original topological space which contains V^C, U is an open set in the original topological space which contains K, and $W \setminus \{\infty\}$ and U are disjoint.

Now for each $x \in K$, let U_x be a basic open set whose closure is compact and such that

$$x \in U_x \subseteq U.$$

Thus $\overline{U_x}$ must have empty intersection with V^C because the open set, $W \setminus \{\infty\}$ contains no points of U_x. Since K is compact, there are finitely many of these sets, $U_{x_1}, U_{x_2}, \cdots, U_{x_n}$ which cover K. Now let $H \equiv \cup_{i=1}^n U_{x_i}$.

Claim: $\overline{H} = \cup_{i=1}^n \overline{U_{x_i}}$

Proof of claim: Suppose $p \in \overline{H}$. If $p \notin \cup_{i=1}^n \overline{U_{x_i}}$ then if follows $p \notin \overline{U_{x_i}}$ for each i. Therefore, there exists an open set, R_i containing p such that R_i contains no other points of U_{x_i}. Therefore, $R \equiv \cap_{i=1}^n R_i$ is an open set containing p which contains no other points of $\cup_{i=1}^n U_{x_i} = W$, a contradiction. Therefore, $\overline{H} \subseteq \cup_{i=1}^n \overline{U_{x_i}}$. On the other hand, if $p \in \overline{U_{x_i}}$ then p is obviously in $\overline{H}$ so this proves the claim.

From the claim, $K \subseteq H \subseteq \overline{H} \subseteq V$ and $\overline{H}$ is compact because it is the finite union of compact sets. Repeating the same argument, there exists an open set, I such that $\overline{H} \subseteq I \subseteq \overline{I} \subseteq V$ with $\overline{I}$ compact. Now $(\overline{I}, \tau_I)$ is a compact topological space where τ_I is the topology which is obtained by taking intersections of open sets in X with $\overline{I}$. Therefore, by Urysohn's lemma, there exists $f : \overline{I} \to [0,1]$ such that f is continuous at every point of $\overline{I}$ and also $f(K) = 1$ while $f(\overline{I} \setminus H) = 0$. Extending f to equal 0 on $\overline{I}^C$, it follows that f is continuous on X, has values in $[0,1]$, and satisfies $f(K) = 1$ and $\operatorname{spt}(f)$ is a compact subset contained in $\overline{I} \subseteq V$. This proves the theorem.

In fact, the conclusion of the above theorem could be used to prove that the topological space is locally compact. However, this is not needed here.

Definition 8.16 *Define* $\operatorname{spt}(f)$ *(support of f) to be the closure of the set $\{x : f(x) \neq 0\}$. If V is an open set, $C_c(V)$ will be the set of continuous functions f, defined on Ω having* $\operatorname{spt}(f) \subseteq V$. *Thus in Theorem 8.15, $f \in C_c(V)$.*

Definition 8.17 *If K is a compact subset of an open set, V, then $K \prec \phi \prec V$ if*

$$\phi \in C_c(V),\ \phi(K) = \{1\},\ \phi(\Omega) \subseteq [0,1],$$

where Ω denotes the whole topological space considered. Also for $\phi \in C_c(\Omega)$, $K \prec \phi$ if

$$\phi(\Omega) \subseteq [0,1] \text{ and } \phi(K) = 1.$$

and $\phi \prec V$ if

$$\phi(\Omega) \subseteq [0,1] \text{ and } \operatorname{spt}(\phi) \subseteq V.$$

Theorem 8.18 *(Partition of unity) Let K be a compact subset of a locally compact Hausdorff topological space satisfying Theorem 8.15 and suppose*

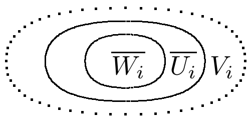
$$K \subseteq V = \cup_{i=1}^n V_i,\ V_i \text{ open.}$$

Then there exist $\psi_i \prec V_i$ with

$$\sum_{i=1}^n \psi_i(x) = 1$$

for all $x \in K$.

Proof: Let $K_1 = K \setminus \cup_{i=2}^n V_i$. Thus K_1 is compact and $K_1 \subseteq V_1$. Let $K_1 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1$ with $\overline{W}_1$compact. To obtain W_1, use Theorem 8.15 to get f such that $K_1 \prec f \prec V_1$ and let $W_1 \equiv \{x : f(x) \neq 0\}$. Thus $W_1, V_2, \cdots V_n$ covers K and $\overline{W}_1 \subseteq V_1$. Let $K_2 = K \setminus (\cup_{i=3}^n V_i \cup W_1)$. Then K_2 is compact and $K_2 \subseteq V_2$. Let $K_2 \subseteq W_2 \subseteq \overline{W}_2 \subseteq V_2$ $\overline{W}_2$ compact. Continue this way finally obtaining $W_1, \cdots, W_n$, $K \subseteq W_1 \cup \cdots \cup W_n$, and $\overline{W}_i \subseteq V_i$ $\overline{W}_i$ compact. Now let $\overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i$,$\overline{U}_i$ compact.



By Theorem 8.15, let $\overline{U}_i \prec \phi_i \prec V_i$, $\cup_{i=1}^n \overline{W}_i \prec \gamma \prec \cup_{i=1}^n U_i$. Define

$$\psi_i(x) = \begin{cases} \gamma(x)\phi_i(x)/\sum_{j=1}^n \phi_j(x) \text{ if } \sum_{j=1}^n \phi_j(x) \neq 0, \\ 0 \text{ if } \sum_{j=1}^n \phi_j(x) = 0. \end{cases}$$

If x is such that $\sum_{j=1}^n \phi_j(x) = 0$, then $x \notin \cup_{i=1}^n \overline{U}_i$. Consequently $\gamma(y) = 0$ for all y near x and so $\psi_i(y) = 0$ for all y near x. Hence ψ_i is continuous at such x. If $\sum_{j=1}^n \phi_j(x) \neq 0$, this situation persists near x and so ψ_i is continuous at such points. Therefore ψ_i is continuous. If $x \in K$, then $\gamma(x) = 1$ and so $\sum_{j=1}^n \psi_j(x) = 1$. Clearly $0 \leq \psi_i(x) \leq 1$ and $\operatorname{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The following corollary won't be needed immediately but is of considerable interest later.

Corollary 8.19 *If H is a compact subset of V_i, there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Theorem 8.18.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H.

8.4 Positive Linear Functionals

Definition 8.20 *Let (Ω, τ) be a topological space. $L : C_c(\Omega) \to \mathbb{C}$ is called a positive linear functional if L is linear,*

$$L(af_1 + bf_2) = aLf_1 + bLf_2,$$

and if $Lf \geq 0$ whenever $f \geq 0$.

Theorem 8.21 *(Riesz representation theorem) Let (Ω, τ) be a locally compact Hausdorff space and let L be a positive linear functional on $C_c(\Omega)$. Then there exists a σ algebra $\mathcal{S}$ containing the Borel sets and a unique measure μ, defined on $\mathcal{S}$, such that*

$$\mu \text{ is complete,} \tag{8.10}$$
$$\mu(K) < \infty \text{ for all } K \text{ compact,} \tag{8.11}$$

$$\mu(F) = \sup\{\mu(K) : K \subseteq F, \ K \text{ compact}\},$$

for all F open and for all $F \in \mathcal{S}$ with $\mu(F) < \infty$,

$$\mu(F) = \inf\{\mu(V) : V \supseteq F, \ V \text{ open}\}$$

for all $F \in \mathcal{S}$, and

$$\int f d\mu = Lf \quad \text{for all } f \in C_c(\Omega). \tag{8.12}$$

The plan is to define an outer measure and then to show that it, together with the σ algebra of sets measurable in the sense of Caratheodory, satisfies the conclusions of the theorem. Always, K will be a compact set and V will be an open set.

Definition 8.22 $\mu(V) \equiv \sup\{Lf : f \prec V\}$ *for V open,* $\mu(\emptyset) = 0$. $\mu(E) \equiv \inf\{\mu(V) : V \supseteq E\}$ *for arbitrary sets E.*

Lemma 8.23 *μ is a well-defined outer measure.*

Proof: First it is necessary to verify μ is well defined because there are two descriptions of it on open sets. Suppose then that $\mu_1(V) \equiv \inf\{\mu(U) : U \supseteq V$ and U is open$\}$. It is required to verify that $\mu_1(V) = \mu(V)$ where μ is given as $\sup\{Lf : f \prec V\}$. If $U \supseteq V$, then $\mu(U) \geq \mu(V)$ directly from the definition. Hence from the definition of μ_1, it follows $\mu_1(V) \geq \mu(V)$. On the other hand, $V \supseteq V$ and so $\mu_1(V) \leq \mu(V)$. This verifies μ is well defined.

It remains to show that μ is an outer measure. Let $V = \cup_{i=1}^{\infty} V_i$ and let $f \prec V$. Then $\mathrm{spt}(f) \subseteq \cup_{i=1}^{n} V_i$ for some n. Let $\psi_i \prec V_i$, $\sum_{i=1}^{n} \psi_i = 1$ on $\mathrm{spt}(f)$.

$$Lf = \sum_{i=1}^{n} L(f\psi_i) \leq \sum_{i=1}^{n} \mu(V_i) \leq \sum_{i=1}^{\infty} \mu(V_i).$$

Hence

$$\mu(V) \leq \sum_{i=1}^{\infty} \mu(V_i)$$

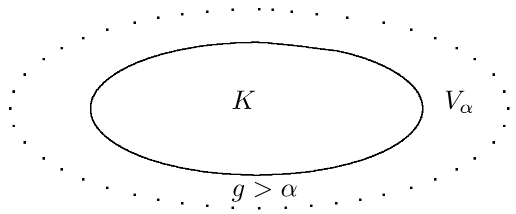
since $f \prec V$ is arbitrary. Now let $E = \cup_{i=1}^{\infty} E_i$. Is $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$? Without loss of generality, it can be assumed $\mu(E_i) < \infty$ for each i since if not so, there is nothing to prove. Let $V_i \supseteq E_i$ with $\mu(E_i) + \varepsilon 2^{-i} > \mu(V_i)$.

$$\mu(E) \leq \mu(\cup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq \varepsilon + \sum_{i=1}^{\infty} \mu(E_i).$$

Since ε was arbitrary, $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$ which proves the lemma.

Lemma 8.24 *Let K be compact, $g \geq 0$, $g \in C_c(\Omega)$, and $g = 1$ on K. Then $\mu(K) \leq Lg$. Also $\mu(K) < \infty$ whenever K is compact.*

Proof: Let $\alpha \in (0, 1)$ and $V_\alpha = \{x : g(x) > \alpha\}$ so $V_\alpha \supseteq K$ and let $h \prec V_\alpha$.



Then $h \leq 1$ on V_α while $g\alpha^{-1} \geq 1$ on V_α and so $g\alpha^{-1} \geq h$ which implies $L(g\alpha^{-1}) \geq Lh$ and that therefore, since L is linear,

$$Lg \geq \alpha Lh.$$

Since $h \prec V_\alpha$ is arbitrary, and $K \subseteq V_\alpha$,

$$Lg \geq \alpha\mu(V_\alpha) \geq \alpha\mu(K).$$

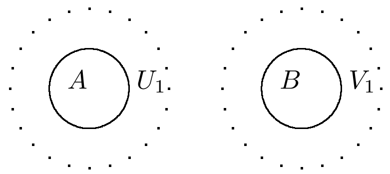
Letting $\alpha \uparrow 1$ yields $Lg \geq \mu(K)$. This proves the first part of the lemma. The second assertion follows from this and Theorem 8.15. If K is given, let

$$K \prec g \prec \Omega$$

and so from what was just shown, $\mu(K) \leq Lg < \infty$. This proves the lemma.

Lemma 8.25 *If A and B are disjoint compact subsets of Ω, then $\mu(A \cup B) = \mu(A) + \mu(B)$.*

Proof: By Theorem 8.15, there exists $h \in C_c(\Omega)$ such that $A \prec h \prec B^C$. Let $U_1 = h^{-1}((\frac{1}{2}, 1])$, $V_1 = h^{-1}([0, \frac{1}{2}))$. Then $A \subseteq U_1, B \subseteq V_1$ and $U_1 \cap V_1 = \emptyset$.



From Lemma 8.24 $\mu(A \cup B) < \infty$ and so there exists an open set, W such that

$$W \supseteq A \cup B,\ \mu(A \cup B) + \varepsilon > \mu(W).$$

Now let $U = U_1 \cap W$ and $V = V_1 \cap W$. Then

$$U \supseteq A,\ V \supseteq B,\ U \cap V = \emptyset, \text{and } \mu(A \cup B) + \varepsilon \geq \mu(W) \geq \mu(U \cup V).$$

Let $A \prec f \prec U,\ B \prec g \prec V$. Then by Lemma 8.24,

$$\mu(A \cup B) + \varepsilon \geq \mu(U \cup V) \geq L(f + g) = Lf + Lg \geq \mu(A) + \mu(B).$$

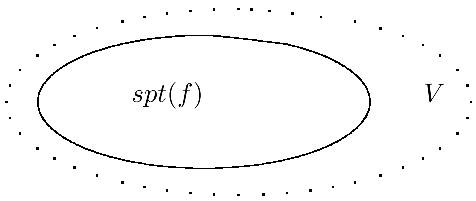
Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

From Lemma 8.24 the following lemma is obtained.

Lemma 8.26 *Let $f \in C_c(\Omega)$, $f(\Omega) \subseteq [0, 1]$. Then $\mu(\mathrm{spt}(f)) \geq Lf$. Also, every open set, V satisfies*

$$\mu(V) = \sup\{\mu(K) : K \subseteq V\}.$$

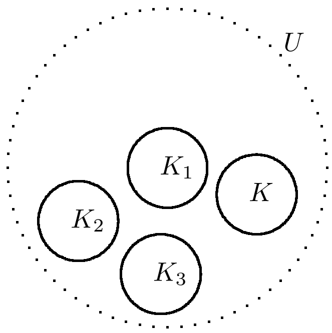
Proof: Let $V \supseteq \mathrm{spt}(f)$ and let $\mathrm{spt}(f) \prec g \prec V$. Then $Lf \leq Lg \leq \mu(V)$ because $f \leq g$. Since this holds for all $V \supseteq \mathrm{spt}(f)$, $Lf \leq \mu(\mathrm{spt}(f))$ by definition of μ.



Finally, let V be open and let $l < \mu(V)$. Then from the definition of μ, there exists $f \prec V$ such that $L(f) > l$. Therefore, $l < \mu(\mathrm{spt}(f)) \leq \mu(V)$ and so this shows the claim about inner regularity of the measure on an open set.

Lemma 8.27 *If K is compact there exists V open, $V \supseteq K$, such that $\mu(V \setminus K) \leq \varepsilon$. If V is open with $\mu(V) < \infty$, then there exists a compact set, $K \subseteq V$ with $\mu(V \setminus K) \leq \varepsilon$.*

Proof: Let K be compact. Then from the definition of μ, there exists an open set U, with $\mu(U) < \infty$ and $U \supseteq K$. Suppose for every open set, V, containing $K, \mu(V \setminus K) > \varepsilon$. Then there exists $f \prec U \setminus K$ with $Lf > \varepsilon$. Consequently, $\mu((f)) > Lf > \varepsilon$. Let $K_1 = \mathrm{spt}(f)$ and repeat the construction with $U \setminus K_1$ in place of U.



Continuing in this way yields a sequence of disjoint compact sets, $K, K_1, \cdots$ contained in U such that $\mu(K_i) > \varepsilon$. By Lemma 8.25

$$\mu(U) \geq \mu(K \cup \cup_{i=1}^{r} K_i) = \mu(K) + \sum_{i=1}^{r} \mu(K_i) \geq r\varepsilon$$

for all r, contradicting $\mu(U) < \infty$. This demonstrates the first part of the lemma.

To show the second part, employ a similar construction. Suppose $\mu(V \setminus K) > \varepsilon$ for all $K \subseteq V$. Then $\mu(V) > \varepsilon$ so there exists $f \prec V$ with $Lf > \varepsilon$. Let $K_1 = \mathrm{spt}(f)$ so $\mu(\mathrm{spt}(f)) > \varepsilon$. If $K_1 \cdots K_n$, disjoint, compact subsets of V have been chosen, there must exist $g \prec (V \setminus \cup_{i=1}^{n} K_i)$ be such that $Lg > \varepsilon$. Hence $\mu(\mathrm{spt}(g)) > \varepsilon$. Let $K_{n+1} = \mathrm{spt}(g)$. In this way there exists a sequence of disjoint compact subsets of V, $\{K_i\}$ with $\mu(K_i) > \varepsilon$. Thus for any m, $K_1 \cdots K_m$ are all contained in V and are disjoint and compact. By Lemma 8.25

$$\mu(V) \geq \mu(\cup_{i=1}^{m} K_i) = \sum_{i=1}^{m} \mu(K_i) > m\varepsilon$$

for all m, a contradiction to $\mu(V) < \infty$. This proves the second part.

Lemma 8.28 *Let $\mathcal{S}$ be the σ algebra of μ measurable sets in the sense of Caratheodory. Then $\mathcal{S} \supseteq$ Borel sets and μ is inner regular on every open set and for every $E \in \mathcal{S}$ with $\mu(E) < \infty$.*

Proof: Define

$$\mathcal{S}_1 = \{E \subseteq \Omega : E \cap K \in \mathcal{S}\}$$

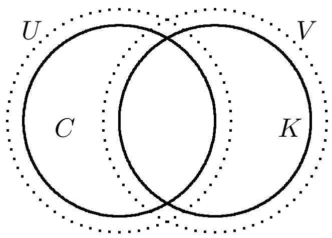
for all compact K.

Let C be a compact set. The idea is to show that $C \in \mathcal{S}$. From this it will follow that the closed sets are in $\mathcal{S}_1$ because if C is only closed, $C \cap K$ is compact. Hence $C \cap K = (C \cap K) \cap K \in \mathcal{S}$. The steps are to first show the compact sets are in $\mathcal{S}$ and this implies the closed sets are in $\mathcal{S}_1$. Then you show $\mathcal{S}_1$ is a σ algebra and so it contains the Borel sets. Finally, it is shown that $\mathcal{S}_1 = \mathcal{S}$ and then the inner regularity conclusion is established.

Let V be an open set with $\mu(V) < \infty$. I will show that

$$\mu(V) \geq \mu(V \setminus C) + \mu(V \cap C).$$

By Lemma 8.27, there exists an open set U containing C and a compact subset of V, K, such that $\mu(V \setminus K) < \varepsilon$ and $\mu(U \setminus C) < \varepsilon$.



Then by Lemma 8.25,

$$\begin{aligned}
\mu(V) &\geq \mu(K) \geq \mu((K \setminus U) \cup (K \cap C)) \\
&= \mu(K \setminus U) + \mu(K \cap C) \\
&\geq \mu(V \setminus C) + \mu(V \cap C) - 3\varepsilon
\end{aligned}$$

Since ε is arbitrary,

$$\mu(V) = \mu(V \setminus C) + \mu(V \cap C) \tag{8.13}$$

whenever C is compact and V is open. (If $\mu(V) = \infty$, it is obvious that $\mu(V) \geq \mu(V \setminus C) + \mu(V \cap C)$ and it is always the case that $\mu(V) \leq \mu(V \setminus C) + \mu(V \cap C)$.)

Of course 8.13 is exactly what needs to be shown for arbitrary S in place of V. It suffices to consider only S having $\mu(S) < \infty$. If $S \subseteq \Omega$, with $\mu(S) < \infty$, let $V \supseteq S$, $\mu(S) + \varepsilon > \mu(V)$. Then from what was just shown, if C is compact,

$$\begin{aligned}
\varepsilon + \mu(S) &> \mu(V) = \mu(V \setminus C) + \mu(V \cap C) \\
&\geq \mu(S \setminus C) + \mu(S \cap C).
\end{aligned}$$

Since ε is arbitrary, this shows the compact sets are in $\mathcal{S}$. As discussed above, this verifies the closed sets are in $\mathcal{S}_1$.

Therefore, $\mathcal{S}_1$ contains the closed sets and $\mathcal{S}$ contains the compact sets. Therefore, if $E \in \mathcal{S}$ and K is a compact set, it follows $K \cap E \in \mathcal{S}$ and so $\mathcal{S}_1 \supseteq \mathcal{S}$.

To see that $\mathcal{S}_1$ is closed with respect to taking complements, let $E \in \mathcal{S}_1$.

$$K = (E^C \cap K) \cup (E \cap K).$$

Then from the fact, just established, that the compact sets are in $\mathcal{S}$,

$$E^C \cap K = K \setminus (E \cap K) \in \mathcal{S}.$$

Similarly $\mathcal{S}_1$ is closed under countable unions. Thus $\mathcal{S}_1$ is a σ algebra which contains the Borel sets since it contains the closed sets.

The next task is to show $\mathcal{S}_1 = \mathcal{S}$. Let $E \in \mathcal{S}_1$ and let V be an open set with $\mu(V) < \infty$ and choose $K \subseteq V$ such that $\mu(V \setminus K) < \varepsilon$. Then since $E \in \mathcal{S}_1$, it follows $E \cap K \in \mathcal{S}$ and

$$\begin{aligned} \mu(V) &= \mu(V \setminus (K \cap E)) + \mu(V \cap (K \cap E)) \\ &\geq \mu(V \setminus E) + \mu(V \cap E) - \varepsilon \end{aligned}$$

because

$$\mu (V \cap (K \cap E)) + \overbrace{\mu (V \setminus K)}^{<\varepsilon} \geq \mu (V \cap E)$$

Since ε is arbitrary,

$$\mu(V) = \mu(V \setminus E) + \mu(V \cap E).$$

Now let $S \subseteq \Omega$. If $\mu(S) = \infty$, then $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$. If $\mu(S) < \infty$, let

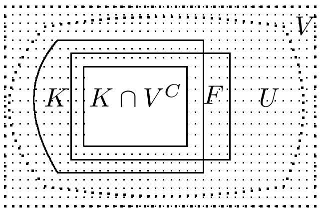
$$V \supseteq S, \ \mu(S) + \varepsilon \geq \mu(V).$$

Then

$$\mu(S) + \varepsilon \geq \mu(V) = \mu(V \setminus E) + \mu(V \cap E) \geq \mu(S \setminus E) + \mu(S \cap E).$$

Since ε is arbitrary, this shows that $E \in \mathcal{S}$ and so $\mathcal{S}_1 = \mathcal{S}$. Thus $\mathcal{S} \supseteq$ Borel sets as claimed.

From Lemma 8.26 and the definition of μ it follows μ is inner regular on all open sets. It remains to show that $\mu(F) = \sup\{\mu(K) : K \subseteq F\}$ for all $F \in \mathcal{S}$ with $\mu(F) < \infty$. It might help to refer to the following crude picture to keep things straight.



In this picture the shaded area is V.

Let U be an open set, $U \supseteq F$, $\mu(U) < \infty$. Let V be open, $V \supseteq U \setminus F$, and $\mu(V \setminus (U \setminus F)) < \varepsilon$. This can be obtained because μ is a measure on $\mathcal{S}$. Thus from outer regularity there exists $V \supseteq U \setminus F$ such that $\mu (U \setminus F) + \varepsilon > \mu (V)$. Then

$$\mu (V \setminus (U \setminus F)) + \mu (U \setminus F) = \mu (V)$$

and so

$$\mu\left(V\setminus\left(U\setminus F\right)\right)=\mu\left(V\right)-\mu\left(U\setminus F\right)<\varepsilon.$$

Also,

$$\begin{aligned} V\setminus\left(U\setminus F\right) &= V\cap\left(U\cap F^C\right)^C \\ &= V\cap\left[U^C\cup F\right] \\ &= \left(V\cap F\right)\cup\left(V\cap U^C\right) \\ &\supseteq V\cap F \end{aligned}$$

and so

$$\mu(V\cap F)\leq\mu\left(V\setminus\left(U\setminus F\right)\right)<\varepsilon.$$

Since $V\supseteq U\cap F^C$, $V^C\subseteq U^C\cup F$ so $U\cap V^C\subseteq U\cap F=F$. Hence $U\cap V^C$ is a subset of F. Now let $K\subseteq U$, $\mu(U\setminus K)<\varepsilon$. Thus $K\cap V^C$ is a compact subset of F and

$$\begin{aligned} \mu(F) &= \mu(V\cap F)+\mu(F\setminus V) \\ &< \varepsilon+\mu(F\setminus V)\leq\varepsilon+\mu(U\cap V^C)\leq 2\varepsilon+\mu(K\cap V^C). \end{aligned}$$

Since ε is arbitrary, this proves the second part of the lemma. Formula 8.11 of this theorem was established earlier.

It remains to show μ satisfies 8.12.

Lemma 8.29 *$\int f d\mu = Lf$ for all $f\in C_c(\Omega)$.*

Proof: Let $f\in C_c(\Omega)$, f real-valued, and suppose $f(\Omega)\subseteq[a,b]$. Choose $t_0<a$ and let $t_0<t_1<\cdots<t_n=b$, $t_i-t_{i-1}<\varepsilon$. Let

$$E_i=f^{-1}((t_{i-1},t_i])\cap\operatorname{spt}(f). \tag{8.14}$$

Note that $\cup_{i=1}^n E_i$ is a closed set and in fact

$$\cup_{i=1}^n E_i=\operatorname{spt}(f) \tag{8.15}$$

since $\Omega=\cup_{i=1}^n f^{-1}((t_{i-1},t_i])$. Let $V_i\supseteq E_i, V_i$ is open and let V_i satisfy

$$f\left(x\right)<t_i+\varepsilon \text{ for all } x\in V_i, \tag{8.16}$$

$$\mu(V_i\setminus E_i)<\varepsilon/n.$$

By Theorem 8.18 there exists $h_i\in C_c(\Omega)$ such that

$$h_i\prec V_i,\ \sum_{i=1}^n h_i(x)=1 \text{ on } \operatorname{spt}(f).$$

Now note that for each i,

$$f(x)h_i(x)\leq h_i(x)(t_i+\varepsilon).$$

(If $x \in V_i$, this follows from Formula 8.16. If $x \notin V_i$ both sides equal 0.) Therefore,

$$\begin{aligned} Lf &= L(\sum_{i=1}^{n} fh_i) \leq L(\sum_{i=1}^{n} h_i(t_i+\varepsilon)) \\ &= \sum_{i=1}^{n}(t_i+\varepsilon)L(h_i) \\ &= \sum_{i=1}^{n}(|t_0|+t_i+\varepsilon)L(h_i) - |t_0|L\left(\sum_{i=1}^{n} h_i\right). \end{aligned}$$

Now note that $|t_0|+t_i+\varepsilon \geq 0$ and so from the definition of μ and Lemma 8.24, this is no larger than

$$\sum_{i=1}^{n}(|t_0|+t_i+\varepsilon)\mu(V_i) - |t_0|\mu(\text{spt}(f))$$

$$\leq \sum_{i=1}^{n}(|t_0|+t_i+\varepsilon)(\mu(E_i)+\varepsilon/n) - |t_0|\mu(\text{spt}(f))$$

$$\leq |t_0|\sum_{i=1}^{n}\mu(E_i) + |t_0|\varepsilon + \sum_{i=1}^{n} t_i\mu(E_i) + \varepsilon(|t_0|+|b|)$$
$$+\varepsilon\sum_{i=1}^{n}\mu(E_i) + \varepsilon^2 - |t_0|\mu(\text{spt}(f)).$$

From 8.15 and 8.14, the first and last terms cancel. Therefore this is no larger than

$$(2|t_0|+|b|+\mu(\text{spt}(f))+\varepsilon)\varepsilon + \sum_{i=1}^{n} t_{i-1}\mu(E_i) + \varepsilon\mu(\text{spt}(f))$$

$$\leq \int f d\mu + (2|t_0|+|b|+2\mu(\text{spt}(f))+\varepsilon)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$Lf \leq \int f d\mu \tag{8.17}$$

for all $f \in C_c(\Omega)$, f real. Hence equality holds in 8.17 because $L(-f) \leq -\int f d\mu$ so $L(f) \geq \int f d\mu$. Thus $Lf = \int f d\mu$ for all $f \in C_c(\Omega)$. Just apply the result for real functions to the real and imaginary parts of f. This proves the Lemma.

This gives the existence part of the Riesz representation theorem.

It only remains to prove uniqueness. Suppose both μ_1 and μ_2 are measures on $\mathcal{S}$ satisfying the conclusions of the theorem. Then if K is compact and $V \supseteq K$, let $K \prec f \prec V$. Then

$$\mu_1(K) \leq \int f d\mu_1 = Lf = \int f d\mu_2 \leq \mu_2(V).$$

Thus $\mu_1(K) \leq \mu_2(K)$ for all K. Similarly, the inequality can be reversed and so it follows the two measures are equal on compact sets. By the assumption of inner regularity on open sets, the two measures are also equal on all open sets. By outer regularity, they are equal on all sets of $\mathcal{S}$. This proves the theorem.

An important example of a locally compact Hausdorff space is any metric space in which the closures of balls are compact. For example, $\mathbb{R}^n$ with the usual metric is an example of this. Not surprisingly, more can be said in this important special case.

Theorem 8.30 *Let (Ω, τ) be a metric space in which the closures of the balls are compact and let L be a positive linear functional defined on $C_c(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 8.15 and in addition the property that μ is regular. The same conclusion follows if (Ω, τ) is a compact Hausdorff space.*

Proof: Let μ and $\mathcal{S}$ be as described in Theorem 8.21. The outer regularity comes automatically as a conclusion of Theorem 8.21. It remains to verify inner regularity. Let $F \in \mathcal{S}$ and let $l < k < \mu(F)$. Now let $z \in \Omega$ and $\Omega_n = \overline{B(z,n)}$ for $n \in \mathbb{N}$. Thus $F \cap \Omega_n \uparrow F$. It follows that for n large enough,

$$k < \mu(F \cap \Omega_n) \leq \mu(F).$$

Since $\mu(F \cap \Omega_n) < \infty$ it follows there exists a compact set, K such that $K \subseteq F \cap \Omega_n \subseteq F$ and

$$l < \mu(K) \leq \mu(F).$$

This proves inner regularity. In case (Ω, τ) is a compact Hausdorff space, the conclusion of inner regularity follows from Theorem 8.21. This proves the theorem.

The proof of the above yields the following corollary.

Corollary 8.31 *Let (Ω, τ) be a locally compact Hausdorff space and suppose μ defined on a σ algebra, $\mathcal{S}$ represents the positive linear functional L where L is defined on $C_c(\Omega)$ in the sense of Theorem 8.15. Suppose also that there exist $\Omega_n \in \mathcal{S}$ such that $\Omega = \cup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty$. Then μ is regular.*

The following is on the uniqueness of the σ algebra in some cases.

Definition 8.32 *Let (Ω, τ) be a locally compact Hausdorff space and let L be a positive linear functional defined on $C_c(\Omega)$ such that the complete measure defined by the Riesz representation theorem for positive linear functionals is inner regular. Then this is called a Radon measure. Thus a Radon measure is complete, and regular.*

Corollary 8.33 *Let (Ω, τ) be a locally compact Hausdorff space which is also σ compact meaning*

$$\Omega = \cup_{n=1}^{\infty} \Omega_n, \ \Omega_n \text{ is compact,}$$

and let L be a positive linear functional defined on $C_c(\Omega)$. Then if $(\mu_1, \mathcal{S}_1)$, and $(\mu_2, \mathcal{S}_2)$ are two Radon measures, together with their σ algebras which represent L then the two σ algebras are equal and the two measures are equal.

Proof: Suppose $(\mu_1, \mathcal{S}_1)$ and $(\mu_2, \mathcal{S}_2)$ both work. It will be shown the two measures are equal on every compact set. Let K be compact and let V be an open set containing K. Then let $K \prec f \prec V$. Then

$$\mu_1(K) = \int_K d\mu_1 \leq \int f d\mu_1 = L(f) = \int f d\mu_2 \leq \mu_2(V).$$

Therefore, taking the infimum over all V containing K implies $\mu_1(K) \leq \mu_2(K)$. Reversing the argument shows $\mu_1(K) = \mu_2(K)$. This also implies the two measures are equal on all open sets because they are both inner regular on open sets. It is being assumed the two measures are regular. Now let $F \in \mathcal{S}_1$ with $\mu_1(F) < \infty$. Then there exist sets, H, G such that $H \subseteq F \subseteq G$ such that H is the countable union of compact sets and G is a countable intersection of open sets such that $\mu_1(G) = \mu_1(H)$ which implies $\mu_1(G \setminus H) = 0$. Now $G \setminus H$ can be written as the countable intersection of sets of the form $V_k \setminus K_k$ where V_k is open, $\mu_1(V_k) < \infty$ and K_k is compact. From what was just shown, $\mu_2(V_k \setminus K_k) = \mu_1(V_k \setminus K_k)$ so it follows $\mu_2(G \setminus H) = 0$ also. Since μ_2 is complete, and G and H are in $\mathcal{S}_2$, it follows $F \in \mathcal{S}_2$ and $\mu_2(F) = \mu_1(F)$. Now for arbitrary F possibly having $\mu_1(F) = \infty$, consider $F \cap \Omega_n$. From what was just shown, this set is in $\mathcal{S}_2$ and $\mu_2(F \cap \Omega_n) = \mu_1(F \cap \Omega_n)$. Taking the union of these $F \cap \Omega_n$ gives $F \in \mathcal{S}_2$ and also $\mu_1(F) = \mu_2(F)$. This shows $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Similarly, $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

The following lemma is often useful.

Lemma 8.34 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where Ω is a metric space having closed balls compact or more generally a topological space. Suppose μ is a Radon measure and f is measurable with respect to $\mathcal{F}$. Then there exists a Borel measurable function, g, such that $g = f$ a.e.*

Proof: Assume without loss of generality that $f \geq 0$. Then let $s_n \uparrow f$ pointwise. Say

$$s_n(\omega) = \sum_{k=1}^{P_n} c_k^n \mathcal{X}_{E_k^n}(\omega)$$

where $E_k^n \in \mathcal{F}$. By the outer regularity of μ, there exists a Borel set, $F_k^n \supseteq E_k^n$ such that $\mu(F_k^n) = \mu(E_k^n)$. In fact F_k^n can be assumed to be a G_δ set. Let

$$t_n(\omega) \equiv \sum_{k=1}^{P_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Then t_n is Borel measurable and $t_n(\omega) = s_n(\omega)$ for all $\omega \notin N_n$ where $N_n \in \mathcal{F}$ is a set of measure zero. Now let $N \equiv \cup_{n=1}^{\infty} N_n$. Then N is a set of measure zero and if $\omega \notin N$, then $t_n(\omega) \to f(\omega)$. Let $N' \supseteq N$ where N' is a Borel set and $\mu(N') = 0$. Then $t_n \mathcal{X}_{(N')^C}$ converges pointwise to a Borel measurable function, g, and $g(\omega) = f(\omega)$ for all $\omega \notin N'$. Therefore, $g = f$ a.e. and this proves the lemma.

8.5 One Dimensional Lebesgue Measure

To obtain one dimensional Lebesgue measure, you use the positive linear functional L given by

$$Lf = \int f(x)\,dx$$

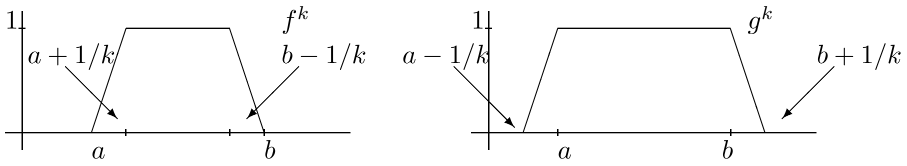
whenever $f \in C_c(\mathbb{R})$. Lebesgue measure, denoted by m is the measure obtained from the Riesz representation theorem such that

$$\int f dm = Lf = \int f(x)\,dx.$$

From this it is easy to verify that

$$m([a,b]) = m((a,b)) = b - a. \tag{8.18}$$

This will be done in general a little later but for now, consider the following picture of functions, f^k and g^k converging pointwise as $k \to \infty$ to $\mathcal{X}_{[a,b]}$.



Then the following estimate follows.

$$\begin{aligned}\left(b - a - \frac{2}{k}\right) &\leq \int f^k dx = \int f^k dm \leq m((a,b)) \leq m([a,b]) \\ &= \int \mathcal{X}_{[a,b]} dm \leq \int g^k dm = \int g^k dx \leq \left(b - a + \frac{2}{k}\right).\end{aligned}$$

From this the claim in 8.18 follows.

8.6 The Distribution Function

There is an interesting connection between the Lebesgue integral of a nonnegative function with something called the distribution function.

Definition 8.35 *Let $f \geq 0$ and suppose f is measurable. The distribution function is the function defined by*

$$t \to \mu([t < f]).$$

Lemma 8.36 *If $\{f_n\}$ is an increasing sequence of functions converging pointwise to f then*

$$\mu([f>t]) = \lim_{n\to\infty} \mu([f_n>t])$$

Proof: The sets, $[f_n>t]$ are increasing and their union is $[f>t]$ because if $f(\omega)>t$, then for all n large enough, $f_n(\omega)>t$ also. Therefore, from Theorem 7.5 on Page 126 the desired conclusion follows.

Lemma 8.37 *Suppose $s\geq 0$ is a measurable simple function,*

$$s(\omega) \equiv \sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(\omega)$$

where the a_k are the distinct nonzero values of $s, a_1<a_2<\cdots<a_n$. Suppose ϕ is a C^1 function defined on $[0,\infty)$ which has the property that $\phi(0)=0, \phi'(t)>0$ for all t. Then

$$\int_0^\infty \phi'(t)\,\mu([s>t])\,dm = \int \phi(s)\,d\mu.$$

Proof: First note that if $\mu(E_k)=\infty$ for any k then both sides equal ∞ and so without loss of generality, assume $\mu(E_k)<\infty$ for all k. Letting $a_0\equiv 0$, the left side equals

$$\begin{aligned}
\sum_{k=1}^{n}\int_{a_{k-1}}^{a_k} \phi'(t)\,\mu([s>t])\,dm &= \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k} \phi'(t)\sum_{i=k}^{n}\mu(E_i)\,dm \\
&= \sum_{k=1}^{n}\sum_{i=k}^{n}\mu(E_i)\int_{a_{k-1}}^{a_k}\phi'(t)\,dm \\
&= \sum_{k=1}^{n}\sum_{i=k}^{n}\mu(E_i)\left(\phi(a_k)-\phi(a_{k-1})\right) \\
&= \sum_{i=1}^{n}\mu(E_i)\sum_{k=1}^{i}\left(\phi(a_k)-\phi(a_{k-1})\right) \\
&= \sum_{i=1}^{n}\mu(E_i)\,\phi(a_i) = \int \phi(s)\,d\mu.
\end{aligned}$$

This proves the lemma.

With this lemma the next theorem which is the main result follows easily.

Theorem 8.38 *Let $f\geq 0$ be measurable and let ϕ be a C^1 function defined on $[0,\infty)$ which satisfies $\phi'(t)>0$ for all $t>0$ and $\phi(0)=0$. Then*

$$\int \phi(f)\,d\mu = \int_0^\infty \phi'(t)\,\mu([f>t])\,dt.$$

Proof: By Theorem 7.24 on Page 139 there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ which converges pointwise to f. By the monotone convergence theorem and Lemma 8.36,

$$\begin{aligned}\int \phi(f)\,d\mu &= \lim_{n\to\infty}\int \phi(s_n)\,d\mu = \lim_{n\to\infty}\int_0^\infty \phi'(t)\,\mu([s_n>t])\,dm \\ &= \int_0^\infty \phi'(t)\,\mu([f>t])\,dm\end{aligned}$$

This proves the theorem.

8.7 Product Measures

Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be two complete measure spaces. In this section consider the problem of defining a product measure, $\overline{\mu\times\nu}$ which is defined on a σ algebra of sets of $X\times Y$ such that $(\overline{\mu\times\nu})(E\times F)=\mu(E)\,\nu(F)$ whenever $E\in\mathcal{S}$ and $F\in\mathcal{T}$. I found the following approach to product measures in [17] and they say they got it from [19].

Definition 8.39 *Let $\mathcal{R}$ denote the set of countable unions of sets of the form $A\times B$, where $A\in\mathcal{S}$ and $B\in\mathcal{T}$ (Sets of the form $A\times B$ are referred to as measurable rectangles) and also let*

$$\rho(A\times B)=\mu(A)\,\nu(B) \tag{8.19}$$

More generally, define

$$\rho(E)\equiv\int\int \mathcal{X}_E(x,y)\,d\mu d\nu \tag{8.20}$$

whenever E is such that

$$x\to\mathcal{X}_E(x,y)\ \text{ is } \mu \text{ measurable for all } y \tag{8.21}$$

and

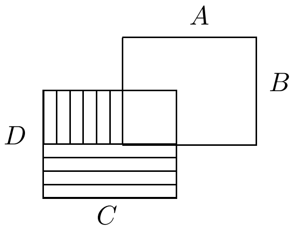
$$y\to\int \mathcal{X}_E(x,y)\,d\mu \text{ is } \nu \text{ measurable.} \tag{8.22}$$

Note that if $E=A\times B$ as above, then

$$\begin{aligned}\int\int \mathcal{X}_E(x,y)\,d\mu d\nu &= \int\int \mathcal{X}_{A\times B}(x,y)\,d\mu d\nu \\ &= \int\int \mathcal{X}_A(x)\,\mathcal{X}_B(y)\,d\mu d\nu = \mu(A)\,\nu(B)=\rho(E)\end{aligned}$$

and so there is no contradiction between 8.20 and 8.19.

The first goal is to show that for $Q\in\mathcal{R}$, 8.21 and 8.22 both hold. That is, $x\to\mathcal{X}_Q(x,y)$ is μ measurable for all y and $y\to\int\mathcal{X}_Q(x,y)\,d\mu$ is ν measurable. This is done so that it is possible to speak of $\rho(Q)$. The following lemma will be the fundamental result which will make this possible. First here is a picture.



Lemma 8.40 *Given $C \times D$ and $\{A_i \times B_i\}_{i=1}^n$, there exist finitely many disjoint rectangles, $\{C_i' \times D_i'\}_{i=1}^p$ such that none of these sets intersect any of the $A_i \times B_i$, each set is contained in $C \times D$ and*

$$(\cup_{i=1}^n A_i \times B_i) \cup (\cup_{k=1}^p C_k' \times D_k') = (C \times D) \cup (\cup_{i=1}^n A_i \times B_i).$$

Proof: From the above picture, you see that

$$(C \times D) \setminus (A_1 \times B_1) = C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)$$

and these last two sets are disjoint, have empty intersection with $A_1 \times B_1$, and

$$(C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)) \cup (\cup_{i=1}^n A_i \times B_i) = (C \times D) \cup (\cup_{i=1}^n A_i \times B_i)$$

Now suppose disjoint sets, $\left\{\widetilde{C}_i \times \widetilde{D}_i\right\}_{i=1}^m$ have been obtained, each being a subset of $C \times D$ such that

$$(\cup_{i=1}^n A_i \times B_i) \cup \left(\cup_{k=1}^m \widetilde{C}_k \times \widetilde{D}_k\right) = (\cup_{i=1}^n A_i \times B_i) \cup (C \times D)$$

and for all k, $\widetilde{C}_k \times \widetilde{D}_k$ has empty intersection with each set of $\{A_i \times B_i\}_{i=1}^p$. Then using the same procedure, replace each of $\widetilde{C}_k \times \widetilde{D}_k$ with finitely many disjoint rectangles such that none of these intersect $A_{p+1} \times B_{p+1}$ while preserving the union of all the sets involved. The process stops when you have gotten to n. This proves the lemma.

Lemma 8.41 *If $Q = \cup_{i=1}^\infty A_i \times B_i \in \mathcal{R}$, then there exist disjoint sets, of the form $A_i' \times B_i'$ such that $Q = \cup_{i=1}^\infty A_i' \times B_i'$, each $A_i' \times B_i'$ is a subset of some $A_i \times B_i$, and $A_i' \in \mathcal{S}$ while $B_i' \in \mathcal{T}$. Also, the intersection of finitely many sets of $\mathcal{R}$ is a set of $\mathcal{R}$. For ρ defined in 8.20, it follows that 8.21 and 8.22 hold for any element of $\mathcal{R}$. Furthermore,*

$$\rho(Q) = \sum_i \mu(A_i') \nu(B_i') = \sum_i \rho(A_i' \times B_i').$$

Proof: Let Q be given as above. Let $A_1' \times B_1' = A_1 \times B_1$. By Lemma 8.40, it is possible to replace $A_2 \times B_2$ with finitely many disjoint rectangles, $\{A_i' \times B_i'\}_{i=2}^{m_2}$ such that none of these rectangles intersect $A_1' \times B_1'$, each is a subset of $A_2 \times B_2$, and

$$\cup_{i=1}^\infty A_i \times B_i = (\cup_{i=1}^{m_2} A_i' \times B_i') \cup (\cup_{k=3}^\infty A_k \times B_k)$$

Now suppose disjoint rectangles, $\{A_i' \times B_i'\}_{i=1}^{m_p}$ have been obtained such that each rectangle is a subset of $A_k \times B_k$ for some $k \leq p$ and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_p} A_i' \times B_i'\right) \cup \left(\cup_{k=p+1}^{\infty} A_k \times B_k\right).$$

By Lemma 8.40 again, there exist disjoint rectangles $\{A_i' \times B_i'\}_{i=m_p+1}^{m_{p+1}}$ such that each is contained in $A_{p+1} \times B_{p+1}$, none have intersection with any of $\{A_i' \times B_i'\}_{i=1}^{m_p}$ and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_{p+1}} A_i' \times B_i'\right) \cup \left(\cup_{k=p+2}^{\infty} A_k \times B_k\right).$$

Note that no change is made in $\{A_i' \times B_i'\}_{i=1}^{m_p}$. Continuing this way proves the existence of the desired sequence of disjoint rectangles, each of which is a subset of at least one of the original rectangles and such that

$$Q = \cup_{i=1}^{\infty} A_i' \times B_i'.$$

It remains to verify $x \to \mathcal{X}_Q(x,y)$ is μ measurable for all y and $y \to \int \mathcal{X}_Q(x,y)\,d\mu$ is ν measurable whenever $Q \in \mathcal{R}$. Let $Q \equiv \cup_{i=1}^{\infty} A_i \times B_i \in \mathcal{R}$. Then by the first part of this lemma, there exists $\{A_i' \times B_i'\}_{i=1}^{\infty}$ such that the sets are disjoint and $\cup_{i=1}^{\infty} A_i' \times B_i' = Q$. Therefore, since the sets are disjoint,

$$\mathcal{X}_Q(x,y) = \sum_{i=1}^{\infty} \mathcal{X}_{A_i' \times B_i'}(x,y) = \sum_{i=1}^{\infty} \mathcal{X}_{A_i'}(x)\,\mathcal{X}_{B_i'}(y).$$

It follows $x \to \mathcal{X}_Q(x,y)$ is measurable. Now by the monotone convergence theorem,

$$\begin{aligned}
\int \mathcal{X}_Q(x,y)\,d\mu &= \int \sum_{i=1}^{\infty} \mathcal{X}_{A_i'}(x)\,\mathcal{X}_{B_i'}(y)\,d\mu \\
&= \sum_{i=1}^{\infty} \mathcal{X}_{B_i'}(y) \int \mathcal{X}_{A_i'}(x)\,d\mu \\
&= \sum_{i=1}^{\infty} \mathcal{X}_{B_i'}(y)\,\mu(A_i').
\end{aligned}$$

It follows $y \to \int \mathcal{X}_Q(x,y)\,d\mu$ is measurable and so by the monotone convergence theorem again,

$$\begin{aligned}
\int\int \mathcal{X}_Q(x,y)\,d\mu d\nu &= \int \sum_{i=1}^{\infty} \mathcal{X}_{B_i'}(y)\,\mu(A_i')\,d\nu \\
&= \sum_{i=1}^{\infty} \int \mathcal{X}_{B_i'}(y)\,\mu(A_i')\,d\nu \\
&= \sum_{i=1}^{\infty} \nu(B_i')\,\mu(A_i'). \qquad (8.23)
\end{aligned}$$

This shows the measurability conditions, 8.21 and 8.22 hold for $Q \in \mathcal{R}$ and also establishes the formula for $\rho(Q)$, 8.23.

If $\cup_i A_i \times B_i$ and $\cup_j C_j \times D_j$ are two sets of $\mathcal{R}$, then their intersection is

$$\cup_i \cup_j (A_i \cap C_j) \times (B_i \cap D_j)$$

a countable union of measurable rectangles. Thus finite intersections of sets of $\mathcal{R}$ are in $\mathcal{R}$. This proves the lemma.

Now note that from the definition of $\mathcal{R}$ if you have a sequence of elements of $\mathcal{R}$ then their union is also in $\mathcal{R}$. The next lemma will enable the definition of an outer measure.

Lemma 8.42 *Suppose $\{R_i\}_{i=1}^{\infty}$ is a sequence of sets of $\mathcal{R}$ then*

$$\rho\left(\cup_{i=1}^{\infty} R_i\right) \leq \sum_{i=1}^{\infty} \rho(R_i).$$

Proof: Let $R_i = \cup_{j=1}^{\infty} A_j^i \times B_j^i$. Using Lemma 8.41, let $\{A'_m \times B'_m\}_{m=1}^{\infty}$ be a sequence of disjoint rectangles each of which is contained in some $A_j^i \times B_j^i$ for some i, j such that

$$\cup_{i=1}^{\infty} R_i = \cup_{m=1}^{\infty} A'_m \times B'_m.$$

Now define

$$S_i \equiv \left\{m : A'_m \times B'_m \subseteq A_j^i \times B_j^i \text{ for some } j\right\}.$$

It is not important to consider whether some m might be in more than one S_i. The important thing to notice is that

$$\cup_{m \in S_i} A'_m \times B'_m \subseteq \cup_{j=1}^{\infty} A_j^i \times B_j^i = R_i.$$

Then by Lemma 8.41,

$$\begin{aligned}
\rho\left(\cup_{i=1}^{\infty} R_i\right) &= \sum_m \rho(A'_m \times B'_m) \\
&\leq \sum_{i=1}^{\infty} \sum_{m \in S_i} \rho(A'_m \times B'_m) \\
&\leq \sum_{i=1}^{\infty} \rho\left(\cup_{m \in S_i} A'_m \times B'_m\right) \leq \sum_{i=1}^{\infty} \rho(R_i).
\end{aligned}$$

This proves the lemma.

So far, there is no measure and no σ algebra. However, the next step is to define an outer measure which will lead to a measure on a σ algebra of measurable sets from the Caratheodory procedure. When this is done, it will be shown that this measure can be computed using ρ which implies the important Fubini theorem.

Now it is possible to define an outer measure.

Definition 8.43 *For $S \subseteq X \times Y$, define*

$$\left(\overline{\mu \times \nu}\right)(S) \equiv \inf \{\rho(R) : S \subseteq R,\ R \in \mathcal{R}\}. \tag{8.24}$$

The following proposition is interesting but is not needed in the development which follows. It gives a different description of $\left(\overline{\mu \times \nu}\right)$.

Proposition 8.44 $\left(\overline{\mu \times \nu}\right)(S) = \inf \{\sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i) : S \subseteq \cup_{i=1}^{\infty} A_i \times B_i\}$

Proof: Let $\lambda(S) \equiv \inf \{\sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i) : S \subseteq \cup_{i=1}^{\infty} A_i \times B_i\}$. Suppose $S \subseteq \cup_{i=1}^{\infty} A_i \times B_i \equiv Q \in \mathcal{R}$. Then by Lemma 8.41, $Q = \cup_i A_i' \times B_i'$ where these rectangles are disjoint. Thus by this lemma, $\rho(Q) = \sum_{i=1}^{\infty} \mu(A_i')\,\nu(B_i') \geq \lambda(S)$ and so $\lambda(S) \leq \left(\overline{\mu \times \nu}\right)(S)$. If $\lambda(S) = \infty$, this shows $\lambda(S) = \left(\overline{\mu \times \nu}\right)(S)$. Suppose then that $\lambda(S) < \infty$ and $\lambda(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i)$ where $Q = \cup_{i=1}^{\infty} A_i \times B_i \supseteq S$. Then by Lemma 8.41 again, $\cup_{i=1}^{\infty} A_i \times B_i = \cup_{i=1}^{\infty} A_i' \times B_i'$ where the primed rectangles are disjoint, each is a subset of some $A_i \times B_i$ and so

$$\lambda(S) + \varepsilon \geq \sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i) \geq \sum_{i=1}^{\infty} \mu(A_i')\,\nu(B_i') = \rho(Q) \geq \left(\overline{\mu \times \nu}\right)(S).$$

Since ε is arbitrary, this shows $\lambda(S) \geq \left(\overline{\mu \times \nu}\right)(S)$ and this proves the proposition.

Lemma 8.45 $\overline{\mu \times \nu}$ *is an outer measure on $X \times Y$ and for $R \in \mathcal{R}$*

$$\left(\overline{\mu \times \nu}\right)(R) = \rho(R). \tag{8.25}$$

Proof: First consider 8.25. Since $R \supseteq R$, it follows $\rho(R) \geq \left(\overline{\mu \times \nu}\right)(R)$. On the other hand, if $Q \in \mathcal{R}$ and $Q \supseteq R$, then $\rho(Q) \geq \rho(R)$ and so, taking the infimum on the left yields $\left(\overline{\mu \times \nu}\right)(R) \geq \rho(R)$. This shows 8.25.

It is necessary to show that if $S \subseteq T$, then

$$\left(\overline{\mu \times \nu}\right)(S) \leq \left(\overline{\mu \times \nu}\right)(T), \tag{8.26}$$

$$\left(\overline{\mu \times \nu}\right)(\cup_{i=1}^{\infty} S_i) \leq \sum_{i=1}^{\infty} \left(\overline{\mu \times \nu}\right)(S_i). \tag{8.27}$$

To do this, note that 8.26 is obvious. To verify 8.27, note that it is obvious if $\left(\overline{\mu \times \nu}\right)(S_i) = \infty$ for any i. Therefore, assume $\left(\overline{\mu \times \nu}\right)(S_i) < \infty$. Then letting $\varepsilon > 0$ be given, there exist $R_i \in \mathcal{R}$ such that

$$\left(\overline{\mu \times \nu}\right)(S_i) + \frac{\varepsilon}{2^i} > \rho(R_i),\ R_i \supseteq S_i.$$

Then by Lemma 8.42, 8.25, and the observation that $\cup_{i=1}^{\infty} R_i \in \mathcal{R}$,

$$\begin{aligned}
\left(\overline{\mu \times \nu}\right)\left(\cup_{i=1}^{\infty} S_i\right) &\leq \left(\overline{\mu \times \nu}\right)\left(\cup_{i=1}^{\infty} R_i\right) \\
&= \rho\left(\cup_{i=1}^{\infty} R_i\right) \leq \sum_{i=1}^{\infty} \rho\left(R_i\right) \\
&\leq \sum_{i=1}^{\infty} \left(\left(\overline{\mu \times \nu}\right)\left(S_i\right) + \frac{\varepsilon}{2^i}\right) \\
&= \left(\sum_{i=1}^{\infty} \left(\overline{\mu \times \nu}\right)\left(S_i\right)\right) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves the lemma.

By Caratheodory's procedure, it follows there is a σ algebra of subsets of $X \times Y$, denoted here by $\overline{\mathcal{S} \times \mathcal{T}}$ such that $\left(\overline{\mu \times \nu}\right)$ is a complete measure on this σ algebra. The first thing to note is that every rectangle is in this σ algebra.

Lemma 8.46 *Every rectangle is* $\left(\overline{\mu \times \nu}\right)$ *measurable.*

Proof: Let $S \subseteq X \times Y$. The following inequality must be established.

$$\left(\overline{\mu \times \nu}\right)(S) \geq \left(\overline{\mu \times \nu}\right)\left(S \cap (A \times B)\right) + \left(\overline{\mu \times \nu}\right)\left(S \setminus (A \times B)\right). \tag{8.28}$$

The following claim will be used to establish this inequality.

Claim: Let $P, A \times B \in \mathcal{R}$. Then

$$\rho\left(P \cap (A \times B)\right) + \rho\left(P \setminus (A \times B)\right) = \rho(P).$$

Proof of the claim: From Lemma 8.41, $P = \cup_{i=1}^{\infty} A_i' \times B_i'$ where the $A_i' \times B_i'$ are disjoint. Therefore,

$$P \cap (A \times B) = \bigcup_{i=1}^{\infty} \left(A \cap A_i'\right) \times \left(B \cap B_i'\right)$$

while

$$P \setminus (A \times B) = \bigcup_{i=1}^{\infty} \left(A_i' \setminus A\right) \times B_i' \cup \bigcup_{i=1}^{\infty} \left(A \cap A_i'\right) \times \left(B_i' \setminus B\right).$$

Since all of the sets in the above unions are disjoint,

$$\rho\left(P \cap (A \times B)\right) + \rho\left(P \setminus (A \times B)\right) =$$

$$\int\int \sum_{i=1}^{\infty} \mathcal{X}_{\left(A \cap A_i'\right)}(x)\, \mathcal{X}_{B \cap B_i'}(y)\, d\mu d\nu + \int\int \sum_{i=1}^{\infty} \mathcal{X}_{\left(A_i' \setminus A\right)}(x)\, \mathcal{X}_{B_i'}(y)\, d\mu d\nu$$

$$+ \int\int \sum_{i=1}^{\infty} \mathcal{X}_{A \cap A_i'}(x)\, \mathcal{X}_{B_i' \setminus B}(y)\, d\mu d\nu$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} \mu\left(A \cap A_i'\right) \nu\left(B \cap B_i'\right) + \mu\left(A_i' \setminus A\right) \nu\left(B_i'\right) + \mu\left(A \cap A_i'\right) \nu\left(B_i' \setminus B\right) \\
&= \sum_{i=1}^{\infty} \mu\left(A \cap A_i'\right) \nu\left(B_i'\right) + \mu\left(A_i' \setminus A\right) \nu\left(B_i'\right) = \sum_{i=1}^{\infty} \mu\left(A_i'\right) \nu\left(B_i'\right) = \rho\left(P\right).
\end{aligned}$$

This proves the claim.

Now continuing to verify 8.28, without loss of generality, $\left(\overline{\mu \times \nu}\right)(S)$ can be assumed finite. Let $P \supseteq S$ for $P \in \mathcal{R}$ and

$$\left(\overline{\mu \times \nu}\right)(S) + \varepsilon > \rho\left(P\right).$$

Then from the claim,

$$\begin{aligned}
\left(\overline{\mu \times \nu}\right)(S) + \varepsilon &> \rho\left(P\right) = \rho\left(P \cap (A \times B)\right) + \rho\left(P \setminus (A \times B)\right) \\
&\geq \left(\overline{\mu \times \nu}\right)\left(S \cap (A \times B)\right) + \left(\overline{\mu \times \nu}\right)\left(S \setminus (A \times B)\right).
\end{aligned}$$

Since $\varepsilon > 0$ this shows $A \times B$ is $\overline{\mu \times \nu}$ measurable as claimed.

Lemma 8.47 *Let $\mathcal{R}_1$ be defined as the set of all countable intersections of sets of $\mathcal{R}$. Then if $S \subseteq X \times Y$, there exists $R \in \mathcal{R}_1$ for which it makes sense to write $\rho(R)$ because 8.21 and 8.22 hold such that*

$$\left(\overline{\mu \times \nu}\right)(S) = \rho\left(R\right). \tag{8.29}$$

Also, every element of $\mathcal{R}_1$ is $\overline{\mu \times \nu}$ measurable.

Proof: Consider 8.29. Let $S \subseteq X \times Y$. If $\left(\overline{\mu \times \nu}\right)(S) = \infty$, let $R = X \times Y$ and it follows $\rho\left(X \times Y\right) = \infty = \left(\overline{\mu \times \nu}\right)(S)$. Assume then that $\left(\overline{\mu \times \nu}\right)(S) < \infty$.

Therefore, there exists $P_n \in \mathcal{R}$ such that $P_n \supseteq S$ and

$$\left(\overline{\mu \times \nu}\right)(S) \leq \rho\left(P_n\right) < \left(\overline{\mu \times \nu}\right)(S) + 1/n. \tag{8.30}$$

Let $Q_n = \cap_{i=1}^{n} P_i \in \mathcal{R}$. Define

$$P \equiv \cap_{i=1}^{\infty} Q_i \supseteq S.$$

Then 8.30 holds with Q_n in place of P_n. It is clear that

$$x \to \mathcal{X}_P\left(x, y\right) \text{ is } \mu \text{ measurable}$$

because this function is the pointwise limit of functions for which this is so. It remains to consider whether $y \to \int \mathcal{X}_P\left(x, y\right) d\mu$ is ν measurable. First observe $Q_n \supseteq Q_{n+1}$, $\mathcal{X}_{Q_i} \leq \mathcal{X}_{P_i}$, and

$$\rho\left(Q_1\right) = \rho\left(P_1\right) = \int \int \mathcal{X}_{P_1}\left(x, y\right) d\mu d\nu < \infty. \tag{8.31}$$

Therefore, there exists a set of ν measure 0, N, such that if $y \notin N$, then

$$\int \mathcal{X}_{P_1}(x,y)\,d\mu < \infty.$$

It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \mathcal{X}_{N^C}(y) \int \mathcal{X}_{Q_n}(x,y)\,d\mu = \mathcal{X}_{N^C}(y) \int \mathcal{X}_P(x,y)\,d\mu$$

and so

$$y \to \mathcal{X}_{N^C}(y) \int \mathcal{X}_P(x,y)\,d\mu$$

is also measurable. By completeness of ν,

$$y \to \int \mathcal{X}_P(x,y)\,d\mu$$

must also be ν measurable and so it makes sense to write

$$\int\int \mathcal{X}_P(x,y)\,d\mu d\nu$$

for every $P \in \mathcal{R}_1$. Also, by the dominated convergence theorem,

$$\begin{aligned}
\int\int \mathcal{X}_P(x,y)\,d\mu d\nu &= \int \mathcal{X}_{N^C}(y) \int \mathcal{X}_P(x,y)\,d\mu d\nu \\
&= \lim_{n\to\infty} \int \mathcal{X}_{N^C}(y) \int \mathcal{X}_{Q_n}(x,y)\,d\mu d\nu \\
&= \lim_{n\to\infty} \int\int \mathcal{X}_{Q_n}(x,y)\,d\mu d\nu \\
&= \lim_{n\to\infty} \rho(Q_n) \in [(\overline{\mu\times\nu})(S), (\overline{\mu\times\nu})(S) + 1/n]
\end{aligned}$$

for all n. Therefore,

$$\rho(P) \equiv \int\int \mathcal{X}_P(x,y)\,d\mu d\nu = (\overline{\mu\times\nu})(S).$$

The sets of $\mathcal{R}_1$ are $\overline{\mu\times\nu}$ measurable because these sets are countable intersections of countable unions of rectangles and Lemma 8.46 verifies the rectangles are $\overline{\mu\times\nu}$ measurable. This proves the Lemma.

The following theorem is the main result.

Theorem 8.48 *Let $E \subseteq X \times Y$ be $\overline{\mu\times\nu}$ measurable and suppose $(\overline{\mu\times\nu})(E) < \infty$. Then*

$$x \to \mathcal{X}_E(x,y) \text{ is } \mu \text{ measurable a.e. } y.$$

Modifying $\mathcal{X}_E$ on a set of measure zero, it is possible to write

$$\int \mathcal{X}_E(x,y)\,d\mu.$$

The function,

$$y \to \int \mathcal{X}_E(x,y)\, d\mu$$

is ν measurable and

$$\left(\overline{\mu \times \nu}\right)(E) = \int \int \mathcal{X}_E(x,y)\, d\mu d\nu.$$

Similarly,

$$\left(\overline{\mu \times \nu}\right)(E) = \int \int \mathcal{X}_E(x,y)\, d\nu d\mu.$$

Proof: By Lemma 8.47, there exists $R \in \mathcal{R}_1$ such that

$$\rho(R) = \left(\overline{\mu \times \nu}\right)(E),\ R \supseteq E.$$

Therefore, since R is $\overline{\mu \times \nu}$ measurable and $\rho(R) = \left(\overline{\mu \times \nu}\right)(R)$, it follows

$$\left(\overline{\mu \times \nu}\right)(R \setminus E) = 0.$$

By Lemma 8.47 again, there exists $P \supseteq R \setminus E$ with $P \in \mathcal{R}_1$ and

$$\rho(P) = \left(\overline{\mu \times \nu}\right)(R \setminus E) = 0.$$

Thus

$$\int \int \mathcal{X}_P(x,y)\, d\mu d\nu = 0. \tag{8.32}$$

Since $P \in \mathcal{R}_1$ Lemma 8.47 implies $x \to \mathcal{X}_P(x,y)$ is μ measurable and it follows from the above there exists a set of ν measure zero, N such that if $y \notin N$, then $\int \mathcal{X}_P(x,y)\, d\mu = 0$. Therefore, by completeness of ν,

$$x \to \mathcal{X}_{N^C}(y)\, \mathcal{X}_{R \setminus E}(x,y)$$

is μ measurable and

$$\int \mathcal{X}_{N^C}(y)\, \mathcal{X}_{R \setminus E}(x,y)\, d\mu = 0. \tag{8.33}$$

Now also

$$\mathcal{X}_{N^C}(y)\, \mathcal{X}_R(x,y) = \mathcal{X}_{N^C}(y)\, \mathcal{X}_{R \setminus E}(x,y) + \mathcal{X}_{N^C}(y)\, \mathcal{X}_E(x,y) \tag{8.34}$$

and this shows that

$$x \to \mathcal{X}_{N^C}(y)\, \mathcal{X}_E(x,y)$$

is μ measurable because it is the difference of two functions with this property. Then by 8.33 it follows

$$\int \mathcal{X}_{N^C}(y)\, \mathcal{X}_E(x,y)\, d\mu = \int \mathcal{X}_{N^C}(y)\, \mathcal{X}_R(x,y)\, d\mu.$$

The right side of this equation equals a ν measurable function and so the left side which equals it is also a ν measurable function. It follows from completeness of ν that $y \to \int \mathcal{X}_E(x, y)\, d\mu$ is ν measurable because for y outside of a set of ν measure zero, N it equals $\int \mathcal{X}_R(x, y)\, d\mu$. Therefore,

$$\begin{aligned}
\int\int \mathcal{X}_E(x, y)\, d\mu d\nu &= \int\int \mathcal{X}_{N^C}(y)\, \mathcal{X}_E(x, y)\, d\mu d\nu \\
&= \int\int \mathcal{X}_{N^C}(y)\, \mathcal{X}_R(x, y)\, d\mu d\nu \\
&= \int\int \mathcal{X}_R(x, y)\, d\mu d\nu \\
&= \rho(R) = \left(\overline{\mu \times \nu}\right)(E).
\end{aligned}$$

In all the above there would be no change in writing $d\nu d\mu$ instead of $d\mu d\nu$. The same result would be obtained. This proves the theorem.

Now let $f : X \times Y \to [0, \infty]$ be $\overline{\mu \times \nu}$ measurable and

$$\int f d\left(\overline{\mu \times \nu}\right) < \infty. \tag{8.35}$$

Let $s(x, y) \equiv \sum_{i=1}^{m} c_i \mathcal{X}_{E_i}(x, y)$ be a nonnegative simple function with c_i being the nonzero values of s and suppose

$$0 \leq s \leq f.$$

Then from the above theorem,

$$\int s d\left(\overline{\mu \times \nu}\right) = \int\int s d\mu d\nu$$

In which

$$\int s d\mu = \int \mathcal{X}_{N^C}(y)\, s d\mu$$

for N a set of ν measure zero such that $y \to \int \mathcal{X}_{N^C}(y)\, s d\mu$ is ν measurable. This follows because 8.35 implies $\left(\overline{\mu \times \nu}\right)(E_i) < \infty$. Now let $s_n \uparrow f$ where s_n is a nonnegative simple function and

$$\int s_n d\left(\overline{\mu \times \nu}\right) = \int\int \mathcal{X}_{N_n^C}(y)\, s_n(x, y)\, d\mu d\nu$$

where

$$y \to \int \mathcal{X}_{N_n^C}(y)\, s_n(x, y)\, d\mu$$

is ν measurable. Then let $N \equiv \cup_{n=1}^{\infty} N_n$. It follows N is a set of ν measure zero. Thus

$$\int s_n d\left(\overline{\mu \times \nu}\right) = \int\int \mathcal{X}_{N^C}(y)\, s_n(x, y)\, d\mu d\nu$$

and letting $n \to \infty$, the monotone convergence theorem implies

$$\begin{aligned}\int f d\left(\overline{\mu \times \nu}\right) &= \int \int \mathcal{X}_{N^C}(y) f(x,y)\, d\mu d\nu \\ &= \int \int f(x,y)\, d\mu d\nu\end{aligned}$$

because of completeness of the measures, μ and ν. This proves Fubini's theorem.

Theorem 8.49 *(Fubini) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete measure spaces and let*

$$\left(\overline{\mu \times \nu}\right)(E) \equiv \inf \left\{ \int \int \mathcal{X}_R(x,y)\, d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}^{2}$$

where $A_i \in \mathcal{S}$ and $B_i \in \mathcal{T}$. Then $\overline{\mu \times \nu}$ is an outer measure on the subsets of $X \times Y$ and the σ algebra of $\overline{\mu \times \nu}$ measurable sets, $\overline{\mathcal{S} \times \mathcal{T}}$, contains all measurable rectangles. If $f \geq 0$ is a $\overline{\mu \times \nu}$ measurable function satisfying

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) < \infty, \tag{8.36}$$

then

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) = \int_Y \int_X f d\mu d\nu,$$

where the iterated integral on the right makes sense because for ν a.e. y, $x \to f(x,y)$ is μ measurable and $y \to \int f(x,y)\, d\mu$ is ν measurable. Similarly,

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) = \int_X \int_Y f d\nu d\mu.$$

In the case where $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are both σ finite, it is not necessary to assume 8.36.

Corollary 8.50 *(Fubini) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete measure spaces such that $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are both σ finite and let*

$$\left(\overline{\mu \times \nu}\right)(E) \equiv \inf \left\{ \int \int \mathcal{X}_R(x,y)\, d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}$$

where $A_i \in \mathcal{S}$ and $B_i \in \mathcal{T}$. Then $\overline{\mu \times \nu}$ is an outer measure. If $f \geq 0$ is a $\overline{\mu \times \nu}$ measurable function then

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) = \int_Y \int_X f d\mu d\nu,$$

[2] Recall this is the same as

$$\inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\, \nu(B_i) : E \subseteq \cup_{i=1}^{\infty} A_i \times B_i \right\}$$

in which the A_i and B_i are measurable.

where the iterated integral on the right makes sense because for ν a.e. y, $x \to f(x,y)$ is μ measurable and $y \to \int f(x,y)\,d\mu$ is ν measurable. Similarly,

$$\int_{X\times Y} f d\left(\overline{\mu\times\nu}\right) = \int_X \int_Y f d\nu d\mu.$$

Proof: Let $\cup_{n=1}^{\infty} X_n = X$ and $\cup_{n=1}^{\infty} Y_n = Y$ where $X_n \in \mathcal{S}$, $Y_n \in \mathcal{T}$, $X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1}$ for all n and $\mu(X_n) < \infty, \nu(Y_n) < \infty$. From Theorem 8.49 applied to X_n, Y_n and $f_m \equiv \min(f, m)$,

$$\int_{X_n\times Y_n} f_m d\left(\overline{\mu\times\nu}\right) = \int_{Y_n}\int_{X_n} f_m d\mu d\nu$$

Now take $m \to \infty$ and use the monotone convergence theorem to obtain

$$\int_{X_n\times Y_n} f d\left(\overline{\mu\times\nu}\right) = \int_{Y_n}\int_{X_n} f d\mu d\nu.$$

Then use the monotone convergence theorem again letting $n \to \infty$ to obtain the desired conclusion. The argument for the other order of integration is similar.

Corollary 8.51 *If $f \in L^1(X\times Y)$, then*

$$\int f d\left(\overline{\mu\times\nu}\right) = \int\int f(x,y)\,d\mu d\nu = \int\int f(x,y)\,d\nu d\mu.$$

If μ and ν are σ finite, then if f is $\overline{\mu\times\nu}$ measurable having complex values and either $\int\int |f|\,d\mu d\nu < \infty$ or $\int\int |f|\,d\nu d\mu < \infty$, then $\int |f|\,d\left(\overline{\mu\times\nu}\right) < \infty$ so $f \in L^1(X\times Y)$.

Proof: Without loss of generality, it can be assumed that f has real values. Then

$$f = \frac{|f| + f - (|f| - f)}{2}$$

and both $f^+ \equiv \frac{|f|+f}{2}$ and $f^- \equiv \frac{|f|-f}{2}$ are nonnegative and are less than $|f|$. Therefore, $\int g d\left(\overline{\mu\times\nu}\right) < \infty$ for $g = f^+$ and $g = f^-$ so the above theorem applies and

$$\begin{aligned}
\int f d\left(\overline{\mu\times\nu}\right) &\equiv \int f^+ d\left(\overline{\mu\times\nu}\right) - \int f^- d\left(\overline{\mu\times\nu}\right) \\
&= \int\int f^+ d\mu d\nu - \int\int f^- d\mu d\nu \\
&= \int\int f d\mu d\nu.
\end{aligned}$$

It remains to verify the last claim. Suppose s is a simple function,

$$s(x,y) \equiv \sum_{i=1}^{m} c_i \mathcal{X}_{E_i} \le |f|(x,y)$$

where the c_i are the nonzero values of s. Then

$$s\mathcal{X}_{R_n} \leq |f| \, \mathcal{X}_{R_n}$$

where $R_n \equiv X_n \times Y_n$ where $X_n \uparrow X$ and $Y_n \uparrow Y$ with $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. It follows, since the nonzero values of $s\mathcal{X}_{R_n}$ are achieved on sets of finite measure,

$$\int s\mathcal{X}_{R_n} d\left(\overline{\mu \times \nu}\right) = \int \int s\mathcal{X}_{R_n} d\mu d\nu.$$

Letting $n \to \infty$ and applying the monotone convergence theorem, this yields

$$\int sd\left(\overline{\mu \times \nu}\right) = \int \int sd\mu d\nu. \tag{8.37}$$

Now let $s_n \uparrow |f|$ where s_n is a nonnegative simple function. From 8.37,

$$\int s_n d\left(\overline{\mu \times \nu}\right) = \int \int s_n d\mu d\nu.$$

Letting $n \to \infty$ and using the monotone convergence theorem, yields

$$\int |f| \, d\left(\overline{\mu \times \nu}\right) = \int \int |f| \, d\mu d\nu < \infty$$

8.8 Disturbing Examples

There are examples which help to define what can be expected of product measures and Fubini type theorems. Three such examples are given in Rudin [36]. The theorems given above are more general than those in this reference but the same examples are still useful for showing that the hypotheses of the above theorems are all necessary.

Example 8.52 *Let $\{a_n\}$ be an increasing sequence of numbers in $(0,1)$ which converges to 1. Let $g_n \in C_c(a_n, a_{n+1})$ such that $\int g_n dx = 1$. Now for $(x,y) \in [0,1) \times [0,1)$ define*

$$f(x,y) \equiv \sum_{k=1}^{\infty} g_n(y)\left(g_n(x) - g_{n+1}(x)\right).$$

Note this is actually a finite sum for each such (x,y). Therefore, this is a continuous function on $[0,1) \times [0,1)$. Now for a fixed y,

$$\int_0^1 f(x,y)\, dx = \sum_{k=1}^{\infty} g_n(y) \int_0^1 \left(g_n(x) - g_{n+1}(x)\right) dx = 0$$

showing that $\int_0^1 \int_0^1 f(x,y)\, dxdy = \int_0^1 0 dy = 0$. Next fix x.

$$\int_0^1 f(x,y)\, dy = \sum_{k=1}^{\infty} \left(g_n(x) - g_{n+1}(x)\right) \int_0^1 g_n(y)\, dy = g_1(x).$$

Hence $\int_0^1 \int_0^1 f(x,y)\,dydx = \int_0^1 g_1(x)\,dx = 1$. *The iterated integrals are not equal. Note the function, g is not nonnegative even though it is measurable. In addition, neither* $\int_0^1 \int_0^1 |f(x,y)|\,dxdy$ *nor* $\int_0^1 \int_0^1 |f(x,y)|\,dydx$ *is finite and so you can't apply Corollary 8.51. The problem here is the function is not nonnegative and is not absolutely integrable.*

Example 8.53 *This time let* $\mu = m$, *Lebesgue measure on* $[0,1]$ *and let* ν *be counting measure on* $[0,1]$, *in this case, the* σ *algebra is* $\mathcal{P}([0,1])$. *Let* l *denote the line segment in* $[0,1] \times [0,1]$ *which goes from* $(0,0)$ *to* $(1,1)$. *Thus* $l = (x,x)$ *where* $x \in [0,1]$. *Consider the outer measure of* l *in* $\overline{m \times \nu}$. *Let* $l \subseteq \cup_k A_k \times B_k$ *where* A_k *is Lebesgue measurable and* B_k *is a subset of* $[0,1]$. *Let* $\mathcal{B} \equiv \{k \in \mathbb{N} : \nu(B_k) = \infty\}$. *If* $m(\cup_{k\in\mathcal{B}} A_k)$ *has measure zero, then there are uncountably many points of* $[0,1]$ *outside of* $\cup_{k\in\mathcal{B}} A_k$. *For* p *one of these points,* $(p,p) \in A_i \times B_i$ *and* $i \notin \mathcal{B}$. *Thus each of these points is in* $\cup_{i\notin\mathcal{B}} B_i$, *a countable set because these* B_i *are each finite. But this is a contradiction because there need to be uncountably many of these points as just indicated. Thus* $m(A_k) > 0$ *for some* $k \in \mathcal{B}$ *and so* $\overline{m \times \nu}(A_k \times B_k) = \infty$. *It follows* $\overline{m \times \nu}(l) = \infty$ *and so* l *is* $\overline{m \times \nu}$ *measurable. Thus* $\int \mathcal{X}_l(x,y)\,d\,\overline{m \times \nu} = \infty$ *and so you cannot apply Fubini's theorem, Theorem 8.49. Since* ν *is not* σ *finite, you cannot apply the corollary to this theorem either. Thus there is no contradiction to the above theorems in the following observation.*

$$\int \int \mathcal{X}_l(x,y)\,d\nu dm = \int 1 dm = 1, \quad \int \int \mathcal{X}_l(x,y)\,dm d\nu = \int 0 d\nu = 0.$$

The problem here is that you have neither $\int f d\,\overline{m \times \nu} < \infty$ *not* σ *finite measure spaces.*

The next example is far more exotic. It concerns the case where both iterated integrals make perfect sense but are unequal. In 1877 Cantor conjectured that the cardinality of the real numbers is the next size of infinity after countable infinity. This hypothesis is called the continuum hypothesis and it has never been proved or disproved[3]. Assuming this continuum hypothesis will provide the basis for the following example. It is due to Sierpinski.

Example 8.54 *Let* X *be an uncountable set. It follows from the well ordering theorem which says every set can be well ordered which is presented in the appendix that* X *can be well ordered. Let* $\omega \in X$ *be the first element of* X *which is preceded by uncountably many points of* X. *Let* Ω *denote* $\{x \in X : x < \omega\}$. *Then* Ω *is uncountable but there is no smaller uncountable set. Thus by the continuum hypothesis, there exists a one to one and onto mapping,* j *which maps* $[0,1]$ *onto* Ω. *Thus, for* $x \in [0,1]$, $j(x)$ *is preceeded by countably many points. Let*

[3] In 1940 it was shown by Godel that the continuum hypothesis cannot be disproved. In 1963 it was shown by Cohen that the continuum hypothesis cannot be proved. These assertions are based on the axiom of choice and the Zermelo Frankel axioms of set theory. This topic is far outside the scope of this book and this is only a hopefully interesting historical observation.

$Q \equiv \left\{(x,y) \in [0,1]^2 : j(x) < j(y)\right\}$ *and let* $f(x,y) = \mathcal{X}_Q(x,y)$. *Then*

$$\int_0^1 f(x,y)\,dy = 1,\ \int_0^1 f(x,y)\,dx = 0$$

In each case, the integrals make sense. In the first, for fixed x, $f(x,y) = 1$ *for all but countably many* y *so the function of* y *is Borel measurable. In the second where* y *is fixed,* $f(x,y) = 0$ *for all but countably many* x. *Thus*

$$\int_0^1 \int_0^1 f(x,y)\,dydx = 1,\ \int_0^1 \int_0^1 f(x,y)\,dxdy = 0.$$

The problem here must be that f *is not* $\overline{m \times m}$ *measurable.*

8.9 Exercises

1. Let $\Omega = \mathbb{N}$, the natural numbers and let $d(p,q) = |p-q|$, the usual distance in $\mathbb{R}$. Show that (Ω, d) the closures of the balls are compact. Now let $\Lambda f \equiv \sum_{k=1}^{\infty} f(k)$ whenever $f \in C_c(\Omega)$. Show this is a well defined positive linear functional on the space $C_c(\Omega)$. Describe the measure of the Riesz representation theorem which results from this positive linear functional. What if $\Lambda(f) = f(1)$? What measure would result from this functional? Which functions are measurable?

2. Verify that $\overline{\mu}$ defined in Lemma 8.7 is an outer measure.

3. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Let $\Lambda f \equiv \int f dF$ where the integral is the Riemann Stieltjes integral of $f \in C_c(\mathbb{R})$. Show the measure μ from the Riesz representation theorem satisfies

$$\begin{aligned} \mu([a,b]) &= F(b) - F(a-), \mu((a,b]) = F(b) - F(a), \\ \mu([a,a]) &= F(a) - F(a-). \end{aligned}$$

   Hint: You might want to review the material on Riemann Stieltjes integrals presented in the Preliminary part of the notes.

4. Let Ω be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of Ω which is finite on compact sets. Show there exists a unique Radon measure, $\overline{\mu}$ which equals μ on the Borel sets.

5. ↑ Random vectors are measurable functions, $\mathbf{X}$, mapping a probability space, $(\Omega, P, \mathcal{F})$ to $\mathbb{R}^n$. Thus $\mathbf{X}(\omega) \in \mathbb{R}^n$ for each $\omega \in \Omega$ and P is a probability measure defined on the sets of $\mathcal{F}$, a σ algebra of subsets of Ω. For E a Borel set in $\mathbb{R}^n$, define

$$\mu(E) \equiv P\left(\mathbf{X}^{-1}(E)\right) \equiv \text{ probability that } \mathbf{X} \in E.$$

Show this is a well defined measure on the Borel sets of $\mathbb{R}^n$ and use Problem 4 to obtain a Radon measure, $\lambda_{\mathbf{X}}$ defined on a σ algebra of sets of $\mathbb{R}^n$ including the Borel sets such that for E a Borel set, $\lambda_{\mathbf{X}}(E)$ =Probability that ($\mathbf{X} \in E$).

6. Suppose X and Y are metric spaces having compact closed balls. Show
$$(X \times Y, d_{X \times Y})$$
is also a metric space which has the closures of balls compact. Here
$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) \equiv \max(d(x_1, x_2), d(y_1, y_2)).$$
Let
$$\mathcal{A} \equiv \{E \times F : E \text{ is a Borel set in } X, F \text{ is a Borel set in } Y\}.$$
Show $\sigma(\mathcal{A})$, the smallest σ algebra containing $\mathcal{A}$ contains the Borel sets. **Hint:** Show every open set in a metric space which has closed balls compact can be obtained as a countable union of compact sets. Next show this implies every open set can be obtained as a countable union of open sets of the form $U \times V$ where U is open in X and V is open in Y.

7. Suppose $(\Omega, \mathcal{S}, \mu)$ is a measure space which may not be complete. Could you obtain a complete measure space, $(\Omega, \overline{\mathcal{S}}, \mu_1)$ by simply letting $\overline{\mathcal{S}}$ consist of all sets of the form E where there exists $F \in \mathcal{S}$ such that $(F \setminus E) \cup (E \setminus F) \subseteq N$ for some $N \in \mathcal{S}$ which has measure zero and then let $\mu(E) = \mu_1(F)$? Explain.

8. Let $(\Omega, \mathcal{S}, \mu)$ be a σ finite measure space and let $f : \Omega \to [0, \infty)$ be measurable. Define
$$A \equiv \{(x, y) : y < f(x)\}$$
Verify that A is $\overline{\mu \times m}$ measurable. Show that
$$\int f d\mu = \int \int \mathcal{X}_A(x, y) \, d\mu dm = \int \mathcal{X}_A d\overline{\mu \times m}.$$

9. For f a nonnegative measurable function, it was shown that
$$\int f d\mu = \int \mu([f > t]) \, dt.$$
Would it work the same if you used $\int \mu([f \geq t]) \, dt$? Explain.

10. The Riemann integral is only defined for functions which are bounded which are also defined on a bounded interval. If either of these two criteria are not satisfied, then the integral is not the Riemann integral. Suppose f is Riemann integrable on a bounded interval, $[a, b]$. Show that it must also be Lebesgue integrable with respect to one dimensional Lebesgue measure and the two integrals coincide. Give a theorem in which the improper Riemann integral coincides with a suitable Lebesgue integral. (There are many such situations just find one.) Note that $\int_0^\infty \frac{\sin x}{x} dx$ is a valid improper Riemann integral but is not a Lebesgue integral. Why?

Lebesgue Measure

9.1 Basic Properties

Definition 9.1 *Define the following positive linear functional for $f \in C_c(\mathbb{R}^n)$.*

$$\Lambda f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x})\, dx_1 \cdots dx_n.$$

Then the measure representing this functional is Lebesgue measure.

The following lemma will help in understanding Lebesgue measure.

Lemma 9.2 *Every open set in $\mathbb{R}^n$ is the countable disjoint union of half open boxes of the form*

$$\prod_{i=1}^{n}(a_i, a_i + 2^{-k}]$$

where $a_i = l2^{-k}$ for some integers, l, k. The sides of these boxes are equal.

Proof: Let

$$\mathcal{C}_k = \{\text{All half open boxes } \prod_{i=1}^{n}(a_i, a_i + 2^{-k}] \text{ where}$$

$$a_i = l2^{-k} \text{ for some integer } l.\}$$

Thus $\mathcal{C}_k$ consists of a countable disjoint collection of boxes whose union is $\mathbb{R}^n$. This is sometimes called a tiling of $\mathbb{R}^n$. Think of tiles on the floor of a bathroom and you will get the idea. Note that each box has diameter no larger than $2^{-k}\sqrt{n}$. This is because if

$$\mathbf{x}, \mathbf{y} \in \prod_{i=1}^{n}(a_i, a_i + 2^{-k}],$$

then $|x_i - y_i| \leq 2^{-k}$. Therefore,

$$|\mathbf{x} - \mathbf{y}| \leq \left(\sum_{i=1}^{n} \left(2^{-k}\right)^2\right)^{1/2} = 2^{-k}\sqrt{n}.$$

Let U be open and let $\mathcal{B}_1 \equiv$ all sets of $\mathcal{C}_1$ which are contained in U. If $\mathcal{B}_1, \cdots, \mathcal{B}_k$ have been chosen, $\mathcal{B}_{k+1} \equiv$ all sets of $\mathcal{C}_{k+1}$ contained in

$$U \setminus \cup \left(\cup_{i=1}^{k} \mathcal{B}_i \right).$$

Let $\mathcal{B}_\infty = \cup_{i=1}^{\infty} \mathcal{B}_i$. In fact $\cup \mathcal{B}_\infty = U$. Clearly $\cup \mathcal{B}_\infty \subseteq U$ because every box of every $\mathcal{B}_i$ is contained in U. If $p \in U$, let k be the smallest integer such that p is contained in a box from $\mathcal{C}_k$ which is also a subset of U. Thus

$$p \in \cup \mathcal{B}_k \subseteq \cup \mathcal{B}_\infty.$$

Hence $\mathcal{B}_\infty$ is the desired countable disjoint collection of half open boxes whose union is U. This proves the lemma.

Now what does Lebesgue measure do to a rectangle, $\prod_{i=1}^{n} (a_i, b_i]$?

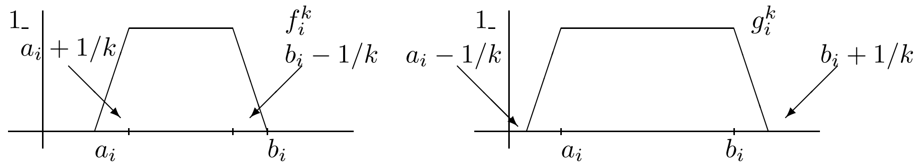
Lemma 9.3 *Let* $R = \prod_{i=1}^{n} [a_i, b_i]$, $R_0 = \prod_{i=1}^{n} (a_i, b_i)$. *Then*

$$m_n(R_0) = m_n(R) = \prod_{i=1}^{n} (b_i - a_i).$$

Proof: Let k be large enough that

$$a_i + 1/k < b_i - 1/k$$

for $i = 1, \cdots, n$ and consider functions g_i^k and f_i^k having the following graphs.



Let

$$g^k(\mathbf{x}) = \prod_{i=1}^{n} g_i^k(x_i), \quad f^k(\mathbf{x}) = \prod_{i=1}^{n} f_i^k(x_i).$$

Then by elementary calculus along with the definition of Λ,

$$\prod_{i=1}^{n} (b_i - a_i + 2/k) \geq \Lambda g^k = \int g^k dm_n \geq m_n(R) \geq m_n(R_0)$$

$$\geq \int f^k dm_n = \Lambda f^k \geq \prod_{i=1}^{n} (b_i - a_i - 2/k).$$

Letting $k \to \infty$, it follows that

$$m_n(R) = m_n(R_0) = \prod_{i=1}^{n} (b_i - a_i).$$

This proves the lemma.

Lemma 9.4 *Let U be an open or closed set. Then $m_n(U) = m_n(\mathbf{x}+U)$.*

Proof: By Lemma 9.2 there is a sequence of disjoint half open rectangles, $\{R_i\}$ such that $\cup_i R_i = U$. Therefore, $\mathbf{x}+U = \cup_i (\mathbf{x}+R_i)$ and the $\mathbf{x}+R_i$ are also disjoint rectangles which are identical to the R_i but translated. From Lemma 9.3, $m_n(U) = \sum_i m_n(R_i) = \sum_i m_n(\mathbf{x}+R_i) = m_n(\mathbf{x}+U)$.

It remains to verify the lemma for a closed set. Let H be a closed bounded set first. Then $H \subseteq B(\mathbf{0},R)$ for some R large enough. First note that $\mathbf{x}+H$ is a closed set. Thus

$$\begin{aligned}
m_n(B(\mathbf{x},R)) &= m_n(\mathbf{x}+H) + m_n((B(\mathbf{0},R)+\mathbf{x}) \setminus (\mathbf{x}+H)) \\
&= m_n(\mathbf{x}+H) + m_n((B(\mathbf{0},R) \setminus H) + \mathbf{x}) \\
&= m_n(\mathbf{x}+H) + m_n((B(\mathbf{0},R) \setminus H)) \\
&= m_n(B(\mathbf{0},R)) - m_n(H) + m_n(\mathbf{x}+H) \\
&= m_n(B(\mathbf{x},R)) - m_n(H) + m_n(\mathbf{x}+H)
\end{aligned}$$

the last equality because of the first part of the lemma which implies $m_n(B(\mathbf{x},R)) = m_n(B(\mathbf{0},R))$. Therefore, $m_n(\mathbf{x}+H) = m_n(H)$ as claimed. If H is not bounded, consider $H_m \equiv \overline{B(\mathbf{0},m)} \cap H$. Then $m_n(\mathbf{x}+H_m) = m_n(H_m)$. Passing to the limit as $m \to \infty$ yields the result in general.

Theorem 9.5 *Lebesgue measure is translation invariant. That is*

$$m_n(E) = m_n(\mathbf{x}+E)$$

for all E Lebesgue measurable.

Proof: Suppose $m_n(E) < \infty$. By regularity of the measure, there exist sets G, H such that G is a countable intersection of open sets, H is a countable union of compact sets, $m_n(G \setminus H) = 0$, and $G \supseteq E \supseteq H$. Now $m_n(G) = m_n(G+\mathbf{x})$ and $m_n(H) = m_n(H+\mathbf{x})$ which follows from Lemma 9.4 applied to the sets which are either intersected to form G or unioned to form H. Now

$$\mathbf{x}+H \subseteq \mathbf{x}+E \subseteq \mathbf{x}+G$$

and both $\mathbf{x}+H$ and $\mathbf{x}+G$ are measurable because they are either countable unions or countable intersections of measurable sets. Furthermore,

$$m_n(\mathbf{x}+G \setminus \mathbf{x}+H) = m_n(\mathbf{x}+G) - m_n(\mathbf{x}+H) = m_n(G) - m_n(H) = 0$$

and so by completeness of the measure, $\mathbf{x}+E$ is measurable. It follows

$$\begin{aligned}
m_n(E) &= m_n(H) = m_n(\mathbf{x}+H) \le m_n(\mathbf{x}+E) \\
&\le m_n(\mathbf{x}+G) = m_n(G) = m_n(E).
\end{aligned}$$

If $m_n(E)$ is not necessarily less than ∞, consider $E_m \equiv B(\mathbf{0},m) \cap E$. Then $m_n(E_m) = m_n(E_m+\mathbf{x})$ by the above. Letting $m \to \infty$ it follows $m_n(E_m) = m_n(E_m+\mathbf{x})$. This proves the theorem.

Corollary 9.6 *Let D be an $n \times n$ diagonal matrix and let U be an open set. Then*

$$m_n(DU) = |\det(D)| \, m_n(U).$$

Proof: If any of the diagonal entries of D equals 0 there is nothing to prove because then both sides equal zero. Therefore, it can be assumed none are equal to zero. Suppose these diagonal entries are $k_1, \cdots, k_n$. From Lemma 9.2 there exist half open boxes, $\{R_i\}$ having all sides equal such that $U = \cup_i R_i$. Suppose one of these is $R_i = \prod_{j=1}^n (a_j, b_j]$, where $b_j - a_j = l_i$. Then $DR_i = \prod_{j=1}^n I_j$ where $I_j = (k_j a_j, k_j b_j]$ if $k_j > 0$ and $I_j = [k_j b_j, k_j a_j)$ if $k_j < 0$. Then the rectangles, DR_i are disjoint because D is one to one and their union is DU. Also,

$$m_n(DR_i) = \prod_{j=1}^n |k_j| \, l_i = |\det D| \, m_n(R_i).$$

Therefore,

$$m_n(DU) = \sum_{i=1}^\infty m_n(DR_i) = |\det(D)| \sum_{i=1}^\infty m_n(R_i) = |\det(D)| \, m_n(U).$$

and this proves the corollary.

From this the following corollary is obtained.

Corollary 9.7 *Let $M > 0$. Then $m_n(B(\mathbf{a}, Mr)) = M^n m_n(B(\mathbf{0}, r))$.*

Proof: By Lemma 9.4 there is no loss of generality in taking $\mathbf{a} = \mathbf{0}$. Let D be the diagonal matrix which has M in every entry of the main diagonal so $|\det(D)| = M^n$. Note that $DB(\mathbf{0}, r) = B(\mathbf{0}, Mr)$. By Corollary 9.6

$$m_n(B(\mathbf{0}, Mr)) = m_n(DB(\mathbf{0}, r)) = M^n m_n(B(\mathbf{0}, r)).$$

There are many norms on $\mathbb{R}^n$. Other common examples are

$$||\mathbf{x}||_\infty \equiv \max\{|x_k| : \mathbf{x} = (x_1, \cdots, x_n)\}$$

or

$$||\mathbf{x}||_p \equiv \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

With $||\cdot||$ any norm for $\mathbb{R}^n$ you can define a corresponding ball in terms of this norm.

$$B(\mathbf{a}, r) \equiv \{\mathbf{x} \in \mathbb{R}^n \text{ such that } ||\mathbf{x} - \mathbf{a}|| < r\}$$

It follows from general considerations involving metric spaces presented earlier that these balls are open sets. Therefore, Corollary 9.7 has an obvious generalization.

Corollary 9.8 *Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Then for $M > 0$, $m_n(B(\mathbf{a}, Mr)) = M^n m_n(B(\mathbf{0}, r))$ where these balls are defined in terms of the norm $||\cdot||$.*

9.2 The Vitali Covering Theorem

The Vitali covering theorem is concerned with the situation in which a set is contained in the union of balls. You can imagine that it might be very hard to get disjoint balls from this collection of balls which would cover the given set. However, it is possible to get disjoint balls from this collection of balls which have the property that if each ball is enlarged appropriately, the resulting enlarged balls do cover the set. When this result is established, it is used to prove another form of this theorem in which the disjoint balls do not cover the set but they only miss a set of measure zero.

Recall the Hausdorff maximal principle, Theorem 1.13 on Page 18 which is proved to be equivalent to the axiom of choice in the appendix. For convenience, here it is:

Theorem 9.9 *(Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

I will use this Hausdorff maximal principle to give a very short and elegant proof of the Vitali covering theorem. This follows the treatment in Evans and Gariepy [17] which they got from another book. I am not sure who first did it this way but it is very nice because it is so short. In the following lemma and theorem, the balls will be either open or closed and determined by some norm on $\mathbb{R}^n$. When pictures are drawn, I shall draw them as though the norm is the usual norm but the results are unchanged for any norm. Also, I will write (in this section only) $B(\mathbf{a}, r)$ to indicate a set which satisfies

$$\{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < r\} \subseteq B(\mathbf{a}, r) \subseteq \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| \leq r\}$$

and $\widehat{B}(\mathbf{a}, r)$ to indicate the usual ball but with radius 5 times as large,

$$\{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < 5r\}.$$

Lemma 9.10 *Let $||\cdot||$ be a norm on $\mathbb{R}^n$ and let $\mathcal{F}$ be a collection of balls determined by this norm. Suppose*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\text{if } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \tag{9.1}$$

$$\text{if } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \tag{9.2}$$

$$\mathcal{G} \text{ is maximal with respect to 9.1 and 9.2.}$$

Note that if there is no ball of $\mathcal{F}$ which has radius larger than k then $\mathcal{G} = \emptyset$.

Proof: Let $\mathcal{H} = \{\mathcal{B} \subseteq \mathcal{F}$ such that 9.1 and 9.2 hold$\}$. If there are no balls with radius larger than k then $\mathcal{H} = \emptyset$ and you let $\mathcal{G} = \emptyset$. In the other case, $\mathcal{H} \neq \emptyset$ because there exists $B(\mathbf{p}, r) \in \mathcal{F}$ with $r > k$. In this case, partially order $\mathcal{H}$ by set inclusion and use the Hausdorff maximal principle (see the appendix on set theory) to let $\mathcal{C}$ be a maximal chain in $\mathcal{H}$. Clearly $\cup\mathcal{C}$ satisfies 9.1 and 9.2 because if B_1 and B_2 are two balls from $\cup\mathcal{C}$ then since $\mathcal{C}$ is a chain, it follows there is some element of $\mathcal{C}, \mathcal{B}$ such that both B_1 and B_2 are elements of $\mathcal{B}$ and $\mathcal{B}$ satisfies 9.1 and 9.2. If $\cup\mathcal{C}$ is not maximal with respect to these two properties, then $\mathcal{C}$ was not a maximal chain because then there would exist $\mathcal{B} \supsetneq \cup\mathcal{C}$, that is, $\mathcal{B}$ contains $\mathcal{C}$ as a proper subset and $\{\mathcal{C}, \mathcal{B}\}$ would be a strictly larger chain in $\mathcal{H}$. Let $\mathcal{G} = \cup\mathcal{C}$.

Theorem 9.11 *(Vitali) Let $\mathcal{F}$ be a collection of balls and let*

$$A \equiv \cup\{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widehat{B} : B \in \mathcal{G}\}.$$

Proof: Using Lemma 9.10, there exists $\mathcal{G}_1 \subseteq \mathcal{F} \equiv \mathcal{F}_0$ which satisfies

$$B(\mathbf{p}, r) \in \mathcal{G}_1 \text{ implies } r > \frac{M}{2}, \tag{9.3}$$

$$B_1, B_2 \in \mathcal{G}_1 \text{ implies } B_1 \cap B_2 = \emptyset, \tag{9.4}$$

$$\mathcal{G}_1 \text{ is maximal with respect to 9.3, and 9.4.}$$

Suppose $\mathcal{G}_1, \cdots, \mathcal{G}_m$ have been chosen, $m \geq 1$. Let

$$\mathcal{F}_m \equiv \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \cup\{\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m\}\}.$$

Using Lemma 9.10, there exists $\mathcal{G}_{m+1} \subseteq \mathcal{F}_m$ such that

$$B(\mathbf{p}, r) \in \mathcal{G}_{m+1} \text{ implies } r > \frac{M}{2^{m+1}}, \tag{9.5}$$

$$B_1, B_2 \in \mathcal{G}_{m+1} \text{ implies } B_1 \cap B_2 = \emptyset, \tag{9.6}$$

$$\mathcal{G}_{m+1} \text{ is a maximal subset of } \mathcal{F}_m \text{ with respect to 9.5 and 9.6.}$$

Note it might be the case that $\mathcal{G}_{m+1} = \emptyset$ which happens if $\mathcal{F}_m = \emptyset$. Define

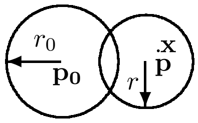
$$\mathcal{G} \equiv \cup_{k=1}^{\infty} \mathcal{G}_k.$$

Thus $\mathcal{G}$ is a collection of disjoint balls in $\mathcal{F}$. I must show $\{\widehat{B} : B \in \mathcal{G}\}$ covers A.

Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$ and let

$$\frac{M}{2^m} < r \leq \frac{M}{2^{m-1}}.$$

Then $B(\mathbf{p}, r)$ must intersect some set, $B(\mathbf{p}_0, r_0) \in \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ since otherwise, $\mathcal{G}_m$ would fail to be maximal. Then $r_0 > \frac{M}{2^m}$ because all balls in $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ satisfy this inequality.



Then for $\mathbf{x} \in B(\mathbf{p}, r)$, the following chain of inequalities holds because $r \leq \frac{M}{2^{m-1}}$ and $r_0 > \frac{M}{2^m}$

$$\begin{aligned} |\mathbf{x} - \mathbf{p}_0| &\leq |\mathbf{x} - \mathbf{p}| + |\mathbf{p} - \mathbf{p}_0| \leq r + r_0 + r \\ &\leq \frac{2M}{2^{m-1}} + r_0 = \frac{4M}{2^m} + r_0 < 5r_0. \end{aligned}$$

Thus $B(\mathbf{p}, r) \subseteq \widehat{B}(\mathbf{p}_0, r_0)$ and this proves the theorem.

9.3 The Vitali Covering Theorem (Elementary Version)

The proof given here is from Basic Analysis [28]. It first considers the case of open balls and then generalizes to balls which may be neither open nor closed or closed.

Lemma 9.12 *Let $\mathcal{F}$ be a countable collection of balls satisfying*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and let $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\text{If } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \tag{9.7}$$

$$\text{If } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \tag{9.8}$$

$$\mathcal{G} \text{ is maximal with respect to 9.7 and 9.8.} \tag{9.9}$$

Proof: If no ball of $\mathcal{F}$ has radius larger than k, let $\mathcal{G} = \emptyset$. Assume therefore, that some balls have radius larger than k. Let $\mathcal{F} \equiv \{B_i\}_{i=1}^{\infty}$. Now let B_{n_1} be the first ball in the list which has radius greater than k. If every ball having radius larger than k intersects this one, then stop. The maximal set is just B_{n_1}. Otherwise, let B_{n_2} be the next ball having radius larger than k which is disjoint from B_{n_1}. Continue this way obtaining $\{B_{n_i}\}_{i=1}^{\infty}$, a finite or infinite sequence of disjoint balls having radius larger than k. Then let $\mathcal{G} \equiv \{B_{n_i}\}$. To see that $\mathcal{G}$ is maximal with respect to 9.7 and 9.8, suppose $B \in \mathcal{F}$, B has radius larger than k, and $\mathcal{G} \cup \{B\}$ satisfies 9.7 and 9.8. Then at some point in the process, B would have been chosen because it would be the ball of radius larger than k which has the smallest index. Therefore, $B \in \mathcal{G}$ and this shows $\mathcal{G}$ is maximal with respect to 9.7 and 9.8.

For the next lemma, for an open ball, $B = B(\mathbf{x}, r)$, denote by $\widetilde{B}$ the open ball, $B(\mathbf{x}, 4r)$.

Lemma 9.13 *Let $\mathcal{F}$ be a collection of open balls, and let*

$$A \equiv \cup \{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup \{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widetilde{B} : B \in \mathcal{G}\}.$$

Proof: Without loss of generality assume $\mathcal{F}$ is countable. This is because there is a countable subset of $\mathcal{F}$, $\mathcal{F}'$ such that $\cup\mathcal{F}' = A$. To see this, consider the set of balls having rational radii and centers having all components rational. This is a countable set of balls and you should verify that every open set is the union of balls of this form. Therefore, you can consider the subset of this set of balls consisting of those which are contained in some open set of $\mathcal{F}$, G so $\cup G = A$ and use the axiom of choice to define a subset of $\mathcal{F}$ consisting of a single set from $\mathcal{F}$ containing each set of G. Then this is $\mathcal{F}'$. The union of these sets equals A . Then consider $\mathcal{F}'$ instead of $\mathcal{F}$. Therefore, assume at the outset $\mathcal{F}$ is countable. By Lemma 9.12, there exists $\mathcal{G}_1 \subseteq \mathcal{F}$ which satisfies 9.7, 9.8, and 9.9 with $k = \frac{2M}{3}$.

Suppose $\mathcal{G}_1, \cdots, \mathcal{G}_{m-1}$ have been chosen for $m \geq 2$. Let

$$\mathcal{F}_m = \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \overbrace{\cup\{\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{m-1}\}}^{\text{union of the balls in these } \mathcal{G}_j} \}$$

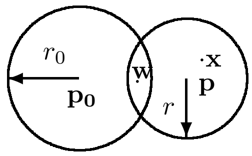
and using Lemma 9.12, let $\mathcal{G}_m$ be a maximal collection of disjoint balls from $\mathcal{F}_m$ with the property that each ball has radius larger than $\left(\frac{2}{3}\right)^m M$. Let $\mathcal{G} \equiv \cup_{k=1}^{\infty}\mathcal{G}_k$. Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$. Choose m such that

$$\left(\frac{2}{3}\right)^m M < r \leq \left(\frac{2}{3}\right)^{m-1} M$$

Then $B(\mathbf{p}, r)$ must have nonempty intersection with some ball from $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ because if it didn't, then $\mathcal{G}_m$ would fail to be maximal. Denote by $B(\mathbf{p}_0, r_0)$ a ball in $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ which has nonempty intersection with $B(\mathbf{p}, r)$. Thus

$$r_0 > \left(\frac{2}{3}\right)^m M.$$

Consider the picture, in which $\mathbf{w} \in B(\mathbf{p}_0, r_0) \cap B(\mathbf{p}, r)$.



Then

$$\begin{aligned}
|\mathbf{x}-\mathbf{p}_0| &\leq |\mathbf{x}-\mathbf{p}|+|\mathbf{p}-\mathbf{w}|+\overbrace{|\mathbf{w}-\mathbf{p}_0|}^{<r_0} \\
&< r+r+r_0 \leq 2\overbrace{\left(\frac{2}{3}\right)^{m-1}M}^{<\frac{3}{2}r_0}+r_0 \\
&< 2\left(\frac{3}{2}\right)r_0+r_0=4r_0.
\end{aligned}$$

This proves the lemma since it shows $B(\mathbf{p},r)\subseteq B(\mathbf{p}_0,4r_0)$.

With this Lemma consider a version of the Vitali covering theorem in which the balls do not have to be open. A ball centered at $\mathbf{x}$ of radius r will denote something which contains the open ball, $B(\mathbf{x},r)$ and is contained in the closed ball, $\overline{B(\mathbf{x},r)}$. Thus the balls could be open or they could contain some but not all of their boundary points.

Definition 9.14 *Let B be a ball centered at $\mathbf{x}$ having radius r. Denote by $\widehat{B}$ the open ball, $B(\mathbf{x},5r)$.*

Theorem 9.15 *(Vitali) Let $\mathcal{F}$ be a collection of balls, and let*

$$A\equiv\cup\{B:B\in\mathcal{F}\}.$$

Suppose

$$\infty>M\equiv\sup\{r:B(\mathbf{p},r)\in\mathcal{F}\}>0.$$

Then there exists $\mathcal{G}\subseteq\mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A\subseteq\cup\{\widehat{B}:B\in\mathcal{G}\}.$$

Proof: For B one of these balls, say $\overline{B(\mathbf{x},r)}\supseteq B\supseteq B(\mathbf{x},r)$, denote by B_1, the ball $B\left(\mathbf{x},\frac{5r}{4}\right)$. Let $\mathcal{F}_1\equiv\{B_1:B\in\mathcal{F}\}$ and let A_1 denote the union of the balls in $\mathcal{F}_1$. Apply Lemma 9.13 to $\mathcal{F}_1$ to obtain

$$A_1\subseteq\cup\{\widetilde{B_1}:B_1\in\mathcal{G}_1\}$$

where $\mathcal{G}_1$ consists of disjoint balls from $\mathcal{F}_1$. Now let $\mathcal{G}\equiv\{B\in\mathcal{F}:B_1\in\mathcal{G}_1\}$. Thus $\mathcal{G}$ consists of disjoint balls from $\mathcal{F}$ because they are contained in the disjoint open balls, $\mathcal{G}_1$. Then

$$A\subseteq A_1\subseteq\cup\{\widetilde{B_1}:B_1\in\mathcal{G}_1\}=\cup\{\widehat{B}:B\in\mathcal{G}\}$$

because for $B_1=B\left(\mathbf{x},\frac{5r}{4}\right)$, it follows $\widetilde{B_1}=B(\mathbf{x},5r)=\widehat{B}$. This proves the theorem.

9.4 Vitali Coverings

There is another version of the Vitali covering theorem which is also of great importance. In this one, balls from the original set of balls almost cover the set,leaving out only a set of measure zero. It is like packing a truck with stuff. You keep trying to fill in the holes with smaller and smaller things so as to not waste space. It is remarkable that you can avoid wasting any space at all when you are dealing with balls of any sort provided you can use arbitrarily small balls.

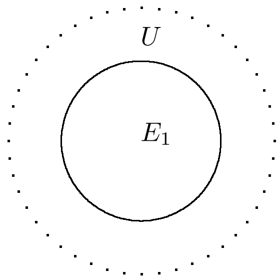
Definition 9.16 *Let $\mathcal{F}$ be a collection of balls that cover a set, E, which have the property that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Such a collection covers E in the sense of Vitali.*

In the following covering theorem, $\overline{m_n}$ denotes the outer measure determined by n dimensional Lebesgue measure.

Theorem 9.17 *Let $E \subseteq \mathbb{R}^n$ and suppose $0 < \overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$ be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: From the definition of outer measure there exists a Lebesgue measurable set, $E_1 \supseteq E$ such that $m_n(E_1) = \overline{m_n}(E)$. Now by outer regularity of Lebesgue measure, there exists U, an open set which satisfies

$$m_n(E_1) > (1 - 10^{-n})m_n(U),\ U \supseteq E_1.$$



Each point of E is contained in balls of $\mathcal{F}$ of arbitrarily small radii and so there exists a covering of E with balls of $\mathcal{F}$ which are themselves contained in U. Therefore, by the Vitali covering theorem, there exist disjoint balls, $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{\infty} \widehat{B}_j,\ B_j \subseteq U.$$

Therefore,

$$\begin{aligned} m_n(E_1) &= \overline{m_n}(E) \le m_n\left(\cup_{j=1}^{\infty}\widehat{B}_j\right) \le \sum_j m_n\left(\widehat{B}_j\right) \\ &= 5^n \sum_j m_n(B_j) = 5^n m_n\left(\cup_{j=1}^{\infty} B_j\right) \end{aligned}$$

Then

$$\begin{gathered} m_n(E_1) > (1-10^{-n})m_n(U) \\ \ge (1-10^{-n})[m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) + m_n(\cup_{j=1}^{\infty} B_j)] \\ \ge (1-10^{-n})[m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) + 5^{-n} \overbrace{\overline{m_n}(E)}^{=m_n(E_1)}]. \end{gathered}$$

and so

$$\left(1-\left(1-10^{-n}\right)5^{-n}\right) m_n(E_1) \ge (1-10^{-n}) m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j)$$

which implies

$$m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) \le \frac{(1-(1-10^{-n})\,5^{-n})}{(1-10^{-n})} m_n(E_1)$$

Now a short computation shows

$$0 < \frac{(1-(1-10^{-n})\,5^{-n})}{(1-10^{-n})} < 1$$

Hence, denoting by θ_n a number such that

$$\frac{(1-(1-10^{-n})\,5^{-n})}{(1-10^{-n})} < \theta_n < 1,$$

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) \le m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) < \theta_n m_n(E_1) = \theta_n \overline{m_n}(E)$$

Now pick N_1 large enough that

$$\theta_n \overline{m_n}(E) \ge m_n(E_1 \setminus \cup_{j=1}^{N_1} B_j) \ge \overline{m_n}(E \setminus \cup_{j=1}^{N_1} B_j) \tag{9.10}$$

Let $\mathcal{F}_1 = \{B \in \mathcal{F} : B_j \cap B = \emptyset,\ j = 1, \cdots, N_1\}$. If $E \setminus \cup_{j=1}^{N_1} B_j = \emptyset$, then $\mathcal{F}_1 = \emptyset$ and

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{N_1} B_j\right) = 0$$

Therefore, in this case let $B_k = \emptyset$ for all $k > N_1$. Consider the case where

$$E \setminus \cup_{j=1}^{N_1} B_j \neq \emptyset.$$

In this case, $\mathcal{F}_1 \neq \emptyset$ and covers $E \setminus \cup_{j=1}^{N_1} B_j$ in the sense of Vitali. Repeat the same argument, letting $E \setminus \cup_{j=1}^{N_1} B_j$ play the role of E and letting $U \setminus \cup_{j=1}^{N_1} B_j$ play the role of U. (You pick a different E_1 whose measure equals the outer measure of $E \setminus \cup_{j=1}^{N_1} B_j$.) Then choosing B_j for $j = N_1 + 1, \cdots, N_2$ as in the above argument,

$$\theta_n \overline{m_n}(E \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_2} B_j)$$

and so from 9.10,

$$\theta_n^2 \overline{m_n}(E) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_2} B_j).$$

Continuing this way

$$\theta_n^k \overline{m_n}(E) \geq \overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right).$$

If it is ever the case that $E \setminus \cup_{j=1}^{N_k} B_j = \emptyset$, then, as in the above argument,

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right) = 0.$$

Otherwise, the process continues and

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) \leq \overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right) \leq \theta_n^k \overline{m_n}(E)$$

for every $k \in \mathbb{N}$. Therefore, the conclusion holds in this case also. This proves the Theorem.

There is an obvious corollary which removes the assumption that $0 < \overline{m_n}(E)$.

Corollary 9.18 *Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: If $0 = \overline{m_n}(E)$ you simply pick any ball from $\mathcal{F}$ for your collection of disjoint balls.

It is also not hard to remove the assumption that $\overline{m_n}(E) < \infty$.

Corollary 9.19 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: Let $R_m \equiv (-m, m)^n$ be the open rectangle having sides of length $2m$ which is centered at $\mathbf{0}$ and let $R_0 = \emptyset$. Let $H_m \equiv \overline{R_m} \setminus R_m$. Since both $\overline{R_m}$ and R_m have the same measure, $(2m)^n$, it follows $m_n(H_m) = 0$. Now for all $k \in \mathbb{N}$, $R_k \subseteq \overline{R_k} \subseteq R_{k+1}$. Consider the disjoint open sets, $U_k \equiv R_{k+1} \setminus \overline{R_k}$. Thus $\mathbb{R}^n = \cup_{k=0}^{\infty} U_k \cup N$ where N is a set of measure zero equal to the union of the H_k. Let $\mathcal{F}_k$ denote those balls of $\mathcal{F}$ which are contained in U_k and let $E_k \equiv U_k \cap E$. Then from Theorem 9.17, there exists a sequence of disjoint balls, $D_k \equiv \left\{B_i^k\right\}_{i=1}^{\infty}$

of $\mathcal{F}_k$ such that $\overline{m_n}(E_k \setminus \cup_{j=1}^{\infty} B_j^k) = 0$. Letting $\{B_i\}_{i=1}^{\infty}$ be an enumeration of all the balls of $\cup_k D_k$, it follows that

$$\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) \leq m_n(N) + \sum_{k=1}^{\infty} \overline{m_n}(E_k \setminus \cup_{j=1}^{\infty} B_j^k) = 0.$$

Also, you don't have to assume the balls are closed.

Corollary 9.20 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: Let $\overline{\mathcal{F}}$ be the collection of closures of balls in $\mathcal{F}$. Then $\overline{\mathcal{F}}$ covers E in the sense of Vitali and so from Corollary 9.19 there exists a sequence of disjoint closed balls from $\overline{\mathcal{F}}$ satisfying $\overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) = 0$. Now boundaries of the balls, B_i have measure zero and so $\{B_i\}$ is a sequence of disjoint open balls satisfying $\overline{m_n}(E \setminus \cup_{i=1}^{\infty} B_i) = 0$. The reason for this is that

$$(E \setminus \cup_{i=1}^{\infty} B_i) \setminus \left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) \subseteq \cup_{i=1}^{\infty} \overline{B_i} \setminus \cup_{i=1}^{\infty} B_i \subseteq \cup_{i=1}^{\infty} \overline{B_i} \setminus B_i,$$

a set of measure zero. Therefore,

$$E \setminus \cup_{i=1}^{\infty} B_i \subseteq \left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) \cup \left(\cup_{i=1}^{\infty} \overline{B_i} \setminus B_i\right)$$

and so

$$\begin{aligned} \overline{m_n}(E \setminus \cup_{i=1}^{\infty} B_i) &\leq \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) + m_n\left(\cup_{i=1}^{\infty} \overline{B_i} \setminus B_i\right) \\ &= \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) = 0. \end{aligned}$$

This implies you can fill up an open set with balls which cover the open set in the sense of Vitali.

Corollary 9.21 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{F}$ be a collection of closed or even open balls of bounded radii contained in U such that $\mathcal{F}$ covers U in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(U \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

9.5 Change Of Variables For Linear Maps

To begin with certain kinds of functions map measurable sets to measurable sets. It will be assumed that U is an open set in $\mathbb{R}^n$ and that $\mathbf{h} : U \to \mathbb{R}^n$ satisfies

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in U, \tag{9.11}$$

Lemma 9.22 *Let $\mathbf{h}$ satisfy 9.11. If $T \subseteq U$ and $m_n(T) = 0$, then $m_n(\mathbf{h}(T)) = 0$.*

Proof: Let
$$T_k \equiv \{\mathbf{x} \in T : ||D\mathbf{h}(\mathbf{x})|| < k\}$$
and let $\varepsilon > 0$ be given. Now by outer regularity, there exists an open set, V, containing T_k which is contained in U such that $m_n(V) < \varepsilon$. Let $\mathbf{x} \in T_k$. Then by differentiability,
$$\mathbf{h}(\mathbf{x}+\mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$
and so there exist arbitrarily small $r_{\mathbf{x}} < 1$ such that $B(\mathbf{x},5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| \leq r_{\mathbf{x}}, |o(\mathbf{v})| < k|\mathbf{v}|$. Thus
$$\mathbf{h}(B(\mathbf{x}, r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 2kr_{\mathbf{x}}).$$

From the Vitali covering theorem there exists a countable disjoint sequence of these sets, $\{B(\mathbf{x}_i, r_i)\}_{i=1}^{\infty}$ such that $\{B(\mathbf{x}_i, 5r_i)\}_{i=1}^{\infty} = \left\{\widehat{B_i}\right\}_{i=1}^{\infty}$ covers T_k Then letting $\overline{m_n}$ denote the outer measure determined by m_n,
$$\overline{m_n}(\mathbf{h}(T_k)) \leq \overline{m_n}\left(\mathbf{h}\left(\cup_{i=1}^{\infty}\widehat{B}_i\right)\right)$$
$$\leq \sum_{i=1}^{\infty} \overline{m_n}\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^{\infty} m_n(B(\mathbf{h}(\mathbf{x}_i), 2kr_{\mathbf{x}_i}))$$
$$\begin{aligned} &= \sum_{i=1}^{\infty} m_n(B(\mathbf{x}_i, 2kr_{\mathbf{x}_i})) = (2k)^n \sum_{i=1}^{\infty} m_n(B(\mathbf{x}_i, r_{\mathbf{x}_i})) \\ &\leq (2k)^n m_n(V) \leq (2k)^n \varepsilon. \end{aligned}$$
Since $\varepsilon > 0$ is arbitrary, this shows $m_n(\mathbf{h}(T_k)) = 0$. Now
$$m_n(\mathbf{h}(T)) = \lim_{k\to\infty} m_n(\mathbf{h}(T_k)) = 0.$$
This proves the lemma.

Lemma 9.23 *Let* $\mathbf{h}$ *satisfy 9.11. If* S *is a Lebesgue measurable subset of* U*, then* $\mathbf{h}(S)$ *is Lebesgue measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k), k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F, which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that
$$F \cup T = S_k.$$
Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 9.22, $m_n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is Lebesgue measurable because of completeness of Lebesgue measure. Now $\mathbf{h}(S) = \cup_{k=1}^{\infty}\mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is Lebesgue measurable. This proves the lemma.

In particular, this proves the following corollary.

Corollary 9.24 *Suppose A is an $n \times n$ matrix. Then if S is a Lebesgue measurable set, it follows AS is also a Lebesgue measurable set.*

Lemma 9.25 *Let R be unitary ($R^*R = RR^* = I$) and let V be a an open or closed set. Then $m_n(RV) = m_n(V)$.*

Proof: First assume V is a bounded open set. By Corollary 9.21 there is a disjoint sequence of closed balls, $\{B_i\}$ such that $U = \cup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and let r_i be the radius of B_i. Then by Lemma 9.22 $m_n(RV) = \sum_{i=1}^{\infty} m_n(RB_i)$. Now by invariance of translation of Lebesgue measure, this equals $\sum_{i=1}^{\infty} m_n(RB_i - R\mathbf{x}_i) = \sum_{i=1}^{\infty} m_n(RB(\mathbf{0}, r_i))$. Since R is unitary, it preserves all distances and so $RB(\mathbf{0}, r_i) = B(\mathbf{0}, r_i)$ and therefore,

$$m_n(RV) = \sum_{i=1}^{\infty} m_n(B(\mathbf{0}, r_i)) = \sum_{i=1}^{\infty} m_n(B_i) = m_n(V).$$

This proves the lemma in the case that V is bounded. Suppose now that V is just an open set. Let $V_k = V \cap B(\mathbf{0}, k)$. Then $m_n(RV_k) = m_n(V_k)$. Letting $k \to \infty$, this yields the desired conclusion. This proves the lemma in the case that V is open.

Suppose now that H is a closed and bounded set. Let $B(\mathbf{0}, R) \supseteq H$. Then letting $B = B(\mathbf{0}, R)$ for short,

$$\begin{aligned} m_n(RH) &= m_n(RB) - m_n(R(B \setminus H)) \\ &= m_n(B) - m_n(B \setminus H) = m_n(H). \end{aligned}$$

In general, let $H_m = H \cap \overline{B(\mathbf{0}, m)}$. Then from what was just shown, $m_n(RH_m) = m_n(H_m)$. Now let $m \to \infty$ to get the conclusion of the lemma in general. This proves the lemma.

Lemma 9.26 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let R be unitary. Then $m_n(RE) = m_n(E)$.*

Proof: First suppose E is bounded. Then there exist sets, G and H such that $H \subseteq E \subseteq G$ and H is the countable union of closed sets while G is the countable intersection of open sets such that $m_n(G \setminus H) = 0$. By Lemma 9.25 applied to these sets whose union or intersection equals H or G respectively, it follows

$$m_n(RG) = m_n(G) = m_n(H) = m_n(RH).$$

Therefore,

$$m_n(H) = m_n(RH) \le m_n(RE) \le m_n(RG) = m_n(G) = m_n(E) = m_n(H).$$

In the general case, let $E_m = E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \to \infty$.

Lemma 9.27 *Let V be an open or closed set in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Then $m_n(AV) = |\det(A)|\, m_n(V)$.*

Proof: Let RU be the right polar decomposition (Theorem 3.59 on Page 69) of A and let V be an open set. Then from Lemma 9.26,

$$m_n(AV) = m_n(RUV) = m_n(UV).$$

Now $U = Q^* DQ$ where D is a diagonal matrix such that $|\det(D)| = |\det(A)|$ and Q is unitary. Therefore,

$$m_n(AV) = m_n(Q^* DQV) = m_n(DQV).$$

Now QV is an open set and so by Corollary 9.6 on Page 200 and Lemma 9.25,

$$m_n(AV) = |\det(D)|\, m_n(QV) = |\det(D)|\, m_n(V) = |\det(A)|\, m_n(V).$$

This proves the lemma in case V is open.

Now let H be a closed set which is also bounded. First suppose $\det(A) = 0$. Then letting V be an open set containing H,

$$m_n(AH) \leq m_n(AV) = |\det(A)|\, m_n(V) = 0$$

which shows the desired equation is obvious in the case where $\det(A) = 0$. Therefore, assume A is one to one. Since H is bounded, $H \subseteq B(\mathbf{0}, R)$ for some $R > 0$. Then letting $B = B(\mathbf{0}, R)$ for short,

$$\begin{aligned} m_n(AH) &= m_n(AB) - m_n(A(B \setminus H)) \\ &= |\det(A)|\, m_n(B) - |\det(A)|\, m_n(B \setminus H) = |\det(A)|\, m_n(H). \end{aligned}$$

If H is not bounded, apply the result just obtained to $H_m \equiv H \cap \overline{B(\mathbf{0}, m)}$ and then let $m \to \infty$.

With this preparation, the main result is the following theorem.

Theorem 9.28 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Then $m_n(AE) = |\det(A)|\, m_n(E)$.*

Proof: First suppose E is bounded. Then there exist sets, G and H such that $H \subseteq E \subseteq G$ and H is the countable union of closed sets while G is the countable intersection of open sets such that $m_n(G \setminus H) = 0$. By Lemma 9.27 applied to these sets whose union or intersection equals H or G respectively, it follows

$$m_n(AG) = |\det(A)|\, m_n(G) = |\det(A)|\, m_n(H) = m_n(AH).$$

Therefore,

$$\begin{aligned} |\det(A)|\, m_n(E) &= |\det(A)|\, m_n(H) = m_n(AH) \leq m_n(AE) \\ &\leq m_n(AG) = |\det(A)|\, m_n(G) = |\det(A)|\, m_n(E). \end{aligned}$$

In the general case, let $E_m = E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \to \infty$.

9.6 Change Of Variables For C^1 Functions

In this section theorems are proved which generalize the above to C^1 functions. More general versions can be seen in Kuttler [28], Kuttler [29], and Rudin [36]. There is also a very different approach to this theorem given in [28]. The more general version in [28] follows [36] and both are based on the Brouwer fixed point theorem and a very clever lemma presented in Rudin [36]. This same approach will be used later in this book to prove a different sort of change of variables theorem in which the functions are only Lipschitz. The proof will be based on a sequence of easy lemmas.

Lemma 9.29 *Let U and V be bounded open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $f \in C_c(V)$. Then*

$$\int_V f(\mathbf{y})\, dy = \int_U f(\mathbf{h}(\mathbf{x})) \left|\det(D\mathbf{h}(\mathbf{x}))\right| dx$$

Proof: Let $\mathbf{x} \in U$. By the assumption that $\mathbf{h}$ and $\mathbf{h}^{-1}$ are C^1,

$$\begin{aligned}
\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) &= D\mathbf{h}(\mathbf{x})\,\mathbf{v} + \mathbf{o}(\mathbf{v}) \\
&= D\mathbf{h}(\mathbf{x})\left(\mathbf{v} + D\mathbf{h}^{-1}(\mathbf{h}(\mathbf{x}))\,\mathbf{o}(\mathbf{v})\right) \\
&= D\mathbf{h}(\mathbf{x})(\mathbf{v} + \mathbf{o}(\mathbf{v}))
\end{aligned}$$

and so if $r > 0$ is small enough then $B(\mathbf{x}, r)$ is contained in U and

$$\mathbf{h}(B(\mathbf{x}, r)) - \mathbf{h}(\mathbf{x}) = \mathbf{h}(\mathbf{x}+B(\mathbf{0},r)) - \mathbf{h}(\mathbf{x}) \subseteq D\mathbf{h}(\mathbf{x})(B(\mathbf{0},(1+\varepsilon)r)). \tag{9.12}$$

Making r still smaller if necessary, one can also obtain

$$|f(\mathbf{y}) - f(\mathbf{h}(\mathbf{x}))| < \varepsilon \tag{9.13}$$

for any $\mathbf{y} \in \mathbf{h}(B(\mathbf{x}, r))$ and

$$\left|f(\mathbf{h}(x_1))\left|\det(D\mathbf{h}(\mathbf{x}_1))\right| - f(\mathbf{h}(\mathbf{x}))\left|\det(D\mathbf{h}(\mathbf{x}))\right|\right| < \varepsilon \tag{9.14}$$

whenever $\mathbf{x}_1 \in B(\mathbf{x}, r)$. The collection of such balls is a Vitali cover of U. By Corollary 9.21 there is a sequence of disjoint closed balls $\{B_i\}$ such that $U = \cup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and r_i the radius. Then by Lemma 9.22, the monotone convergence theorem, and 9.12 - 9.14,

$$\begin{gathered}
\int_V f(\mathbf{y})\, dy = \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} f(\mathbf{y})\, dy \\
\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} f(\mathbf{h}(\mathbf{x}_i))\, dy \\
\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(\mathbf{h}(\mathbf{x}_i))\, m_n(\mathbf{h}(B_i)) \\
\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(\mathbf{h}(\mathbf{x}_i))\, m_n(D\mathbf{h}(\mathbf{x}_i)(B(\mathbf{0},(1+\varepsilon)r_i))) \\
= \varepsilon m_n(V) + (1+\varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(\mathbf{h}(\mathbf{x}_i)) \left|\det(D\mathbf{h}(\mathbf{x}_i))\right| dx \\
\leq \varepsilon m_n(V) + (1+\varepsilon)^n \sum_{i=1}^{\infty} \left(\int_{B_i} f(\mathbf{h}(\mathbf{x})) \left|\det(D\mathbf{h}(\mathbf{x}))\right| dx + \varepsilon m_n(B_i)\right) \\
\leq \varepsilon m_n(V) + (1+\varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(\mathbf{h}(\mathbf{x})) \left|\det(D\mathbf{h}(\mathbf{x}))\right| dx + (1+\varepsilon)^n \varepsilon m_n(U) \\
= \varepsilon m_n(V) + (1+\varepsilon)^n \int_U f(\mathbf{h}(\mathbf{x})) \left|\det(D\mathbf{h}(\mathbf{x}))\right| dx + (1+\varepsilon)^n \varepsilon m_n(U)
\end{gathered}$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\int_V f(\mathbf{y})\,dy \leq \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx \tag{9.15}$$

whenever $f \in C_c(V)$. Now $\mathbf{x} \to f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|$ is in $C_c(U)$ and so using the same argument with U and V switching roles and replacing $\mathbf{h}$ with $\mathbf{h}^{-1}$,

$$\begin{aligned} & \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx \\ \leq & \int_V f\left(\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\left|\det\left(D\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\right|\left|\det\left(D\mathbf{h}^{-1}(\mathbf{y})\right)\right|dy \\ = & \int_V f(\mathbf{y})\,dy \end{aligned}$$

by the chain rule. This with 9.15 proves the lemma.

Corollary 9.30 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $f \in C_c(V)$. Then*

$$\int_V f(\mathbf{y})\,dy = \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx$$

Proof: Choose m large enough that $\operatorname{spt}(f) \subseteq B(\mathbf{0},m) \cap V \equiv V_m$. Then let $\mathbf{h}^{-1}(V_m) = U_m$. From Lemma 9.29,

$$\begin{aligned} \int_V f(\mathbf{y})\,dy &= \int_{V_m} f(\mathbf{y})\,dy = \int_{U_m} f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx \\ &= \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx. \end{aligned}$$

This proves the corollary.

Corollary 9.31 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $E \subseteq V$ be measurable. Then*

$$\int_V \mathcal{X}_E(\mathbf{y})\,dy = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx.$$

Proof: Let $E_m = E \cap V_m$ where V_m and U_m are as in Corollary 9.30. By regularity of the measure there exist sets, K_k, G_k such that $K_k \subseteq E_m \subseteq G_k$, G_k is open, K_k is compact, and $m_n(G_k \setminus K_k) < 2^{-k}$. Let $K_k \prec f_k \prec G_k$. Then $f_k(\mathbf{y}) \to \mathcal{X}_{E_m}(\mathbf{y})$ a.e. because if $\mathbf{y}$ is such that convergence fails, it must be the case that $\mathbf{y}$ is in $G_k \setminus K_k$ infinitely often and $\sum_k m_n(G_k \setminus K_k) < \infty$. Let $N = \cap_m \cup_{k=m}^{\infty} G_k \setminus K_k$, the set of $\mathbf{y}$ which is in infinitely many of the $G_k \setminus K_k$. Then $f_k(\mathbf{h}(\mathbf{x}))$ must converge to $\mathcal{X}_E(\mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \notin \mathbf{h}^{-1}(N)$, a set of measure zero by Lemma 9.22. By Corollary 9.30

$$\int_{V_m} f_k(\mathbf{y})\,dy = \int_{U_m} f_k(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx.$$

By the dominated convergence theorem using a dominating function, $\mathcal{X}_{V_m}$ in the integral on the left and $\mathcal{X}_{U_m} |\det (D\mathbf{h})|$ on the right,

$$\int_{V_m} \mathcal{X}_{E_m} (\mathbf{y}) \, dy = \int_{U_m} \mathcal{X}_{E_m} (\mathbf{h} (\mathbf{x})) \, |\det (D\mathbf{h} (\mathbf{x}))| \, dx.$$

Therefore,

$$\begin{aligned} \int_V \mathcal{X}_{E_m} (\mathbf{y}) \, dy &= \int_{V_m} \mathcal{X}_{E_m} (\mathbf{y}) \, dy = \int_{U_m} \mathcal{X}_{E_m} (\mathbf{h} (\mathbf{x})) \, |\det (D\mathbf{h} (\mathbf{x}))| \, dx \\ &= \int_U \mathcal{X}_{E_m} (\mathbf{h} (\mathbf{x})) \, |\det (D\mathbf{h} (\mathbf{x}))| \, dx \end{aligned}$$

Let $m \to \infty$ and use the monotone convergence theorem to obtain the conclusion of the corollary.

With this corollary, the main theorem follows.

Theorem 9.32 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h} (U) = V$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_V g (\mathbf{y}) \, dy = \int_U g (\mathbf{h} (\mathbf{x})) \, |\det (D\mathbf{h} (\mathbf{x}))| \, dx. \tag{9.16}$$

Proof: From Corollary 9.31, 9.16 holds for any nonnegative simple function in place of g. In general, let $\{s_k\}$ be an increasing sequence of simple functions which converges to g pointwise. Then from the monotone convergence theorem

$$\begin{aligned} \int_V g (\mathbf{y}) \, dy &= \lim_{k \to \infty} \int_V s_k dy = \lim_{k \to \infty} \int_U s_k (\mathbf{h} (\mathbf{x})) \, |\det (D\mathbf{h} (\mathbf{x}))| \, dx \\ &= \int_U g (\mathbf{h} (\mathbf{x})) \, |\det (D\mathbf{h} (\mathbf{x}))| \, dx. \end{aligned}$$

This proves the theorem.

This is a pretty good theorem but it isn't too hard to generalize it. In particular, it is not necessary to assume $\mathbf{h}^{-1}$ is C^1.

Lemma 9.33 *Suppose V is an $n-1$ dimensional subspace of $\mathbb{R}^n$ and K is a compact subset of V. Then letting*

$$K_\varepsilon \equiv \cup_{\mathbf{x} \in K} B (\mathbf{x}, \varepsilon) = K + B (\mathbf{0}, \varepsilon),$$

it follows that

$$m_n (K_\varepsilon) \leq 2^n \varepsilon \left(\operatorname{diam} (K) + \varepsilon\right)^{n-1}.$$

Proof: Let an orthonormal basis for V be

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$$

and let

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n\}$$

be an orthonormal basis for $\mathbb{R}^n$. Now define a linear transformation, Q by $Q\mathbf{v}_i = \mathbf{e}_i$. Thus $QQ^* = Q^*Q = I$ and Q preserves all distances because

$$\left|Q\sum_i a_i \mathbf{e}_i\right|^2 = \left|\sum_i a_i \mathbf{v}_i\right|^2 = \sum_i |a_i|^2 = \left|\sum_i a_i \mathbf{e}_i\right|^2.$$

Letting $\mathbf{k}_0 \in K$, it follows $K \subseteq B(\mathbf{k}_0, \mathrm{diam}(K))$ and so,

$$QK \subseteq B^{n-1}(Q\mathbf{k}_0, \mathrm{diam}(QK)) = B^{n-1}(Q\mathbf{k}_0, \mathrm{diam}(K))$$

where B^{n-1} refers to the ball taken with respect to the usual norm in $\mathbb{R}^{n-1}$. Every point of K_ε is within ε of some point of K and so it follows that every point of QK_ε is within ε of some point of QK. Therefore,

$$QK_\varepsilon \subseteq B^{n-1}(Q\mathbf{k}_0, \mathrm{diam}(QK) + \varepsilon) \times (-\varepsilon, \varepsilon),$$

To see this, let $\mathbf{x} \in QK_\varepsilon$. Then there exists $\mathbf{k} \in QK$ such that $|\mathbf{k} - \mathbf{x}| < \varepsilon$. Therefore, $|(x_1, \cdots, x_{n-1}) - (k_1, \cdots, k_{n-1})| < \varepsilon$ and $|x_n - k_n| < \varepsilon$ and so $\mathbf{x}$ is contained in the set on the right in the above inclusion because $k_n = 0$. However, the measure of the set on the right is smaller than

$$[2(\mathrm{diam}(QK) + \varepsilon)]^{n-1}(2\varepsilon) = 2^n [(\mathrm{diam}(K) + \varepsilon)]^{n-1} \varepsilon.$$

This proves the lemma.

Note this is a very sloppy estimate. You can certainly do much better but this estimate is sufficient to prove Sard's lemma which follows.

Definition 9.34 *In any metric space, if $\mathbf{x}$ is a point of the metric space and S is a nonempty subset,*

$$\mathrm{dist}(\mathbf{x}, S) \equiv \inf\{d(x, s) : s \in S\}.$$

More generally, if T, S are two nonempty sets,

$$\mathrm{dist}(S, T) \equiv \inf\{d(t, s) : s \in S, t \in T\}.$$

Lemma 9.35 *The function $\mathbf{x} \to \mathrm{dist}(\mathbf{x}, S)$ is continuous.*

Proof: Let $\mathbf{x}, \mathbf{y}$ be given. Suppose $\mathrm{dist}(\mathbf{x}, S) \geq \mathrm{dist}(\mathbf{y}, S)$ and pick $\mathbf{s} \in S$ such that $\mathrm{dist}(\mathbf{y}, S) + \varepsilon \geq d(\mathbf{y}, \mathbf{s})$. Then

$$\begin{aligned} 0 &\leq \mathrm{dist}(\mathbf{x}, S) - \mathrm{dist}(\mathbf{y}, S) \leq \mathrm{dist}(\mathbf{x}, S) - (d(\mathbf{y}, \mathbf{s}) - \varepsilon) \\ &\leq d(\mathbf{x}, \mathbf{s}) - d(\mathbf{y}, \mathbf{s}) + \varepsilon \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{s}) - d(\mathbf{y}, \mathbf{s}) + \varepsilon = d(\mathbf{x}, \mathbf{y}) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $|\mathrm{dist}(\mathbf{x}, S) - \mathrm{dist}(\mathbf{y}, S)| \leq d(\mathbf{x}, \mathbf{y})$. This proves the lemma.

Lemma 9.36 *Let* $\mathbf{h}$ *be a* C^1 *function defined on an open set,* U *and let* K *be a compact subset of* U*. Then if* $\varepsilon > 0$ *is given, there exits* $r_1 > 0$ *such that if* $|\mathbf{v}| \leq r_1$*, then for all* $\mathbf{x} \in K$*,*

$$|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon |\mathbf{v}| .$$

Proof: Let $0 < \delta < \text{dist}\left(K, U^C\right)$. Such a positive number exists because if there exists a sequence of points in K, $\{\mathbf{k}_k\}$ and points in U^C, $\{\mathbf{s}_k\}$ such that $|\mathbf{k}_k - \mathbf{s}_k| \to 0$, then you could take a subsequence, still denoted by k such that $\mathbf{k}_k \to \mathbf{k} \in K$ and then $\mathbf{s}_k \to \mathbf{k}$ also. But U^C is closed so $\mathbf{k} \in K \cap U^C$, a contradiction. Then

$$\begin{aligned} \frac{|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\leq \frac{\left|\int_0^1 D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v}dt - D\mathbf{h}(\mathbf{x})\mathbf{v}\right|}{|\mathbf{v}|} \\ &\leq \frac{\int_0^1 |D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}|\, dt}{|\mathbf{v}|}. \end{aligned}$$

Now from uniform continuity of $D\mathbf{h}$ on the compact set, $\{\mathbf{x} : \text{dist}(\mathbf{x},K) \leq \delta\}$ it follows there exists $r_1 < \delta$ such that if $|\mathbf{v}| \leq r_1$, then $||D\mathbf{h}(\mathbf{x}+t\mathbf{v}) - D\mathbf{h}(\mathbf{x})|| < \varepsilon$ for every $\mathbf{x} \in K$. From the above formula, it follows that if $|\mathbf{v}| \leq r_1$,

$$\frac{|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} \leq \frac{\int_0^1 |D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}|\, dt}{|\mathbf{v}|} < \frac{\int_0^1 \varepsilon |\mathbf{v}|\, dt}{|\mathbf{v}|} = \varepsilon.$$

This proves the lemma.

A different proof of the following is in [28]. See also [29].

Lemma 9.37 *(Sard) Let* U *be an open set in* $\mathbb{R}^n$ *and let* $\mathbf{h} : U \to \mathbb{R}^n$ *be* C^1*. Let*

$$Z \equiv \{\mathbf{x} \in U : \det D\mathbf{h}(\mathbf{x}) = 0\} .$$

Then $m_n(\mathbf{h}(Z)) = 0$*.*

Proof: Let $\{U_k\}_{k=1}^\infty$ be an increasing sequence of open sets whose closures are compact and whose union equals U and let $Z_k \equiv Z \cap \overline{U_k}$. To obtain such a sequence, let $U_k = \left\{\mathbf{x} \in U : \text{dist}\left(\mathbf{x},U^C\right) < \frac{1}{k}\right\} \cap B(\mathbf{0}, k)$. First it is shown that $\mathbf{h}(Z_k)$ has measure zero. Let W be an open set contained in U_{k+1} which contains Z_k and satisfies

$$m_n(Z_k) + \varepsilon > m_n(W)$$

where here and elsewhere, $\varepsilon < 1$. Let

$$r = \text{dist}\left(\overline{U_k}, U_{k+1}^C\right)$$

and let $r_1 > 0$ be a constant as in Lemma 9.36 such that whenever $\mathbf{x} \in \overline{U_k}$ and $0 < |\mathbf{v}| \leq r_1$,

$$|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon |\mathbf{v}| . \tag{9.17}$$

Now the closures of balls which are contained in W and which have the property that their diameters are less than r_1 yield a Vitali covering of W. Therefore, by Corollary 9.21 there is a disjoint sequence of these closed balls, $\left\{\widetilde{B}_i\right\}$ such that

$$W = \cup_{i=1}^{\infty} \widetilde{B}_i \cup N$$

where N is a set of measure zero. Denote by $\{B_i\}$ those closed balls in this sequence which have nonempty intersection with Z_k, let d_i be the diameter of B_i, and let $\mathbf{z}_i$ be a point in $B_i \cap Z_k$. Since $\mathbf{z}_i \in Z_k$, it follows $D\mathbf{h}(\mathbf{z}_i) B(\mathbf{0},d_i) = D_i$ where D_i is contained in a subspace, V which has dimension $n-1$ and the diameter of D_i is no larger than $2C_k d_i$ where

$$C_k \geq \max\{||D\mathbf{h}(\mathbf{x})|| : \mathbf{x} \in Z_k\}$$

Then by 9.17, if $\mathbf{z} \in B_i$,

$$\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{z}_i) \in D_i + B(\mathbf{0}, \varepsilon d_i) \subseteq \overline{D_i} + B(\mathbf{0},\varepsilon d_i).$$

Thus

$$\mathbf{h}(B_i) \subseteq \mathbf{h}(\mathbf{z}_i) + \overline{D_i} + B(\mathbf{0},\varepsilon d_i)$$

By Lemma 9.33

$$\begin{aligned} m_n(\mathbf{h}(B_i)) &\leq 2^n (2C_k d_i + \varepsilon d_i)^{n-1} \varepsilon d_i \\ &\leq d_i^n \left(2^n [2C_k + \varepsilon]^{n-1}\right) \varepsilon \\ &\leq C_{n,k} m_n(B_i) \varepsilon. \end{aligned}$$

Therefore, by Lemma 9.22

$$\begin{aligned} m_n(\mathbf{h}(Z_k)) &\leq m_n(W) = \sum_i m_n(\mathbf{h}(B_i)) \leq C_{n,k}\varepsilon \sum_i m_n(B_i) \\ &\leq \varepsilon C_{n,k} m_n(W) \leq \varepsilon C_{n,k}(m_n(Z_k) + \varepsilon) \end{aligned}$$

Since ε is arbitrary, this shows $m_n(\mathbf{h}(Z_k)) = 0$ and so $0 = \lim_{k\to\infty} m_n(\mathbf{h}(Z_k)) = m_n(\mathbf{h}(Z))$.

With this important lemma, here is a generalization of Theorem 9.32.

Theorem 9.38 *Let U be an open set and let $\mathbf{h}$ be a $1-1$, C^1 function with values in $\mathbb{R}^n$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_{\mathbf{h}(U)} g(\mathbf{y})\, dy = \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx. \tag{9.18}$$

Proof: Let $Z = \{\mathbf{x} : \det(D\mathbf{h}(\mathbf{x})) = 0\}$. Then by the inverse function theorem, $\mathbf{h}^{-1}$ is C^1 on $\mathbf{h}(U \setminus Z)$ and $\mathbf{h}(U \setminus Z)$ is an open set. Therefore, from Lemma 9.37 and Theorem 9.32,

$$\begin{aligned} \int_{\mathbf{h}(U)} g(\mathbf{y})\, dy &= \int_{\mathbf{h}(U\setminus Z)} g(\mathbf{y})\, dy = \int_{U\setminus Z} g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx \\ &= \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx. \end{aligned}$$

This proves the theorem.

Of course the next generalization considers the case when $\mathbf{h}$ is not even one to one.

9.7 Mappings Which Are Not One To One

Now suppose $\mathbf{h}$ is only C^1, not necessarily one to one. For

$$U_+ \equiv \{\mathbf{x} \in U : |\det D\mathbf{h}(x)| > 0\}$$

and Z the set where $|\det D\mathbf{h}(\mathbf{x})| = 0$, Lemma 9.37 implies $m_n(\mathbf{h}(Z)) = 0$. For $\mathbf{x} \in U_+$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}}$ such that $\mathbf{x} \in B_{\mathbf{x}} \subseteq U_+$, $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\{B_i\}$ be a countable subset of $\{B_{\mathbf{x}}\}_{\mathbf{x}\in U_+}$ such that $U_+ = \cup_{i=1}^{\infty} B_i$. Let $E_1 = B_1$. If $E_1, \cdots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \cup_{i=1}^{k} E_i$. Thus

$$\cup_{i=1}^{\infty} E_i = U_+, \ \ \mathbf{h} \text{ is one to one on } E_i, \ \ E_i \cap E_j = \emptyset,$$

and each E_i is a Borel set contained in the open set B_i. Now define

$$n(\mathbf{y}) \equiv \sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y}).$$

The set, $\mathbf{h}(E_i), \mathbf{h}(Z)$ are measurable by Lemma 9.23. Thus $n(\cdot)$ is measurable.

Lemma 9.39 *Let $F \subseteq \mathbf{h}(U)$ be measurable. Then*

$$\int_{\mathbf{h}(U)} n(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx.$$

Proof: Using Lemma 9.37 and the Monotone Convergence Theorem or Fubini's Theorem,

$$\begin{aligned}
\int_{\mathbf{h}(U)} n(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy &= \int_{\mathbf{h}(U)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \overbrace{\mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{m_n(\mathbf{h}(Z))=0} \right) \mathcal{X}_F(\mathbf{y})dy \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(U)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy \\
&= \sum_{i=1}^{\infty} \int_{B_i} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx \\
&= \sum_{i=1}^{\infty} \int_{U} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx \\
&= \int_{U} \sum_{i=1}^{\infty} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx
\end{aligned}$$

$$= \int_{U_+} \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx.$$

This proves the lemma.

Definition 9.40 *For $\mathbf{y} \in \mathbf{h}(U)$, define a function, $\#$, according to the formula*

$$\#(\mathbf{y}) \equiv \text{number of elements in } \mathbf{h}^{-1}(\mathbf{y}).$$

Observe that

$$\#(\mathbf{y}) = n(\mathbf{y}) \quad \text{a.e.} \tag{9.19}$$

because $n(\mathbf{y}) = \#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of measure 0. Therefore, $\#$ is a measurable function.

Theorem 9.41 *Let $g \geq 0$, g measurable, and let $\mathbf{h}$ be $C^1(U)$. Then*

$$\int_{\mathbf{h}(U)} \#(\mathbf{y})g(\mathbf{y})dy = \int_U g(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx. \tag{9.20}$$

Proof: From 9.19 and Lemma 9.39, 9.20 holds for all g, a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, g, with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 9.20 for an arbitrary nonnegative measurable function, g. This proves the theorem.

9.8 Lebesgue Measure And Iterated Integrals

The following is the main result.

Theorem 9.42 *Let $f \geq 0$ and suppose f is a Lebesgue measurable function defined on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} f dm_n < \infty$. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

This will be accomplished by Fubini's theorem, Theorem 8.49 and the following lemma.

Lemma 9.43 $\overline{m_k \times m_{n-k}} = m_n$ *on the m_n measurable sets.*

Proof: First of all, let $R = \prod_{i=1}^n (a_i, b_i]$ be a measurable rectangle and let $R_k = \prod_{i=1}^k (a_i, b_i]$, $R_{n-k} = \prod_{i=k+1}^n (a_i, b_i]$. Then by Fubini's theorem,

$$\begin{aligned} \int \mathcal{X}_R d\left(\overline{m_k \times m_{n-k}}\right) &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \mathcal{X}_{R_k} \mathcal{X}_{R_{n-k}} dm_k dm_{n-k} \\ &= \int_{\mathbb{R}^k} \mathcal{X}_{R_k} dm_k \int_{\mathbb{R}^{n-k}} \mathcal{X}_{R_{n-k}} dm_{n-k} \\ &= \int \mathcal{X}_R dm_n \end{aligned}$$

and so $\overline{m_k \times m_{n-k}}$ and m_n agree on every half open rectangle. By Lemma 9.2 these two measures agree on every open set. Now if K is a compact set, then $K = \cap_{k=1}^{\infty} U_k$ where U_k is the open set, $K + B\left(\mathbf{0}, \frac{1}{k}\right)$. Another way of saying this is $U_k \equiv \left\{\mathbf{x} : \text{dist}\left(\mathbf{x}, K\right) < \frac{1}{k}\right\}$ which is obviously open because $\mathbf{x} \to \text{dist}\left(\mathbf{x}, K\right)$ is a continuous function. Since K is the countable intersection of these decreasing open sets, each of which has finite measure with respect to either of the two measures, it follows that $\overline{m_k \times m_{n-k}}$ and m_n agree on all the compact sets. Now let E be a bounded Lebesgue measurable set. Then there are sets, H and G such that H is a countable union of compact sets, G a countable intersection of open sets, $H \subseteq E \subseteq G$, and $m_n\left(G \setminus H\right) = 0$. Then from what was just shown about compact and open sets, the two measures agree on G and on H. Therefore,

$$\begin{aligned} m_n\left(H\right) &= \overline{m_k \times m_{n-k}}\left(H\right) \leq \overline{m_k \times m_{n-k}}\left(E\right) \\ &\leq \overline{m_k \times m_{n-k}}\left(G\right) = m_n\left(E\right) = m_n\left(H\right) \end{aligned}$$

By completeness of the measure space for $\overline{m_k \times m_{n-k}}$, it follows that E is $\overline{m_k \times m_{n-k}}$ measurable and $\overline{m_k \times m_{n-k}}\left(E\right) = m_n\left(E\right)$. This proves the lemma.

You could also show that the two σ algebras are the same. However, this is not needed for the lemma or the theorem.

Proof of Theorem 9.42: By the lemma and Fubini's theorem, Theorem 8.49,

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^n} f d\left(\overline{m_k \times m_{n-k}}\right) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

Corollary 9.44 *Let f be a nonnegative real valued measurable function. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

Proof: Let $S_p \equiv \left\{\mathbf{x} \in \mathbb{R}^n : 0 \leq f\left(\mathbf{x}\right) \leq p\right\} \cap B\left(\mathbf{0}, p\right)$. Then $\int_{\mathbb{R}^n} f \mathcal{X}_{S_p} dm_n < \infty$. Therefore, from Theorem 9.42,

$$\int_{\mathbb{R}^n} f \mathcal{X}_{S_p} dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \mathcal{X}_{S_p} f dm_{n-k} dm_k.$$

Now let $p \to \infty$ and use the Monotone convergence theorem and the Fubini Theorem 8.49 on Page 191.

Not surprisingly, the following corollary follows from this.

Corollary 9.45 *Let $f \in L^1\left(\mathbb{R}^n\right)$ where the measure is m_n. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

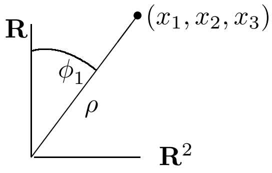
Proof: Apply Corollary 9.44 to the postive and negative parts of the real and imaginary parts of f.

9.9 Spherical Coordinates In Many Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

$$\begin{aligned} y_1 &= \rho \cos \theta \\ y_2 &= \rho \sin \theta \end{aligned}$$

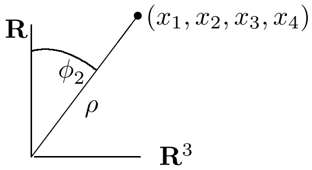
where $\rho > 0$ and $\theta \in [0, 2\pi)$. Here I am writing ρ in place of r to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing y_1 and y_2 instead of the more usual x and y. Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.



From this picture, you see that $y_3 = \rho \cos \phi_1$. Also the distance between (y_1, y_2) and $(0, 0)$ is $\rho \sin (\phi_1)$. Therefore, using polar coordinates to write (y_1, y_2) in terms of θ and this distance,

$$\begin{aligned} y_1 &= \rho \sin \phi_1 \cos \theta, \\ y_2 &= \rho \sin \phi_1 \sin \theta, \\ y_3 &= \rho \cos \phi_1. \end{aligned}$$

where $\phi_1 \in [0, \pi]$. What was done is to replace ρ with $\rho \sin \phi_1$ and then to add in $y_3 = \rho \cos \phi_1$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:



From this picture, you see that $y_4 = \rho \cos \phi_2$. Also the distance between (y_1, y_2, y_3) and $(0, 0, 0)$ is $\rho \sin (\phi_2)$. Therefore, using polar coordinates to write (y_1, y_2, y_3) in terms of θ, ϕ_1, and this distance,

$$\begin{aligned}
y_1 &= \rho \sin \phi_2 \sin \phi_1 \cos \theta, \\
y_2 &= \rho \sin \phi_2 \sin \phi_1 \sin \theta, \\
y_3 &= \rho \sin \phi_2 \cos \phi_1, \\
y_4 &= \rho \cos \phi_2
\end{aligned}$$

where $\phi_2 \in [0, \pi]$.

Continuing this way, given spherical coordinates in $\mathbb{R}^n$, to get the spherical coordinates in $\mathbb{R}^{n+1}$, you let $y_{n+1} = \rho \cos \phi_{n-1}$ and then replace every occurance of ρ with $\rho \sin \phi_{n-1}$ to obtain $y_1 \cdots y_n$ in terms of $\phi_1, \phi_2, \cdots, \phi_{n-1}, \theta$, and ρ.

It is always the case that ρ measures the distance from the point in $\mathbb{R}^n$ to the origin in $\mathbb{R}^n$, $\mathbf{0}$. Each $\phi_i \in [0, \pi]$, and $\theta \in [0, 2\pi)$. It can be shown using math induction that these coordinates map $\prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi) \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Theorem 9.46 *Let $\mathbf{y} = \mathbf{h}(\boldsymbol{\phi}, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^n$. Then letting $A = \prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi)$, it follows $\mathbf{h}$ maps $A \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Also $|\det D\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)|$ will always be of the form*

$$|\det D\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)| = \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta). \tag{9.21}$$

where Φ is a continuous function of $\boldsymbol{\phi}$ and θ.[1] *Furthermore whenever f is Lebesgue measurable and nonnegative,*

$$\int_{\mathbb{R}^n} f(\mathbf{y})\, dy = \int_0^\infty \rho^{n-1} \int_A f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho))\, \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta d\rho \tag{9.22}$$

where here $d\boldsymbol{\phi}\, d\theta$ denotes dm_{n-1} on A. The same formula holds if $f \in L^1(\mathbb{R}^n)$.

Proof: Formula 9.21 is obvious from the definition of the spherical coordinates. The first claim is also clear from the definition and math induction. It remains to verify 9.22. Let $A_0 \equiv \prod_{i=1}^{n-2} (0, \pi) \times (0, 2\pi)$. Then it is clear that $(A \setminus A_0) \times (0, \infty) \equiv N$ is a set of measure zero in $\mathbb{R}^n$. Therefore, from Lemma 9.22 it follows $\mathbf{h}(N)$ is also a set of measure zero. Therefore, using the change of variables theorem, Corollary 9.44, and Sard's lemma,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\mathbf{y})\, dy &= \int_{\mathbb{R}^n \setminus \{\mathbf{0}\}} f(\mathbf{y})\, dy = \int_{\mathbb{R}^n \setminus (\{\mathbf{0}\} \cup \mathbf{h}(N))} f(\mathbf{y})\, dy \\
&= \int_{A_0 \times (0,\infty)} f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho))\, \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta)\, dm_n \\
&= \int \mathcal{X}_{A \times (0,\infty)}(\boldsymbol{\phi}, \theta, \rho)\, f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho))\, \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta)\, dm_n \\
&= \int_0^\infty \rho^{n-1} \left(\int_A f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho))\, \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta \right) d\rho.
\end{aligned}$$

[1] Actually it is only a function of the first but this is not important in what follows.

Now the claim about $f \in L^1$ follows routinely from considering the positive and negative parts of the real and imaginary parts of f in the usual way. This proves the theorem.

Notation 9.47 *Often this is written differently. Note that from the spherical coordinate formulas, $f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) = f(\rho\boldsymbol{\omega})$ where $|\boldsymbol{\omega}| = 1$. Letting S^{n-1} denote the unit sphere, $\{\boldsymbol{\omega} \in \mathbb{R}^n : |\boldsymbol{\omega}| = 1\}$, the inside integral in the above formula is sometimes written as*

$$\int_{S^{n-1}} f(\rho\boldsymbol{\omega})\, d\sigma$$

where σ is a measure on S^{n-1}. See [28] for another description of this measure. It isn't an important issue here. Later in the book when integration on manifolds is discussed, more general considerations will be dealt with. Either 9.22 or the formula

$$\int_0^\infty \rho^{n-1} \left(\int_{S^{n-1}} f(\rho\boldsymbol{\omega})\, d\sigma \right) d\rho$$

will be referred to as polar coordinates and is very useful in establishing estimates. Here $\sigma\left(S^{n-1}\right) \equiv \int_A \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta$.

Example 9.48 *For what values of s is the integral $\int_{B(\mathbf{0},R)} \left(1 + |\mathbf{x}|^2\right)^s dy$ bounded independent of R? Here $B(\mathbf{0}, R)$ is the ball, $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$.*

I think you can see immediately that s must be negative but exactly how negative? It turns out it depends on n and using polar coordinates, you can find just exactly what is needed. From the polar coordinats formula above,

$$\begin{aligned} \int_{B(\mathbf{0},R)} \left(1 + |\mathbf{x}|^2\right)^s dy &= \int_0^R \int_{S^{n-1}} \left(1 + \rho^2\right)^s \rho^{n-1} d\sigma d\rho \\ &= C_n \int_0^R \left(1 + \rho^2\right)^s \rho^{n-1} d\rho \end{aligned}$$

Now the very hard problem has been reduced to considering an easy one variable problem of finding when

$$\int_0^R \rho^{n-1} \left(1 + \rho^2\right)^s d\rho$$

is bounded independent of R. You need $2s + (n-1) < -1$ so you need $s < -n/2$.

9.10 The Brouwer Fixed Point Theorem

This seems to be a good place to present a short proof of one of the most important of all fixed point theorems. There are many approaches to this but the easiest and shortest I have ever seen is the one in Dunford and Schwartz [15]. This is what is presented here. In Evans [18] there is a different proof which depends on integration theory. A good reference for an introduction to various kinds of fixed point theorems is the book by Smart [39]. This book also gives an entirely different approach to the Brouwer fixed point theorem.

The proof given here is based on the following lemma. Recall that the mixed partial derivatives of a C^2 function are equal. In the following lemma, and elsewhere, a comma followed by an index indicates the partial derivative with respect to the indicated variable. Thus, $f_{,j}$ will mean $\frac{\partial f}{\partial x_j}$. Also, write $D\mathbf{g}$ for the Jacobian matrix which is the matrix of $D\mathbf{g}$ taken with respect to the usual basis vectors in $\mathbb{R}^n$. Recall that for A an $n \times n$ matrix, $\operatorname{cof}(A)_{ij}$ is the determinant of the matrix which results from deleting the i^{th} row and the j^{th} column and multiplying by $(-1)^{i+j}$.

Lemma 9.49 *Let $\mathbf{g} : U \to \mathbb{R}^n$ be C^2 where U is an open subset of $\mathbb{R}^n$. Then*

$$\sum_{j=1}^{n} \operatorname{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\operatorname{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Proof: From the cofactor expansion theorem,

$$\det(D\mathbf{g}) = \sum_{i=1}^{n} g_{i,j} \operatorname{cof}(D\mathbf{g})_{ij}$$

and so

$$\frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}} = \operatorname{cof}(D\mathbf{g})_{ij} \tag{9.23}$$

which shows the last claim of the lemma. Also

$$\delta_{kj} \det(D\mathbf{g}) = \sum_i g_{i,k} (\operatorname{cof}(D\mathbf{g}))_{ij} \tag{9.24}$$

because if $k \neq j$ this is just the cofactor expansion of the determinant of a matrix in which the k^{th} and j^{th} columns are equal. Differentiate 9.24 with respect to x_j and sum on j. This yields

$$\sum_{r,s,j} \delta_{kj} \frac{\partial(\det D\mathbf{g})}{\partial g_{r,s}} g_{r,sj} = \sum_{ij} g_{i,kj} (\operatorname{cof}(D\mathbf{g}))_{ij} + \sum_{ij} g_{i,k} \operatorname{cof}(D\mathbf{g})_{ij,j}.$$

Hence, using $\delta_{kj} = 0$ if $j \neq k$ and 9.23,

$$\sum_{rs} (\operatorname{cof}(D\mathbf{g}))_{rs} g_{r,sk} = \sum_{rs} g_{r,ks} (\operatorname{cof}(D\mathbf{g}))_{rs} + \sum_{ij} g_{i,k} \operatorname{cof}(D\mathbf{g})_{ij,j}.$$

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

$$\sum_i g_{i,k} \left(\sum_j (\operatorname{cof}(D\mathbf{g}))_{ij,j} \right) = 0.$$

If $\det(g_{i,k}) \neq 0$ so that $(g_{i,k})$ is invertible, this shows $\sum_j (\operatorname{cof}(D\mathbf{g}))_{ij,j} = 0$. If $\det(D\mathbf{g}) = 0$, let

$$g_k = g + \varepsilon_k I$$

where $\varepsilon_k \to 0$ and $\det(D\mathbf{g} + \varepsilon_k I) \equiv \det(D\mathbf{g}_k) \neq 0$. Then

$$\sum_j (\operatorname{cof}(D\mathbf{g}))_{ij,j} = \lim_{k\to\infty} \sum_j (\operatorname{cof}(D\mathbf{g}_k))_{ij,j} = 0$$

and this proves the lemma.

To prove the Brouwer fixed point theorem, first consider a version of it valid for C^2 mappings. This is the following lemma.

Lemma 9.50 *Let $B_r = \overline{B(\mathbf{0},r)}$ and suppose $\mathbf{g}$ is a C^2 function defined on $\mathbb{R}^n$ which maps B_r to B_r. Then $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ for some $\mathbf{x} \in B_r$.*

Proof: Suppose not. Then $|\mathbf{g}(\mathbf{x}) - \mathbf{x}|$ must be bounded away from zero on B_r. Let $a(\mathbf{x})$ be the larger of the two roots of the equation,

$$|\mathbf{x}+a(\mathbf{x})(\mathbf{x}-\mathbf{g}(\mathbf{x}))|^2 = r^2. \tag{9.25}$$

Thus

$$a(\mathbf{x}) = \frac{-(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) + \sqrt{(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))^2 + \left(r^2 - |\mathbf{x}|^2\right)|\mathbf{x}-\mathbf{g}(\mathbf{x})|^2}}{|\mathbf{x}-\mathbf{g}(\mathbf{x})|^2} \tag{9.26}$$

The expression under the square root sign is always nonnegative and it follows from the formula that $a(\mathbf{x}) \geq 0$. Therefore, $(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) \geq 0$ for all $\mathbf{x} \in B_r$. The reason for this is that $a(\mathbf{x})$ is the larger zero of a polynomial of the form $p(z) = |\mathbf{x}|^2 + z^2|\mathbf{x}-g(\mathbf{x})|^2 - 2z(\mathbf{x},\mathbf{x}-\mathbf{g}(\mathbf{x}))$ and from the formula above, it is nonnegative. $-2(\mathbf{x},\mathbf{x}-\mathbf{g}(\mathbf{x}))$ is the slope of the tangent line to $p(z)$ at $z=0$. If $\mathbf{x} \neq \mathbf{0}$, then $|\mathbf{x}|^2 > 0$ and so this slope needs to be negative for the larger of the two zeros to be positive. If $\mathbf{x} = \mathbf{0}$, then $(\mathbf{x},\mathbf{x}-\mathbf{g}(\mathbf{x})) = 0$.

Now define for $t \in [0,1]$,

$$\mathbf{f}(t,\mathbf{x}) \equiv \mathbf{x}+ta(\mathbf{x})(\mathbf{x}-\mathbf{g}(\mathbf{x})).$$

The important properties of $\mathbf{f}(t,\mathbf{x})$ and $a(\mathbf{x})$ are that

$$a(\mathbf{x}) = 0 \text{ if } |\mathbf{x}| = r. \tag{9.27}$$

and

$$|\mathbf{f}(t,\mathbf{x})| = r \text{ for all } |\mathbf{x}| = r \tag{9.28}$$

These properties follow immediately from 9.26 and the above observation that for $\mathbf{x} \in B_r$, it follows $(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) \geq 0$.

Also from 9.26, a is a C^2 function near B_r. This is obvious from 9.26 as long as $|\mathbf{x}| < r$. However, even if $|\mathbf{x}| = r$ it is still true. To show this, it suffices to verify the expression under the square root sign is positive. If this expression were not positive for some $|\mathbf{x}| = r$, then $(\mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x}))) = 0$. Then also, since $\mathbf{g}(\mathbf{x}) \neq \mathbf{x}$,

$$\left|\frac{\mathbf{g}(\mathbf{x}) + \mathbf{x}}{2}\right| < r$$

and so

$$r^2 > \left(\mathbf{x}, \frac{\mathbf{g}(\mathbf{x}) + \mathbf{x}}{2}\right) = \frac{1}{2}(\mathbf{x}, \mathbf{g}(\mathbf{x})) + \frac{r^2}{2} = \frac{|\mathbf{x}|^2}{2} + \frac{r^2}{2} = r^2,$$

a contradiction. Therefore, the expression under the square root in 9.26 is always positive near B_r and so a is a C^2 function near B_r as claimed because the square root function is C^2 away from zero.

Now define

$$I(t) \equiv \int_{B_r} \det(D_2\mathbf{f}(t, \mathbf{x}))\, dx.$$

Then

$$I(0) = \int_{B_r} dx = m_n(B_r) > 0. \tag{9.29}$$

Using the dominated convergence theorem one can differentiate $I(t)$ as follows.

$$\begin{aligned} I'(t) &= \int_{B_r} \sum_{ij} \frac{\partial \det(D_2\mathbf{f}(t, \mathbf{x}))}{\partial f_{i,j}} \frac{\partial f_{i,j}}{\partial t} dx \\ &= \int_{B_r} \sum_{ij} \operatorname{cof}(D_2\mathbf{f})_{ij} \frac{\partial (a(\mathbf{x})(x_i - g_i(\mathbf{x})))}{\partial x_j} dx. \end{aligned}$$

Now from 9.27 $a(\mathbf{x}) = 0$ when $|\mathbf{x}| = r$ and so integration by parts and Lemma 9.49 yields

$$\begin{aligned} I'(t) &= \int_{B_r} \sum_{ij} \operatorname{cof}(D_2\mathbf{f})_{ij} \frac{\partial (a(\mathbf{x})(x_i - g_i(\mathbf{x})))}{\partial x_j} dx \\ &= -\int_{B_r} \sum_{ij} \operatorname{cof}(D_2\mathbf{f})_{ij,j}\, a(\mathbf{x})(x_i - g_i(\mathbf{x}))\, dx = 0. \end{aligned}$$

Therefore, $I(1) = I(0)$. However, from 9.25 it follows that for $t = 1$,

$$\sum_i f_i f_i = r^2$$

and so, $\sum_i f_{i,j} f_i = 0$ which implies since $|\mathbf{f}(1, \mathbf{x})| = r$ by 9.25, that $\det(f_{i,j}) = \det(D_2\mathbf{f}(1, \mathbf{x})) = 0$ and so $I(1) = 0$, a contradiction to 9.29 since $I(1) = I(0)$. This proves the lemma.

The following theorem is the Brouwer fixed point theorem for a ball.

Theorem 9.51 *Let B_r be the above closed ball and let $\mathbf{f} : B_r \to B_r$ be continuous. Then there exists $\mathbf{x} \in B_r$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof: Let $\mathbf{f}_k(\mathbf{x}) \equiv \frac{\mathbf{f}(\mathbf{x})}{1+k^{-1}}$. Thus $||\mathbf{f}_k - \mathbf{f}|| < \frac{r}{1+k}$ where

$$||\mathbf{h}|| \equiv \max\{|\mathbf{h}(\mathbf{x})| : \mathbf{x} \in B_r\}.$$

Using the Weierstrass approximation theorem, there exists a polynomial $\mathbf{g}_k$ such that $||\mathbf{g}_k - \mathbf{f}_k|| < \frac{r}{k+1}$. Then if $\mathbf{x} \in B_r$, it follows

$$\begin{aligned} |\mathbf{g}_k(\mathbf{x})| &\leq |\mathbf{g}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{x})| + |\mathbf{f}_k(\mathbf{x})| \\ &< \frac{r}{1+k} + \frac{kr}{1+k} = r \end{aligned}$$

and so $\mathbf{g}_k$ maps B_r to B_r. By Lemma 9.50 each of these $\mathbf{g}_k$ has a fixed point, $\mathbf{x}_k$ such that $\mathbf{g}_k(\mathbf{x}_k) = \mathbf{x}_k$. The sequence of points, $\{\mathbf{x}_k\}$ is contained in the compact set, B_r and so there exists a convergent subsequence still denoted by $\{\mathbf{x}_k\}$ which converges to a point, $\mathbf{x} \in B_r$. Then

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{x}| &\leq |\mathbf{f}(\mathbf{x}) - \mathbf{f}_k(\mathbf{x})| + |\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{x}_k)| + \left|\mathbf{f}_k(\mathbf{x}_k) - \overbrace{\mathbf{g}_k(\mathbf{x}_k)}^{=\mathbf{x}_k}\right| + |\mathbf{x}_k - \mathbf{x}| \\ &\leq \frac{r}{1+k} + |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_k)| + \frac{r}{1+k} + |\mathbf{x}_k - \mathbf{x}|. \end{aligned}$$

Now let $k \to \infty$ in the right side to conclude $\mathbf{f}(\mathbf{x}) = \mathbf{x}$. This proves the theorem.

It is not surprising that the ball does not need to be centered at $\mathbf{0}$.

Corollary 9.52 *Let $\mathbf{f} : \overline{B(\mathbf{a}, r)} \to \overline{B(\mathbf{a}, r)}$ be continuous. Then there exists $\mathbf{x} \in \overline{B(\mathbf{a}, r)}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof: Let $\mathbf{g} : B_r \to B_r$ be defined by $\mathbf{g}(\mathbf{y}) \equiv \mathbf{f}(\mathbf{y} + \mathbf{a}) - \mathbf{a}$. Then $\mathbf{g}$ is a continuous map from B_r to B_r. Therefore, there exists $\mathbf{y} \in B_r$ such that $\mathbf{g}(\mathbf{y}) = \mathbf{y}$. Therefore, $\mathbf{f}(\mathbf{y} + \mathbf{a}) - \mathbf{a} = \mathbf{y}$ and so letting $\mathbf{x} = \mathbf{y} + \mathbf{a}$, $\mathbf{f}$ also has a fixed point as claimed.

9.11 Exercises

1. Let $R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Show that R preserves distances if and only if $RR^* = R^*R = I$.

2. Let f be a nonnegative strictly decreasing function defined on $[0, \infty)$. For $0 \leq y \leq f(0)$, let $f^{-1}(y) = x$ where $y \in [f(x+), f(x-)]$. (Draw a picture. f could have jump discontinuities.) Show that f^{-1} is nonincreasing and that

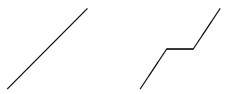
$$\int_0^\infty f(t)\,dt = \int_0^{f(0)} f^{-1}(y)\,dy.$$

Hint: Use the distribution function description.

3. Let X be a metric space and let $Y \subseteq X$, so Y is also a metric space. Show the Borel sets of Y are the Borel sets of X intersected with the set, Y.

4. Consider the following nested sequence of compact sets, $\{P_n\}$. We let $P_1 = [0,1]$, $P_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, etc. To go from P_n to P_{n+1}, delete the open interval which is the middle third of each closed interval in P_n. Let $P = \cap_{n=1}^{\infty} P_n$. Since P is the intersection of nested nonempty compact sets, it follows from advanced calculus that $P \neq \emptyset$. Show $m(P) = 0$. Show there is a one to one onto mapping of $[0,1]$ to P. The set P is called the Cantor set. Thus, although P has measure zero, it has the same number of points in it as $[0,1]$ in the sense that there is a one to one and onto mapping from one to the other. **Hint:** There are various ways of doing this last part but the most enlightenment is obtained by exploiting the construction of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the theorems related to the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0,1]$. If you do this right and remember the theorems about characterizations of compact metric spaces, you may get a pretty good idea why every compact metric space is the continuous image of the Cantor set which is a really interesting theorem in topology.

5. $\uparrow$ Consider the sequence of functions defined in the following way. Let $f_1(x) = x$ on $[0,1]$. To get from f_n to f_{n+1}, let $f_{n+1} = f_n$ on all intervals where f_n is constant. If f_n is nonconstant on $[a,b]$, let $f_{n+1}(a) = f_n(a)$, $f_{n+1}(b) = f_n(b)$, f_{n+1} is piecewise linear and equal to $\frac{1}{2}(f_n(a) + f_n(b))$ on the middle third of $[a,b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.



Show $\{f_n\}$ converges uniformly on $[0,1]$. If $f(x) = \lim_{n\to\infty} f_n(x)$, show that $f(0) = 0$, $f(1) = 1$, f is continuous, and $f'(x) = 0$ for all $x \notin P$ where P is the Cantor set. This function is called the Cantor function.It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1. **Hint:** This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.

6. Let $m(W) > 0$, W is measurable, $W \subseteq [a,b]$. Show there exists a nonmeasurable subset of W. **Hint**: Let $x \sim y$ if $x - y \in \mathbb{Q}$. Observe that $\sim$ is an equivalence relation on $\mathbb{R}$. See Definition 1.9 on Page 17 for a review of this terminology. Let $\mathcal{C}$ be the set of equivalence classes and let $\mathcal{D} \equiv \{C \cap W : C \in \mathcal{C}$ and $C \cap W \neq \emptyset\}$. By the axiom of choice, there exists a set, A, consisting of

exactly one point from each of the nonempty sets which are the elements of $\mathcal{D}$. Show

$$W \subseteq \cup_{r\in\mathbb{Q}} A + r \tag{a.}$$

$$A + r_1 \cap A + r_2 = \emptyset \text{ if } r_1 \neq r_2,\ r_i \in \mathbb{Q}. \tag{b.}$$

Observe that since $A \subseteq [a,b]$, then $A+r \subseteq [a-1,b+1]$ whenever $|r| < 1$. Use this to show that if $m(A) = 0$, or if $m(A) > 0$ a contradiction results.Show there exists some set, S such that $\overline{m}(S) < \overline{m}(S\cap A) + \overline{m}(S \setminus A)$ where $\overline{m}$ is the outer measure determined by m.

7. $\uparrow$ This problem gives a very interesting example found in the book by McShane [32]. Let $g(x) = x + f(x)$ where f is the strange function of Problem 5. Let P be the Cantor set of Problem 4. Let $[0,1] \setminus P = \cup_{j=1}^{\infty} I_j$ where I_j is open and $I_j \cap I_k = \emptyset$ if $j \neq k$. These intervals are the connected components of the complement of the Cantor set. Show $m(g(I_j)) = m(I_j)$ so

$$m(g(\cup_{j=1}^{\infty} I_j)) = \sum_{j=1}^{\infty} m(g(I_j)) = \sum_{j=1}^{\infty} m(I_j) = 1.$$

Thus $m(g(P)) = 1$ because $g([0,1]) = [0,2]$. By Problem 6 there exists a set, $A \subseteq g(P)$ which is non measurable. Define $\phi(x) = \mathcal{X}_A(g(x))$. Thus $\phi(x) = 0$ unless $x \in P$. Tell why ϕ is measurable. (Recall $m(P) = 0$ and Lebesgue measure is complete.) Now show that $\mathcal{X}_A(y) = \phi(g^{-1}(y))$ for $y \in [0,2]$. Tell why g^{-1} is continuous but $\phi \circ g^{-1}$ is not measurable. (This is an example of measurable $\circ$ continuous $\neq$ measurable.) Show there exist Lebesgue measurable sets which are not Borel measurable. **Hint**: The function, ϕ is Lebesgue measurable. Now recall that Borel $\circ$ measurable $=$ measurable.

8. If A is $m\lfloor S$ measurable, it does not follow that A is m measurable. Give an example to show this is the case.

9. Let $f(y) = g(y) = |y|^{-1/2}$if $y \in (-1,0) \cup (0,1)$ and $f(y) = g(y) = 0$ if $y \notin (-1,0) \cup (0,1)$. For which values of x does it make sense to write the integral $\int_{\mathbb{R}} f(x-y)\, g(y)\, dy$?

10. $\uparrow$ Let $f \in L^1(\mathbb{R}),\ g \in L^1(\mathbb{R})$. Wherever the integral makes sense, define

$$(f * g)(x) \equiv \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Show the above integral makes sense for a.e. x and that if $f * g\,(x)$ is defined to equal 0 at every point where the above integral does not make sense, it follows that $|(f*g)(x)| < \infty$ a.e. and

$$||f*g||_{L^1} \leq ||f||_{L^1}||g||_{L^1}. \text{ Here } ||f||_{L^1} \equiv \int |f|dx.$$

11. $\uparrow$ Let $f : [0, \infty) \to \mathbb{R}$ be in $L^1(\mathbb{R}, m)$. The Laplace transform is given by $\widehat{f}(x) = \int_0^\infty e^{-xt} f(t)dt$. Let f, g be in $L^1(\mathbb{R}, m)$, and let $h(x) = \int_0^x f(x-t)g(t)dt$. Show $h \in L^1$, and $\widehat{h} = \widehat{f}\widehat{g}$.

12. Show that the function $\sin(x)/x$ is not in $L^1(0, \infty)$. Even though this function is not in $L^1(0, \infty)$, show $\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}$. This limit is sometimes called the Cauchy principle value and it is often the case that this is what is found when you use methods of contour integrals to evaluate improper integrals. **Hint:** Use $\frac{1}{x} = \int_0^\infty e^{-xt} dt$ and Fubini's theorem.

13. Let E be a countable subset of $\mathbb{R}$. Show $m(E) = 0$. **Hint:** Let the set be $\{e_i\}_{i=1}^\infty$ and let e_i be the center of an open interval of length $\varepsilon/2^i$.

14. $\uparrow$ If S is an uncountable set of irrational numbers, is it necessary that S has a rational number as a limit point? **Hint:** Consider the proof of Problem 13 when applied to the rational numbers. (This problem was shown to me by Lee Erlebach.)

The L^p Spaces

10.1 Basic Inequalities And Properties

One of the main applications of the Lebesgue integral is to the study of various sorts of functions space. These are vector spaces whose elements are functions of various types. One of the most important examples of a function space is the space of measurable functions whose absolute values are p^{th} power integrable where $p \geq 1$. These spaces, referred to as L^p spaces, are very useful in applications. In the chapter $(\Omega, \mathcal{S}, \mu)$ will be a measure space.

Definition 10.1 *Let $1 \leq p < \infty$. Define*

$$L^p(\Omega) \equiv \{f \ : f \ \text{ is measurable and } \int_\Omega |f(\omega)|^p d\mu < \infty\}$$

In terms of the distribution function,

$$L^p(\Omega) = \{f \ : f \ \text{ is measurable and } \int_0^\infty pt^{p-1}\mu([|f| > t])\, dt < \infty\}$$

For each $p > 1$ define q by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Often one uses p' instead of q in this context.

$L^p(\Omega)$ is a vector space and has a norm. This is similar to the situation for $\mathbb{R}^n$ but the proof requires the following fundamental inequality. .

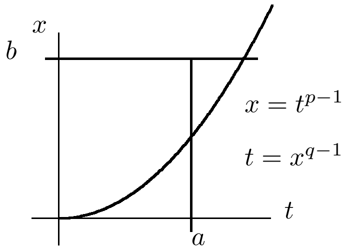
Theorem 10.2 *(Holder's inequality) If f and g are measurable functions, then if $p > 1$,*

$$\int |f|\, |g|\, d\mu \leq \left(\int |f|^p d\mu\right)^{\frac{1}{p}} \left(\int |g|^q d\mu\right)^{\frac{1}{q}}. \tag{10.1}$$

Proof: First here is a proof of Young's inequality .

Lemma 10.3 *If $p > 1$, and $0 \leq a, b$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.*

Proof: Consider the following picture:



From this picture, the sum of the area between the x axis and the curve added to the area between the t axis and the curve is at least as large as ab. Using beginning calculus, this is equivalent to the following inequality.

$$ab \leq \int_0^a t^{p-1} dt + \int_0^b x^{q-1} dx = \frac{a^p}{p} + \frac{b^q}{q}.$$

The above picture represents the situation which occurs when $p > 2$ because the graph of the function is concave up. If $2 \geq p > 1$ the graph would be concave down or a straight line. You should verify that the same argument holds in these cases just as well. In fact, the only thing which matters in the above inequality is that the function $x = t^{p-1}$ be strictly increasing.

Note equality occurs when $a^p = b^q$.

Here is an alternate proof.

Lemma 10.4 *For $a, b \geq 0$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and equality occurs when if and only if $a^p = b^q$.

Proof: If $b = 0$, the inequality is obvious. Fix $b > 0$ and consider $f(a) \equiv \frac{a^p}{p} + \frac{b^q}{q} - ab$. Then $f'(a) = a^{p-1} - b$. This is negative when $a < b^{1/(p-1)}$ and is positive when $a > b^{1/(p-1)}$. Therefore, f has a minimum when $a = b^{1/(p-1)}$. In other words, when $a^p = b^{p/(p-1)} = b^q$ since $1/p + 1/q = 1$. Thus the minimum value of f is

$$\frac{b^q}{p} + \frac{b^q}{q} - b^{1/(p-1)} b = b^q - b^q = 0.$$

It follows $f \geq 0$ and this yields the desired inequality.

Proof of Holder's inequality: If either $\int |f|^p d\mu$ or $\int |g|^p d\mu$ equals ∞, the inequality 10.1 is obviously valid because $\infty \geq$ anything. If either $\int |f|^p d\mu$ or $\int |g|^p d\mu$ equals 0, then $f = 0$ a.e. or that $g = 0$ a.e. and so in this case the left side of the inequality equals 0 and so the inequality is therefore true. Therefore assume both

$\int |f|^p d\mu$ and $\int |g|^p d\mu$ are less than ∞ and not equal to 0. Let $\left(\int |f|^p d\mu\right)^{1/p} = I(f)$ and let $\left(\int |g|^p d\mu\right)^{1/q} = I(g)$. Then using the lemma,

$$\int \frac{|f|}{I(f)} \frac{|g|}{I(g)} \, d\mu \leq \frac{1}{p} \int \frac{|f|^p}{I(f)^p} \, d\mu + \frac{1}{q} \int \frac{|g|^q}{I(g)^q} \, d\mu = 1.$$

Hence,

$$\int |f| \, |g| \, d\mu \leq I(f) \, I(g) = \left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{1/q}.$$

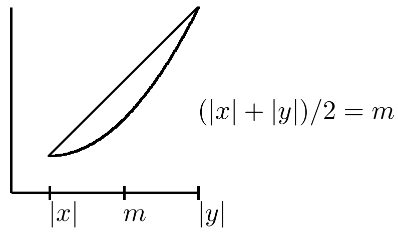
This proves Holder's inequality.

The following lemma will be needed.

Lemma 10.5 *Suppose $x, y \in \mathbb{C}$. Then*

$$|x + y|^p \leq 2^{p-1} \left(|x|^p + |y|^p\right).$$

Proof: The function $f(t) = t^p$ is concave up for $t \geq 0$ because $p > 1$. Therefore, the secant line joining two points on the graph of this function must lie above the graph of the function. This is illustrated in the following picture.



Now as shown above,

$$\left(\frac{|x| + |y|}{2}\right)^p \leq \frac{|x|^p + |y|^p}{2}$$

which implies

$$|x + y|^p \leq (|x| + |y|)^p \leq 2^{p-1} \left(|x|^p + |y|^p\right)$$

and this proves the lemma.

Note that if $y = \phi(x)$ is any function for which the graph of ϕ is concave up, you could get a similar inequality by the same argument.

Corollary 10.6 *(Minkowski inequality) Let $1 \leq p < \infty$. Then*

$$\left(\int |f + g|^p \, d\mu\right)^{1/p} \leq \left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p}. \tag{10.2}$$

Proof: If $p = 1$, this is obvious because it is just the triangle inequality. Let $p > 1$. Without loss of generality, assume

$$\left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p} < \infty$$

and $\left(\int |f+g|^p \, d\mu\right)^{1/p} \neq 0$ or there is nothing to prove. Therefore, using the above lemma,

$$\int |f+g|^p d\mu \leq 2^{p-1}\left(\int |f|^p + |g|^p d\mu\right) < \infty.$$

Now $|f(\omega)+g(\omega)|^p \leq |f(\omega)+g(\omega)|^{p-1}(|f(\omega)|+|g(\omega)|)$. Also, it follows from the definition of p and q that $p-1=\frac{p}{q}$. Therefore, using this and Holder's inequality,

$$\int |f+g|^p d\mu \leq$$

$$\begin{aligned}
& \int |f+g|^{p-1}|f|d\mu + \int |f+g|^{p-1}|g|d\mu \\
= \; & \int |f+g|^{\frac{p}{q}}|f|d\mu + \int |f+g|^{\frac{p}{q}}|g|d\mu \\
\leq \; & (\int |f+g|^p d\mu)^{\frac{1}{q}}(\int |f|^p d\mu)^{\frac{1}{p}} + (\int |f+g|^p d\mu)^{\frac{1}{q}}(\int |g|^p d\mu)^{\frac{1}{p}}.
\end{aligned}$$

Dividing both sides by $(\int |f+g|^p d\mu)^{\frac{1}{q}}$ yields 10.2. This proves the corollary.

The following follows immediately from the above.

Corollary 10.7 *Let $f_i \in L^p(\Omega)$ for $i = 1, 2, \cdots, n$. Then*

$$\left(\int \left|\sum_{i=1}^n f_i\right|^p d\mu\right)^{1/p} \leq \sum_{i=1}^n \left(\int |f_i|^p\right)^{1/p}.$$

This shows that if $f, g \in L^p$, then $f + g \in L^p$. Also, it is clear that if a is a constant and $f \in L^p$, then $af \in L^p$ because

$$\int |af|^p \, d\mu = |a|^p \int |f|^p \, d\mu < \infty.$$

Thus L^p is a vector space and

a.) $\left(\int |f|^p \, d\mu\right)^{1/p} \geq 0, \left(\int |f|^p \, d\mu\right)^{1/p} = 0$ if and only if $f = 0$ a.e.

b.) $\left(\int |af|^p \, d\mu\right)^{1/p} = |a| \left(\int |f|^p \, d\mu\right)^{1/p}$ if a is a scalar.

c.) $\left(\int |f+g|^p \, d\mu\right)^{1/p} \leq \left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p}$.

$f \to \left(\int |f|^p \, d\mu\right)^{1/p}$ would define a norm if $\left(\int |f|^p \, d\mu\right)^{1/p} = 0$ implied $f = 0$. Unfortunately, this is not so because if $f = 0$ a.e. but is nonzero on a set of measure zero, $\left(\int |f|^p \, d\mu\right)^{1/p} = 0$ and this is not allowed. However, all the other properties of a norm are available and so a little thing like a set of measure zero will not prevent the consideration of L^p as a normed vector space if two functions in L^p which differ only on a set of measure zero are considered the same. That is, an element of L^p is really an equivalence class of functions where two functions are equivalent if they are equal a.e. With this convention, here is a definition.

Definition 10.8 *Let $f \in L^p(\Omega)$. Define*

$$||f||_p \equiv ||f||_{L^p} \equiv \left(\int |f|^p \, d\mu\right)^{1/p}.$$

Then with this definition and using the convention that elements in L^p are considered to be the same if they differ only on a set of measure zero, $||\,||_p$ is a norm on $L^p(\Omega)$ because if $||f||_p = 0$ then $f = 0$ a.e. and so f is considered to be the zero function because it differs from 0 only on a set of measure zero.

The following is an important definition.

Definition 10.9 *A complete normed linear space is called a Banach*[1] *space.*

L^p is a Banach space. This is the next big theorem.

Theorem 10.10 *The following hold for $L^p(\Omega)$*

a.) $L^p(\Omega)$ is complete.

b.) If $\{f_n\}$ is a Cauchy sequence in $L^p(\Omega)$, then there exists $f \in L^p(\Omega)$ and a subsequence which converges a.e. to $f \in L^p(\Omega)$, and $||f_n - f||_p \to 0$.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega)$. This means that for every $\varepsilon > 0$ there exists N such that if $n, m \geq N$, then $||f_n - f_m||_p < \varepsilon$. Now select a subsequence as follows. Let n_1 be such that $||f_n - f_m||_p < 2^{-1}$ whenever $n, m \geq n_1$. Let n_2 be such that $n_2 > n_1$ and $||f_n - f_m||_p < 2^{-2}$ whenever $n, m \geq n_2$. If $n_1, \cdots, n_k$ have been chosen, let $n_{k+1} > n_k$ and whenever $n, m \geq n_{k+1}$, $||f_n - f_m||_p < 2^{-(k+1)}$. The subsequence just mentioned is $\{f_{n_k}\}$. Thus, $||f_{n_k} - f_{n_{k+1}}||_p < 2^{-k}$. Let

$$g_{k+1} = f_{n_{k+1}} - f_{n_k}.$$

[1] These spaces are named after Stefan Banach, 1892-1945. Banach spaces are the basic item of study in the subject of functional analysis and will be considered later in this book.

There is a recent biography of Banach, R. Katuża, *The Life of Stefan Banach,* (A. Kostant and W. Woyczyński, translators and editors) Birkhauser, Boston (1996). More information on Banach can also be found in a recent short article written by Douglas Henderson who is in the department of chemistry and biochemistry at BYU.

Banach was born in Austria, worked in Poland and died in the Ukraine but never moved. This is because borders kept changing. There is a rumor that he died in a German concentration camp which is apparently not true. It seems he died after the war of lung cancer.

He was an interesting character. He hated taking examinations so much that he did not receive his undergraduate university degree. Nevertheless, he did become a professor of mathematics due to his important research. He and some friends would meet in a cafe called the Scottish cafe where they wrote on the marble table tops until Banach's wife supplied them with a notebook which became the "Scotish notebook" and was eventually published.

Then by the corollary to Minkowski's inequality,

$$\infty > \sum_{k=1}^{\infty} ||g_{k+1}||_p \geq \sum_{k=1}^{m} ||g_{k+1}||_p \geq \left|\left|\sum_{k=1}^{m} |g_{k+1}|\right|\right|_p$$

for all m. It follows that

$$\int \left(\sum_{k=1}^{m} |g_{k+1}|\right)^p d\mu \leq \left(\sum_{k=1}^{\infty} ||g_{k+1}||_p\right)^p < \infty \tag{10.3}$$

for all m and so the monotone convergence theorem implies that the sum up to m in 10.3 can be replaced by a sum up to ∞. Thus,

$$\int \left(\sum_{k=1}^{\infty} |g_{k+1}|\right)^p d\mu < \infty$$

which requires

$$\sum_{k=1}^{\infty} |g_{k+1}(x)| < \infty \text{ a.e. } x.$$

Therefore, $\sum_{k=1}^{\infty} g_{k+1}(x)$ converges for a.e. x because the functions have values in a complete space, $\mathbb{C}$, and this shows the partial sums form a Cauchy sequence. Now let x be such that this sum is finite. Then define

$$f(x) \equiv f_{n_1}(x) + \sum_{k=1}^{\infty} g_{k+1}(x) = \lim_{m\to\infty} f_{n_m}(x)$$

since $\sum_{k=1}^{m} g_{k+1}(x) = f_{n_{m+1}}(x) - f_{n_1}(x)$. Therefore there exists a set, E having measure zero such that

$$\lim_{k\to\infty} f_{n_k}(x) = f(x)$$

for all $x \notin E$. Redefine f_{n_k} to equal 0 on E and let $f(x) = 0$ for $x \in E$. It then follows that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for all x. By Fatou's lemma, and the Minkowski inequality,

$$||f - f_{n_k}||_p = \left(\int |f - f_{n_k}|^p \, d\mu\right)^{1/p} \leq$$

$$\liminf_{m\to\infty} \left(\int |f_{n_m} - f_{n_k}|^p \, d\mu\right)^{1/p} = \liminf_{m\to\infty} ||f_{n_m} - f_{n_k}||_p \leq$$

$$\liminf_{m\to\infty} \sum_{j=k}^{m-1} \left|\left|f_{n_{j+1}} - f_{n_j}\right|\right|_p \leq \sum_{i=k}^{\infty} \left|\left|f_{n_{i+1}} - f_{n_i}\right|\right|_p \leq 2^{-(k-1)}. \tag{10.4}$$

Therefore, $f \in L^p(\Omega)$ because

$$||f||_p \leq ||f - f_{n_k}||_p + ||f_{n_k}||_p < \infty,$$

and $\lim_{k\to\infty} ||f_{n_k} - f||_p = 0$. This proves b.).

This has shown f_{n_k} converges to f in $L^p(\Omega)$. It follows the original Cauchy sequence also converges to f in $L^p(\Omega)$. This is a general fact that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence. You should give a proof of this. This proves the theorem.

In working with the L^p spaces, the following inequality also known as Minkowski's inequality is very useful. It is similar to the Minkowski inequality for sums. To see this, replace the integral, $\int_X$ with a finite summation sign and you will see the usual Minkowski inequality or rather the version of it given in Corollary 10.7.

To prove this theorem first consider a special case of it in which technical considerations which shed no light on the proof are excluded.

Lemma 10.11 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be finite complete measure spaces and let f be $\overline{\mu\times\lambda}$ measurable and uniformly bounded. Then the following inequality is valid for $p \geq 1$.*

$$\int_X \left(\int_Y |f(x,y)|^p \, d\lambda\right)^{\frac{1}{p}} d\mu \geq \left(\int_Y (\int_X |f(x,y)| \, d\mu)^p d\lambda\right)^{\frac{1}{p}}. \tag{10.5}$$

Proof: Since f is bounded and $\mu(X), \lambda(X) < \infty$,

$$\left(\int_Y (\int_X |f(x,y)|d\mu)^p d\lambda\right)^{\frac{1}{p}} < \infty.$$

Let

$$J(y) = \int_X |f(x,y)|d\mu.$$

Note there is no problem in writing this for a.e. y because f is product measurable. Then by Fubini's theorem,

$$\begin{aligned}
\int_Y \left(\int_X |f(x,y)|d\mu\right)^p d\lambda &= \int_Y J(y)^{p-1} \int_X |f(x,y)|d\mu \, d\lambda \\
&= \int_X \int_Y J(y)^{p-1} |f(x,y)| d\lambda \, d\mu
\end{aligned}$$

Now apply Holder's inequality in the last integral above and recall $p - 1 = \frac{p}{q}$. This yields

$$\begin{aligned}
&\int_Y \left(\int_X |f(x,y)|d\mu\right)^p d\lambda \\
\leq\ &\int_X \left(\int_Y J(y)^p d\lambda\right)^{\frac{1}{q}} \left(\int_Y |f(x,y)|^p d\lambda\right)^{\frac{1}{p}} d\mu \\
=\ &\left(\int_Y J(y)^p d\lambda\right)^{\frac{1}{q}} \int_X \left(\int_Y |f(x,y)|^p d\lambda\right)^{\frac{1}{p}} d\mu
\end{aligned}$$

$$= \left(\int_Y (\int_X |f(x,y)| d\mu)^p d\lambda \right)^{\frac{1}{q}} \int_X \left(\int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu. \tag{10.6}$$

Therefore, dividing both sides by the first factor in the above expression,

$$\left(\int_Y \left(\int_X |f(x,y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu. \tag{10.7}$$

Note that 10.7 holds even if the first factor of 10.6 equals zero. This proves the lemma.

Now consider the case where f is not assumed to be bounded and where the measure spaces are σ finite.

Theorem 10.12 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be σ-finite measure spaces and let f be product measurable. Then the following inequality is valid for $p \geq 1$.*

$$\int_X \left(\int_Y |f(x,y)|^p \, d\lambda \right)^{\frac{1}{p}} d\mu \geq \left(\int_Y (\int_X |f(x,y)| \, d\mu)^p d\lambda \right)^{\frac{1}{p}}. \tag{10.8}$$

Proof: Since the two measure spaces are σ finite, there exist measurable sets, X_m and Y_k such that $X_m \subseteq X_{m+1}$ for all m, $Y_k \subseteq Y_{k+1}$ for all k, and $\mu(X_m), \lambda(Y_k) < \infty$. Now define

$$f_n(x,y) \equiv \begin{cases} f(x,y) \text{ if } |f(x,y)| \leq n \\ n \text{ if } |f(x,y)| > n. \end{cases}$$

Thus f_n is uniformly bounded and product measurable. By the above lemma,

$$\int_{X_m} \left(\int_{Y_k} |f_n(x,y)|^p \, d\lambda \right)^{\frac{1}{p}} d\mu \geq \left(\int_{Y_k} (\int_{X_m} |f_n(x,y)| \, d\mu)^p d\lambda \right)^{\frac{1}{p}}. \tag{10.9}$$

Now observe that $|f_n(x,y)|$ increases in n and the pointwise limit is $|f(x,y)|$. Therefore, using the monotone convergence theorem in 10.9 yields the same inequality with f replacing f_n. Next let $k \to \infty$ and use the monotone convergence theorem again to replace Y_k with Y. Finally let $m \to \infty$ in what is left to obtain 10.8. This proves the theorem.

Note that the proof of this theorem depends on two manipulations, the interchange of the order of integration and Holder's inequality. Note that there is nothing to check in the case of double sums. Thus if $a_{ij} \geq 0$, it is always the case that

$$\left(\sum_j \left(\sum_i a_{ij} \right)^p \right)^{1/p} \leq \sum_i \left(\sum_j a_{ij}^p \right)^{1/p}$$

because the integrals in this case are just sums and $(i,j) \to a_{ij}$ is measurable.

The L^p spaces have many important properties.

10.2 Density Considerations

Theorem 10.13 *Let $p \geq 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a measure space. Then the simple functions are dense in $L^p(\Omega)$.*

Proof: Recall that a function, f, having values in $\mathbb{R}$ can be written in the form $f = f^+ - f^-$ where

$$f^+ = \max(0, f), \ f^- = \max(0, -f).$$

Therefore, an arbitrary complex valued function, f is of the form

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i\left(\operatorname{Im} f^+ - \operatorname{Im} f^-\right).$$

If each of these nonnegative functions is approximated by a simple function, it follows f is also approximated by a simple function. Therefore, there is no loss of generality in assuming at the outset that $f \geq 0$.

Since f is measurable, Theorem 7.24 implies there is an increasing sequence of simple functions, $\{s_n\}$, converging pointwise to $f(x)$. Now

$$|f(x) - s_n(x)| \leq |f(x)|.$$

By the Dominated Convergence theorem,

$$0 = \lim_{n\to\infty} \int |f(x) - s_n(x)|^p d\mu.$$

Thus simple functions are dense in L^p.

Recall that for Ω a topological space, $C_c(\Omega)$ is the space of continuous functions with compact support in Ω. Also recall the following definition.

Definition 10.14 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose (Ω, τ) is also a topological space. Then $(\Omega, \mathcal{S}, \mu)$ is called a regular measure space if the σ algebra of Borel sets is contained in $\mathcal{S}$ and for all $E \in \mathcal{S}$,*

$$\mu(E) = \inf\{\mu(V) : V \supseteq E \text{ and } V \text{ open}\}$$

and if $\mu(E) < \infty$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact }\}$$

and $\mu(K) < \infty$ for any compact set, K.

For example Lebesgue measure is an example of such a measure.

Lemma 10.15 *Let Ω be a metric space in which the closed balls are compact and let K be a compact subset of V, an open set. Then there exists a continuous function $f : \Omega \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$ and $\operatorname{spt}(f)$ is a compact subset of V. That is, $K \prec f \prec V$.*

Proof: Let $K \subseteq W \subseteq \overline{W} \subseteq V$ and $\overline{W}$ is compact. To obtain this list of inclusions consider a point in K, x, and take $B(x, r_x)$ a ball containing x such that $\overline{B(x, r_x)}$ is a compact subset of V. Next use the fact that K is compact to obtain the existence of a list, $\{B(x_i, r_{x_i}/2)\}_{i=1}^m$ which covers K. Then let

$$W \equiv \cup_{i=1}^m B\left(x_i, \frac{r_{x_i}}{2}\right).$$

It follows since this is a finite union that

$$\overline{W} = \cup_{i=1}^m \overline{B\left(x_i, \frac{r_{x_i}}{2}\right)}$$

and so $\overline{W}$, being a finite union of compact sets is itself a compact set. Also, from the construction

$$\overline{W} \subseteq \cup_{i=1}^m B(x_i, r_{x_i}).$$

Define f by

$$f(x) = \frac{\operatorname{dist}(x, W^C)}{\operatorname{dist}(x, K) + \operatorname{dist}(x, W^C)}.$$

It is clear that f is continuous if the denominator is always nonzero. But this is clear because if $x \in W^C$ there must be a ball $B(x, r)$ such that this ball does not intersect K. Otherwise, x would be a limit point of K and since K is closed, $x \in K$. However, $x \notin K$ because $K \subseteq W$.

It is not necessary to be in a metric space to do this. You can accomplish the same thing using Urysohn's lemma.

Theorem 10.16 *Let $(\Omega, \mathcal{S}, \mu)$ be a regular measure space as in Definition 10.14 where the conclusion of Lemma 10.15 holds. Then $C_c(\Omega)$ is dense in $L^p(\Omega)$.*

Proof: First consider a measurable set, E where $\mu(E) < \infty$. Let $K \subseteq E \subseteq V$ where $\mu(V \setminus K) < \varepsilon$. Now let $K \prec h \prec V$. Then

$$\int |h - \mathcal{X}_E|^p \, d\mu \leq \int \mathcal{X}_{V \setminus K}^p d\mu = \mu(V \setminus K) < \varepsilon.$$

It follows that for each s a simple function in $L^p(\Omega)$, there exists $h \in C_c(\Omega)$ such that $||s - h||_p < \varepsilon$. This is because if

$$s(x) = \sum_{i=1}^m c_i \mathcal{X}_{E_i}(x)$$

is a simple function in L^p where the c_i are the distinct nonzero values of s each $\mu(E_i) < \infty$ since otherwise $s \notin L^p$ due to the inequality

$$\int |s|^p \, d\mu \geq |c_i|^p \, \mu(E_i).$$

By Theorem 10.13, simple functions are dense in $L^p(\Omega)$,and so this proves the Theorem.

10.3 Separability

Theorem 10.17 *For $p \geq 1$ and μ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $||f - g||_p < \varepsilon$.*

Proof: Let Q be all functions of the form $c\mathcal{X}_{[\mathbf{a},\mathbf{b})}$ where

$$[\mathbf{a}, \mathbf{b}) \equiv [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

and both a_i, b_i are rational, while c has rational real and imaginary parts. Let $\mathcal{D}$ be the set of all finite sums of functions in Q. Thus, $\mathcal{D}$ is countable. In fact $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n, \mu)$. To prove this it is necessary to show that for every $f \in L^p(\mathbb{R}^n, \mu)$, there exists an element of $\mathcal{D}$, s such that $||s - f||_p < \varepsilon$. If it can be shown that for every $g \in C_c(\mathbb{R}^n)$ there exists $h \in \mathcal{D}$ such that $||g - h||_p < \varepsilon$, then this will suffice because if $f \in L^p(\mathbb{R}^n)$ is arbitrary, Theorem 10.16 implies there exists $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_p \leq \frac{\varepsilon}{2}$ and then there would exist $h \in C_c(\mathbb{R}^n)$ such that $||h - g||_p < \frac{\varepsilon}{2}$. By the triangle inequality,

$$||f - h||_p \leq ||h - g||_p + ||g - f||_p < \varepsilon.$$

Therefore, assume at the outset that $f \in C_c(\mathbb{R}^n)$.

Let $\mathcal{P}_m$ consist of all sets of the form $[\mathbf{a}, \mathbf{b}) \equiv \prod_{i=1}^n [a_i, b_i)$ where $a_i = j2^{-m}$and $b_i = (j+1)2^{-m}$ for j an integer. Thus $\mathcal{P}_m$ consists of a tiling of $\mathbb{R}^n$ into half open rectangles having diameters $2^{-m}n^{\frac{1}{2}}$. There are countably many of these rectangles; so, let $\mathcal{P}_m = \{[\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^{\infty}$ and $\mathbb{R}^n = \cup_{i=1}^{\infty}[\mathbf{a}_i, \mathbf{b}_i)$. Let c_i^mbe complex numbers with rational real and imaginary parts satisfying

$$|f(\mathbf{a}_i) - c_i^m| < 2^{-m},$$
$$|c_i^m| \leq |f(\mathbf{a}_i)|. \tag{10.10}$$

Let

$$s_m(\mathbf{x}) = \sum_{i=1}^{\infty} c_i^m \mathcal{X}_{[\mathbf{a}_i, \mathbf{b}_i)}(\mathbf{x}).$$

Since $f(\mathbf{a}_i) = 0$ except for finitely many values of i, the above is a finite sum. Then 10.10 implies $s_m \in \mathcal{D}$. If s_m converges uniformly to f then it follows $||s_m - f||_p \to 0$ because $|s_m| \leq |f|$ and so

$$\begin{aligned} ||s_m - f||_p &= \left(\int |s_m - f|^p \, d\mu \right)^{1/p} \\ &= \left(\int_{\text{spt}(f)} |s_m - f|^p \, d\mu \right)^{1/p} \\ &\leq [\varepsilon m_n (\text{spt}(f))]^{1/p} \end{aligned}$$

whenever m is large enough.

Since $f \in C_c(\mathbb{R}^n)$ it follows that f is uniformly continuous and so given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon/2$. Now let m be large enough that every box in $\mathcal{P}_m$ has diameter less than δ and also that $2^{-m} < \varepsilon/2$. Then if $[\mathbf{a}_i, \mathbf{b}_i)$ is one of these boxes of $\mathcal{P}_m$, and $\mathbf{x} \in [\mathbf{a}_i, \mathbf{b}_i)$,

$$|f(\mathbf{x}) - f(\mathbf{a}_i)| < \varepsilon/2$$

and

$$|f(\mathbf{a}_i) - c_i^m| < 2^{-m} < \varepsilon/2.$$

Therefore, using the triangle inequality, it follows that $|f(\mathbf{x}) - c_i^m| = |s_m(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$ and since $\mathbf{x}$ is arbitrary, this establishes uniform convergence. This proves the theorem.

Here is an easier proof if you know the Weierstrass approximation theorem.

Theorem 10.18 *For $p \geq 1$ and μ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $||f - g||_p < \varepsilon$.*

Proof: Let $\mathcal{P}$ denote the set of all polynomials which have rational coefficients. Then $\mathcal{P}$ is countable. Let $\tau_k \in C_c\left((-(k+1), (k+1))^n\right)$ such that $\overline{[-k,k]^n} \prec \tau_k \prec (-(k+1), (k+1))^n$. Let $\mathcal{D}_k$ denote the functions which are of the form, $p\tau_k$ where $p \in \mathcal{P}$. Thus $\mathcal{D}_k$ is also countable. Let $\mathcal{D} \equiv \cup_{k=1}^{\infty} \mathcal{D}_k$. It follows each function in $\mathcal{D}$ is in $C_c(\mathbb{R}^n)$ and so it in $L^p(\mathbb{R}^n, \mu)$. Let $f \in L^p(\mathbb{R}^n, \mu)$. By regularity of μ there exists $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_{L^p(\mathbb{R}^n,\mu)} < \frac{\varepsilon}{3}$. Let k be such that $\operatorname{spt}(g) \subseteq (-k, k)^n$. Now by the Weierstrass approximation theorem there exists a polynomial q such that

$$\begin{aligned} ||g - q||_{[-(k+1),k+1]^n} &\equiv \sup\{|g(\mathbf{x}) - q(\mathbf{x})| : \mathbf{x} \in [-(k+1), (k+1)]^n\} \\ &< \frac{\varepsilon}{3\mu\left((-(k+1), k+1)^n\right)}. \end{aligned}$$

It follows

$$||g - \tau_k q||_{[-(k+1),k+1]^n} = ||\tau_k g - \tau_k q||_{[-(k+1),k+1]^n} < \frac{\varepsilon}{3\mu\left((-(k+1), k+1)^n\right)}.$$

Without loss of generality, it can be assumed this polynomial has all rational coefficients. Therefore, $\tau_k q \in \mathcal{D}$.

$$\begin{aligned} ||g - \tau_k q||_{L^p(\mathbb{R}^n)}^p &= \int_{(-(k+1),k+1)^n} |g(\mathbf{x}) - \tau_k(\mathbf{x}) q(\mathbf{x})|^p \, d\mu \\ &\leq \left(\frac{\varepsilon}{3\mu\left((-(k+1), k+1)^n\right)}\right)^p \mu\left((-(k+1), k+1)^n\right) \\ &< \left(\frac{\varepsilon}{3}\right)^p. \end{aligned}$$

It follows

$$||f - \tau_k q||_{L^p(\mathbb{R}^n,\mu)} \leq ||f - g||_{L^p(\mathbb{R}^n,\mu)} + ||g - \tau_k q||_{L^p(\mathbb{R}^n,\mu)}^p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

This proves the theorem.

Corollary 10.19 *Let Ω be any μ measurable subset of $\mathbb{R}^n$ and let μ be a Radon measure. Then $L^p(\Omega, \mu)$ is separable. Here the σ algebra of measurable sets will consist of all intersections of measurable sets with Ω and the measure will be μ restricted to these sets.*

Proof: Let $\widetilde{\mathcal{D}}$ be the restrictions of $\mathcal{D}$ to Ω. If $f \in L^p(\Omega)$, let F be the zero extension of f to all of $\mathbb{R}^n$. Let $\varepsilon > 0$ be given. By Theorem 10.17 or 10.18 there exists $s \in \mathcal{D}$ such that $||F - s||_p < \varepsilon$. Thus

$$||s - f||_{L^p(\Omega,\mu)} \leq ||s - F||_{L^p(\mathbb{R}^n,\mu)} < \varepsilon$$

and so the countable set $\widetilde{\mathcal{D}}$ is dense in $L^p(\Omega)$.

10.4 Continuity Of Translation

Definition 10.20 *Let f be a function defined on $U \subseteq \mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$. Then $f_{\mathbf{w}}$ will be the function defined on $\mathbf{w} + U$ by*

$$f_{\mathbf{w}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{w}).$$

Theorem 10.21 *(Continuity of translation in L^p) Let $f \in L^p(\mathbb{R}^n)$ with the measure being Lebesgue measure. Then*

$$\lim_{||\mathbf{w}|| \to 0} ||f_{\mathbf{w}} - f||_p = 0.$$

Proof: Let $\varepsilon > 0$ be given and let $g \in C_c(\mathbb{R}^n)$ with $||g - f||_p < \frac{\varepsilon}{3}$. Since Lebesgue measure is translation invariant ($m_n(\mathbf{w} + E) = m_n(E)$),

$$||g_{\mathbf{w}} - f_{\mathbf{w}}||_p = ||g - f||_p < \frac{\varepsilon}{3}.$$

You can see this from looking at simple functions and passing to the limit or you could use the change of variables formula to verify it.

Therefore

$$\begin{aligned} ||f - f_{\mathbf{w}}||_p &\leq ||f - g||_p + ||g - g_{\mathbf{w}}||_p + ||g_{\mathbf{w}} - f_{\mathbf{w}}|| \\ &< \frac{2\varepsilon}{3} + ||g - g_{\mathbf{w}}||_p. \end{aligned} \tag{10.11}$$

But $\lim_{|\mathbf{w}| \to 0} g_{\mathbf{w}}(\mathbf{x}) = g(\mathbf{x})$ uniformly in $\mathbf{x}$ because g is uniformly continuous. Now let B be a large ball containing $\operatorname{spt}(g)$ and let δ_1 be small enough that $B(\mathbf{x}, \delta) \subseteq B$ whenever $\mathbf{x} \in \operatorname{spt}(g)$. If $\varepsilon > 0$ is given there exists $\delta < \delta_1$ such that if $|\mathbf{w}| < \delta$, it follows that $|g(\mathbf{x} - \mathbf{w}) - g(\mathbf{x})| < \varepsilon/3 \left(1 + m_n(B)^{1/p}\right)$. Therefore,

$$\begin{aligned} ||g - g_{\mathbf{w}}||_p &= \left(\int_B |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{w})|^p \, dm_n \right)^{1/p} \\ &\leq \varepsilon \frac{m_n(B)^{1/p}}{3\left(1 + m_n(B)^{1/p}\right)} < \frac{\varepsilon}{3}. \end{aligned}$$

Therefore, whenever $|\mathbf{w}| < \delta$, it follows $||g - g_{\mathbf{w}}||_p < \frac{\varepsilon}{3}$ and so from 10.11 $||f - f_{\mathbf{w}}||_p < \varepsilon$. This proves the theorem.

Part of the argument of this theorem is significant enough to be stated as a corollary.

Corollary 10.22 *Suppose $g \in C_c(\mathbb{R}^n)$ and μ is a Radon measure on $\mathbb{R}^n$. Then*

$$\lim_{\mathbf{w} \to \mathbf{0}} ||g - g_{\mathbf{w}}||_p = 0.$$

Proof: The proof of this follows from the last part of the above argument simply replacing m_n with μ. Translation invariance of the measure is not needed to draw this conclusion because of uniform continuity of g.

10.5 Mollifiers And Density Of Smooth Functions

Definition 10.23 *Let U be an open subset of $\mathbb{R}^n$. $C_c^\infty(U)$ is the vector space of all infinitely differentiable functions which equal zero for all $\mathbf{x}$ outside of some compact set contained in U. Similarly, $C_c^m(U)$ is the vector space of all functions which are m times continuously differentiable and whose support is a compact subset of U.*

Example 10.24 *Let $U = B(\mathbf{z}, 2r)$*

$$\psi(\mathbf{x}) = \begin{cases} \exp\left[\left(|\mathbf{x} - \mathbf{z}|^2 - r^2\right)^{-1}\right] & \text{if } |\mathbf{x} - \mathbf{z}| < r, \\ 0 \text{ if } |\mathbf{x} - \mathbf{z}| \geq r. \end{cases}$$

Then a little work shows $\psi \in C_c^\infty(U)$. The following also is easily obtained.

Lemma 10.25 *Let U be any open set. Then $C_c^\infty(U) \neq \emptyset$.*

Proof: Pick $\mathbf{z} \in U$ and let r be small enough that $B(\mathbf{z}, 2r) \subseteq U$. Then let $\psi \in C_c^\infty(B(\mathbf{z}, 2r)) \subseteq C_c^\infty(U)$ be the function of the above example.

Definition 10.26 *Let $U = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$. A sequence $\{\psi_m\} \subseteq C_c^\infty(U)$ is called a mollifier (sometimes an approximate identity) if*

$$\psi_m(\mathbf{x}) \geq 0,\ \psi_m(\mathbf{x}) = 0,\ \text{if } |\mathbf{x}| \geq \frac{1}{m},$$

and $\int \psi_m(\mathbf{x}) = 1$. Sometimes it may be written as $\{\psi_\varepsilon\}$ where ψ_ε satisfies the above conditions except $\psi_\varepsilon(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq \varepsilon$. In other words, ε takes the place of $1/m$ and in everything that follows $\varepsilon \to 0$ instead of $m \to \infty$.

As before, $\int f(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \to f(\mathbf{x}, \mathbf{y})$ is being integrated. To make the notation more familiar, dx is written instead of $dm_n(x)$.

Example 10.27 *Let*

$$\psi \in C_c^\infty(B(0,1)) \ \ (B(0,1) = \{\mathbf{x} : |\mathbf{x}| < 1\})$$

with $\psi(\mathbf{x}) \geq 0$ *and* $\int \psi dm = 1$. *Let* $\psi_m(\mathbf{x}) = c_m \psi(m\mathbf{x})$ *where* c_m *is chosen in such a way that* $\int \psi_m dm = 1$. *By the change of variables theorem* $c_m = m^n$.

Definition 10.28 *A function,* f*, is said to be in* $L^1_{loc}(\mathbb{R}^n, \mu)$ *if* f *is* μ *measurable and if* $|f|\mathcal{X}_K \in L^1(\mathbb{R}^n, \mu)$ *for every compact set,* K*. Here* μ *is a Radon measure on* $\mathbb{R}^n$*. Usually* $\mu = m_n$*, Lebesgue measure. When this is so, write* $L^1_{loc}(\mathbb{R}^n)$ *or* $L^p(\mathbb{R}^n)$*, etc. If* $f \in L^1_{loc}(\mathbb{R}^n, \mu)$*, and* $g \in C_c(\mathbb{R}^n)$*,*

$$f * g(\mathbf{x}) \equiv \int f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) d\mu.$$

The following lemma will be useful in what follows. It says that one of these very unregular functions in $L^1_{loc}(\mathbb{R}^n, \mu)$ is smoothed out by convolving with a mollifier.

Lemma 10.29 *Let* $f \in L^1_{loc}(\mathbb{R}^n, \mu)$*, and* $g \in C_c^\infty(\mathbb{R}^n)$*. Then* $f * g$ *is an infinitely differentiable function. Here* μ *is a Radon measure on* $\mathbb{R}^n$*.*

Proof: Consider the difference quotient for calculating a partial derivative of $f * g$.

$$\frac{f * g(\mathbf{x} + t\mathbf{e}_j) - f * g(\mathbf{x})}{t} = \int f(\mathbf{y}) \frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x} - \mathbf{y})}{t} d\mu(y).$$

Using the fact that $g \in C_c^\infty(\mathbb{R}^n)$, the quotient,

$$\frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x} - \mathbf{y})}{t},$$

is uniformly bounded. To see this easily, use Theorem 4.9 on Page 79 to get the existence of a constant, M depending on

$$\max\{||Dg(\mathbf{x})|| : \mathbf{x} \in \mathbb{R}^n\}$$

such that

$$|g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x} - \mathbf{y})| \leq M|t|$$

for any choice of $\mathbf{x}$ and $\mathbf{y}$. Therefore, there exists a dominating function for the integrand of the above integral which is of the form $C|f(\mathbf{y})|\mathcal{X}_K$ where K is a compact set containing the support of g. It follows the limit of the difference quotient above passes inside the integral as $t \to 0$ and

$$\frac{\partial}{\partial x_j}(f * g)(\mathbf{x}) = \int f(\mathbf{y}) \frac{\partial}{\partial x_j} g(\mathbf{x} - \mathbf{y}) d\mu(y).$$

Now letting $\frac{\partial}{\partial x_j} g$ play the role of g in the above argument, partial derivatives of all orders exist. This proves the lemma.

Theorem 10.30 *Let K be a compact subset of an open set, U. Then there exists a function, $h \in C_c^\infty(U)$, such that $h(\mathbf{x}) = 1$ for all $\mathbf{x} \in K$ and $h(\mathbf{x}) \in [0,1]$ for all $\mathbf{x}$.*

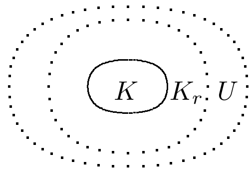
Proof: Let $r > 0$ be small enough that $K + B(\mathbf{0}, 3r) \subseteq U$. The symbol, $K + B(\mathbf{0}, 3r)$ means

$$\{\mathbf{k} + \mathbf{x} : \mathbf{k} \in K \text{ and } \mathbf{x} \in B(\mathbf{0}, 3r)\}.$$

Thus this is simply a way to write

$$\cup \{B(\mathbf{k}, 3r) : \mathbf{k} \in K\}.$$

Think of it as fattening up the set, K. Let $K_r = K + B(0, r)$. A picture of what is happening follows.



Consider $\mathcal{X}_{K_r} * \psi_m$ where ψ_m is a mollifier. Let m be so large that $\frac{1}{m} < r$. Then from the definition of what is meant by a convolution, and using that ψ_m has support in $B\left(\mathbf{0}, \frac{1}{m}\right)$, $\mathcal{X}_{K_r} * \psi_m = 1$ on K and that its support is in $K + B(\mathbf{0}, 3r)$. Now using Lemma 10.29, $\mathcal{X}_{K_r} * \psi_m$ is also infinitely differentiable. Therefore, let $h = \mathcal{X}_{K_r} * \psi_m$.

The following corollary will be used later.

Corollary 10.31 *Let K be a compact set in $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^\infty$ be an open cover of K. Then there exist functions, $\psi_k \in C_c^\infty(U_i)$ such that $\psi_i \prec U_i$ and*

$$\sum_{i=1}^{\infty} \psi_i(\mathbf{x}) = 1.$$

If K_1 is a compact subset of U_1 there exist such functions such that also $\psi_1(\mathbf{x}) = 1$ for all $\mathbf{x} \in K_1$.

Proof: This follows from a repeat of the proof of Theorem 8.18 on Page 168, replacing the lemma used in that proof with Theorem 10.30.

Theorem 10.32 *For each $p \geq 1$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Here the measure is Lebesgue measure.*

Proof: Let $f \in L^p(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given. Choose $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_p < \frac{\varepsilon}{2}$. This can be done by using Theorem 10.16. Now let

$$g_m(\mathbf{x}) = g * \psi_m(\mathbf{x}) \equiv \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) \, dm_n(y) = \int g(\mathbf{y}) \psi_m(\mathbf{x} - \mathbf{y}) \, dm_n(y)$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 10.29 $g_m \in C_c^\infty(\mathbb{R}^n)$. It vanishes if $\mathbf{x} \notin \text{spt}(g) + B(0, \frac{1}{m})$.

$$
\begin{aligned}
||g - g_m||_p &= \left(\int |g(\mathbf{x}) - \int g(\mathbf{x}-\mathbf{y})\psi_m(\mathbf{y})dm_n(\mathbf{y})|^p dm_n(\mathbf{x})\right)^{\frac{1}{p}} \\
&\leq \left(\int (\int |g(\mathbf{x}) - g(\mathbf{x}-\mathbf{y})|\psi_m(\mathbf{y})dm_n(\mathbf{y}))^p dm_n(\mathbf{x})\right)^{\frac{1}{p}} \\
&\leq \int \left(\int |g(\mathbf{x}) - g(\mathbf{x}-\mathbf{y})|^p dm_n(\mathbf{x})\right)^{\frac{1}{p}} \psi_m(\mathbf{y})dm_n(\mathbf{y}) \\
&= \int_{B(0,\frac{1}{m})} ||g - g_{\mathbf{y}}||_p \psi_m(\mathbf{y})dm_n(\mathbf{y}) < \frac{\varepsilon}{2}
\end{aligned}
$$

whenever m is large enough. This follows from Corollary 10.22. Theorem 10.12 was used to obtain the third inequality. There is no measurability problem because the function

$$(\mathbf{x}, \mathbf{y}) \to |g(\mathbf{x}) - g(\mathbf{x}-\mathbf{y})|\psi_m(\mathbf{y})$$

is continuous. Thus when m is large enough,

$$||f - g_m||_p \leq ||f - g||_p + ||g - g_m||_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the theorem.

This is a very remarkable result. Functions in $L^p(\mathbb{R}^n)$ don't need to be continuous anywhere and yet every such function is very close in the L^p norm to one which is infinitely differentiable having compact support.

Another thing should probably be mentioned. If you have had a course in complex analysis, you may be wondering whether these infinitely differentiable functions having compact support have anything to do with analytic functions which also have infinitely many derivatives. The answer is no! Recall that if an analytic function has a limit point in the set of zeros then it is identically equal to zero. Thus these functions in $C_c^\infty(\mathbb{R}^n)$ are not analytic. This is a strictly real analysis phenomenon and has absolutely nothing to do with the theory of functions of a complex variable.

10.6 Exercises

1. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set
$$E - E = \{x - y : x \in E, y \in E\}.$$
Show that $E - E$ contains an interval. **Hint:** Let
$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x+t)dt.$$
Note f is continuous at 0 and $f(0) > 0$ and use continuity of translation in L^p.

2. Give an example of a sequence of functions in $L^p(\mathbb{R})$ which converges to zero in L^p but does not converge pointwise to 0. Does this contradict the proof of the theorem that L^p is complete? You don't have to be real precise, just describe it.

3. Let K be a bounded subset of $L^p(\mathbb{R}^n)$ and suppose that for each $\varepsilon > 0$ there exists G such that $\overline{G}$ is compact with
$$\int_{\mathbb{R}^n \setminus \overline{G}} |u(\mathbf{x})|^p \, dx < \varepsilon^p$$
and for all $\varepsilon > 0$, there exist a $\delta > 0$ and such that if $|\mathbf{h}| < \delta$, then
$$\int |u(\mathbf{x}+\mathbf{h}) - u(\mathbf{x})|^p \, dx < \varepsilon^p$$
for all $u \in K$. Show that K is precompact in $L^p(\mathbb{R}^n)$. **Hint:** Let ϕ_k be a mollifier and consider
$$K_k \equiv \{u * \phi_k : u \in K\}.$$
Verify the conditions of the Ascoli Arzela theorem for these functions defined on $\overline{G}$ and show there is an ε net for each $\varepsilon > 0$. Can you modify this to let an arbitrary open set take the place of $\mathbb{R}^n$? This is a very important result.

4. Let (Ω, d) be a metric space and suppose also that $(\Omega, \mathcal{S}, \mu)$ is a regular measure space such that $\mu(\Omega) < \infty$ and let $f \in L^1(\Omega)$ where f has complex values. Show that for every $\varepsilon > 0$, there exists an open set of measure less than ε, denoted here by V and a continuous function, g defined on Ω such that $f = g$ on V^C. Thus, aside from a set of small measure, f is continuous. If $|f(\omega)| \leq M$, show that we can also assume $|g(\omega)| \leq M$. This is called Lusin's theorem. **Hint:** Use Theorems 10.16 and 10.10 to obtain a sequence of functions in $C_c(\Omega), \{g_n\}$ which converges pointwise a.e. to f and then use Egoroff's theorem to obtain a small set, W of measure less than $\varepsilon/2$ such that convergence is uniform on W^C. Now let F be a closed subset of W^C such that $\mu\left(W^C \setminus F\right) < \varepsilon/2$. Let $V = F^C$. Thus $\mu(V) < \varepsilon$ and on $F = V^C$, the convergence of $\{g_n\}$ is uniform showing that the restriction of f to V^C is continuous. Now use the Tietze extension theorem.

5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. This means
$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$
whenever $\lambda \in [0,1]$. Verify that if $x < y < z$, then $\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$ and that $\frac{\phi(z)-\phi(x)}{z-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$. Show if $s \in \mathbb{R}$ there exists λ such that $\phi(s) \leq \phi(t) + \lambda(s-t)$ for all t. Show that if ϕ is convex, then ϕ is continuous.

6. ↑ Prove Jensen's inequality. If $\phi : \mathbb{R} \to \mathbb{R}$ is convex, $\mu(\Omega) = 1$, and $f : \Omega \to \mathbb{R}$ is in $L^1(\Omega)$, then $\phi(\int_\Omega f\ du) \leq \int_\Omega \phi(f)d\mu$. **Hint:** Let $s = \int_\Omega f\ d\mu$ and use Problem 5.

7. Let $\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1,$ let $f \in L^p(\mathbb{R}),\ g \in L^{p'}(\mathbb{R})$. Show $f * g$ is uniformly continuous on $\mathbb{R}$ and $|(f * g)(x)| \leq ||f||_{L^p} ||g||_{L^{p'}}$. **Hint:** You need to consider why $f * g$ exists and then this follows from the definition of convolution and continuity of translation in L^p.

8. $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt$ for $p, q > 0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p + 1) = p\Gamma(p)$; b.) $\Gamma(p)\Gamma(q) = B(p, q)\Gamma(p + q)$.

9. Let $f \in C_c(0, \infty)$ and define $F(x) = \frac{1}{x}\int_0^x f(t)dt$. Show
$$||F||_{L^p(0,\infty)} \leq \frac{p}{p-1}||f||_{L^p(0,\infty)} \text{ whenever } p > 1.$$
Hint: Argue there is no loss of generality in assuming $f \geq 0$ and then assume this is so. Integrate $\int_0^\infty |F(x)|^p dx$ by parts as follows:
$$\int_0^\infty F^p dx = \overbrace{xF^p|_0^\infty}^{\text{show} = 0} - p\int_0^\infty xF^{p-1}F'dx.$$
Now show $xF' = f - F$ and use this in the last integral. Complete the argument by using Holder's inequality and $p - 1 = p/q$.

10. ↑ Now suppose $f \in L^p(0, \infty),\ p > 1$, and f not necessarily in $C_c(0, \infty)$. Show that $F(x) = \frac{1}{x}\int_0^x f(t)dt$ still makes sense for each $x > 0$. Show the inequality of Problem 9 is still valid. This inequality is called Hardy's inequality. **Hint:** To show this, use the above inequality along with the density of $C_c(0, \infty)$ in $L^p(0, \infty)$.

11. Suppose $f, g \geq 0$. When does equality hold in Holder's inequality?

12. Prove Vitali's Convergence theorem: Let $\{f_n\}$ be uniformly integrable and complex valued, $\mu(\Omega) < \infty,\ f_n(x) \to f(x)$ a.e. where f is measurable. Then $f \in L^1$ and $\lim_{n\to\infty} \int_\Omega |f_n - f|d\mu = 0$. **Hint:** Use Egoroff's theorem to show $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$. This yields a different and easier proof than what was done earlier. See Theorem 7.46 on Page 152.

13. ↑ Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces but there exist examples where the Vitali convergence theorem works and the dominated convergence theorem does not.

14. Suppose $f \in L^\infty \cap L^1$. Show $\lim_{p\to\infty} ||f||_{L^p} = ||f||_\infty$. **Hint:**
$$(||f||_\infty - \varepsilon)^p \mu([|f| > ||f||_\infty - \varepsilon]) \leq \int_{[|f|>||f||_\infty - \varepsilon]} |f|^p d\mu \leq$$

$$\int |f|^p \, d\mu = \int |f|^{p-1} |f| \, d\mu \leq ||f||_\infty^{p-1} \int |f| \, d\mu.$$

Now raise both ends to the $1/p$ power and take lim inf and lim sup as $p \to \infty$. You should get $||f||_\infty - \varepsilon \leq \liminf ||f||_p \leq \limsup ||f||_p \leq ||f||_\infty$

15. Suppose $\mu(\Omega) < \infty$. Show that if $1 \leq p < q$, then $L^q(\Omega) \subseteq L^p(\Omega)$. **Hint** Use Holder's inequality.

16. Show $L^1(\mathbb{R}) \nsubseteq L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \nsubseteq L^1(\mathbb{R})$ if Lebesgue measure is used. **Hint:** Consider $1/\sqrt{x}$ and $1/x$.

17. Suppose that $\theta \in [0,1]$ and $r, s, q > 0$ with

$$\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}.$$

show that

$$(\int |f|^q d\mu)^{1/q} \leq ((\int |f|^r d\mu)^{1/r})^\theta ((\int |f|^s d\mu)^{1/s})^{1-\theta}.$$

If $q, r, s \geq 1$ this says that

$$||f||_q \leq ||f||_r^\theta ||f||_s^{1-\theta}.$$

Using this, show that

$$\ln\left(||f||_q\right) \leq \theta \ln\left(||f||_r\right) + (1-\theta) \ln\left(||f||_s\right).$$

Hint:

$$\int |f|^q d\mu = \int |f|^{q\theta} |f|^{q(1-\theta)} d\mu.$$

Now note that $1 = \frac{\theta q}{r} + \frac{q(1-\theta)}{s}$ and use Holder's inequality.

18. Suppose f is a function in $L^1(\mathbb{R})$ and f is infinitely differentiable. Does it follow that $f' \in L^1(\mathbb{R})$? **Hint:** What if $\phi \in C_c^\infty(0,1)$ and $f(x) = \phi(2^n(x-n))$ for $x \in (n, n+1)$, $f(x) = 0$ if $x < 0$?

Banach Spaces

11.1 Theorems Based On Baire Category

11.1.1 Baire Category Theorem

Some examples of Banach spaces that have been discussed up to now are $\mathbb{R}^n, \mathbb{C}^n$, and $L^p(\Omega)$. Theorems about general Banach spaces are proved in this chapter. The main theorems to be presented here are the uniform boundedness theorem, the open mapping theorem, the closed graph theorem, and the Hahn Banach Theorem. The first three of these theorems come from the Baire category theorem which is about to be presented. They are topological in nature. The Hahn Banach theorem has nothing to do with topology. Banach spaces are all normed linear spaces and as such, they are all metric spaces because a normed linear space may be considered as a metric space with $d(x, y) \equiv ||x - y||$. You can check that this satisfies all the axioms of a metric. As usual, if every Cauchy sequence converges, the metric space is called complete.

Definition 11.1 *A complete normed linear space is called a Banach space.*

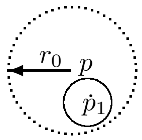
The following remarkable result is called the Baire category theorem. To get an idea of its meaning, imagine you draw a line in the plane. The complement of this line is an open set and is dense because every point, even those on the line, are limit points of this open set. Now draw another line. The complement of the two lines is still open and dense. Keep drawing lines and looking at the complements of the union of these lines. You always have an open set which is dense. Now what if there were countably many lines? The Baire category theorem implies the complement of the union of these lines is dense. In particular it is nonempty. Thus you cannot write the plane as a countable union of lines. This is a rather rough description of this very important theorem. The precise statement and proof follow.

Theorem 11.2 *Let (X, d) be a complete metric space and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open subsets of X satisfying $\overline{U_n} = X$ (U_n is dense). Then $D \equiv \cap_{n=1}^{\infty} U_n$ is a dense subset of X.*

Proof: Let $p \in X$ and let $r_0 > 0$. I need to show $D \cap B(p, r_0) \neq \emptyset$. Since U_1 is dense, there exists $p_1 \in U_1 \cap B(p, r_0)$, an open set. Let $p_1 \in B(p_1, r_1) \subseteq \overline{B(p_1, r_1)} \subseteq U_1 \cap B(p, r_0)$ and $r_1 < 2^{-1}$. This is possible because $U_1 \cap B(p, r_0)$ is an open set and so there exists r_1 such that $B(p_1, 2r_1) \subseteq U_1 \cap B(p, r_0)$. But

$$B(p_1, r_1) \subseteq \overline{B(p_1, r_1)} \subseteq B(p_1, 2r_1)$$

because $\overline{B(p_1, r_1)} = \{x \in X : d(x, p) \leq r_1\}$. (Why?)



There exists $p_2 \in U_2 \cap B(p_1, r_1)$ because U_2 is dense. Let

$$p_2 \in B(p_2, r_2) \subseteq \overline{B(p_2, r_2)} \subseteq U_2 \cap B(p_1, r_1) \subseteq U_1 \cap U_2 \cap B(p, r_0).$$

and let $r_2 < 2^{-2}$. Continue in this way. Thus

$$r_n < 2^{-n},$$

$$\overline{B(p_n, r_n)} \subseteq U_1 \cap U_2 \cap ... \cap U_n \cap B(p, r_0),$$

$$\overline{B(p_n, r_n)} \subseteq B(p_{n-1}, r_{n-1}).$$

The sequence, $\{p_n\}$ is a Cauchy sequence because all terms of $\{p_k\}$ for $k \geq n$ are contained in $B(p_n, r_n)$, a set whose diameter is no larger than 2^{-n}. Since X is complete, there exists p_∞ such that

$$\lim_{n \to \infty} p_n = p_\infty.$$

Since all but finitely many terms of $\{p_n\}$ are in $\overline{B(p_m, r_m)}$, it follows that $p_\infty \in \overline{B(p_m, r_m)}$ for each m. Therefore,

$$p_\infty \in \cap_{m=1}^{\infty} \overline{B(p_m, r_m)} \subseteq \cap_{i=1}^{\infty} U_i \cap B(p, r_0).$$

This proves the theorem.

The following corollary is also called the Baire category theorem.

Corollary 11.3 *Let X be a complete metric space and suppose $X = \cup_{i=1}^{\infty} F_i$ where each F_i is a closed set. Then for some i, interior $F_i \neq \emptyset$.*

Proof: If all F_i has empty interior, then F_i^C would be a dense open set. Therefore, from Theorem 11.2, it would follow that

$$\emptyset = \left(\cup_{i=1}^{\infty} F_i\right)^C = \cap_{i=1}^{\infty} F_i^C \neq \emptyset.$$

The set D of Theorem 11.2 is called a G_δ set because it is the countable intersection of open sets. Thus D is a dense G_δ set.

Recall that a norm satisfies:

a.) $||x|| \geq 0$, $||x|| = 0$ if and only if $x = 0$.

b.) $||x + y|| \leq ||x|| + ||y||$.

c.) $||cx|| = |c|\ ||x||$ if c is a scalar and $x \in X$.

From the definition of continuity, it follows easily that a function is continuous if

$$\lim_{n\to\infty} x_n = x$$

implies

$$\lim_{n\to\infty} f(x_n) = f(x).$$

Theorem 11.4 *Let X and Y be two normed linear spaces and let $L : X \to Y$ be linear ($L(ax + by) = aL(x) + bL(y)$ for a, b scalars and $x, y \in X$). The following are equivalent*

a.) L is continuous at 0

b.) L is continuous

c.) There exists $K > 0$ such that $||Lx||_Y \leq K\ ||x||_X$ for all $x \in X$ (L is bounded).

Proof: a.)⇒b.) Let $x_n \to x$. It is necessary to show that $Lx_n \to Lx$. But $(x_n - x) \to 0$ and so from continuity at 0, it follows

$$L(x_n - x) = Lx_n - Lx \to 0$$

so $Lx_n \to Lx$. This shows a.) implies b.).

b.)⇒c.) Since L is continuous, L is continuous at 0. Hence $||Lx||_Y < 1$ whenever $||x||_X \leq \delta$ for some δ. Therefore, suppressing the subscript on the $||\ ||$,

$$||L\left(\frac{\delta x}{||x||}\right)|| \leq 1.$$

Hence

$$||Lx|| \leq \frac{1}{\delta}||x||.$$

c.)⇒a.) follows from the inequality given in c.).

Definition 11.5 *Let $L : X \to Y$ be linear and continuous where X and Y are normed linear spaces. Denote the set of all such continuous linear maps by $\mathcal{L}(X, Y)$ and define*

$$||L|| = \sup\{||Lx|| : ||x|| \leq 1\}. \tag{11.1}$$

This is called the operator norm.

Note that from Theorem 11.4 $||L||$ is well defined because of part c.) of that Theorem.

The next lemma follows immediately from the definition of the norm and the assumption that L is linear.

Lemma 11.6 *With $||L||$ defined in 11.1, $\mathcal{L}(X,Y)$ is a normed linear space. Also $||Lx|| \leq ||L||\,||x||$.*

Proof: Let $x \neq 0$ then $x/\,||x||$ has norm equal to 1 and so

$$\left|\left|L\left(\frac{x}{||x||}\right)\right|\right| \leq ||L||\,.$$

Therefore, multiplying both sides by $||x||$, $||Lx|| \leq ||L||\,||x||$. This is obviously a linear space. It remains to verify the operator norm really is a norm. First of all, if $||L|| = 0$, then $Lx = 0$ for all $||x|| \leq 1$. It follows that for any $x \neq 0, 0 = L\left(\frac{x}{||x||}\right)$ and so $Lx = 0$. Therefore, $L = 0$. Also, if c is a scalar,

$$||cL|| = \sup_{||x|| \leq 1} ||cL\,(x)|| = |c| \sup_{||x|| \leq 1} ||Lx|| = |c|\,||L||\,.$$

It remains to verify the triangle inequality. Let $L, M \in \mathcal{L}\,(X,Y)$.

$$\begin{aligned} ||L+M|| &\equiv \sup_{||x|| \leq 1} ||(L+M)\,(x)|| \leq \sup_{||x|| \leq 1} (||Lx|| + ||Mx||) \\ &\leq \sup_{||x|| \leq 1} ||Lx|| + \sup_{||x|| \leq 1} ||Mx|| = ||L|| + ||M||\,. \end{aligned}$$

This shows the operator norm is really a norm as hoped. This proves the lemma.

For example, consider the space of linear transformations defined on $\mathbb{R}^n$ having values in $\mathbb{R}^m$. The fact the transformation is linear automatically imparts continuity to it. You should give a proof of this fact. Recall that every such linear transformation can be realized in terms of matrix multiplication.

Thus, in finite dimensions the algebraic condition that an operator is linear is sufficient to imply the topological condition that the operator is continuous. The situation is not so simple in infinite dimensional spaces such as $C\,(X;\mathbb{R}^n)$. This explains the imposition of the topological condition of continuity as a criterion for membership in $\mathcal{L}\,(X,Y)$ in addition to the algebraic condition of linearity.

Theorem 11.7 *If Y is a Banach space, then $\mathcal{L}(X,Y)$ is also a Banach space.*

Proof: Let $\{L_n\}$ be a Cauchy sequence in $\mathcal{L}(X,Y)$ and let $x \in X$.

$$||L_n x - L_m x|| \leq ||x||\,||L_n - L_m||.$$

Thus $\{L_n x\}$ is a Cauchy sequence. Let

$$Lx = \lim_{n\to\infty} L_n x.$$

Then, clearly, L is linear because if x_1, x_2 are in X, and a, b are scalars, then

$$\begin{aligned} L(ax_1 + bx_2) &= \lim_{n\to\infty} L_n(ax_1 + bx_2) \\ &= \lim_{n\to\infty} (aL_n x_1 + bL_n x_2) \\ &= aLx_1 + bLx_2. \end{aligned}$$

Also L is continuous. To see this, note that $\{||L_n||\}$ is a Cauchy sequence of real numbers because $|||L_n|| - ||L_m||| \leq ||L_n - L_m||$. Hence there exists $K > \sup\{||L_n|| : n \in \mathbb{N}\}$. Thus, if $x \in X$,

$$||Lx|| = \lim_{n\to\infty} ||L_n x|| \leq K||x||.$$

This proves the theorem.

11.1.2 Uniform Boundedness Theorem

The next big result is sometimes called the Uniform Boundedness theorem, or the Banach-Steinhaus theorem. This is a very surprising theorem which implies that for a collection of bounded linear operators, if they are bounded pointwise, then they are also bounded uniformly. As an example of a situation in which pointwise bounded does not imply uniformly bounded, consider the functions $f_\alpha(x) \equiv \mathcal{X}_{(\alpha,1)}(x)\, x^{-1}$ for $\alpha \in (0,1)$. Clearly each function is bounded and the collection of functions is bounded at each point of $(0,1)$, but there is no bound for all these functions taken together. One problem is that $(0,1)$ is not a Banach space. Therefore, the functions cannot be linear.

Theorem 11.8 *Let X be a Banach space and let Y be a normed linear space. Let $\{L_\alpha\}_{\alpha\in\Lambda}$ be a collection of elements of $\mathcal{L}(X,Y)$. Then one of the following happens.*
a.) $\sup\{||L_\alpha|| : \alpha \in \Lambda\} < \infty$
b.) There exists a dense G_δ set, D, such that for all $x \in D$,

$$\sup\{||L_\alpha x|| \; \alpha \in \Lambda\} = \infty.$$

Proof: For each $n \in \mathbb{N}$, define

$$U_n = \{x \in X : \sup\{||L_\alpha x|| : \alpha \in \Lambda\} > n\}.$$

Then U_n is an open set because if $x \in U_n$, then there exists $\alpha \in \Lambda$ such that

$$||L_\alpha x|| > n$$

But then, since L_α is continuous, this situation persists for all y sufficiently close to x, say for all $y \in B(x,\delta)$. Then $B(x,\delta) \subseteq U_n$ which shows U_n is open.

Case b.) is obtained from Theorem 11.2 if each U_n is dense.

The other case is that for some n, U_n is not dense. If this occurs, there exists x_0 and $r > 0$ such that for all $x \in B(x_0, r)$, $||L_\alpha x|| \leq n$ for all α. Now if $y \in$

$B(0,r)$, $x_0 + y \in B(x_0,r)$. Consequently, for all such y, $||L_\alpha(x_0 + y)|| \leq n$. This implies that for all $\alpha \in \Lambda$ and $||y|| < r$,

$$||L_\alpha y|| \leq n + ||L_\alpha(x_0)|| \leq 2n.$$

Therefore, if $||y|| \leq 1$, $\left|\left|\frac{r}{2}y\right|\right| < r$ and so for all α,

$$||L_\alpha \left(\frac{r}{2}y\right)|| \leq 2n.$$

Now multiplying by $r/2$ it follows that whenever $||y|| \leq 1$, $||L_\alpha (y)|| \leq 4n/r$. Hence case a.) holds.

11.1.3 Open Mapping Theorem

Another remarkable theorem which depends on the Baire category theorem is the open mapping theorem. Unlike Theorem 11.8 it requires both X and Y to be Banach spaces.

Theorem 11.9 *Let X and Y be Banach spaces, let $L \in \mathcal{L}(X,Y)$, and suppose L is onto. Then L maps open sets onto open sets.*

To aid in the proof, here is a lemma.

Lemma 11.10 *Let a and b be positive constants and suppose*

$$B(0,a) \subseteq \overline{L(B(0,b))}.$$

Then

$$\overline{L(B(0,b))} \subseteq L(B(0,2b)).$$

Proof of Lemma 11.10: Let $y \in \overline{L(B(0,b))}$. There exists $x_1 \in B(0,b)$ such that $||y - Lx_1|| < \frac{a}{2}$. Now this implies

$$2y - 2Lx_1 \in B(0,a) \subseteq \overline{L(B(0,b))}.$$

Thus $2y - 2Lx_1 \in \overline{L(B(0,b))}$ just like y was. Therefore, there exists $x_2 \in B(0,b)$ such that $||2y - 2Lx_1 - Lx_2|| < a/2$. Hence $||4y - 4Lx_1 - 2Lx_2|| < a$, and there exists $x_3 \in B\left(0,b\right)$ such that $||4y - 4Lx_1 - 2Lx_2 - Lx_3|| < a/2$. Continuing in this way, there exist $x_1, x_2, x_3, x_4, ...$ in $B(0,b)$ such that

$$||2^n y - \sum_{i=1}^{n} 2^{n-(i-1)} L(x_i)|| < a$$

which implies

$$||y - \sum_{i=1}^{n} 2^{-(i-1)} L(x_i)|| = ||y - L\left(\sum_{i=1}^{n} 2^{-(i-1)}(x_i)\right)|| < 2^{-n}a \tag{11.2}$$

Now consider the partial sums of the series, $\sum_{i=1}^{\infty} 2^{-(i-1)} x_i$.

$$||\sum_{i=m}^{n} 2^{-(i-1)} x_i|| \leq b \sum_{i=m}^{\infty} 2^{-(i-1)} = b\, 2^{-m+2}.$$

Therefore, these partial sums form a Cauchy sequence and so since X is complete, there exists $x = \sum_{i=1}^{\infty} 2^{-(i-1)} x_i$. Letting $n \to \infty$ in 11.2 yields $||y - Lx|| = 0$. Now

$$||x|| = \lim_{n\to\infty} ||\sum_{i=1}^{n} 2^{-(i-1)} x_i||$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} 2^{-(i-1)} ||x_i|| < \lim_{n\to\infty} \sum_{i=1}^{n} 2^{-(i-1)} b = 2b.$$

This proves the lemma.

Proof of Theorem 11.9: $Y = \cup_{n=1}^{\infty} \overline{L(B(0,n))}$. By Corollary 11.3, the set, $\overline{L(B(0,n_0))}$ has nonempty interior for some n_0. Thus $B(y,r) \subseteq \overline{L(B(0,n_0))}$ for some y and some $r > 0$. Since L is linear $B(-y,r) \subseteq \overline{L(B(0,n_0))}$ also. Here is why. If $z \in B(-y,r)$, then $-z \in B(y,r)$ and so there exists $x_n \in B(0,n_0)$ such that $Lx_n \to -z$. Therefore, $L(-x_n) \to z$ and $-x_n \in B(0,n_0)$ also. Therefore $z \in \overline{L(B(0,n_0))}$. Then it follows that

$$\begin{aligned} B(0,r) &\subseteq B(y,r) + B(-y,r) \\ &\equiv \{y_1 + y_2 : y_1 \in B(y,r) \text{ and } y_2 \in B(-y,r)\} \\ &\subseteq \overline{L(B(0,2n_0))} \end{aligned}$$

The reason for the last inclusion is that from the above, if $y_1 \in B(y,r)$ and $y_2 \in B(-y,r)$, there exists $x_n, z_n \in B(0,n_0)$ such that

$$Lx_n \to y_1,\ Lz_n \to y_2.$$

Therefore,

$$||x_n + z_n|| \leq 2n_0$$

and so $(y_1 + y_2) \in \overline{L(B(0,2n_0))}$.

By Lemma 11.10, $\overline{L(B(0,2n_0))} \subseteq L(B(0,4n_0))$ which shows

$$B(0,r) \subseteq L(B(0,4n_0)).$$

Letting $a = r(4n_0)^{-1}$, it follows, since L is linear, that $B(0,a) \subseteq L(B(0,1))$. It follows since L is linear,

$$L(B(0,r)) \supseteq B(0,ar). \tag{11.3}$$

Now let U be open in X and let $x + B(0,r) = B(x,r) \subseteq U$. Using 11.3,

$$L(U) \supseteq L(x + B(0,r))$$

$$= Lx + L(B(0,r)) \supseteq Lx + B(0,ar) = B(Lx,ar).$$

Hence

$$Lx \in B(Lx, ar) \subseteq L(U).$$

which shows that every point, $Lx \in LU$, is an interior point of LU and so LU is open. This proves the theorem.

This theorem is surprising because it implies that if $|\cdot|$ and $||\cdot||$ are two norms with respect to which a vector space X is a Banach space such that $|\cdot| \leq K ||\cdot||$, then there exists a constant k, such that $||\cdot|| \leq k |\cdot|$. This can be useful because sometimes it is not clear how to compute k when all that is needed is its existence. To see the open mapping theorem implies this, consider the identity map $\operatorname{id} x = x$. Then $\operatorname{id} : (X, ||\cdot||) \to (X, |\cdot|)$ is continuous and onto. Hence id is an open map which implies id^{-1} is continuous. Theorem 11.4 gives the existence of the constant k.

11.1.4 Closed Graph Theorem

Definition 11.11 *Let $f : D \to E$. The set of all ordered pairs of the form $\{(x, f(x)) : x \in D\}$ is called the graph of f.*

Definition 11.12 *If X and Y are normed linear spaces, make $X \times Y$ into a normed linear space by using the norm $||(x, y)|| = \max(||x||, ||y||)$ along with component-wise addition and scalar multiplication. Thus $a(x, y) + b(z, w) \equiv (ax + bz, ay + bw)$.*

There are other ways to give a norm for $X \times Y$. For example, you could define $||(x, y)|| = ||x|| + ||y||$

Lemma 11.13 *The norm defined in Definition 11.12 on $X \times Y$ along with the definition of addition and scalar multiplication given there make $X \times Y$ into a normed linear space.*

Proof: The only axiom for a norm which is not obvious is the triangle inequality. Therefore, consider

$$\begin{aligned} ||(x_1, y_1) + (x_2, y_2)|| &= ||(x_1 + x_2, y_1 + y_2)|| \\ &= \max(||x_1 + x_2||, ||y_1 + y_2||) \\ &\leq \max(||x_1|| + ||x_2||, ||y_1|| + ||y_2||) \end{aligned}$$

$$\begin{aligned} &\leq \max(||x_1||, ||y_1||) + \max(||x_2||, ||y_2||) \\ &= ||(x_1, y_1)|| + ||(x_2, y_2)||. \end{aligned}$$

It is obvious $X \times Y$ is a vector space from the above definition. This proves the lemma.

Lemma 11.14 *If X and Y are Banach spaces, then $X \times Y$ with the norm and vector space operations defined in Definition 11.12 is also a Banach space.*

Proof: The only thing left to check is that the space is complete. But this follows from the simple observation that $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$ if and only if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Thus if $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$, it follows there exist x and y such that $x_n \to x$ and $y_n \to y$. But then from the definition of the norm, $(x_n, y_n) \to (x, y)$.

Lemma 11.15 *Every closed subspace of a Banach space is a Banach space.*

Proof: If $F \subseteq X$ where X is a Banach space and $\{x_n\}$ is a Cauchy sequence in F, then since X is complete, there exists a unique $x \in X$ such that $x_n \to x$. However this means $x \in \overline{F} = F$ since F is closed.

Definition 11.16 *Let X and Y be Banach spaces and let $D \subseteq X$ be a subspace. A linear map $L : D \to Y$ is said to be closed if its graph is a closed subspace of $X \times Y$. Equivalently, L is closed if $x_n \to x$ and $Lx_n \to y$ implies $x \in D$ and $y = Lx$.*

Note the distinction between closed and continuous. If the operator is closed the assertion that $y = Lx$ only follows if it is known that the sequence $\{Lx_n\}$ converges. In the case of a continuous operator, the convergence of $\{Lx_n\}$ follows from the assumption that $x_n \to x$. It is not always the case that a mapping which is closed is necessarily continuous. Consider the function $f(x) = \tan(x)$ if x is not an odd multiple of $\frac{\pi}{2}$ and $f(x) \equiv 0$ at every odd multiple of $\frac{\pi}{2}$. Then the graph is closed and the function is defined on $\mathbb{R}$ but it clearly fails to be continuous. Of course this function is not linear. You could also consider the map,

$$\frac{d}{dx} : \left\{y \in C^1([0,1]) : y(0) = 0\right\} \equiv D \to C([0,1]).$$

where the norm is the uniform norm on $C([0,1])$, $||y||_\infty$. If $y \in D$, then

$$y(x) = \int_0^x y'(t)\, dt.$$

Therefore, if $\frac{dy_n}{dx} \to f \in C([0,1])$ and if $y_n \to y$ in $C([0,1])$ it follows that

$$\begin{array}{ccc} y_n(x) & = & \int_0^x \frac{dy_n(t)}{dx} dt \\ \downarrow & & \downarrow \\ y(x) & = & \int_0^x f(t)\, dt \end{array}$$

and so by the fundamental theorem of calculus $f(x) = y'(x)$ and so the mapping is closed. It is obviously not continuous because it takes $y(x)$ and $y(x) + \frac{1}{n}\sin(nx)$ to two functions which are far from each other even though these two functions are very close in $C([0,1])$. Furthermore, it is not defined on the whole space, $C([0,1])$.

The next theorem, the closed graph theorem, gives conditions under which closed implies continuous.

Theorem 11.17 *Let X and Y be Banach spaces and suppose $L : X \to Y$ is closed and linear. Then L is continuous.*

Proof: Let G be the graph of L. $G = \{(x, Lx) : x \in X\}$. By Lemma 11.15 it follows that G is a Banach space. Define $P : G \to X$ by $P(x, Lx) = x$. P maps the Banach space G onto the Banach space X and is continuous and linear. By the open mapping theorem, P maps open sets onto open sets. Since P is also one to one, this says that P^{-1} is continuous. Thus $||P^{-1}x|| \leq K||x||$. Hence

$$||Lx|| \leq \max(||x||, ||Lx||) \leq K||x||$$

By Theorem 11.4 on Page 255, this shows L is continuous and proves the theorem.

The following corollary is quite useful. It shows how to obtain a new norm on the domain of a closed operator such that the domain with this new norm becomes a Banach space.

Corollary 11.18 *Let $L : D \subseteq X \to Y$ where X, Y are a Banach spaces, and L is a closed operator. Then define a new norm on D by*

$$||x||_D \equiv ||x||_X + ||Lx||_Y .$$

Then D with this new norm is a Banach space.

Proof: If $\{x_n\}$ is a Cauchy sequence in D with this new norm, it follows both $\{x_n\}$ and $\{Lx_n\}$ are Cauchy sequences and therefore, they converge. Since L is closed, $x_n \to x$ and $Lx_n \to Lx$ for some $x \in D$. Thus $||x_n - x||_D \to 0$.

11.2 Hahn Banach Theorem

The closed graph, open mapping, and uniform boundedness theorems are the three major topological theorems in functional analysis. The other major theorem is the Hahn-Banach theorem which has nothing to do with topology. Before presenting this theorem, here are some preliminaries about partially ordered sets.

Definition 11.19 *Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that*

$$x \leq x \text{ for all } x \in \mathcal{F}.$$

$$\text{If } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 11.20 *(Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

Definition 11.21 *Let X be a real vector space $\rho : X \to \mathbb{R}$ is called a gauge function if*

$$\begin{aligned} \rho(x+y) &\leq \rho(x)+\rho(y), \\ \rho(ax) &= a\rho(x) \text{ if } a \geq 0. \end{aligned} \tag{11.4}$$

Suppose M is a subspace of X and $z \notin M$. Suppose also that f is a linear real-valued function having the property that $f(x) \leq \rho(x)$ for all $x \in M$. Consider the problem of extending f to $M \oplus \mathbb{R}z$ such that if F is the extended function, $F(y) \leq \rho(y)$ for all $y \in M \oplus \mathbb{R}z$ and F is linear. Since F is to be linear, it suffices to determine how to define $F(z)$. Letting $a > 0$, it is required to define $F(z)$ such that the following hold for all $x, y \in M$.

$$\begin{aligned} \overbrace{F(x)}^{f(x)} + aF(z) &= F(x+az) \leq \rho(x+az), \\ \overbrace{F(y)}^{f(y)} - aF(z) &= F(y-az) \leq \rho(y-az). \end{aligned} \tag{11.5}$$

Now if these inequalities hold for all y/a, they hold for all y because M is given to be a subspace. Therefore, multiplying by a^{-1} 11.4 implies that what is needed is to choose $F(z)$ such that for all $x, y \in M$,

$$f(x)+F(z) \leq \rho(x+z),\ f(y)-\rho(y-z) \leq F(z)$$

and that if $F(z)$ can be chosen in this way, this will satisfy 11.5 for all x, y and the problem of extending f will be solved. Hence it is necessary to choose $F(z)$ such that for all $x, y \in M$

$$f(y)-\rho(y-z) \leq F(z) \leq \rho(x+z)-f(x). \tag{11.6}$$

Is there any such number between $f(y)-\rho(y-z)$ and $\rho(x+z)-f(x)$ for every pair $x, y \in M$? This is where $f(x) \leq \rho(x)$ on M and that f is linear is used. For $x, y \in M$,

$$\begin{aligned} &\rho(x+z)-f(x)-[f(y)-\rho(y-z)] \\ &= \rho(x+z)+\rho(y-z)-(f(x)+f(y)) \\ &\geq \rho(x+y)-f(x+y) \geq 0. \end{aligned}$$

Therefore there exists a number between

$$\sup\{f(y)-\rho(y-z):y\in M\}$$

and

$$\inf\{\rho(x+z)-f(x):x\in M\}$$

Choose $F(z)$ to satisfy 11.6. This has proved the following lemma.

Lemma 11.22 *Let M be a subspace of X, a real linear space, and let ρ be a gauge function on X. Suppose $f:M\to\mathbb{R}$ is linear, $z\notin M$, and $f(x)\le\rho(x)$ for all $x\in M$. Then f can be extended to $M\oplus\mathbb{R}z$ such that, if F is the extended function, F is linear and $F(x)\le\rho(x)$ for all $x\in M\oplus\mathbb{R}z$.*

With this lemma, the Hahn Banach theorem can be proved.

Theorem 11.23 *(Hahn Banach theorem) Let X be a real vector space, let M be a subspace of X, let $f:M\to\mathbb{R}$ be linear, let ρ be a gauge function on X, and suppose $f(x)\le\rho(x)$ for all $x\in M$. Then there exists a linear function, $F:X\to\mathbb{R}$, such that*

a.) $F(x)=f(x)$ for all $x\in M$

b.) $F(x)\le\rho(x)$ for all $x\in X$.

Proof: Let $\mathcal{F}=\{(V,g):V\supseteq M,\ V$ is a subspace of $X,\ g:V\to\mathbb{R}$ is linear, $g(x)=f(x)$ for all $x\in M$, and $g(x)\le\rho(x)$ for $x\in V\}$. Then $(M,f)\in\mathcal{F}$ so $\mathcal{F}\neq\emptyset$. Define a partial order by the following rule.

$$(V,g)\le(W,h)$$

means

$$V\subseteq W \text{ and } h(x)=g(x) \text{ if } x\in V.$$

By Theorem 11.20, there exists a maximal chain, $\mathcal{C}\subseteq\mathcal{F}$. Let $Y=\cup\{V:(V,g)\in\mathcal{C}\}$ and let $h:Y\to\mathbb{R}$ be defined by $h(x)=g(x)$ where $x\in V$ and $(V,g)\in\mathcal{C}$. This is well defined because if $x\in V_1$ and V_2 where (V_1,g_1) and (V_2,g_2) are both in the chain, then since $\mathcal{C}$ is a chain, the two element related. Therefore, $g_1(x)=g_2(x)$. Also h is linear because if $ax+by\in Y$, then $x\in V_1$ and $y\in V_2$ where (V_1,g_1) and (V_2,g_2) are elements of $\mathcal{C}$. Therefore, letting V denote the larger of the two V_i, and g be the function that goes with V, it follows $ax+by\in V$ where $(V,g)\in\mathcal{C}$. Therefore,

$$\begin{aligned} h(ax+by) &= g(ax+by)\\ &= ag(x)+bg(y)\\ &= ah(x)+bh(y).\end{aligned}$$

Also, $h(x)=g(x)\le\rho(x)$ for any $x\in Y$ because for such x, $x\in V$ where $(V,g)\in\mathcal{C}$.

Is $Y=X$? If not, there exists $z\in X\setminus Y$ and there exists an extension of h to $Y\oplus\mathbb{R}z$ using Lemma 11.22. Letting $\overline{h}$ denote this extended function, contradicts the maximality of $\mathcal{C}$. Indeed, $\mathcal{C} \cup \{(Y \oplus \mathbb{R}z, \overline{h})\}$ would be a longer chain. This proves the Hahn Banach theorem.

This is the original version of the theorem. There is also a version of this theorem for complex vector spaces which is based on a trick.

Corollary 11.24 *(Hahn Banach) Let M be a subspace of a complex normed linear space, X, and suppose $f : M \to \mathbb{C}$ is linear and satisfies $|f(x)| \leq K||x||$ for all $x \in M$. Then there exists a linear function, F, defined on all of X such that $F(x) = f(x)$ for all $x \in M$ and $|F(x)| \leq K||x||$ for all x.*

Proof: First note $f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x)$ and so

$$\operatorname{Re} f(ix) + i \operatorname{Im} f(ix) = f(ix) = if(x) = i \operatorname{Re} f(x) - \operatorname{Im} f(x).$$

Therefore, $\operatorname{Im} f(x) = -\operatorname{Re} f(ix)$, and

$$f(x) = \operatorname{Re} f(x) - i \operatorname{Re} f(ix).$$

This is important because it shows it is only necessary to consider $\operatorname{Re} f$ in understanding f. Now it happens that $\operatorname{Re} f$ is linear with respect to real scalars so the above version of the Hahn Banach theorem applies. This is shown next.

If c is a real scalar

$$\operatorname{Re} f(cx) - i \operatorname{Re} f(icx) = cf(x) = c \operatorname{Re} f(x) - ic \operatorname{Re} f(ix).$$

Thus $\operatorname{Re} f(cx) = c \operatorname{Re} f(x)$. Also,

$$\begin{aligned} \operatorname{Re} f(x+y) - i \operatorname{Re} f(i(x+y)) &= f(x+y) \\ &= f(x) + f(y) \end{aligned}$$

$$= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) + \operatorname{Re} f(y) - i \operatorname{Re} f(iy).$$

Equating real parts, $\operatorname{Re} f(x+y) = \operatorname{Re} f(x) + \operatorname{Re} f(y)$. Thus $\operatorname{Re} f$ is linear with respect to real scalars as hoped.

Consider X as a real vector space and let $\rho(x) \equiv K||x||$. Then for all $x \in M$,

$$|\operatorname{Re} f(x)| \leq |f(x)| \leq K||x|| = \rho(x).$$

From Theorem 11.23, $\operatorname{Re} f$ may be extended to a function, h which satisfies

$$\begin{aligned} h(ax+by) &= ah(x) + bh(y) \text{ if } a, b \in \mathbb{R} \\ h(x) &\leq K||x|| \text{ for all } x \in X. \end{aligned}$$

Actually, $|h(x)| \leq K||x||$. The reason for this is that $h(-x) = -h(x) \leq K||-x|| = K||x||$ and therefore, $h(x) \geq -K||x||$. Let

$$F(x) \equiv h(x) - ih(ix).$$

By arguments similar to the above, F is linear.

$$\begin{aligned} F(ix) &= h(ix) - ih(-x) \\ &= ih(x) + h(ix) \\ &= i(h(x) - ih(ix)) = iF(x). \end{aligned}$$

If c is a real scalar,

$$\begin{aligned} F(cx) &= h(cx) - ih(icx) \\ &= ch(x) - cih(ix) = cF(x) \end{aligned}$$

Now

$$\begin{aligned} F(x+y) &= h(x+y) - ih(i(x+y)) \\ &= h(x) + h(y) - ih(ix) - ih(iy) \\ &= F(x) + F(y). \end{aligned}$$

Thus

$$\begin{aligned} F((a+ib)x) &= F(ax) + F(ibx) \\ &= aF(x) + ibF(x) \\ &= (a+ib)F(x). \end{aligned}$$

This shows F is linear as claimed.

Now $wF(x) = |F(x)|$ for some $|w| = 1$. Therefore

$$\begin{aligned} |F(x)| &= wF(x) = h(wx) - \overbrace{ih(iwx)}^{\text{must equal zero}} = h(wx) \\ &= |h(wx)| \leq K||wx|| = K\,||x||. \end{aligned}$$

This proves the corollary.

Definition 11.25 *Let X be a Banach space. Denote by X' the space of continuous linear functions which map X to the field of scalars. Thus $X' = \mathcal{L}(X, \mathbb{F})$. By Theorem 11.7 on Page 256, X' is a Banach space. Remember with the norm defined on $\mathcal{L}(X, \mathbb{F})$,*

$$||f|| = \sup\{|f(x)| : ||x|| \leq 1\}$$

X' is called the dual space.

Definition 11.26 *Let X and Y be Banach spaces and suppose $L \in \mathcal{L}(X, Y)$. Then define the adjoint map in $\mathcal{L}(Y', X')$, denoted by L^*, by*

$$L^* y^*(x) \equiv y^*(Lx)$$

for all $y^* \in Y'$.

The following diagram is a good one to help remember this definition.

$$\begin{array}{ccc} & L^* & \\ X' & \leftarrow & Y' \\ & \rightarrow & \\ X & L & Y \end{array}$$

This is a generalization of the adjoint of a linear transformation on an inner product space. Recall

$$(Ax, y) = (x, A^*y)$$

What is being done here is to generalize this algebraic concept to arbitrary Banach spaces. There are some issues which need to be discussed relative to the above definition. First of all, it must be shown that $L^*y^* \in X'$. Also, it will be useful to have the following lemma which is a useful application of the Hahn Banach theorem.

Lemma 11.27 *Let X be a normed linear space and let $x \in X$. Then there exists $x^* \in X'$ such that $||x^*|| = 1$ and $x^*(x) = ||x||$.*

Proof: Let $f : \mathbb{F}x \to \mathbb{F}$ be defined by $f(\alpha x) = \alpha||x||$. Then for $y = \alpha x \in \mathbb{F}x$,

$$|f(y)| = |f(\alpha x)| = |\alpha| \, ||x|| = |y| \, .$$

By the Hahn Banach theorem, there exists $x^* \in X'$ such that $x^*(\alpha x) = f(\alpha x)$ and $||x^*|| \le 1$. Since $x^*(x) = ||x||$ it follows that $||x^*|| = 1$ because

$$||x^*|| \ge \left| x^* \left(\frac{x}{||x||} \right) \right| = \frac{||x||}{||x||} = 1.$$

This proves the lemma.

Theorem 11.28 *Let $L \in \mathcal{L}(X, Y)$ where X and Y are Banach spaces. Then*
a.) $L^* \in \mathcal{L}(Y', X')$ as claimed and $||L^*|| = ||L||$.
b.) If L maps one to one onto a closed subspace of Y, then L^* is onto.
c.) If L maps onto a dense subset of Y, then L^* is one to one.

Proof: It is routine to verify L^*y^* and L^* are both linear. This follows immediately from the definition. As usual, the interesting thing concerns continuity.

$$||L^*y^*|| = \sup_{||x|| \le 1} |L^*y^*(x)| = \sup_{||x|| \le 1} |y^*(Lx)| \le ||y^*|| \, ||L|| \, .$$

Thus L^* is continuous as claimed and $||L^*|| \le ||L||$.

By Lemma 11.27, there exists $y_x^* \in Y'$ such that $||y_x^*|| = 1$ and $y_x^*(Lx) = ||Lx||$ .Therefore,

$$\begin{aligned} ||L^*|| &= \sup_{||y^*|| \le 1} ||L^*y^*|| = \sup_{||y^*|| \le 1} \sup_{||x|| \le 1} |L^*y^*(x)| \\ &= \sup_{||y^*|| \le 1} \sup_{||x|| \le 1} |y^*(Lx)| = \sup_{||x|| \le 1} \sup_{||y^*|| \le 1} |y^*(Lx)| \ge \sup_{||x|| \le 1} |y_x^*(Lx)| = \sup_{||x|| \le 1} ||Lx|| = ||L|| \end{aligned}$$

showing that $||L^*|| \geq ||L||$ and this shows part a.).

If L is one to one and onto a closed subset of Y, then $L(X)$ being a closed subspace of a Banach space, is itself a Banach space and so the open mapping theorem implies $L^{-1} : L(X) \to X$ is continuous. Hence

$$||x|| = ||L^{-1}Lx|| \leq \left|\left|L^{-1}\right|\right| ||Lx||$$

Now let $x^* \in X'$ be given. Define $f \in \mathcal{L}(L(X), \mathbb{C})$ by $f(Lx) = x^*(x)$. The function, f is well defined because if $Lx_1 = Lx_2$, then since L is one to one, it follows $x_1 = x_2$ and so $f(L(x_1)) = x^*(x_1) = x^*(x_2) = f(L(x_1))$. Also, f is linear because

$$\begin{aligned} f(aL(x_1) + bL(x_2)) &= f(L(ax_1 + bx_2)) \\ &\equiv x^*(ax_1 + bx_2) \\ &= ax^*(x_1) + bx^*(x_2) \\ &= af(L(x_1)) + bf(L(x_2)). \end{aligned}$$

In addition to this,

$$|f(Lx)| = |x^*(x)| \leq ||x^*|| \, ||x|| \leq ||x^*|| \left|\left|L^{-1}\right|\right| ||Lx||$$

and so the norm of f on $L(X)$ is no larger than $||x^*|| \left|\left|L^{-1}\right|\right|$. By the Hahn Banach theorem, there exists an extension of f to an element $y^* \in Y'$ such that $||y^*|| \leq ||x^*|| \left|\left|L^{-1}\right|\right|$. Then

$$L^*y^*(x) = y^*(Lx) = f(Lx) = x^*(x)$$

so $L^*y^* = x^*$ because this holds for all x. Since x^* was arbitrary, this shows L^* is onto and proves b.).

Consider the last assertion. Suppose $L^*y^* = 0$. Is $y^* = 0$? In other words is $y^*(y) = 0$ for all $y \in Y$? Pick $y \in Y$. Since $L(X)$ is dense in Y, there exists a sequence, $\{Lx_n\}$ such that $Lx_n \to y$. But then by continuity of y^*, $y^*(y) = \lim_{n\to\infty} y^*(Lx_n) = \lim_{n\to\infty} L^*y^*(x_n) = 0$. Since $y^*(y) = 0$ for all y, this implies $y^* = 0$ and so L^* is one to one.

Corollary 11.29 *Suppose X and Y are Banach spaces, $L \in \mathcal{L}(X,Y)$, and L is one to one and onto. Then L^* is also one to one and onto.*

There exists a natural mapping, called the James map from a normed linear space, X, to the dual of the dual space which is described in the following definition.

Definition 11.30 *Define $J : X \to X''$ by $J(x)(x^*) = x^*(x)$.*

Theorem 11.31 *The map, J, has the following properties.*

a.) J is one to one and linear.

b.) $||Jx|| = ||x||$ and $||J|| = 1$.

c.) $J(X)$ is a closed subspace of X'' if X is complete.

Also if $x^* \in X'$,

$$||x^*|| = \sup\left\{|x^{**}(x^*)| : ||x^{**}|| \leq 1,\ x^{**} \in X''\right\}.$$

Proof:

$$\begin{aligned} J(ax+by)(x^*) &\equiv x^*(ax+by) \\ &= ax^*(x)+bx^*(y) \\ &= (aJ(x)+bJ(y))(x^*). \end{aligned}$$

Since this holds for all $x^* \in X'$, it follows that

$$J(ax+by) = aJ(x)+bJ(y)$$

and so J is linear. If $Jx = 0$, then by Lemma 11.27 there exists x^* such that $x^*(x) = ||x||$ and $||x^*|| = 1$. Then

$$0 = J(x)(x^*) = x^*(x) = ||x||.$$

This shows a.).

To show b.), let $x \in X$ and use Lemma 11.27 to obtain $x^* \in X'$ such that $x^*(x) = ||x||$ with $||x^*|| = 1$. Then

$$\begin{aligned} ||x|| &\geq \sup\{|y^*(x)| : ||y^*|| \leq 1\} \\ &= \sup\{|J(x)(y^*)| : ||y^*|| \leq 1\} = ||Jx|| \\ &\geq |J(x)(x^*)| = |x^*(x)| = ||x|| \end{aligned}$$

Therefore, $||Jx|| = ||x||$ as claimed. Therefore,

$$||J|| = \sup\{||Jx|| : ||x|| \leq 1\} = \sup\{||x|| : ||x|| \leq 1\} = 1.$$

This shows b.).

To verify c.), use b.). If $Jx_n \to y^{**} \in X''$ then by b.), x_n is a Cauchy sequence converging to some $x \in X$ because

$$||x_n - x_m|| = ||Jx_n - Jx_m||$$

and $\{Jx_n\}$ is a Cauchy sequence. Then $Jx = \lim_{n\to\infty} Jx_n = y^{**}$.

Finally, to show the assertion about the norm of x^*, use what was just shown applied to the James map from X' to X''' still referred to as J.

$$||x^*|| = \sup\{|x^*(x)| : ||x|| \leq 1\} = \sup\{|J(x)(x^*)| : ||Jx|| \leq 1\}$$

$$\leq \sup\{|x^{**}(x^*)| : ||x^{**}|| \leq 1\} = \sup\{|J(x^*)(x^{**})| : ||x^{**}|| \leq 1\}$$

$$\equiv ||Jx^*|| = ||x^*||.$$

This proves the theorem.

Definition 11.32 *When J maps X onto X'', X is called reflexive.*

It happens the L^p spaces are reflexive whenever $p > 1$.

11.3 Exercises

1. Is $\mathbb{N}$ a G_δ set? What about $\mathbb{Q}$? What about a countable dense subset of a complete metric space?

2. $\uparrow$ Let $f : \mathbb{R} \to \mathbb{C}$ be a function. Define the oscillation of a function in $B(x,r)$ by $\omega_r f(x) = \sup\{|f(z) - f(y)| : y, z \in B(x,r)\}$. Define the oscillation of the function at the point, x by $\omega f(x) = \lim_{r\to 0} \omega_r f(x)$. Show f is continuous at x if and only if $\omega f(x) = 0$. Then show the set of points where f is continuous is a G_δ set (try $U_n = \{x : \omega f(x) < \frac{1}{n}\}$). Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? **Hint:** Suppose D is any countable set, $D = \{d_i\}_{i=1}^{\infty}$, and define the function, $f_n(x)$ to equal zero for every $x \notin \{d_1, \cdots, d_n\}$ and 2^{-n} for x in this finite set. Then consider $g(x) \equiv \sum_{n=1}^{\infty} f_n(x)$. Show that this series converges uniformly.

3. Let $f \in C([0,1])$ and suppose $f'(x)$ exists. Show there exists a constant, K, such that $|f(x) - f(y)| \leq K|x - y|$ for all $y \in [0,1]$. Let $U_n = \{f \in C([0,1])$ such that for each $x \in [0,1]$ there exists $y \in [0,1]$ such that $|f(x) - f(y)| > n|x-y|\}$. Show that U_n is open and dense in $C([0,1])$ where for $f \in C([0,1])$,
$$||f|| \equiv \sup\{|f(x)| : x \in [0,1]\}.$$
Show that $\cap_n U_n$ is a dense G_δ set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.

4. $\uparrow$ Suppose $f(x) = \sum_{k=1}^{\infty} u_k(x)$ where the convergence is uniform and each u_k is a polynomial. Is it reasonable to conclude that $f'(x) = \sum_{k=1}^{\infty} u_k'(x)$? The answer is no. Use Problem 3 and the Weierstrass approximation theorem do show this.

5. Let X be a normed linear space. $A \subseteq X$ is "weakly bounded" if for each $x^* \in X'$, $\sup\{|x^*(x)| : x \in A\} < \infty$, while A is bounded if $\sup\{||x|| : x \in A\} < \infty$. Show A is weakly bounded if and only if it is bounded.

6. Let f be a 2π periodic locally integrable function on $\mathbb{R}$. The Fourier series for f is given by
$$\sum_{k=-\infty}^{\infty} a_k e^{ikx} \equiv \lim_{n\to\infty} \sum_{k=-n}^{n} a_k e^{ikx} \equiv \lim_{n\to\infty} S_n f(x)$$
where
$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx.$$
Show
$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y)\, dy$$

where

$$D_n(t) = \frac{\sin((n + \frac{1}{2})t)}{2\pi \sin(\frac{t}{2})}.$$

Verify that $\int_{-\pi}^{\pi} D_n(t)\,dt = 1$. Also show that if $g \in L^1(\mathbb{R})$, then

$$\lim_{a\to\infty} \int_{\mathbb{R}} g(x) \sin(ax)\,dx = 0.$$

This last is called the Riemann Lebesgue lemma. **Hint:** For the last part, assume first that $g \in C_c^\infty(\mathbb{R})$ and integrate by parts. Then exploit density of the set of functions in $L^1(\mathbb{R})$.

7. $\uparrow$It turns out that the Fourier series sometimes converges to the function pointwise. Suppose f is 2π periodic and Holder continuous. That is $|f(x) - f(y)| \leq K|x-y|^\theta$ where $\theta \in (0,1]$. Show that if f is like this, then the Fourier series converges to f at every point. Next modify your argument to show that if at every point, x, $|f(x+) - f(y)| \leq K|x-y|^\theta$ for y close enough to x and larger than x and $|f(x-) - f(y)| \leq K|x-y|^\theta$ for every y close enough to x and smaller than x, then $S_n f(x) \to \frac{f(x+)+f(x-)}{2}$, the midpoint of the jump of the function. **Hint:** Use Problem 6.

8. $\uparrow$ Let $Y = \{f$ such that f is continuous, defined on $\mathbb{R}$, and 2π periodic$\}$. Define $||f||_Y = \sup\{|f(x)| : x \in [-\pi,\pi]\}$. Show that $(Y, ||\ ||_Y)$ is a Banach space. Let $x \in \mathbb{R}$ and define $L_n(f) = S_n f(x)$. Show $L_n \in Y'$ but $\lim_{n\to\infty} ||L_n|| = \infty$. Show that for each $x \in \mathbb{R}$, there exists a dense G_δ subset of Y such that for f in this set, $|S_n f(x)|$ is unbounded. Finally, show there is a dense G_δ subset of Y having the property that $|S_n f(x)|$ is unbounded on the rational numbers. **Hint:** To do the first part, let $f(y)$ approximate $\operatorname{sgn}(D_n(x-y))$. Here $\operatorname{sgn} r = 1$ if $r > 0, -1$ if $r < 0$ and 0 if $r = 0$. This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense G_δ sets must also be a dense G_δ set.

9. Let $\alpha \in (0,1]$. Define, for X a compact subset of $\mathbb{R}^p$,

$$C^\alpha(X;\mathbb{R}^n) \equiv \{\mathbf{f} \in C(X;\mathbb{R}^n) : \rho_\alpha(\mathbf{f}) + ||\mathbf{f}|| \equiv ||\mathbf{f}||_\alpha < \infty\}$$

where

$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}$$

and

$$\rho_\alpha(\mathbf{f}) \equiv \sup\{\frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} : \mathbf{x},\mathbf{y} \in X,\ \mathbf{x} \neq \mathbf{y}\}.$$

Show that $(C^\alpha(X;\mathbb{R}^n), ||\cdot||_\alpha)$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha > 1$?

10. ↑Let X be the Holder functions which are periodic of period 2π. Define $L_n f(x) = S_n f(x)$ where $L_n : X \to Y$ for Y given in Problem 8. Show $||L_n||$ is bounded independent of n. Conclude that $L_n f \to f$ in Y for all $f \in X$. In other words, for the Holder continuous and 2π periodic functions, the Fourier series converges to the function uniformly. **Hint:** $L_n f(x)$ is given by

$$L_n f(x) = \int_{-\pi}^{\pi} D_n(y) f(x-y)\, dy$$

where $f(x-y) = f(x) + g(x,y)$ where $|g(x,y)| \leq C|y|^{\alpha}$. Use the fact the Dirichlet kernel integrates to one to write

$$\left| \int_{-\pi}^{\pi} D_n(y) f(x-y)\, dy \right| \leq \overbrace{\left| \int_{-\pi}^{\pi} D_n(y) f(x)\, dy \right|}^{=|f(x)|}$$

$$+C \left| \int_{-\pi}^{\pi} \sin\left(\left(n+\frac{1}{2}\right) y\right) \left(g(x,y) / \sin(y/2)\right) dy \right|$$

Show the functions, $y \to g(x,y)/\sin(y/2)$ are bounded in L^1 independent of x and get a uniform bound on $||L_n||$. Now use a similar argument to show $\{L_n f\}$ is equicontinuous in addition to being uniformly bounded. In doing this you might proceed as follows. Show

$$|L_n f(x) - L_n f(x')| \leq \left| \int_{-\pi}^{\pi} D_n(y) \left(f(x-y) - f(x'-y)\right) dy \right|$$

$$\begin{aligned} \leq \quad & ||f||_{\alpha} |x-x'|^{\alpha} \\ & + \left| \int_{-\pi}^{\pi} \sin\left(\left(n+\frac{1}{2}\right) y\right) \left(\frac{f(x-y) - f(x) - (f(x'-y) - f(x'))}{\sin\left(\frac{y}{2}\right)} \right) dy \right| \end{aligned}$$

Then split this last integral into two cases, one for $|y| < \eta$ and one where $|y| \geq \eta$. If $L_n f$ fails to converge to f uniformly, then there exists $\varepsilon > 0$ and a subsequence, n_k such that $||L_{n_k} f - f||_{\infty} \geq \varepsilon$ where this is the norm in Y or equivalently the sup norm on $[-\pi, \pi]$. By the Arzela Ascoli theorem, there is a further subsequence, $L_{n_{k_l}} f$ which converges uniformly on $[-\pi, \pi]$. But by Problem 7 $L_n f(x) \to f(x)$.

11. Let X be a normed linear space and let M be a convex open set containing 0. Define
$$\rho(x) = \inf\{t > 0 : \frac{x}{t} \in M\}.$$
Show ρ is a gauge function defined on X. This particular example is called a Minkowski functional. It is of fundamental importance in the study of locally convex topological vector spaces. A set, M, is convex if $\lambda x + (1-\lambda) y \in M$ whenever $\lambda \in [0,1]$ and $x, y \in M$.

12. ↑ The Hahn Banach theorem can be used to establish separation theorems. Let M be an open convex set containing 0. Let $x \notin M$. Show there exists $x^* \in X'$ such that $\operatorname{Re} x^*(x) \geq 1 > \operatorname{Re} x^*(y)$ for all $y \in M$. **Hint:** If $y \in M, \rho(y) < 1$. Show this. If $x \notin M$, $\rho(x) \geq 1$. Try $f(\alpha x) = \alpha \rho(x)$ for $\alpha \in \mathbb{R}$. Then extend f to the whole space using the Hahn Banach theorem and call the result F, show F is continuous, then fix it so F is the real part of $x^* \in X'$.

13. A Banach space is said to be strictly convex if whenever $||x|| = ||y||$ and $x \neq y$, then
$$\left|\left|\frac{x+y}{2}\right|\right| < ||x||.$$
$F : X \to X'$ is said to be a duality map if it satisfies the following: a.) $||F(x)|| = ||x||$. b.) $F(x)(x) = ||x||^2$. Show that if X' is strictly convex, then such a duality map exists. The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space. This Riesz map is the map which takes a Hilbert space to its dual defined as follows.
$$R(x)(y) = (y, x)$$
The Riesz representation theorem for Hilbert space says this map is onto. **Hint:** For an arbitrary Banach space, let
$$F(x) \equiv \left\{x^* : ||x^*|| \leq ||x|| \text{ and } x^*(x) = ||x||^2\right\}$$
Show $F(x) \neq \emptyset$ by using the Hahn Banach theorem on $f(\alpha x) = \alpha ||x||^2$. Next show $F(x)$ is closed and convex. Finally show that you can replace the inequality in the definition of $F(x)$ with an equal sign. Now use strict convexity to show there is only one element in $F(x)$.

14. Prove the following theorem which is an improved version of the open mapping theorem, [14]. Let X and Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$. Then the following are equivalent.
$$AX = Y,$$
$$A \text{ is an open map.}$$
Note this gives the equivalence between A being onto and A being an open map. The open mapping theorem says that if A is onto then it is open.

15. Suppose $D \subseteq X$ and D is dense in X. Suppose $L : D \to Y$ is linear and $||Lx|| \leq K||x||$ for all $x \in D$. Show there is a unique extension of L, $\widetilde{L}$, defined on all of X with $||\widetilde{L}x|| \leq K||x||$ and $\widetilde{L}$ is linear. You do not get uniqueness when you use the Hahn Banach theorem. Therefore, in the situation of this problem, it is better to use this result.

16. ↑ A Banach space is uniformly convex if whenever $||x_n||$, $||y_n|| \leq 1$ and $||x_n + y_n|| \to 2$, it follows that $||x_n - y_n|| \to 0$. Show uniform convexity

implies strict convexity (See Problem 13). **Hint:** Suppose it is not strictly convex. Then there exist $||x||$ and $||y||$ both equal to 1 and $\left|\left|\frac{x_n+y_n}{2}\right|\right| = 1$ consider $x_n \equiv x$ and $y_n \equiv y$, and use the conditions for uniform convexity to get a contradiction. It can be shown that L^p is uniformly convex whenever $\infty > p > 1$. See Hewitt and Stromberg [22] or Ray [34].

17. Show that a closed subspace of a reflexive Banach space is reflexive. **Hint:** The proof of this is an exercise in the use of the Hahn Banach theorem. Let Y be the closed subspace of the reflexive space X and let $y^{**} \in Y''$. Then $i^{**}y^{**} \in X''$ and so $i^{**}y^{**} = Jx$ for some $x \in X$ because X is reflexive. Now argue that $x \in Y$ as follows. If $x \notin Y$, then there exists x^* such that $x^*(Y) = 0$ but $x^*(x) \neq 0$. Thus, $i^*x^* = 0$. Use this to get a contradiction. When you know that $x = y \in Y$, the Hahn Banach theorem implies i^* is onto Y' and for all $x^* \in X'$,
$$y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = Jx(x^*) = x^*(x) = x^*(iy) = i^*x^*(y).$$

18. x_n converges weakly to x if for every $x^* \in X'$, $x^*(x_n) \to x^*(x)$. $x_n \rightharpoonup x$ denotes weak convergence. Show that if $||x_n - x|| \to 0$, then $x_n \rightharpoonup x$.

19. $\uparrow$ Show that if X is uniformly convex, then if $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, it follows $||x_n - x|| \to 0$. **Hint:** Use Lemma 11.27 to obtain $f \in X'$ with $||f|| = 1$ and $f(x) = ||x||$. See Problem 16 for the definition of uniform convexity. Now by the weak convergence, you can argue that if $x \neq 0$, $f(x_n/||x_n||) \to f(x/||x||)$. You also might try to show this in the special case where $||x_n|| = ||x|| = 1$.

20. Suppose $L \in \mathcal{L}(X,Y)$ and $M \in \mathcal{L}(Y,Z)$. Show $ML \in \mathcal{L}(X,Z)$ and that $(ML)^* = L^*M^*$.

Hilbert Spaces

12.1 Basic Theory

Definition 12.1 *Let X be a vector space. An inner product is a mapping from $X \times X$ to $\mathbb{C}$ if X is complex and from $X \times X$ to $\mathbb{R}$ if X is real, denoted by (x, y) which satisfies the following.*

$$(x,x) \geq 0, \ (x,x) = 0 \ \text{ if and only if } x = 0, \tag{12.1}$$

$$(x,y) = \overline{(y,x)}. \tag{12.2}$$

For $a, b \in \mathbb{C}$ and $x, y, z \in X$,

$$(ax + by, z) = a(x,z) + b(y,z). \tag{12.3}$$

Note that 12.2 and 12.3 imply $(x, ay + bz) = \overline{a}(x,y) + \overline{b}(x,z)$. Such a vector space is called an inner product space.

The Cauchy Schwarz inequality is fundamental for the study of inner product spaces.

Theorem 12.2 *(Cauchy Schwarz) In any inner product space*

$$|(x,y)| \leq ||x|| \, ||y||.$$

Proof: Let $\omega \in \mathbb{C}$, $|\omega| = 1$, and $\overline{\omega}(x,y) = |(x,y)| = \text{Re}(x, y\omega)$. Let

$$F(t) = (x + ty\omega, x + t\omega y).$$

If $y = 0$ there is nothing to prove because

$$(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0)$$

and so $(x, 0) = 0$. Thus, it can be assumed $y \neq 0$. Then from the axioms of the inner product,

$$F(t) = ||x||^2 + 2t \, \text{Re}(x, \omega y) + t^2 ||y||^2 \geq 0.$$

This yields
$$||x||^2 + 2t|(x,y)| + t^2||y||^2 \geq 0.$$
Since this inequality holds for all $t \in \mathbb{R}$, it follows from the quadratic formula that
$$4|(x,y)|^2 - 4||x||^2||y||^2 \leq 0.$$
This yields the conclusion and proves the theorem.

Proposition 12.3 *For an inner product space,* $||x|| \equiv (x,x)^{1/2}$ *does specify a norm.*

Proof: All the axioms are obvious except the triangle inequality. To verify this,
$$\begin{aligned} ||x+y||^2 &\equiv (x+y, x+y) \equiv ||x||^2 + ||y||^2 + 2\,\mathrm{Re}\,(x,y) \\ &\leq ||x||^2 + ||y||^2 + 2\,|(x,y)| \\ &\leq ||x||^2 + ||y||^2 + 2\,||x||\,||y|| = (||x|| + ||y||)^2. \end{aligned}$$

The following lemma is called the parallelogram identity.

Lemma 12.4 *In an inner product space,*
$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

The proof, a straightforward application of the inner product axioms, is left to the reader.

Lemma 12.5 *For* $x \in H$*, an inner product space,*
$$||x|| = \sup_{||y|| \leq 1} |(x,y)| \tag{12.4}$$

Proof: By the Cauchy Schwarz inequality, if $x \neq 0$,
$$||x|| \geq \sup_{||y|| \leq 1} |(x,y)| \geq \left(x, \frac{x}{||x||}\right) = ||x||\,.$$
It is obvious that 12.4 holds in the case that $x = 0$.

Definition 12.6 *A Hilbert space is an inner product space which is complete. Thus a Hilbert space is a Banach space in which the norm comes from an inner product as described above.*

In Hilbert space, one can define a projection map onto closed convex nonempty sets.

Definition 12.7 *A set,* K*, is convex if whenever* $\lambda \in [0,1]$ *and* $x, y \in K$*,* $\lambda x + (1-\lambda)y \in K$*.*

Theorem 12.8 *Let K be a closed convex nonempty subset of a Hilbert space, H, and let $x \in H$. Then there exists a unique point $Px \in K$ such that $||Px - x|| \leq ||y - x||$ for all $y \in K$.*

Proof: Consider uniqueness. Suppose that z_1 and z_2 are two elements of K such that for $i = 1, 2$,

$$||z_i - x|| \leq ||y - x|| \tag{12.5}$$

for all $y \in K$. Also, note that since K is convex,

$$\frac{z_1 + z_2}{2} \in K.$$

Therefore, by the parallelogram identity,

$$\begin{aligned} ||z_1 - x||^2 &\leq ||\frac{z_1 + z_2}{2} - x||^2 = ||\frac{z_1 - x}{2} + \frac{z_2 - x}{2}||^2 \\ &= 2(||\frac{z_1 - x}{2}||^2 + ||\frac{z_2 - x}{2}||^2) - ||\frac{z_1 - z_2}{2}||^2 \\ &= \frac{1}{2}||z_1 - x||^2 + \frac{1}{2}||z_2 - x||^2 - ||\frac{z_1 - z_2}{2}||^2 \\ &\leq ||z_1 - x||^2 - ||\frac{z_1 - z_2}{2}||^2, \end{aligned}$$

where the last inequality holds because of 12.5 letting $z_i = z_2$ and $y = z_1$. Hence $z_1 = z_2$ and this shows uniqueness.

Now let $\lambda = \inf\{||x - y|| : y \in K\}$ and let y_n be a minimizing sequence. This means $\{y_n\} \subseteq K$ satisfies $\lim_{n\to\infty} ||x - y_n|| = \lambda$. Now the following follows from properties of the norm.

$$||y_n - x + y_m - x||^2 = 4(||\frac{y_n + y_m}{2} - x||^2)$$

Then by the parallelogram identity, and convexity of K, $\frac{y_n + y_m}{2} \in K$, and so

$$\begin{aligned} ||\,(y_n - x) - (y_m - x)\,||^2 &= 2(||y_n - x||^2 + ||y_m - x||^2) - \overbrace{4(||\frac{y_n + y_m}{2} - x||^2)}^{=||y_n - x + y_m - x||^2} \\ &\leq 2(||y_n - x||^2 + ||y_m - x||^2) - 4\lambda^2. \end{aligned}$$

Since $||x - y_n|| \to \lambda$, this shows $\{y_n - x\}$ is a Cauchy sequence. Thus also $\{y_n\}$ is a Cauchy sequence. Since H is complete, $y_n \to y$ for some $y \in H$ which must be in K because K is closed. Therefore

$$||x - y|| = \lim_{n\to\infty} ||x - y_n|| = \lambda.$$

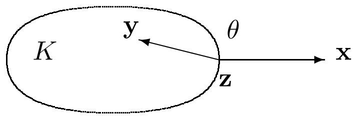
Let $Px = y$.

Corollary 12.9 *Let K be a closed, convex, nonempty subset of a Hilbert space, H, and let $x \in H$. Then for $z \in K$, $z = Px$ if and only if*

$$\operatorname{Re}(x - z, y - z) \leq 0 \tag{12.6}$$

for all $y \in K$.

Before proving this, consider what it says in the case where the Hilbert space is $\mathbb{R}^n$.



Condition 12.6 says the angle, θ, shown in the diagram is always obtuse. Remember from calculus, the sign of $\mathbf{x} \cdot \mathbf{y}$ is the same as the sign of the cosine of the included angle between $\mathbf{x}$ and $\mathbf{y}$. Thus, in finite dimensions, the conclusion of this corollary says that $z = Px$ exactly when the angle of the indicated angle is obtuse. Surely the picture suggests this is reasonable.

The inequality 12.6 is an example of a variational inequality and this corollary characterizes the projection of x onto K as the solution of this variational inequality.

Proof of Corollary: Let $z \in K$ and let $y \in K$ also. Since K is convex, it follows that if $t \in [0, 1]$,

$$z + t(y - z) = (1 - t)\, z + ty \in K.$$

Furthermore, every point of K can be written in this way. (Let $t = 1$ and $y \in K$.) Therefore, $z = Px$ if and only if for all $y \in K$ and $t \in [0, 1]$,

$$||x - (z + t(y - z))||^2 = ||(x - z) - t(y - z)||^2 \geq ||x - z||^2$$

for all $t \in [0, 1]$ and $y \in K$ if and only if for all $t \in [0, 1]$ and $y \in K$

$$||x - z||^2 + t^2\, ||y - z||^2 - 2t \operatorname{Re}(x - z, y - z) \geq ||x - z||^2$$

If and only if for all $t \in [0, 1]$,

$$t^2\, ||y - z||^2 - 2t \operatorname{Re}(x - z, y - z) \geq 0. \tag{12.7}$$

Now this is equivalent to 12.7 holding for all $t \in (0, 1)$. Therefore, dividing by $t \in (0, 1)$, 12.7 is equivalent to

$$t\, ||y - z||^2 - 2 \operatorname{Re}(x - z, y - z) \geq 0$$

for all $t \in (0, 1)$ which is equivalent to 12.6. This proves the corollary.

Corollary 12.10 *Let K be a nonempty convex closed subset of a Hilbert space, H. Then the projection map, P is continuous. In fact,*

$$|Px - Py| \leq |x - y|.$$

Proof: Let $x, x' \in H$. Then by Corollary 12.9,

$$\operatorname{Re}(x' - Px', Px - Px') \leq 0, \ \operatorname{Re}(x - Px, Px' - Px) \leq 0$$

Hence

$$\begin{aligned} 0 &\leq \operatorname{Re}(x - Px, Px - Px') - \operatorname{Re}(x' - Px', Px - Px') \\ &= \operatorname{Re}(x - x', Px - Px') - |Px - Px'|^2 \end{aligned}$$

and so

$$|Px - Px'|^2 \leq |x - x'| |Px - Px'|.$$

This proves the corollary.

The next corollary is a more general form for the Brouwer fixed point theorem.

Corollary 12.11 *Let $\mathbf{f} : K \to K$ where K is a convex compact subset of $\mathbb{R}^n$. Then $\mathbf{f}$ has a fixed point.*

Proof: Let $K \subseteq \overline{B(\mathbf{0}, R)}$ and let P be the projection map onto K. Then consider the map $\mathbf{f} \circ P$ which maps $\overline{B(\mathbf{0}, R)}$ to $\overline{B(\mathbf{0}, R)}$ and is continuous. By the Brouwer fixed point theorem for balls, this map has a fixed point. Thus there exists $\mathbf{x}$ such that

$$\mathbf{f} \circ P(\mathbf{x}) = \mathbf{x}$$

Now the equation also requires $\mathbf{x} \in K$ and so $P(\mathbf{x}) = \mathbf{x}$. Hence $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.

Definition 12.12 *Let H be a vector space and let U and V be subspaces. $U \oplus V = H$ if every element of H can be written as a sum of an element of U and an element of V in a unique way.*

The case where the closed convex set is a closed subspace is of special importance and in this case the above corollary implies the following.

Corollary 12.13 *Let K be a closed subspace of a Hilbert space, H, and let $x \in H$. Then for $z \in K$, $z = Px$ if and only if*

$$(x - z, y) = 0 \tag{12.8}$$

for all $y \in K$. Furthermore, $H = K \oplus K^{\perp}$ where

$$K^{\perp} \equiv \{x \in H : (x, k) = 0 \text{ for all } k \in K\}$$

and

$$||x||^2 = ||x - Px||^2 + ||Px||^2. \tag{12.9}$$

Proof: Since K is a subspace, the condition 12.6 implies $\text{Re}(x-z,y) \leq 0$ for all $y \in K$. Replacing y with $-y$, it follows $\text{Re}(x-z,-y) \leq 0$ which implies $\text{Re}(x-z,y) \geq 0$ for all y. Therefore, $\text{Re}(x-z,y) = 0$ for all $y \in K$. Now let $|\alpha| = 1$ and $\alpha(x-z,y) = |(x-z,y)|$. Since K is a subspace, it follows $\overline{\alpha}y \in K$ for all $y \in K$. Therefore,

$$0 = \text{Re}(x-z,\overline{\alpha}y) = (x-z,\overline{\alpha}y) = \alpha(x-z,y) = |(x-z,y)|.$$

This shows that $z = Px$, if and only if 12.8.

For $x \in H$, $x = x - Px + Px$ and from what was just shown, $x - Px \in K^{\perp}$ and $Px \in K$. This shows that $K^{\perp} + K = H$. Is there only one way to write a given element of H as a sum of a vector in K with a vector in $K^{\perp}$? Suppose $y + z = y_1 + z_1$ where $z, z_1 \in K^{\perp}$ and $y, y_1 \in K$. Then $(y - y_1) = (z_1 - z)$ and so from what was just shown, $(y-y_1, y-y_1) = (y-y_1, z_1-z) = 0$ which shows $y_1 = y$ and consequently $z_1 = z$. Finally, letting $z = Px$,

$$\begin{aligned} ||x||^2 &= (x-z+z, x-z+z) = ||x-z||^2 + (x-z,z) + (z,x-z) + ||z||^2 \\ &= ||x-z||^2 + ||z||^2 \end{aligned}$$

This proves the corollary.

The following theorem is called the Riesz representation theorem for the dual of a Hilbert space. If $z \in H$ then define an element $f \in H'$ by the rule $(x,z) \equiv f(x)$. It follows from the Cauchy Schwarz inequality and the properties of the inner product that $f \in H'$. The Riesz representation theorem says that all elements of H' are of this form.

Theorem 12.14 *Let H be a Hilbert space and let $f \in H'$. Then there exists a unique $z \in H$ such that*

$$f(x) = (x,z) \tag{12.10}$$

for all $x \in H$.

Proof: Letting $y, w \in H$ the assumption that f is linear implies

$$f(yf(w) - f(y)w) = f(w)f(y) - f(y)f(w) = 0$$

which shows that $yf(w) - f(y)w \in f^{-1}(0)$, which is a closed subspace of H since f is continuous. If $f^{-1}(0) = H$, then f is the zero map and $z = 0$ is the unique element of H which satisfies 12.10. If $f^{-1}(0) \neq H$, pick $u \notin f^{-1}(0)$ and let $w \equiv u - Pu \neq 0$. Thus Corollary 12.13 implies $(y,w) = 0$ for all $y \in f^{-1}(0)$. In particular, let $y = xf(w) - f(x)w$ where $x \in H$ is arbitrary. Therefore,

$$0 = (f(w)x - f(x)w, w) = f(w)(x,w) - f(x)||w||^2.$$

Thus, solving for $f(x)$ and using the properties of the inner product,

$$f(x) = (x, \frac{\overline{f(w)}w}{||w||^2})$$

Let $z = \overline{f(w)}w/||w||^2$. This proves the existence of z. If $f(x) = (x, z_i)$ $i = 1, 2$, for all $x \in H$, then for all $x \in H$, then $(x, z_1 - z_2) = 0$ which implies, upon taking $x = z_1 - z_2$ that $z_1 = z_2$. This proves the theorem.

If $R : H \to H'$ is defined by $Rx(y) \equiv (y, x)$, the Riesz representation theorem above states this map is onto. This map is called the Riesz map. It is routine to show R is linear and $|Rx| = |x|$.

12.2 Approximations In Hilbert Space

The Gram Schmidt process applies in any Hilbert space.

Theorem 12.15 *Let $\{x_1, \cdots, x_n\}$ be a basis for M a subspace of H a Hilbert space. Then there exists an orthonormal basis for M, $\{u_1, \cdots, u_n\}$ which has the property that for each $k \leq n$, span$(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$. Also if $\{x_1, \cdots, x_n\} \subseteq H$, then*

$$\text{span}(x_1, \cdots, x_n)$$

is a closed subspace.

Proof: Let $\{x_1, \cdots, x_n\}$ be a basis for M. Let $u_1 \equiv x_1/|x_1|$. Thus for $k = 1$, $\text{span}(u_1) = \text{span}(x_1)$ and $\{u_1\}$ is an orthonormal set. Now suppose for some $k < n$, $u_1, \cdots, u_k$ have been chosen such that $(u_j \cdot u_l) = \delta_{jl}$ and $\text{span}(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$. Then define

$$u_{k+1} \equiv \frac{x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j}{\left| x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j \right|}, \tag{12.11}$$

where the denominator is not equal to zero because the x_j form a basis and so

$$x_{k+1} \notin \text{span}(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$$

Thus by induction,

$$u_{k+1} \in \text{span}(u_1, \cdots, u_k, x_{k+1}) = \text{span}(x_1, \cdots, x_k, x_{k+1}).$$

Also, $x_{k+1} \in \text{span}(u_1, \cdots, u_k, u_{k+1})$ which is seen easily by solving 12.11 for x_{k+1} and it follows

$$\text{span}(x_1, \cdots, x_k, x_{k+1}) = \text{span}(u_1, \cdots, u_k, u_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
(u_{k+1} \cdot u_l) &= C\left((x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j)(u_j \cdot u_l) \right) \\
&= C\left((x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) \delta_{lj} \right) \\
&= C\left((x_{k+1} \cdot u_l) - (x_{k+1} \cdot u_l) \right) = 0.
\end{aligned}$$

The vectors, $\{u_j\}_{j=1}^n$, generated in this way are therefore an orthonormal basis because each vector has unit length.

Consider the second claim about finite dimensional subspaces. Without loss of generality, assume $\{x_1, \cdot\cdot\cdot, x_n\}$ is linearly independent. If it is not, delete vectors until a linearly independent set is obtained. Then by the first part, $\text{span}(x_1, \cdot\cdot\cdot, x_n) = \text{span}(u_1, \cdot\cdot\cdot, u_n) \equiv M$ where the u_i are an orthonormal set of vectors. Suppose $\{y_k\} \subseteq M$ and $y_k \to y \in H$. Is $y \in M$? Let

$$y_k \equiv \sum_{j=1}^n c_j^k u_j$$

Then let $\mathbf{c}^k \equiv \left(c_1^k, \cdot\cdot\cdot, c_n^k\right)^T$. Then

$$\begin{aligned} \left|\mathbf{c}^k - \mathbf{c}^l\right|^2 &\equiv \sum_{j=1}^n \left|c_j^k - c_j^l\right|^2 = \left(\sum_{j=1}^n \left(c_j^k - c_j^l\right) u_j, \sum_{j=1}^n \left(c_j^k - c_j^l\right) u_j\right) \\ &= ||y_k - y_l||^2 \end{aligned}$$

which shows $\left\{\mathbf{c}^k\right\}$ is a Cauchy sequence in $\mathbb{F}^n$ and so it converges to $\mathbf{c} \in \mathbb{F}^n$. Thus

$$y = \lim_{k\to\infty} y_k = \lim_{k\to\infty} \sum_{j=1}^n c_j^k u_j = \sum_{j=1}^n c_j u_j \in M.$$

This completes the proof.

Theorem 12.16 *Let M be the span of $\{u_1, \cdot\cdot\cdot, u_n\}$ in a Hilbert space, H and let $y \in H$. Then Py is given by*

$$Py = \sum_{k=1}^n (y, u_k) u_k \tag{12.12}$$

and the distance is given by

$$\sqrt{|y|^2 - \sum_{k=1}^n |(y, u_k)|^2}. \tag{12.13}$$

Proof:

$$\begin{aligned} \left(y - \sum_{k=1}^n (y, u_k) u_k, u_p\right) &= (y, u_p) - \sum_{k=1}^n (y, u_k)(u_k, u_p) \\ &= (y, u_p) - (y, u_p) = 0 \end{aligned}$$

It follows that

$$\left(y - \sum_{k=1}^n (y, u_k) u_k, u\right) = 0$$

for all $u \in M$ and so by Corollary 12.13 this verifies 12.12.

The square of the distance, d is given by

$$\begin{aligned} d^2 &= \left(y - \sum_{k=1}^n (y, u_k) u_k, y - \sum_{k=1}^n (y, u_k) u_k \right) \\ &= |y|^2 - 2 \sum_{k=1}^n |(y, u_k)|^2 + \sum_{k=1}^n |(y, u_k)|^2 \end{aligned}$$

and this shows 12.13.

What if the subspace is the span of vectors which are not orthonormal? There is a very interesting formula for the distance between a point of a Hilbert space and a finite dimensional subspace spanned by an arbitrary basis.

Definition 12.17 *Let $\{x_1, \cdots, x_n\} \subseteq H$, a Hilbert space. Define*

$$\mathcal{G}(x_1, \cdots, x_n) \equiv \begin{pmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{pmatrix} \tag{12.14}$$

Thus the ij^{th} entry of this matrix is (x_i, x_j). This is sometimes called the Gram matrix. Also define $G(x_1, \cdots, x_n)$ as the determinant of this matrix, also called the Gram determinant.

$$G(x_1, \cdots, x_n) \equiv \begin{vmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{vmatrix} \tag{12.15}$$

The theorem is the following.

Theorem 12.18 *Let $M = \operatorname{span}(x_1, \cdots, x_n) \subseteq H$, a Real Hilbert space where $\{x_1, \cdots, x_n\}$ is a basis and let $y \in H$. Then letting d be the distance from y to M,*

$$d^2 = \frac{G(x_1, \cdots, x_n, y)}{G(x_1, \cdots, x_n)}. \tag{12.16}$$

Proof: By Theorem 12.15 M is a closed subspace of H. Let $\sum_{k=1}^n \alpha_k x_k$ be the element of M which is closest to y. Then by Corollary 12.13,

$$\left(y - \sum_{k=1}^n \alpha_k x_k, x_p \right) = 0$$

for each $p = 1, 2, \cdots, n$. This yields the system of equations,

$$(y, x_p) = \sum_{k=1}^n (x_p, x_k) \alpha_k, p = 1, 2, \cdots, n \tag{12.17}$$

Also by Corollary 12.13,

$$||y||^2 = \overbrace{\left|\left|y - \sum_{k=1}^{n} \alpha_k x_k\right|\right|^2}^{d^2} + \left|\left|\sum_{k=1}^{n} \alpha_k x_k\right|\right|^2$$

and so, using 12.17,

$$\begin{aligned} ||y||^2 &= d^2 + \sum_j \left(\sum_k \alpha_k (x_k, x_j)\right) \alpha_j \\ &= d^2 + \sum_j (y, x_j) \alpha_j \qquad (12.18) \\ &\equiv d^2 + \mathbf{y}_x^T \boldsymbol{\alpha} \qquad (12.19) \end{aligned}$$

in which

$$\mathbf{y}_x^T \equiv ((y, x_1), \cdots, (y, x_n)), \ \boldsymbol{\alpha}^T \equiv (\alpha_1, \cdots, \alpha_n).$$

Then 12.17 and 12.18 imply the following system

$$\begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{0} \\ \mathbf{y}_x^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ d^2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}_x \\ ||y||^2 \end{pmatrix}$$

By Cramer's rule,

$$\begin{aligned} d^2 &= \frac{\det \begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{y}_x \\ \mathbf{y}_x^T & ||y||^2 \end{pmatrix}}{\det \begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{0} \\ \mathbf{y}_x^T & 1 \end{pmatrix}} \\ &= \frac{\det \begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{y}_x \\ \mathbf{y}_x^T & ||y||^2 \end{pmatrix}}{\det (\mathcal{G}(x_1, \cdots, x_n))} \\ &= \frac{\det (\mathcal{G}(x_1, \cdots, x_n, y))}{\det (\mathcal{G}(x_1, \cdots, x_n))} = \frac{G(x_1, \cdots, x_n, y)}{G(x_1, \cdots, x_n)} \end{aligned}$$

and this proves the theorem.

12.3 Orthonormal Sets

The concept of an orthonormal set of vectors is a generalization of the notion of the standard basis vectors of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 12.19 *Let H be a Hilbert space. $S \subseteq H$ is called an orthonormal set if $||x|| = 1$ for all $x \in S$ and $(x, y) = 0$ if $x, y \in S$ and $x \neq y$. For any set, D,*

$$D^{\perp} \equiv \{x \in H : (x, d) = 0 \text{ for all } d \in D\}.$$

If S is a set, $\operatorname{span}(S)$ *is the set of all finite linear combinations of vectors from S.*

You should verify that $D^{\perp}$ is always a closed subspace of H.

Theorem 12.20 *In any separable Hilbert space, H, there exists a countable orthonormal set, $S = \{x_i\}$ such that the span of these vectors is dense in H. Furthermore, if* $\operatorname{span}(S)$ *is dense, then for $x \in H$,*

$$x = \sum_{i=1}^{\infty} (x, x_i)\, x_i \equiv \lim_{n\to\infty} \sum_{i=1}^{n} (x, x_i)\, x_i. \tag{12.20}$$

Proof: Let $\mathcal{F}$ denote the collection of all orthonormal subsets of H. $\mathcal{F}$ is nonempty because $\{x\} \in \mathcal{F}$ where $||x|| = 1$. The set, $\mathcal{F}$ is a partially ordered set with the order given by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathfrak{C}$ in $\mathcal{F}$. Then let $S \equiv \cup\mathfrak{C}$. It follows S must be a maximal orthonormal set of vectors. Why? It remains to verify that S is countable $\operatorname{span}(S)$ is dense, and the condition, 12.20 holds. To see S is countable note that if $x, y \in S$, then

$$||x - y||^2 = ||x||^2 + ||y||^2 - 2\operatorname{Re}(x, y) = ||x||^2 + ||y||^2 = 2.$$

Therefore, the open sets, $B\left(x, \frac{1}{2}\right)$ for $x \in S$ are disjoint and cover S. Since H is assumed to be separable, there exists a point from a countable dense set in each of these disjoint balls showing there can only be countably many of the balls and that consequently, S is countable as claimed.

It remains to verify 12.20 and that $\operatorname{span}(S)$ is dense. If $\overline{\operatorname{span}(S)}$ is not dense, then $\overline{\operatorname{span}(S)}$ is a closed proper subspace of H and letting $y \notin \overline{\operatorname{span}(S)}$,

$$z \equiv \frac{y - Py}{||y - Py||} \in \operatorname{span}(S)^{\perp}.$$

But then $S \cup \{z\}$ would be a larger orthonormal set of vectors contradicting the maximality of S.

It remains to verify 12.20. Let $S = \{x_i\}_{i=1}^{\infty}$ and consider the problem of choosing the constants, c_k in such a way as to minimize the expression

$$\left|\left| x - \sum_{k=1}^{n} c_k x_k \right|\right|^2 =$$

$$||x||^2 + \sum_{k=1}^{n} |c_k|^2 - \sum_{k=1}^{n} \overline{c_k}\,(x, x_k) - \sum_{k=1}^{n} c_k \overline{(x, x_k)}.$$

This equals

$$||x||^2 + \sum_{k=1}^{n} |c_k - (x, x_k)|^2 - \sum_{k=1}^{n} |(x, x_k)|^2$$

and therefore, this minimum is achieved when $c_k = (x, x_k)$ and equals

$$||x||^2 - \sum_{k=1}^{n} |(x, x_k)|^2$$

Now since $\text{span}(S)$ is dense, there exists n large enough that for some choice of constants, c_k,

$$\left|\left|x - \sum_{k=1}^{n} c_k x_k\right|\right|^2 < \varepsilon.$$

However, from what was just shown,

$$\left|\left|x - \sum_{i=1}^{n} (x, x_i) x_i\right|\right|^2 \leq \left|\left|x - \sum_{k=1}^{n} c_k x_k\right|\right|^2 < \varepsilon$$

showing that $\lim_{n\to\infty} \sum_{i=1}^{n} (x, x_i) x_i = x$ as claimed. This proves the theorem.

The proof of this theorem contains the following corollary.

Corollary 12.21 *Let S be any orthonormal set of vectors and let*

$$\{x_1, \cdots, x_n\} \subseteq S.$$

Then if $x \in H$

$$\left|\left|x - \sum_{k=1}^{n} c_k x_k\right|\right|^2 \geq \left|\left|x - \sum_{i=1}^{n} (x, x_i) x_i\right|\right|^2$$

for all choices of constants, c_k. In addition to this, Bessel's inequality

$$||x||^2 \geq \sum_{k=1}^{n} |(x, x_k)|^2 .$$

If S is countable and $\text{span}(S)$ *is dense, then letting $\{x_i\}_{i=1}^{\infty} = S$, 12.20 follows.*

12.4 Fourier Series, An Example

In this section consider the Hilbert space, $L^2(0, 2\pi)$ with the inner product,

$$(f, g) \equiv \int_0^{2\pi} f\overline{g}dm.$$

This is a Hilbert space because of the theorem which states the L^p spaces are complete, Theorem 10.10 on Page 237. An example of an orthonormal set of functions in $L^2(0, 2\pi)$ is

$$\phi_n(x) \equiv \frac{1}{\sqrt{2\pi}} e^{inx}$$

for n an integer. Is it true that the span of these functions is dense in $L^2(0, 2\pi)$?

Theorem 12.22 *Let $S = \{\phi_n\}_{n\in\mathbb{Z}}$. Then* $\text{span}(S)$ *is dense in $L^2(0, 2\pi)$.*

Proof: By regularity of Lebesgue measure, it follows from Theorem 10.16 that $C_c(0,2\pi)$ is dense in $L^2(0,2\pi)$. Therefore, it suffices to show that for $g \in C_c(0,2\pi)$, then for every $\varepsilon > 0$ there exists $h \in \operatorname{span}(S)$ such that $||g-h||_{L^2(0,2\pi)} < \varepsilon$.

Let T denote the points of $\mathbb{C}$ which are of the form e^{it} for $t \in \mathbb{R}$. Let $\mathcal{A}$ denote the algebra of functions consisting of polynomials in z and $1/z$ for $z \in T$. Thus a typical such function would be one of the form

$$\sum_{k=-m}^{m} c_k z^k$$

for m chosen large enough. This algebra separates the points of T because it contains the function, $p(z) = z$. It annihilates no point of t because it contains the constant function 1. Furthermore, it has the property that for $f \in \mathcal{A}$, $\overline{f} \in \mathcal{A}$. By the Stone Weierstrass approximation theorem, Theorem 6.13 on Page 121, $\mathcal{A}$ is dense in $C(T)$. Now for $g \in C_c(0,2\pi)$, extend g to all of $\mathbb{R}$ to be 2π periodic. Then letting $G(e^{it}) \equiv g(t)$, it follows G is well defined and continuous on T. Therefore, there exists $H \in \mathcal{A}$ such that for all $t \in \mathbb{R}$,

$$\left|H\left(e^{it}\right) - G\left(e^{it}\right)\right| < \varepsilon^2/2\pi.$$

Thus $H\left(e^{it}\right)$ is of the form

$$H\left(e^{it}\right) = \sum_{k=-m}^{m} c_k \left(e^{it}\right)^k = \sum_{k=-m}^{m} c_k e^{ikt} \in \operatorname{span}(S).$$

Let $h(t) = \sum_{k=-m}^{m} c_k e^{ikt}$. Then

$$\begin{aligned}
\left(\int_0^{2\pi} |g-h|^2 \, dx\right)^{1/2} &\leq \left(\int_0^{2\pi} \max\{|g(t) - h(t)| : t \in [0,2\pi]\} \, dx\right)^{1/2} \\
&= \left(\int_0^{2\pi} \max\left\{\left|G\left(e^{it}\right) - H\left(e^{it}\right)\right| : t \in [0,2\pi]\right\} dx\right)^{1/2} \\
&< \left(\int_0^{2\pi} \frac{\varepsilon^2}{2\pi}\right)^{1/2} = \varepsilon.
\end{aligned}$$

This proves the theorem.

Corollary 12.23 *For $f \in L^2(0,2\pi)$,*

$$\lim_{m\to\infty} \left\| f - \sum_{k=-m}^{m} (f,\phi_k)\,\phi_k \right\|_{L^2(0,2\pi)}$$

Proof: This follows from Theorem 12.20 on Page 285.

12.5 Exercises

1. For $f, g \in C([0,1])$ let $(f,g) = \int_0^1 f(x)\overline{g(x)}dx$. Is this an inner product space? Is it a Hilbert space? What does the Cauchy Schwarz inequality say in this context?

2. Suppose the following conditions hold.
$$(x,x) \geq 0, \tag{12.21}$$
$$(x,y) = \overline{(y,x)}. \tag{12.22}$$
For $a, b \in \mathbb{C}$ and $x, y, z \in X$,
$$(ax + by, z) = a(x,z) + b(y,z). \tag{12.23}$$
These are the same conditions for an inner product except it is no longer required that $(x,x) = 0$ if and only if $x = 0$. Does the Cauchy Schwarz inequality hold in the following form?
$$|(x,y)| \leq (x,x)^{1/2}(y,y)^{1/2}.$$

3. Let S denote the unit sphere in a Banach space, X,
$$S \equiv \{x \in X : ||x|| = 1\}.$$
Show that if Y is a Banach space, then $A \in \mathcal{L}(X,Y)$ is compact if and only if $A(S)$ is precompact, $\overline{A(S)}$ is compact. $A \in \mathcal{L}(X,Y)$ is said to be compact if whenever B is a bounded subset of X, it follows $\overline{A(B)}$ is a compact subset of Y. In words, A takes bounded sets to precompact sets.

4. $\uparrow$ Show that $A \in \mathcal{L}(X,Y)$ is compact if and only if A^* is compact. **Hint:** Use the result of 3 and the Ascoli Arzela theorem to argue that for S^* the unit ball in X', there is a subsequence, $\{y_n^*\} \subseteq S^*$ such that y_n^* converges uniformly on the compact set, $\overline{A(S)}$. Thus $\{A^*y_n^*\}$ is a Cauchy sequence in X'. To get the other implication, apply the result just obtained for the operators A^* and A^{**}. Then use results about the embedding of a Banach space into its double dual space.

5. Prove the parallelogram identity,
$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$
Next suppose $(X, ||\ ||)$ is a real normed linear space and the parallelogram identity holds. Can it be concluded there exists an inner product $(\cdot,\cdot)$ such that $||x|| = (x,x)^{1/2}$?

6. Let K be a closed, bounded and convex set in $\mathbb{R}^n$ and let $\mathbf{f} : K \to \mathbb{R}^n$ be continuous and let $\mathbf{y} \in \mathbb{R}^n$. Show using the Brouwer fixed point theorem there exists a point $\mathbf{x} \in K$ such that $P(\mathbf{y} - \mathbf{f}(\mathbf{x}) + \mathbf{x}) = \mathbf{x}$. Next show that $(\mathbf{y} - \mathbf{f}(\mathbf{x}), \mathbf{z} - \mathbf{x}) \leq 0$ for all $\mathbf{z} \in K$. The existence of this $\mathbf{x}$ is known as Browder's lemma and it has great significance in the study of certain types of nolinear operators. Now suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies
$$\lim_{|\mathbf{x}| \to \infty} \frac{(\mathbf{f}(\mathbf{x}), \mathbf{x})}{|\mathbf{x}|} = \infty.$$
Show using Browder's lemma that $\mathbf{f}$ is onto.

7. Show that every inner product space is uniformly convex. This means that if x_n, y_n are vectors whose norms are no larger than 1 and if $||x_n + y_n|| \to 2$, then $||x_n - y_n|| \to 0$.

8. Let H be separable and let S be an orthonormal set. Show S is countable. **Hint:** How far apart are two elements of the orthonormal set?

9. Suppose $\{x_1, \cdots, x_m\}$ is a linearly independent set of vectors in a normed linear space. Show $\operatorname{span}(x_1, \cdots, x_m)$ is a closed subspace. Also show every orthonormal set of vectors is linearly independent.

10. Show every Hilbert space, separable or not, has a maximal orthonormal set of vectors.

11. $\uparrow$ Prove Bessel's inequality, which says that if $\{x_n\}_{n=1}^{\infty}$ is an orthonormal set in H, then for all $x \in H$, $||x||^2 \geq \sum_{k=1}^{\infty} |(x, x_k)|^2$. **Hint:** Show that if $M = \operatorname{span}(x_1, \cdots, x_n)$, then $Px = \sum_{k=1}^{n} x_k (x, x_k)$. Then observe $||x||^2 = ||x - Px||^2 + ||Px||^2$.

12. $\uparrow$ Show S is a maximal orthonormal set if and only if $\operatorname{span}(S)$ is dense in H, where $\operatorname{span}(S)$ is defined as
$$\operatorname{span}(S) \equiv \{\text{all finite linear combinations of elements of } S\}.$$

13. $\uparrow$ Suppose $\{x_n\}_{n=1}^{\infty}$ is a maximal orthonormal set. Show that
$$x = \sum_{n=1}^{\infty} (x, x_n) x_n \equiv \lim_{N \to \infty} \sum_{n=1}^{N} (x, x_n) x_n$$
and $||x||^2 = \sum_{i=1}^{\infty} |(x, x_i)|^2$. Also show $(x, y) = \sum_{n=1}^{\infty} (x, x_n)(\overline{y, x_n})$. **Hint:** For the last part of this, you might proceed as follows. Show that
$$((x, y)) \equiv \sum_{n=1}^{\infty} (x, x_n)(\overline{y, x_n})$$
is a well defined inner product on the Hilbert space which delivers the same norm as the original inner product. Then you could verify that there exists a formula for the inner product in terms of the norm and conclude the two inner products, $(\cdot, \cdot)$ and $((\cdot, \cdot))$ must coincide.

14. Suppose X is an infinite dimensional Banach space and suppose

$$\{x_1 \cdots x_n\}$$

are linearly independent with $||x_i|| = 1$. By Problem 9 span $(x_1 \cdots x_n) \equiv X_n$ is a closed linear subspace of X. Now let $z \notin X_n$ and pick $y \in X_n$ such that $||z - y|| \leq 2 \ \text{dist}\,(z, X_n)$ and let

$$x_{n+1} = \frac{z - y}{||z - y||}.$$

Show the sequence $\{x_k\}$ satisfies $||x_n - x_k|| \geq 1/2$ whenever $k < n$. Now show the unit ball $\{x \in X : ||x|| \leq 1\}$ in a normed linear space is compact if and only if X is finite dimensional. **Hint:**

$$\left|\left|\frac{z - y}{||z - y||} - x_k\right|\right| = \left|\left|\frac{z - y - x_k\,||z - y||}{||z - y||}\right|\right|.$$

15. Show that if A is a self adjoint operator on a Hilbert space and $Ay = \lambda y$ for λ a complex number and $y \neq 0$, then λ must be real. Also verify that if A is self adjoint and $Ax = \mu x$ while $Ay = \lambda y$, then if $\mu \neq \lambda$, it must be the case that $(x, y) = 0$.

Representation Theorems

13.1 Radon Nikodym Theorem

This chapter is on various representation theorems. The first theorem, the Radon Nikodym Theorem, is a representation theorem for one measure in terms of another. The approach given here is due to Von Neumann and depends on the Riesz representation theorem for Hilbert space, Theorem 12.14 on Page 280.

Definition 13.1 *Let μ and λ be two measures defined on a σ-algebra, $\mathcal{S}$, of subsets of a set, Ω. λ is absolutely continuous with respect to μ,written as $\lambda \ll \mu$, if $\lambda(E) = 0$ whenever $\mu(E) = 0$.*

It is not hard to think of examples which should be like this. For example, suppose one measure is volume and the other is mass. If the volume of something is zero, it is reasonable to expect the mass of it should also be equal to zero. In this case, there is a function called the density which is integrated over volume to obtain mass. The Radon Nikodym theorem is an abstract version of this notion. Essentially, it gives the existence of the density function.

Theorem 13.2 *(Radon Nikodym) Let λ and μ be finite measures defined on a σ-algebra, $\mathcal{S}$, of subsets of Ω. Suppose $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega, \mu)$ such that $f(x) \geq 0$ and*

$$\lambda(E) = \int_E f \, d\mu.$$

If it is not necessarily the case that $\lambda \ll \mu$, there are two measures, $\lambda_\perp$ and $\lambda_{||}$ such that $\lambda = \lambda_\perp + \lambda_{||}, \lambda_{||} \ll \mu$ and there exists a set of μ measure zero, N such that for all E measurable, $\lambda_\perp(E) = \lambda(E \cap N) = \lambda_\perp(E \cap N)$. In this case the two mesures, $\lambda_\perp$ and $\lambda_{||}$ are unique and the representation of $\lambda = \lambda_\perp + \lambda_{||}$ is called the Lebesgue decomposition of λ. The measure $\lambda_{||}$ is the absolutely continuous part of λ and $\lambda_\perp$ is called the singular part of λ.

Proof: Let $\Lambda : L^2(\Omega, \mu + \lambda) \to \mathbb{C}$ be defined by

$$\Lambda g = \int_\Omega g \, d\lambda.$$

By Holder's inequality,

$$|\Lambda g| \leq \left(\int_\Omega 1^2 d\lambda\right)^{1/2} \left(\int_\Omega |g|^2 \, d(\lambda+\mu)\right)^{1/2} = \lambda(\Omega)^{1/2} \, ||g||_2$$

where $||g||_2$ is the L^2 norm of g taken with respect to $\mu+\lambda$. Therefore, since Λ is bounded, it follows from Theorem 11.4 on Page 255 that $\Lambda \in (L^2(\Omega,\mu+\lambda))'$, the dual space $L^2(\Omega,\mu+\lambda)$. By the Riesz representation theorem in Hilbert space, Theorem 12.14, there exists a unique $h \in L^2(\Omega,\mu+\lambda)$ with

$$\Lambda g = \int_\Omega g \, d\lambda = \int_\Omega hg d(\mu+\lambda). \tag{13.1}$$

The plan is to show h is real and nonnegative at least a.e. Therefore, consider the set where $\operatorname{Im} h$ is positive.

$$E = \{x \in \Omega : \operatorname{Im} h(x) > 0\},$$

Now let $g = \mathcal{X}_E$ and use 13.1 to get

$$\lambda(E) = \int_E (\operatorname{Re} h + i \ \operatorname{Im} h) d(\mu+\lambda). \tag{13.2}$$

Since the left side of 13.2 is real, this shows

$$\begin{aligned} 0 &= \int_E (\operatorname{Im} h) \, d(\mu+\lambda) \\ &\geq \int_{E_n} (\operatorname{Im} h) \, d(\mu+\lambda) \\ &\geq \frac{1}{n} (\mu+\lambda)(E_n) \end{aligned}$$

where

$$E_n \equiv \left\{x : \operatorname{Im} h(x) \geq \frac{1}{n}\right\}$$

Thus $(\mu+\lambda)(E_n) = 0$ and since $E = \cup_{n=1}^\infty E_n$, it follows $(\mu+\lambda)(E) = 0$. A similar argument shows that for

$$E = \{x \in \Omega : \operatorname{Im} h(x) < 0\},$$

$(\mu+\lambda)(E) = 0$. Thus there is no loss of generality in assuming h is real-valued.

The next task is to show h is nonnegative. This is done in the same manner as above. Define the set where it is negative and then show this set has measure zero.

Let $E \equiv \{x : h(x) < 0\}$ and let $E_n \equiv \{x : h(x) < -\frac{1}{n}\}$. Then let $g = \mathcal{X}_{E_n}$. Since $E = \cup_n E_n$, it follows that if $(\mu+\lambda)(E) > 0$ then this is also true for $(\mu+\lambda)(E_n)$ for all n large enough. Then from 13.2

$$\lambda(E_n) = \int_{E_n} h \, d(\mu+\lambda) \leq -(1/n)(\mu+\lambda)(E_n) < 0,$$

a contradiction. Thus it can be assumed $h \geq 0$.

At this point the argument splits into two cases.

Case Where $\lambda \ll \mu$. In this case, $h < 1$.

Let $E = [h \geq 1]$ and let $g = \mathcal{X}_E$. Then

$$\lambda(E) = \int_E h \, d(\mu + \lambda) \geq \mu(E) + \lambda(E).$$

Therefore $\mu(E) = 0$. Since $\lambda \ll \mu$, it follows that $\lambda(E) = 0$ also. Thus it can be assumed

$$0 \leq h(x) < 1$$

for all x.

From 13.1, whenever $g \in L^2(\Omega, \mu + \lambda)$,

$$\int_\Omega g(1-h)d\lambda = \int_\Omega hgd\mu. \tag{13.3}$$

Now let E be a measurable set and define

$$g(x) \equiv \sum_{i=0}^{n} h^i(x)\mathcal{X}_E(x)$$

in 13.3. This yields

$$\int_E (1 - h^{n+1}(x))d\lambda = \int_E \sum_{i=1}^{n+1} h^i(x)d\mu. \tag{13.4}$$

Let $f(x) = \sum_{i=1}^{\infty} h^i(x)$ and use the Monotone Convergence theorem in 13.4 to let $n \to \infty$ and conclude

$$\lambda(E) = \int_E f \, d\mu.$$

$f \in L^1(\Omega, \mu)$ because λ is finite.

The function, f is unique μ a.e. because, if g is another function which also serves to represent λ, consider for each $n \in \mathbb{N}$ the set,

$$E_n \equiv \left[f - g > \frac{1}{n}\right]$$

and conclude that

$$0 = \int_{E_n} (f - g) \, d\mu \geq \frac{1}{n}\mu(E_n).$$

Therefore, $\mu(E_n) = 0$. It follows that

$$\mu([f - g > 0]) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$$

Similarly, the set where g is larger than f has measure zero. This proves the theorem.

Case where it is not necessarily true that $\lambda \ll \mu$.

In this case, let $N = [h \geq 1]$ and let $g = \mathcal{X}_N$. Then

$$\lambda(N) = \int_N h \, d(\mu + \lambda) \geq \mu(N) + \lambda(N).$$

and so $\mu(N) = 0$. Now define a measure, $\lambda_\perp$ by

$$\lambda_\perp(E) \equiv \lambda(E \cap N)$$

so $\lambda_\perp(E \cap N) = \lambda(E \cap N \cap N) \equiv \lambda_\perp(E)$ and let $\lambda_{||} \equiv \lambda - \lambda_\perp$. Then if $\mu(E) = 0$, then $\mu(E) = \mu\left(E \cap N^C\right)$. Also,

$$\lambda_{||}(E) = \lambda(E) - \lambda_\perp(E) \equiv \lambda(E) - \lambda(E \cap N) = \lambda\left(E \cap N^C\right).$$

Therefore, if $\lambda_{||}(E) > 0$, it follows since $h < 1$ on N^C

$$\begin{aligned} 0 &< \lambda_{||}(E) = \lambda\left(E \cap N^C\right) = \int_{E \cap N^C} h \, d(\mu + \lambda) \\ &< \mu\left(E \cap N^C\right) + \lambda\left(E \cap N^C\right) = 0 + \lambda_{||}(E), \end{aligned}$$

a contradiction. Therefore, $\lambda_{||} \ll \mu$.

It only remains to verify the two measures $\lambda_\perp$ and $\lambda_{||}$ are unique. Suppose then that ν_1 and ν_2 play the roles of $\lambda_\perp$ and $\lambda_{||}$ respectively. Let N_1 play the role of N in the definition of ν_1 and let g_1 play the role of g for ν_2. I will show that $g = g_1$ μ a.e. Let $E_k \equiv [g_1 - g > 1/k]$ for $k \in \mathbb{N}$. Then on observing that $\lambda_\perp - \nu_1 = \nu_2 - \lambda_{||}$

$$\begin{aligned} 0 &= (\lambda_\perp - \nu_1)\left(E_n \cap (N_1 \cup N)^C\right) = \int_{E_n \cap (N_1 \cup N)^C} (g_1 - g) \, d\mu \\ &\geq \frac{1}{k}\mu\left(E_k \cap (N_1 \cup N)^C\right) = \frac{1}{k}\mu(E_k). \end{aligned}$$

and so $\mu(E_k) = 0$. Therefore, $\mu([g_1 - g > 0]) = 0$ because $[g_1 - g > 0] = \cup_{k=1}^\infty E_k$. It follows $g_1 \leq g$ μ a.e. Similarly, $g \geq g_1$ μ a.e. Therefore, $\nu_2 = \lambda_{||}$ and so $\lambda_\perp = \nu_1$ also. This proves the theorem.

The f in the theorem for the absolutely continuous case is sometimes denoted by $\frac{d\lambda}{d\mu}$ and is called the Radon Nikodym derivative.

The next corollary is a useful generalization to σ finite measure spaces.

Corollary 13.3 *Suppose $\lambda \ll \mu$ and there exist sets $S_n \in \mathcal{S}$ with*

$$S_n \cap S_m = \emptyset, \ \cup_{n=1}^\infty S_n = \Omega,$$

and $\lambda(S_n)$, $\mu(S_n) < \infty$. Then there exists $f \geq 0$, where f is μ measurable, and

$$\lambda(E) = \int_E f \, d\mu$$

for all $E \in \mathcal{S}$. The function f is $\mu + \lambda$ a.e. unique.

Proof: Define the σ algebra of subsets of S_n,

$$\mathcal{S}_n \equiv \{E \cap S_n : E \in \mathcal{S}\}.$$

Then both λ, and μ are finite measures on $\mathcal{S}_n$, and $\lambda \ll \mu$. Thus, by Theorem 13.2, there exists a nonnegative $\mathcal{S}_n$ measurable function f_n,with $\lambda(E) = \int_E f_n d\mu$ for all $E \in \mathcal{S}_n$. Define $f(x) = f_n(x)$ for $x \in S_n$. Since the S_n are disjoint and their union is all of Ω, this defines f on all of Ω. The function, f is measurable because

$$f^{-1}((a,\infty]) = \cup_{n=1}^{\infty} f_n^{-1}((a,\infty]) \in \mathcal{S}.$$

Also, for $E \in \mathcal{S}$,

$$\begin{aligned}
\lambda(E) &= \sum_{n=1}^{\infty} \lambda(E \cap S_n) = \sum_{n=1}^{\infty} \int \mathcal{X}_{E\cap S_n}(x) f_n(x) d\mu \\
&= \sum_{n=1}^{\infty} \int \mathcal{X}_{E\cap S_n}(x) f(x) d\mu
\end{aligned}$$

By the monotone convergence theorem

$$\begin{aligned}
\sum_{n=1}^{\infty} \int \mathcal{X}_{E\cap S_n}(x) f(x) d\mu &= \lim_{N\to\infty} \sum_{n=1}^{N} \int \mathcal{X}_{E\cap S_n}(x) f(x) d\mu \\
&= \lim_{N\to\infty} \int \sum_{n=1}^{N} \mathcal{X}_{E\cap S_n}(x) f(x) d\mu \\
&= \int \sum_{n=1}^{\infty} \mathcal{X}_{E\cap S_n}(x) f(x) d\mu = \int_E f \, d\mu.
\end{aligned}$$

This proves the existence part of the corollary.

To see f is unique, suppose f_1 and f_2 both work and consider for $n \in \mathbb{N}$

$$E_k \equiv \left[f_1 - f_2 > \frac{1}{k} \right].$$

Then

$$0 = \lambda(E_k \cap S_n) - \lambda(E_k \cap S_n) = \int_{E_k \cap S_n} f_1(x) - f_2(x) d\mu.$$

Hence $\mu(E_k \cap S_n) = 0$ for all n so

$$\mu(E_k) = \lim_{n\to\infty} \mu(E \cap S_n) = 0.$$

Hence $\mu([f_1 - f_2 > 0]) \leq \sum_{k=1}^{\infty} \mu(E_k) = 0$. Therefore, $\lambda([f_1 - f_2 > 0]) = 0$ also. Similarly

$$(\mu + \lambda)([f_1 - f_2 < 0]) = 0.$$

This version of the Radon Nikodym theorem will suffice for most applications, but more general versions are available. To see one of these, one can read the treatment in Hewitt and Stromberg [22]. This involves the notion of decomposable measure spaces, a generalization of σ finite.

Not surprisingly, there is a simple generalization of the Lebesgue decomposition part of Theorem 13.2.

Corollary 13.4 *Let $(\Omega, \mathcal{S})$ be a set with a σ algebra of sets. Suppose λ and μ are two measures defined on the sets of $\mathcal{S}$ and suppose there exists a sequence of disjoint sets of $\mathcal{S}$, $\{\Omega_i\}_{i=1}^{\infty}$ such that $\lambda(\Omega_i), \mu(\Omega_i) < \infty$. Then there is a set of μ measure zero, N and measures $\lambda_\perp$ and $\lambda_{||}$ such that*

$$\lambda_\perp + \lambda_{||} = \lambda,\ \lambda_{||} \ll \mu, \lambda_\perp(E) = \lambda(E \cap N) = \lambda_\perp(E \cap N).$$

Proof: Let $\mathcal{S}_i \equiv \{E \cap \Omega_i : E \in \mathcal{S}\}$ and for $E \in \mathcal{S}_i$, let $\lambda^i(E) = \lambda(E)$ and $\mu^i(E) = \mu(E)$. Then by Theorem 13.2 there exist unique measures $\lambda_\perp^i$ and $\lambda_{||}^i$ such that $\lambda^i = \lambda_\perp^i + \lambda_{||}^i$, a set of μ^i measure zero, $N_i \in \mathcal{S}_i$ such that for all $E \in \mathcal{S}_i$, $\lambda_\perp^i(E) = \lambda^i(E \cap N_i)$ and $\lambda_{||}^i \ll \mu^i$. Define for $E \in \mathcal{S}$

$$\lambda_\perp(E) \equiv \sum_i \lambda_\perp^i(E \cap \Omega_i),\ \lambda_{||}(E) \equiv \sum_i \lambda_{||}^i(E \cap \Omega_i),\ N \equiv \cup_i N_i.$$

First observe that $\lambda_\perp$ and $\lambda_{||}$ are measures.

$$\begin{aligned}
\lambda_\perp\left(\cup_{j=1}^{\infty} E_j\right) &\equiv \sum_i \lambda_\perp^i\left(\cup_{j=1}^{\infty} E_j \cap \Omega_i\right) = \sum_i \sum_j \lambda_\perp^i(E_j \cap \Omega_i) \\
&= \sum_j \sum_i \lambda_\perp^i(E_j \cap \Omega_i) = \sum_j \sum_i \lambda(E_j \cap \Omega_i \cap N_i) \\
&= \sum_j \sum_i \lambda_\perp^i(E_j \cap \Omega_i) = \sum_j \lambda_\perp(E_j).
\end{aligned}$$

The argument for $\lambda_{||}$ is similar. Now

$$\mu(N) = \sum_i \mu(N \cap \Omega_i) = \sum_i \mu^i(N_i) = 0$$

and

$$\begin{aligned}
\lambda_\perp(E) &\equiv \sum_i \lambda_\perp^i(E \cap \Omega_i) = \sum_i \lambda^i(E \cap \Omega_i \cap N_i) \\
&= \sum_i \lambda(E \cap \Omega_i \cap N) = \lambda(E \cap N).
\end{aligned}$$

Also if $\mu(E) = 0$, then $\mu^i(E \cap \Omega_i) = 0$ and so $\lambda_{||}^i(E \cap \Omega_i) = 0$. Therefore,

$$\lambda_{||}(E) = \sum_i \lambda_{||}^i(E \cap \Omega_i) = 0.$$

The decomposition is unique because of the uniqueness of the $\lambda_{||}^i$ and $\lambda_\perp^i$ and the observation that some other decomposition must coincide with the given one on the Ω_i.

13.2 Vector Measures

The next topic will use the Radon Nikodym theorem. It is the topic of vector and complex measures. The main interest is in complex measures although a vector measure can have values in any topological vector space. Whole books have been written on this subject. See for example the book by Diestal and Uhl [13] titled Vector measures.

Definition 13.5 *Let $(V, ||\cdot||)$ be a normed linear space and let $(\Omega, \mathcal{S})$ be a measure space. A function $\mu : \mathcal{S} \to V$ is a vector measure if μ is countably additive. That is, if $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathcal{S}$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Note that it makes sense to take finite sums because it is given that μ has values in a vector space in which vectors can be summed. In the above, $\mu(E_i)$ is a vector. It might be a point in $\mathbb{R}^n$ or in any other vector space. In many of the most important applications, it is a vector in some sort of function space which may be infinite dimensional. The infinite sum has the usual meaning. That is

$$\sum_{i=1}^{\infty} \mu(E_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i)$$

where the limit takes place relative to the norm on V.

Definition 13.6 *Let $(\Omega, \mathcal{S})$ be a measure space and let μ be a vector measure defined on $\mathcal{S}$. A subset, $\pi(E)$, of $\mathcal{S}$ is called a partition of E if $\pi(E)$ consists of finitely many disjoint sets of $\mathcal{S}$ and $\cup\pi(E) = E$. Let*

$$|\mu|(E) = \sup\{\sum_{F\in\pi(E)} ||\mu(F)|| : \pi(E) \text{ is a partition of } E\}.$$

$|\mu|$ is called the total variation of μ.

The next theorem may seem a little surprising. It states that, if finite, the total variation is a nonnegative measure.

Theorem 13.7 *If $|\mu|(\Omega) < \infty$, then $|\mu|$ is a measure on $\mathcal{S}$. Even if $|\mu|(\Omega) = \infty$, $|\mu|(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} |\mu|(E_i)$. That is $|\mu|$ is subadditive and $|\mu|(A) \leq |\mu|(B)$ whenever $A, B \in \mathcal{S}$ with $A \subseteq B$.*

Proof: Consider the last claim. Let $a < |\mu|(A)$ and let $\pi(A)$ be a partition of A such that

$$a < \sum_{F\in\pi(A)} ||\mu(F)||.$$

Then $\pi(A) \cup \{B \setminus A\}$ is a partition of B and

$$|\mu|(B) \geq \sum_{F \in \pi(A)} ||\mu(F)|| + ||\mu(B \setminus A)|| > a.$$

Since this is true for all such a, it follows $|\mu|(B) \geq |\mu|(A)$ as claimed.

Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathcal{S}$ and let $E_\infty = \cup_{j=1}^{\infty} E_j$. Then letting $a < |\mu|(E_\infty)$, it follows from the definition of total variation there exists a partition of E_∞, $\pi(E_\infty) = \{A_1, \cdots, A_n\}$ such that

$$a < \sum_{i=1}^{n} ||\mu(A_i)||.$$

Also,

$$A_i = \cup_{j=1}^{\infty} A_i \cap E_j$$

and so by the triangle inequality, $||\mu(A_i)|| \leq \sum_{j=1}^{\infty} ||\mu(A_i \cap E_j)||$. Therefore, by the above, and either Fubini's theorem or Lemma 7.18 on Page 134

$$\begin{aligned} a &< \sum_{i=1}^{n} \overbrace{\sum_{j=1}^{\infty} ||\mu(A_i \cap E_j)||}^{\geq ||\mu(A_i)||} \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{n} ||\mu(A_i \cap E_j)|| \\ &\leq \sum_{j=1}^{\infty} |\mu|(E_j) \end{aligned}$$

because $\{A_i \cap E_j\}_{i=1}^{n}$ is a partition of E_j.

Since a is arbitrary, this shows

$$|\mu|(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} |\mu|(E_j).$$

If the sets, E_j are not disjoint, let $F_1 = E_1$ and if F_n has been chosen, let $F_{n+1} \equiv E_{n+1} \setminus \cup_{i=1}^{n} E_i$. Thus the sets, F_i are disjoint and $\cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} E_i$. Therefore,

$$|\mu|\left(\cup_{j=1}^{\infty} E_j\right) = |\mu|\left(\cup_{j=1}^{\infty} F_j\right) \leq \sum_{j=1}^{\infty} |\mu|(F_j) \leq \sum_{j=1}^{\infty} |\mu|(E_j)$$

and proves $|\mu|$ is always subadditive as claimed regarless of whether $|\mu|(\Omega) < \infty$.

Now suppose $|\mu|(\Omega) < \infty$ and let E_1 and E_2 be sets of $\mathcal{S}$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A_1^i \cdots A_{n_i}^i\} = \pi(E_i)$, a partition of E_i which is chosen such that

$$|\mu|(E_i) - \varepsilon < \sum_{j=1}^{n_i} ||\mu(A_j^i)|| \quad i = 1, 2.$$

Such a partition exists because of the definition of the total variation. Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ denoted by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|\mu|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} ||\mu(F)|| > |\mu|(E_1) + |\mu|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|\mu|(E_1 \cup E_2) \geq |\mu|(E_1) + |\mu|(E_2). \tag{13.5}$$

Then 13.5 implies that whenever the E_i are disjoint, $|\mu|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |\mu|(E_j)$. Therefore,

$$\sum_{j=1}^{\infty} |\mu|(E_j) \geq |\mu|(\cup_{j=1}^{\infty} E_j) \geq |\mu|(\cup_{j=1}^{n} E_j) \geq \sum_{j=1}^{n} |\mu|(E_j).$$

Since n is arbitrary,

$$|\mu|(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |\mu|(E_j)$$

which shows that $|\mu|$ is a measure as claimed. This proves the theorem.

In the case that μ is a complex measure, it is always the case that $|\mu|(\Omega) < \infty$.

Theorem 13.8 *Suppose μ is a complex measure on $(\Omega, \mathcal{S})$ where $\mathcal{S}$ is a σ algebra of subsets of Ω. That is, whenever, $\{E_i\}$ is a sequence of disjoint sets of $\mathcal{S}$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Then $|\mu|(\Omega) < \infty$.

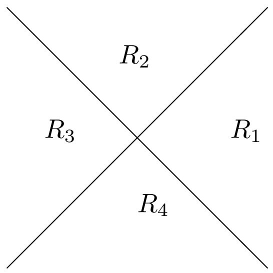
Proof: First here is a claim.

Claim: Suppose $|\mu|(E) = \infty$. Then there are subsets of E, A and B such that $E = A \cup B$, $|\mu(A)|, |\mu(B)| > 1$ and $|\mu|(B) = \infty$.

Proof of the claim: From the definition of $|\mu|$, there exists a partition of $E, \pi(E)$ such that

$$\sum_{F \in \pi(E)} |\mu(F)| > 20(1 + |\mu(E)|). \tag{13.6}$$

Here 20 is just a nice sized number. No effort is made to be delicate in this argument. Also note that $\mu(E) \in \mathbb{C}$ because it is given that μ is a complex measure. Consider the following picture consisting of two lines in the complex plane having slopes 1 and -1 which intersect at the origin, dividing the complex plane into four closed sets, R_1, R_2, R_3, and R_4 as shown.



Let π_i consist of those sets, A of $\pi(E)$ for which $\mu(A) \in R_i$. Thus, some sets, A of $\pi(E)$ could be in two of the π_i if $\mu(A)$ is on one of the intersecting lines. This is not important. The thing which is important is that if $\mu(A) \in R_1$ or R_3, then $\frac{\sqrt{2}}{2}|\mu(A)| \leq |\mathrm{Re}(\mu(A))|$ and if $\mu(A) \in R_2$ or R_4 then $\frac{\sqrt{2}}{2}|\mu(A)| \leq |\mathrm{Im}(\mu(A))|$ and $\mathrm{Re}(z)$ has the same sign for z in R_1 and R_3 while $\mathrm{Im}(z)$ has the same sign for z in R_2 or R_4. Then by 13.6, it follows that for some i,

$$\sum_{F\in\pi_i} |\mu(F)| > 5(1 + |\mu(E)|). \tag{13.7}$$

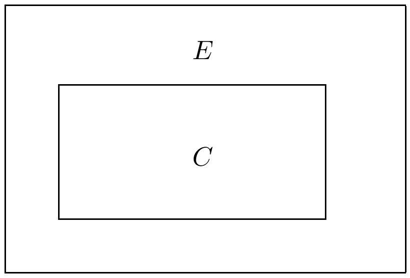
Suppose i equals 1 or 3. A similar argument using the imaginary part applies if i equals 2 or 4. Then,

$$\begin{aligned} \left|\sum_{F\in\pi_i} \mu(F)\right| &\geq \left|\sum_{F\in\pi_i} \mathrm{Re}(\mu(F))\right| = \sum_{F\in\pi_i} |\mathrm{Re}(\mu(F))| \\ &\geq \frac{\sqrt{2}}{2}\sum_{F\in\pi_i} |\mu(F)| > 5\frac{\sqrt{2}}{2}(1 + |\mu(E)|). \end{aligned}$$

Now letting C be the union of the sets in π_i,

$$|\mu(C)| = \left|\sum_{F\in\pi_i} \mu(F)\right| > \frac{5}{2}(1 + |\mu(E)|) > 1. \tag{13.8}$$

Define $D \equiv E \setminus C$.



Then $\mu(C) + \mu(E \setminus C) = \mu(E)$ and so

$$\begin{aligned} \frac{5}{2}(1 + |\mu(E)|) &< |\mu(C)| = |\mu(E) - \mu(E \setminus C)| \\ &= |\mu(E) - \mu(D)| \leq |\mu(E)| + |\mu(D)| \end{aligned}$$

and so

$$1 < \frac{5}{2} + \frac{3}{2}|\mu(E)| < |\mu(D)|.$$

Now since $|\mu|(E) = \infty$, it follows from Theorem 13.8 that $\infty = |\mu|(E) \leq |\mu|(C) + |\mu|(D)$ and so either $|\mu|(C) = \infty$ or $|\mu|(D) = \infty$. If $|\mu|(C) = \infty$, let $B = C$ and $A = D$. Otherwise, let $B = D$ and $A = C$. This proves the claim.

Now suppose $|\mu|(\Omega) = \infty$. Then from the claim, there exist A_1 and B_1 such that $|\mu|(B_1) = \infty, |\mu(B_1)|, |\mu(A_1)| > 1$, and $A_1 \cup B_1 = \Omega$. Let $B_1 \equiv \Omega \setminus A$ play the same role as Ω and obtain $A_2, B_2 \subseteq B_1$ such that $|\mu|(B_2) = \infty, |\mu(B_2)|, |\mu(A_2)| > 1$, and $A_2 \cup B_2 = B_1$. Continue in this way to obtain a sequence of disjoint sets, $\{A_i\}$ such that $|\mu(A_i)| > 1$. Then since μ is a measure,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

but this is impossible because $\lim_{i \to \infty} \mu(A_i) \neq 0$. This proves the theorem.

Theorem 13.9 *Let $(\Omega, \mathcal{S})$ be a measure space and let $\lambda : \mathcal{S} \to \mathbb{C}$ be a complex vector measure. Thus $|\lambda|(\Omega) < \infty$. Let $\mu : \mathcal{S} \to [0, \mu(\Omega)]$ be a finite measure such that $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega)$ such that for all $E \in \mathcal{S}$,*

$$\int_E f d\mu = \lambda(E).$$

Proof: It is clear that $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ are real-valued vector measures on $\mathcal{S}$. Since $|\lambda|(\Omega) < \infty$, it follows easily that $|\operatorname{Re}\lambda|(\Omega)$ and $|\operatorname{Im}\lambda|(\Omega) < \infty$. This is clear because

$$|\lambda(E)| \geq |\operatorname{Re}\lambda(E)|, |\operatorname{Im}\lambda(E)|.$$

Therefore, each of

$$\frac{|\operatorname{Re}\lambda| + \operatorname{Re}\lambda}{2}, \ \frac{|\operatorname{Re}\lambda| - \operatorname{Re}(\lambda)}{2}, \ \frac{|\operatorname{Im}\lambda| + \operatorname{Im}\lambda}{2}, \text{ and } \frac{|\operatorname{Im}\lambda| - \operatorname{Im}(\lambda)}{2}$$

are finite measures on $\mathcal{S}$. It is also clear that each of these finite measures are absolutely continuous with respect to μ and so there exist unique nonnegative functions in $L^1(\Omega)$, f_1, f_2, g_1, g_2 such that for all $E \in \mathcal{S}$,

$$\begin{aligned}
\frac{1}{2}(|\operatorname{Re}\lambda| + \operatorname{Re}\lambda)(E) &= \int_E f_1 d\mu, \\
\frac{1}{2}(|\operatorname{Re}\lambda| - \operatorname{Re}\lambda)(E) &= \int_E f_2 d\mu, \\
\frac{1}{2}(|\operatorname{Im}\lambda| + \operatorname{Im}\lambda)(E) &= \int_E g_1 d\mu, \\
\frac{1}{2}(|\operatorname{Im}\lambda| - \operatorname{Im}\lambda)(E) &= \int_E g_2 d\mu.
\end{aligned}$$

Now let $f = f_1 - f_2 + i(g_1 - g_2)$.

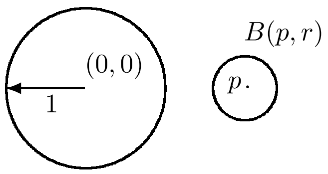
The following corollary is about representing a vector measure in terms of its total variation. It is like representing a complex number in the form $re^{i\theta}$. The proof requires the following lemma.

Lemma 13.10 *Suppose $(\Omega, \mathcal{S}, \mu)$ is a measure space and f is a function in $L^1(\Omega, \mu)$ with the property that*

$$|\int_E f \, d\mu| \leq \mu(E)$$

for all $E \in \mathcal{S}$. Then $|f| \leq 1$ a.e.

Proof of the lemma: Consider the following picture.



where $B(p,r) \cap B(0,1) = \emptyset$. Let $E = f^{-1}(B(p,r))$. In fact $\mu(E) = 0$. If $\mu(E) \neq 0$ then

$$\begin{aligned}
\left|\frac{1}{\mu(E)} \int_E f \, d\mu - p\right| &= \left|\frac{1}{\mu(E)} \int_E (f-p) d\mu\right| \\
&\leq \frac{1}{\mu(E)} \int_E |f - p| d\mu < r
\end{aligned}$$

because on E, $|f(x) - p| < r$. Hence

$$|\frac{1}{\mu(E)} \int_E f d\mu| > 1$$

because it is closer to p than r. (Refer to the picture.) However, this contradicts the assumption of the lemma. It follows $\mu(E) = 0$. Since the set of complex numbers, z such that $|z| > 1$ is an open set, it equals the union of countably many balls, $\{B_i\}_{i=1}^{\infty}$. Therefore,

$$\begin{aligned} \mu\left(f^{-1}(\{z \in \mathbb{C} : |z| > 1\}\right) &= \mu\left(\cup_{k=1}^{\infty} f^{-1}(B_k)\right) \\ &\leq \sum_{k=1}^{\infty} \mu\left(f^{-1}(B_k)\right) = 0. \end{aligned}$$

Thus $|f(x)| \leq 1$ a.e. as claimed. This proves the lemma.

Corollary 13.11 *Let λ be a complex vector measure with $|\lambda|(\Omega) < \infty$[1] Then there exists a unique $f \in L^1(\Omega)$ such that $\lambda(E) = \int_E f d|\lambda|$. Furthermore, $|f| = 1$ for $|\lambda|$ a.e. This is called the polar decomposition of λ.*

Proof: First note that $\lambda \ll |\lambda|$ and so such an L^1 function exists and is unique. It is required to show $|f| = 1$ a.e. If $|\lambda|(E) \neq 0$,

$$\left|\frac{\lambda(E)}{|\lambda|(E)}\right| = \left|\frac{1}{|\lambda|(E)} \int_E f \, d|\lambda|\right| \leq 1.$$

Therefore by Lemma 13.10, $|f| \leq 1$, $|\lambda|$ a.e. Now let

$$E_n = \left[|f| \leq 1 - \frac{1}{n}\right].$$

Let $\{F_1, \cdots, F_m\}$ be a partition of E_n. Then

$$\begin{aligned} \sum_{i=1}^{m} |\lambda(F_i)| &= \sum_{i=1}^{m} \left|\int_{F_i} f d|\lambda|\right| \leq \sum_{i=1}^{m} \int_{F_i} |f| \, d|\lambda| \\ &\leq \sum_{i=1}^{m} \int_{F_i} \left(1 - \frac{1}{n}\right) d|\lambda| = \sum_{i=1}^{m} \left(1 - \frac{1}{n}\right) |\lambda|(F_i) \\ &= |\lambda|(E_n)\left(1 - \frac{1}{n}\right). \end{aligned}$$

Then taking the supremum over all partitions,

$$|\lambda|(E_n) \leq \left(1 - \frac{1}{n}\right) |\lambda|(E_n)$$

which shows $|\lambda|(E_n) = 0$. Hence $|\lambda|([|f| < 1]) = 0$ because $[|f| < 1] = \cup_{n=1}^{\infty} E_n$.This proves Corollary 13.11.

[1] As proved above, the assumption that $|\lambda|(\Omega) < \infty$ is redundant.

Corollary 13.12 *Suppose $(\Omega, \mathcal{S})$ is a measure space and μ is a finite nonnegative measure on $\mathcal{S}$. Then for $h \in L^1(\mu)$, define a complex measure, λ by*

$$\lambda(E) \equiv \int_E h d\mu.$$

Then

$$|\lambda|(E) = \int_E |h| d\mu.$$

Furthermore, $|h| = \overline{g}h$ where $gd|\lambda|$ is the polar decomposition of λ,

$$\lambda(E) = \int_E gd|\lambda|$$

Proof: From Corollary 13.11 there exists g such that $|g| = 1, |\lambda|$ a.e. and for all $E \in \mathcal{S}$

$$\lambda(E) = \int_E gd|\lambda| = \int_E hd\mu.$$

Let s_n be a sequence of simple functions converging pointwise to $\overline{g}$. Then from the above,

$$\int_E gs_n d|\lambda| = \int_E s_n hd\mu.$$

Passing to the limit using the dominated convergence theorem,

$$\int_E d|\lambda| = \int_E \overline{g}hd\mu.$$

It follows $\overline{g}h \geq 0$ a.e. and $|\overline{g}| = 1$. Therefore, $|h| = |\overline{g}h| = \overline{g}h$. It follows from the above, that

$$|\lambda|(E) = \int_E d|\lambda| = \int_E \overline{g}hd\mu = \int_E d|\lambda| = \int_E |h| d\mu$$

and this proves the corollary.

13.3 Representation Theorems For The Dual Space Of L^p

Recall the concept of the dual space of a Banach space in the Chapter on Banach space starting on Page 253. The next topic deals with the dual space of L^p for $p \geq 1$ in the case where the measure space is σ finite or finite. In what follows $q = \infty$ if $p = 1$ and otherwise, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 13.13 *(Riesz representation theorem) Let $p > 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. If $\Lambda \in (L^p(\Omega))'$, then there exists a unique $h \in L^q(\Omega)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that*

$$\Lambda f = \int_\Omega hf d\mu.$$

This function satisfies $||h||_q = ||\Lambda||$ where $||\Lambda||$ is the operator norm of Λ.

Proof: (Uniqueness) If h_1 and h_2 both represent Λ, consider

$$f = |h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}),$$

where $\overline{h}$ denotes complex conjugation. By Holder's inequality, it is easy to see that $f \in L^p(\Omega)$. Thus

$$0 = \Lambda f - \Lambda f =$$

$$\int h_1|h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}) - h_2|h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2})d\mu$$

$$= \int |h_1 - h_2|^q d\mu.$$

Therefore $h_1 = h_2$ and this proves uniqueness.

Now let $\lambda(E) = \Lambda(\mathcal{X}_E)$. Since this is a finite measure space $\mathcal{X}_E$ is an element of $L^p(\Omega)$ and so it makes sense to write $\Lambda(\mathcal{X}_E)$. In fact λ is a complex measure having finite total variation. Let $A_1, \cdots, A_n$ be a partition of Ω.

$$|\Lambda \mathcal{X}_{A_i}| = w_i(\Lambda \mathcal{X}_{A_i}) = \Lambda(w_i \mathcal{X}_{A_i})$$

for some $w_i \in \mathbb{C}$, $|w_i| = 1$. Thus

$$\sum_{i=1}^n |\lambda(A_i)| = \sum_{i=1}^n |\Lambda(\mathcal{X}_{A_i})| = \Lambda(\sum_{i=1}^n w_i \mathcal{X}_{A_i})$$

$$\leq ||\Lambda||(\int |\sum_{i=1}^n w_i \mathcal{X}_{A_i}|^p d\mu)^{\frac{1}{p}} = ||\Lambda||(\int_\Omega d\mu)^{\frac{1}{p}} = ||\Lambda||\mu(\Omega)^{\frac{1}{p}}.$$

This is because if $x \in \Omega$, x is contained in exactly one of the A_i and so the absolute value of the sum in the first integral above is equal to 1. Therefore $|\lambda|(\Omega) < \infty$ because this was an arbitrary partition. Also, if $\{E_i\}_{i=1}^\infty$ is a sequence of disjoint sets of $\mathcal{S}$, let

$$F_n = \cup_{i=1}^n E_i, \ F = \cup_{i=1}^\infty E_i.$$

Then by the Dominated Convergence theorem,

$$||\mathcal{X}_{F_n} - \mathcal{X}_F||_p \to 0.$$

Therefore, by continuity of Λ,

$$\lambda(F) = \Lambda(\mathcal{X}_F) = \lim_{n\to\infty} \Lambda(\mathcal{X}_{F_n}) = \lim_{n\to\infty} \sum_{k=1}^n \Lambda(\mathcal{X}_{E_k}) = \sum_{k=1}^\infty \lambda(E_k).$$

This shows λ is a complex measure with $|\lambda|$ finite.

It is also clear from the definition of λ that $\lambda \ll \mu$. Therefore, by the Radon Nikodym theorem, there exists $h \in L^1(\Omega)$ with

$$\lambda(E) = \int_E h d\mu = \Lambda(\mathcal{X}_E).$$

Actually $h \in L^q$ and satisfies the other conditions above. Let $s = \sum_{i=1}^m c_i \mathcal{X}_{E_i}$ be a simple function. Then since Λ is linear,

$$\Lambda(s) = \sum_{i=1}^m c_i \Lambda(\mathcal{X}_{E_i}) = \sum_{i=1}^m c_i \int_{E_i} h d\mu = \int hs d\mu. \tag{13.9}$$

Claim: If f is uniformly bounded and measurable, then

$$\Lambda(f) = \int hf d\mu.$$

Proof of claim: Since f is bounded and measurable, there exists a sequence of simple functions, $\{s_n\}$ which converges to f pointwise and in $L^p(\Omega)$. This follows from Theorem 7.24 on Page 139 upon breaking f up into positive and negative parts of real and complex parts. In fact this theorem gives uniform convergence. Then

$$\Lambda(f) = \lim_{n\to\infty} \Lambda(s_n) = \lim_{n\to\infty} \int hs_n d\mu = \int hf d\mu,$$

the first equality holding because of continuity of Λ, the second following from 13.9 and the third holding by the dominated convergence theorem.

This is a very nice formula but it still has not been shown that $h \in L^q(\Omega)$.

Let $E_n = \{x : |h(x)| \le n\}$. Thus $|h\mathcal{X}_{E_n}| \le n$. Then

$$|h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n}) \in L^p(\Omega).$$

By the claim, it follows that

$$||h\mathcal{X}_{E_n}||_q^q = \int h|h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n})d\mu = \Lambda(|h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n}))$$

$$\le ||\Lambda|| \, \left|\left||h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n})\right|\right|_p = ||\Lambda|| \, ||h\mathcal{X}_{E_n}||_q^{\frac{q}{p}},$$

the last equality holding because $q - 1 = q/p$ and so

$$\begin{aligned} \left(\int \left||h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n})\right|^p d\mu\right)^{1/p} &= \left(\int \left(|h\mathcal{X}_{E_n}|^{q/p}\right)^p d\mu\right)^{1/p} \\ &= ||h\mathcal{X}_{E_n}||_q^{\frac{q}{p}} \end{aligned}$$

Therefore, since $q - \frac{q}{p} = 1$, it follows that

$$||h\mathcal{X}_{E_n}||_q \le ||\Lambda||.$$

Letting $n \to \infty$, the Monotone Convergence theorem implies

$$||h||_q \le ||\Lambda||. \tag{13.10}$$

Now that h has been shown to be in $L^q(\Omega)$, it follows from 13.9 and the density of the simple functions, Theorem 10.13 on Page 241, that

$$\Lambda f = \int h f d\mu$$

for all $f \in L^p(\Omega)$.

It only remains to verify the last claim.

$$||\Lambda|| = \sup\{\int hf : ||f||_p \leq 1\} \leq ||h||_q \leq ||\Lambda||$$

by 13.10, and Holder's inequality. This proves the theorem.

To represent elements of the dual space of $L^1(\Omega)$, another Banach space is needed.

Definition 13.14 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space. $L^\infty(\Omega)$ is the vector space of measurable functions such that for some $M > 0$, $|f(x)| \leq M$ for all x outside of some set of measure zero ($|f(x)| \leq M$ a.e.). Define $f = g$ when $f(x) = g(x)$ a.e. and $||f||_\infty \equiv \inf\{M : |f(x)| \leq M \text{ a.e.}\}$.*

Theorem 13.15 *$L^\infty(\Omega)$ is a Banach space.*

Proof: It is clear that $L^\infty(\Omega)$ is a vector space. Is $|| \; ||_\infty$ a norm?

Claim: If $f \in L^\infty(\Omega)$, then $|f(x)| \leq ||f||_\infty$ a.e.

Proof of the claim: $\left\{x : |f(x)| \geq ||f||_\infty + n^{-1}\right\} \equiv E_n$ is a set of measure zero according to the definition of $||f||_\infty$. Furthermore, $\{x : |f(x)| > ||f||_\infty\} = \cup_n E_n$ and so it is also a set of measure zero. This verifies the claim.

Now if $||f||_\infty = 0$ it follows that $f(x) = 0$ a.e. Also if $f, g \in L^\infty(\Omega)$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty$$

a.e. and so $||f||_\infty + ||g||_\infty$ serves as one of the constants, M in the definition of $||f + g||_\infty$. Therefore,

$$||f + g||_\infty \leq ||f||_\infty + ||g||_\infty .$$

Next let c be a number. Then $|cf(x)| = |c| |f(x)| \leq |c| ||f||_\infty$ and so $||cf||_\infty \leq |c| ||f||_\infty$. Therefore since c is arbitrary, $||f||_\infty = ||c(1/c) f||_\infty \leq \left|\frac{1}{c}\right| ||cf||_\infty$ which implies $|c| ||f||_\infty \leq ||cf||_\infty$. Thus $|| \; ||_\infty$ is a norm as claimed.

To verify completeness, let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega)$ and use the above claim to get the existence of a set of measure zero, E_{nm} such that for all $x \notin E_{nm}$,

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$$

Let $E = \cup_{n,m} E_{nm}$. Thus $\mu(E) = 0$ and for each $x \notin E$, $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Let

$$f(x) = \left\{ \begin{array}{c} 0 \text{ if } x \in E \\ \lim_{n\to\infty} f_n(x) \text{ if } x \notin E \end{array} \right. = \lim_{n\to\infty} \mathcal{X}_{E^C}(x) f_n(x).$$

Then f is clearly measurable because it is the limit of measurable functions. If

$$F_n = \{x : |f_n(x)| > ||f_n||_\infty\}$$

and $F = \cup_{n=1}^\infty F_n$, it follows $\mu(F) = 0$ and that for $x \notin F \cup E$,

$$|f(x)| \leq \lim \inf_{n\to\infty} |f_n(x)| \leq \lim \inf_{n\to\infty} ||f_n||_\infty < \infty$$

because $\{||f_n||_\infty\}$ is a Cauchy sequence. ($|||f_n||_\infty - ||f_m||_\infty| \leq ||f_n - f_m||_\infty$ by the triangle inequality.) Thus $f \in L^\infty(\Omega)$. Let n be large enough that whenever $m > n$,

$$||f_m - f_n||_\infty < \varepsilon.$$

Then, if $x \notin E$,

$$\begin{aligned} |f(x) - f_n(x)| &= \lim_{m\to\infty} |f_m(x) - f_n(x)| \\ &\leq \lim_{m\to\infty} \inf \ ||f_m - f_n||_\infty < \varepsilon. \end{aligned}$$

Hence $||f - f_n||_\infty < \varepsilon$ for all n large enough. This proves the theorem.

The next theorem is the Riesz representation theorem for $\left(L^1(\Omega)\right)'$.

Theorem 13.16 *(Riesz representation theorem) Let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. If $\Lambda \in (L^1(\Omega))'$, then there exists a unique $h \in L^\infty(\Omega)$ such that*

$$\Lambda(f) = \int_\Omega hf \, d\mu$$

for all $f \in L^1(\Omega)$. If h is the function in $L^\infty(\Omega)$ representing $\Lambda \in (L^1(\Omega))'$, then $||h||_\infty = ||\Lambda||$.

Proof: Just as in the proof of Theorem 13.13, there exists a unique $h \in L^1(\Omega)$ such that for all simple functions, s,

$$\Lambda(s) = \int hs \, d\mu. \tag{13.11}$$

To show $h \in L^\infty(\Omega)$, let $\varepsilon > 0$ be given and let

$$E = \{x : |h(x)| \geq ||\Lambda|| + \varepsilon\}.$$

Let $|k| = 1$ and $hk = |h|$. Since the measure space is finite, $k \in L^1(\Omega)$. As in Theorem 13.13 let $\{s_n\}$ be a sequence of simple functions converging to k in $L^1(\Omega)$, and pointwise. It follows from the construction in Theorem 7.24 on Page 139 that it can be assumed $|s_n| \leq 1$. Therefore

$$\Lambda(k\mathcal{X}_E) = \lim_{n\to\infty} \Lambda(s_n\mathcal{X}_E) = \lim_{n\to\infty} \int_E hs_n d\mu = \int_E hk d\mu$$

where the last equality holds by the Dominated Convergence theorem. Therefore,

$$\begin{aligned}||\Lambda||\mu(E) &\geq |\Lambda(k\mathcal{X}_E)| = |\int_\Omega hk\mathcal{X}_E d\mu| = \int_E |h|d\mu \\ &\geq (||\Lambda|| + \varepsilon)\mu(E).\end{aligned}$$

It follows that $\mu(E) = 0$. Since $\varepsilon > 0$ was arbitrary, $||\Lambda|| \geq ||h||_\infty$. It was shown that $h \in L^\infty(\Omega)$, the density of the simple functions in $L^1(\Omega)$ and 13.11 imply

$$\Lambda f = \int_\Omega hf d\mu \ , \ ||\Lambda|| \geq ||h||_\infty. \tag{13.12}$$

This proves the existence part of the theorem. To verify uniqueness, suppose h_1 and h_2 both represent Λ and let $f \in L^1(\Omega)$ be such that $|f| \leq 1$ and $f(h_1 - h_2) = |h_1 - h_2|$. Then

$$0 = \Lambda f - \Lambda f = \int (h_1 - h_2) f d\mu = \int |h_1 - h_2| d\mu.$$

Thus $h_1 = h_2$. Finally,

$$||\Lambda|| = \sup\{|\int hf d\mu| : ||f||_1 \leq 1\} \leq ||h||_\infty \leq ||\Lambda||$$

by 13.12.

Next these results are extended to the σ finite case.

Lemma 13.17 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose there exists a measurable function, r such that $r(x) > 0$ for all x, there exists M such that $|r(x)| < M$ for all x, and $\int r d\mu < \infty$. Then for*

$$\Lambda \in (L^p(\Omega, \mu))', \ p \geq 1,$$

there exists a unique $h \in L^{p'}(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int hf d\mu.$$

Also $||h|| = ||\Lambda||$. ($||h|| = ||h||_{p'}$ if $p > 1$, $||h||_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proof: Define a new measure $\widetilde{\mu}$, according to the rule

$$\widetilde{\mu}(E) \equiv \int_E r d\mu. \tag{13.13}$$

Thus $\widetilde{\mu}$ is a finite measure on $\mathcal{S}$. Now define a mapping, $\eta : L^p(\Omega, \mu) \to L^p(\Omega, \widetilde{\mu})$ by

$$\eta f = r^{-\frac{1}{p}} f.$$

Then

$$||\eta f||^p_{L^p(\widetilde{\mu})} = \int \left| r^{-\frac{1}{p}} f \right|^p r d\mu = ||f||^p_{L^p(\mu)}$$

and so η is one to one and in fact preserves norms. I claim that also η is onto. To see this, let $g \in L^p(\Omega, \widetilde{\mu})$ and consider the function, $r^{\frac{1}{p}} g$. Then

$$\int \left| r^{\frac{1}{p}} g \right|^p d\mu = \int |g|^p r d\mu = \int |g|^p d\widetilde{\mu} < \infty$$

Thus $r^{\frac{1}{p}} g \in L^p(\Omega, \mu)$ and $\eta \left(r^{\frac{1}{p}} g \right) = g$ showing that η is onto as claimed. Thus η is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which η^* must be one to one and onto.

$$\begin{array}{ccccc} & & \eta^* & & \\ h, L^{p'}(\widetilde{\mu}) & L^p(\widetilde{\mu})', \widetilde{\Lambda} & \rightarrow & L^p(\mu)', \Lambda \\ & & \eta & & \\ & L^p(\widetilde{\mu}) & \leftarrow & L^p(\mu) \end{array}$$

Then for $\Lambda \in L^p(\mu)'$, there exists a unique $\widetilde{\Lambda} \in L^p(\widetilde{\mu})'$ such that $\eta^* \widetilde{\Lambda} = \Lambda$, $\left|\left| \widetilde{\Lambda} \right|\right| = ||\Lambda||$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in L^{p'}(\widetilde{\mu})$ which represents $\widetilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $||h||_{L^{p'}(\widetilde{\mu})} = \left|\left| \widetilde{\Lambda} \right|\right| = ||\Lambda||$ and for all $f \in L^p(\mu)$,

$$\begin{aligned} \Lambda(f) &= \eta^* \widetilde{\Lambda}(f) \equiv \widetilde{\Lambda}(\eta f) = \int h(\eta f) d\widetilde{\mu} = \int rh \left(f^{-\frac{1}{p}} f \right) d\mu \\ &= \int r^{\frac{1}{p'}} h f d\mu. \end{aligned}$$

Now

$$\int \left| r^{\frac{1}{p'}} h \right|^{p'} d\mu = \int |h|^{p'} r d\mu = ||h||^{p'}_{L^{p'}(\widetilde{\mu})} < \infty.$$

Thus $\left|\left| r^{\frac{1}{p'}} h \right|\right|_{L^{p'}(\mu)} = ||h||_{L^{p'}(\widetilde{\mu})} = \left|\left| \widetilde{\Lambda} \right|\right| = ||\Lambda||$ and represents Λ in the appropriate way. If $p = 1$, then $1/p' \equiv 0$. This proves the Lemma.

A situation in which the conditions of the lemma are satisfied is the case where the measure space is σ finite. In fact, you should show this is the only case in which the conditions of the above lemma hold.

Theorem 13.18 *(Riesz representation theorem) Let $(\Omega, \mathcal{S}, \mu)$ be σ finite and let*

$$\Lambda \in (L^p(\Omega, \mu))', \ p \geq 1.$$

Then there exists a unique $h \in L^q(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int h f d\mu.$$

Also $||h|| = ||\Lambda||$. ($||h|| = ||h||_q$ *if* $p > 1$, $||h||_\infty$ *if* $p = 1$). *Here*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proof: Let $\{\Omega_n\}$ be a sequence of disjoint elements of $\mathcal{S}$ having the property that

$$0 < \mu(\Omega_n) < \infty,\ \cup_{n=1}^\infty \Omega_n = \Omega.$$

Define

$$r(x) = \sum_{n=1}^\infty \frac{1}{n^2} \mathcal{X}_{\Omega_n}(x)\ \mu(\Omega_n)^{-1},\ \ \widetilde{\mu}(E) = \int_E r d\mu.$$

Thus

$$\int_\Omega r d\mu = \widetilde{\mu}(\Omega) = \sum_{n=1}^\infty \frac{1}{n^2} < \infty$$

so $\widetilde{\mu}$ is a finite measure. The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

With the Riesz representation theorem, it is easy to show that

$$L^p(\Omega),\ p > 1$$

is a reflexive Banach space. Recall Definition 11.32 on Page 269 for the definition.

Theorem 13.19 *For* $(\Omega, \mathcal{S}, \mu)$ *a* σ *finite measure space and* $p > 1$, $L^p(\Omega)$ *is reflexive.*

Proof: Let $\delta_r : (L^r(\Omega))' \to L^{r'}(\Omega)$ be defined for $\frac{1}{r} + \frac{1}{r'} = 1$ by

$$\int (\delta_r \Lambda) g\ d\mu = \Lambda g$$

for all $g \in L^r(\Omega)$. From Theorem 13.18 δ_r is one to one, onto, continuous and linear. By the open map theorem, δ_r^{-1} is also one to one, onto, and continuous ($\delta_r \Lambda$ equals the representor of Λ). Thus δ_r^* is also one to one, onto, and continuous by Corollary 11.29. Now observe that $J = \delta_p^* \circ \delta_q^{-1}$. To see this, let $z^* \in (L^q)'$, $y^* \in (L^p)'$,

$$\begin{aligned} \delta_p^* \circ \delta_q^{-1}(\delta_q z^*)(y^*) &= (\delta_p^* z^*)(y^*) \\ &= z^*(\delta_p y^*) \\ &= \int (\delta_q z^*)(\delta_p y^*) d\mu, \end{aligned}$$

$$\begin{aligned} J(\delta_q z^*)(y^*) &= y^*(\delta_q z^*) \\ &= \int (\delta_p y^*)(\delta_q z^*) d\mu. \end{aligned}$$

Therefore $\delta_p^* \circ \delta_q^{-1} = J$ on $\delta_q(L^q)' = L^p$. But the two δ maps are onto and so J is also onto.

13.4 The Dual Space Of $C(X)$

Consider the dual space of $C(X)$ where X is a compact Hausdorff space. It will turn out to be a space of measures. To show this, the following lemma will be convenient.

Lemma 13.20 *Suppose λ is a mapping which is defined on the positive continuous functions defined on X, some topological space which satisfies*

$$\lambda(af+bg) = a\lambda(f) + b\lambda(g) \tag{13.14}$$

whenever $a, b \geq 0$ and $f, g \geq 0$. Then there exists a unique extension of λ to all of $C(X)$, Λ such that whenever $f, g \in C(X)$ and $a, b \in \mathbb{C}$, it follows

$$\Lambda(af+bg) = a\Lambda(f) + b\Lambda(g).$$

Proof: Let $C(X;\mathbb{R})$ be the real-valued functions in $C(X)$ and define

$$\Lambda_R(f) = \lambda f^+ - \lambda f^-$$

for $f \in C(X;\mathbb{R})$. Use the identity

$$(f_1+f_2)^+ + f_1^- + f_2^- = f_1^+ + f_2^+ + (f_1+f_2)^-$$

and 13.14 to write

$$\lambda(f_1+f_2)^+ - \lambda(f_1+f_2)^- = \lambda f_1^+ - \lambda f_1^- + \lambda f_2^+ - \lambda f_2^-,$$

it follows that $\Lambda_R(f_1+f_2) = \Lambda_R(f_1) + \Lambda_R(f_2)$. To show that Λ_R is linear, it is necessary to verify that $\Lambda_R(cf) = c\Lambda_R(f)$ for all $c \in \mathbb{R}$. But

$$(cf)^{\pm} = cf^{\pm},$$

if $c \geq 0$ while

$$(cf)^+ = -c(f)^-,$$

if $c < 0$ and

$$(cf)^- = (-c)f^+,$$

if $c < 0$. Thus, if $c < 0$,

$$\Lambda_R(cf) = \lambda(cf)^+ - \lambda(cf)^- = \lambda\left((-c)\,f^-\right) - \lambda\left((-c)f^+\right)$$

$$= -c\lambda(f^-) + c\lambda(f^+) = c(\lambda(f^+) - \lambda(f^-)) = c\Lambda_R(f).$$

A similar formula holds more easily if $c \geq 0$. Now let

$$\Lambda f = \Lambda_R(\operatorname{Re} f) + i\Lambda_R(\operatorname{Im} f)$$

for arbitrary $f \in C(X)$. This is linear as desired. It is obvious that $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$ from the fact that taking the real and imaginary parts are linear operations. The only thing to check is whether you can factor out a complex scalar.

$$\Lambda((a+ib)f) = \Lambda(af) + \Lambda(ibf)$$

$$\equiv \Lambda_R(a \operatorname{Re} f) + i\Lambda_R(a \operatorname{Im} f) + \Lambda_R(-b \operatorname{Im} f) + i\Lambda_R(b \operatorname{Re} f)$$

because $ibf = ib \operatorname{Re} f - b \operatorname{Im} f$ and so $\operatorname{Re}(ibf) = -b \operatorname{Im} f$ and $\operatorname{Im}(ibf) = b \operatorname{Re} f$. Therefore, the above equals

$$\begin{aligned} &= (a+ib)\Lambda_R(\operatorname{Re} f) + i(a+ib)\Lambda_R(\operatorname{Im} f) \\ &= (a+ib)(\Lambda_R(\operatorname{Re} f) + i\Lambda_R(\operatorname{Im} f)) = (a+ib)\Lambda f \end{aligned}$$

The extension is obviously unique. This proves the lemma.

Let $L \in C(X)'$. Also denote by $C^+(X)$ the set of nonnegative continuous functions defined on X. Define for $f \in C^+(X)$

$$\lambda(f) = \sup\{|Lg| : |g| \leq f\}.$$

Note that $\lambda(f) < \infty$ because $|Lg| \leq ||L|| \, ||g|| \leq ||L|| \, ||f||$ for $|g| \leq f$. Then the following lemma is important.

Lemma 13.21 *If $c \geq 0$, $\lambda(cf) = c\lambda(f)$, $f_1 \leq f_2$ implies $\lambda f_1 \leq \lambda f_2$, and*

$$\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2).$$

Proof: The first two assertions are easy to see so consider the third. Let $|g_j| \leq f_j$ and let $\widetilde{g}_j = e^{i\theta_j} g_j$ where θ_j is chosen such that $e^{i\theta_j} L g_j = |Lg_j|$. Thus $L\widetilde{g}_j = |Lg_j|$. Then

$$|\widetilde{g}_1 + \widetilde{g}_2| \leq f_1 + f_2.$$

Hence

$$\begin{aligned} |Lg_1| + |Lg_2| = L\widetilde{g}_1 + L\widetilde{g}_2 = \\ L(\widetilde{g}_1 + \widetilde{g}_2) = |L(\widetilde{g}_1 + \widetilde{g}_2)| \leq \lambda(f_1 + f_2). \end{aligned} \tag{13.15}$$

Choose g_1 and g_2 such that $|Lg_i| + \varepsilon > \lambda(f_i)$. Then 13.15 shows

$$\lambda(f_1) + \lambda(f_2) - 2\varepsilon \leq \lambda(f_1 + f_2).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2). \tag{13.16}$$

Now let $|g| \leq f_1 + f_2$, $|Lg| \geq \lambda(f_1 + f_2) - \varepsilon$. Let

$$h_i(x) = \begin{cases} \frac{f_i(x) g(x)}{f_1(x) + f_2(x)} \text{ if } f_1(x) + f_2(x) > 0, \\ 0 \text{ if } f_1(x) + f_2(x) = 0. \end{cases}$$

Then h_i is continuous and $h_1(x) + h_2(x) = g(x)$, $|h_i| \leq f_i$. Therefore,

$$\begin{aligned} -\varepsilon + \lambda(f_1 + f_2) &\leq |Lg| \leq |Lh_1 + Lh_2| \leq |Lh_1| + |Lh_2| \\ &\leq \lambda(f_1) + \lambda(f_2). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows with 13.16 that

$$\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2)$$

which proves the lemma.

Let Λ be defined in Lemma 13.20. Then Λ is linear by this lemma. Also, if $f \geq 0$,

$$\Lambda f = \Lambda_R f = \lambda(f) \geq 0.$$

Therefore, Λ is a positive linear functional on $C(X)$ $(= C_c(X)$ since X is compact). By Theorem 8.21 on Page 169, there exists a unique Radon measure μ such that

$$\Lambda f = \int_X f \, d\mu$$

for all $f \in C(X)$. Thus $\Lambda(1) = \mu(X)$. What follows is the Riesz representation theorem for $C(X)'$.

Theorem 13.22 *Let $L \in (C(X))'$. Then there exists a Radon measure μ and a function $\sigma \in L^\infty(X, \mu)$ such that*

$$L(f) = \int_X f \, \sigma \, d\mu.$$

Proof: Let $f \in C(X)$. Then there exists a unique Radon measure μ such that

$$|Lf| \leq \Lambda(|f|) = \int_X |f| d\mu = ||f||_1.$$

Since μ is a Radon measure, $C(X)$ is dense in $L^1(X, \mu)$. Therefore L extends uniquely to an element of $(L^1(X, \mu))'$. By the Riesz representation theorem for L^1, there exists a unique $\sigma \in L^\infty(X, \mu)$ such that

$$Lf = \int_X f \, \sigma \, d\mu$$

for all $f \in C(X)$.

13.5 The Dual Space Of $C_0(X)$

It is possible to give a simple generalization of the above theorem. For X a locally compact Hausdorff space, $\widetilde{X}$ denotes the one point compactification of X. Thus, $\widetilde{X} = X \cup \{\infty\}$ and the topology of $\widetilde{X}$ consists of the usual topology of X along with all complements of compact sets which are defined as the open sets containing ∞. Also $C_0(X)$ will denote the space of continuous functions, f, defined on X such that in the topology of $\widetilde{X}$, $\lim_{x\to\infty} f(x) = 0$. For this space of functions, $||f||_0 \equiv \sup\{|f(x)| : x \in X\}$ is a norm which makes this into a Banach space. Then the generalization is the following corollary.

Corollary 13.23 *Let $L \in (C_0(X))'$ where X is a locally compact Hausdorff space. Then there exists $\sigma \in L^\infty(X,\mu)$ for μ a finite Radon measure such that for all $f \in C_0(X)$,*

$$L(f) = \int_X f\sigma d\mu.$$

Proof: Let

$$\widetilde{D} \equiv \left\{ f \in C\left(\widetilde{X}\right) : f(\infty) = 0 \right\}.$$

Thus $\widetilde{D}$ is a closed subspace of the Banach space $C\left(\widetilde{X}\right)$. Let $\theta : C_0(X) \to \widetilde{D}$ be defined by

$$\theta f(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 \text{ if } x = \infty. \end{cases}$$

Then θ is an isometry of $C_0(X)$ and $\widetilde{D}$. ($||\theta u|| = ||u||$.)The following diagram is obtained.

$$\begin{array}{ccccc} C_0(X)' & \overset{\theta^*}{\leftarrow} & \left(\widetilde{D}\right)' & \overset{i^*}{\leftarrow} & C\left(\widetilde{X}\right)' \\ & & & & \\ C_0(X) & \underset{\theta}{\rightarrow} & \widetilde{D} & \underset{i}{\rightarrow} & C\left(\widetilde{X}\right) \end{array}$$

By the Hahn Banach theorem, there exists $L_1 \in C\left(\widetilde{X}\right)'$ such that $\theta^* i^* L_1 = L$. Now apply Theorem 13.22 to get the existence of a finite Radon measure, μ_1, on $\widetilde{X}$ and a function $\sigma \in L^\infty\left(\widetilde{X}, \mu_1\right)$, such that

$$L_1 g = \int_{\widetilde{X}} g\sigma d\mu_1.$$

Letting the σ algebra of μ_1 measurable sets be denoted by $\mathcal{S}_1$, define

$$\mathcal{S} \equiv \{E \setminus \{\infty\} : E \in \mathcal{S}_1\}$$

and let μ be the restriction of μ_1 to $\mathcal{S}$. If $f \in C_0(X)$,

$$Lf = \theta^* i^* L_1 f \equiv L_1 i\theta f = L_1 \theta f = \int_{\widetilde{X}} \theta f \sigma d\mu_1 = \int_X f\sigma d\mu.$$

This proves the corollary.

13.6 More Attractive Formulations

In this section, Corollary 13.23 will be refined and placed in an arguably more attractive form. The measures involved will always be complex Borel measures defined on a σ algebra of subsets of X, a locally compact Hausdorff space.

Definition 13.24 *Let λ be a complex measure. Then $\int f d\lambda \equiv \int f h d|\lambda|$ where $hd|\lambda|$ is the polar decomposition of λ described above. The complex measure, λ is called regular if $|\lambda|$ is regular.*

The following lemma says that the difference of regular complex measures is also regular.

Lemma 13.25 *Suppose $\lambda_i, i = 1, 2$ is a complex Borel measure with total variation finite[2] defined on X, a locally compact Hausdorf space. Then $\lambda_1 - \lambda_2$ is also a regular measure on the Borel sets.*

Proof: Let E be a Borel set. That way it is in the σ algebras associated with both λ_i. Then by regularity of λ_i, there exist K and V compact and open respectively such that $K \subseteq E \subseteq V$ and $|\lambda_i|(V \setminus K) < \varepsilon/2$. Therefore,

$$\begin{aligned}
\sum_{A \in \pi(V \setminus K)} |(\lambda_1 - \lambda_2)(A)| &= \sum_{A \in \pi(V \setminus K)} |\lambda_1(A) - \lambda_2(A)| \\
&\leq \sum_{A \in \pi(V \setminus K)} |\lambda_1(A)| + |\lambda_2(A)| \\
&\leq |\lambda_1|(V \setminus K) + |\lambda_2|(V \setminus K) < \varepsilon.
\end{aligned}$$

Therefore, $|\lambda_1 - \lambda_2|(V \setminus K) \leq \varepsilon$ and this shows $\lambda_1 - \lambda_2$ is regular as claimed.

Theorem 13.26 *Let $L \in C_0(X)'$ Then there exists a unique complex measure, λ with $|\lambda|$ regular and Borel, such that for all $f \in C_0(X)$,*

$$L(f) = \int_X f d\lambda.$$

Furthermore, $||L|| = |\lambda|(X)$.

Proof: By Corollary 13.23 there exists $\sigma \in L^\infty(X, \mu)$ where μ is a Radon measure such that for all $f \in C_0(X)$,

$$L(f) = \int_X f \sigma d\mu.$$

Let a complex Borel measure, λ be given by

$$\lambda(E) \equiv \int_E \sigma d\mu.$$

[2]Recall this is automatic for a complex measure.

This is a well defined complex measure because μ is a finite measure. By Corollary 13.12

$$|\lambda|(E) = \int_E |\sigma| \, d\mu \tag{13.17}$$

and $\sigma = g|\sigma|$ where $gd|\lambda|$ is the polar decomposition for λ. Therefore, for $f \in C_0(X)$,

$$L(f) = \int_X f\sigma d\mu = \int_X fg|\sigma| \, d\mu = \int_X fgd|\lambda| \equiv \int_X fd\lambda. \tag{13.18}$$

From 13.17 and the regularity of μ, it follows that $|\lambda|$ is also regular.

What of the claim about $||L||$? By the regularity of $|\lambda|$, it follows that $C_0(X)$ (In fact, $C_c(X)$) is dense in $L^1(X, |\lambda|)$. Since $|\lambda|$ is finite, $g \in L^1(X, |\lambda|)$. Therefore, there exists a sequence of functions in $C_0(X)$, $\{f_n\}$ such that $f_n \to \overline{g}$ in $L^1(X, |\lambda|)$. Therefore, there exists a subsequence, still denoted by $\{f_n\}$ such that $f_n(x) \to \overline{g}(x)$ $|\lambda|$ a.e. also. But since $|\overline{g}(x)| = 1$ a.e. it follows that $h_n(x) \equiv \frac{f_n(x)}{|f_n(x)| + \frac{1}{n}}$ also converges pointwise $|\lambda|$ a.e. Then from the dominated convergence theorem and 13.18

$$||L|| \geq \lim_{n\to\infty} \int_X h_n gd|\lambda| = |\lambda|(X).$$

Also, if $||f||_{C_0(X)} \leq 1$, then

$$|L(f)| = \left| \int_X fgd|\lambda| \right| \leq \int_X |f| \, d|\lambda| \leq |\lambda|(X) \, ||f||_{C_0(X)}$$

and so $||L|| \leq |\lambda|(X)$. This proves everything but uniqueness.

Suppose λ and λ_1 both work. Then for all $f \in C_0(X)$,

$$0 = \int_X fd(\lambda - \lambda_1) = \int_X fhd|\lambda - \lambda_1|$$

where $hd|\lambda - \lambda_1|$ is the polar decomposition for $\lambda - \lambda_1$. By Lemma 13.25 $\lambda - \lambda_1$ is regular and so, as above, there exists $\{f_n\}$ such that $|f_n| \leq 1$ and $f_n \to \overline{h}$ pointwise. Therefore, $\int_X d|\lambda - \lambda_1| = 0$ so $\lambda = \lambda_1$. This proves the theorem.

13.7 Exercises

1. Suppose μ is a vector measure having values in $\mathbb{R}^n$ or $\mathbb{C}^n$. Can you show that $|\mu|$ must be finite? **Hint:** You might define for each $\mathbf{e}_i$, one of the standard basis vectors, the real or complex measure, $\mu_{\mathbf{e}_i}$ given by $\mu_{\mathbf{e}_i}(E) \equiv \mathbf{e}_i \cdot \mu(E)$. Why would this approach not yield anything for an infinite dimensional normed linear space in place of $\mathbb{R}^n$?

2. The Riesz representation theorem of the L^p spaces can be used to prove a very interesting inequality. Let $r, p, q \in (1, \infty)$ satisfy

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Then

$$\frac{1}{q}=1+\frac{1}{r}-\frac{1}{p}>\frac{1}{r}$$

and so $r>q$. Let $\theta\in(0,1)$ be chosen so that $\theta r=q$. Then also

$$\frac{1}{r}=\left(\overbrace{1-\frac{1}{p'}}^{1/p+1/p'=1}\right)+\frac{1}{q}-1=\frac{1}{q}-\frac{1}{p'}$$

and so

$$\frac{\theta}{q}=\frac{1}{q}-\frac{1}{p'}$$

which implies $p'(1-\theta)=q$. Now let $f\in L^{p}(\mathbb{R}^{n})$, $g\in L^{q}(\mathbb{R}^{n})$, $f,g\geq 0$. Justify the steps in the following argument using what was just shown that $\theta r=q$ and $p'(1-\theta)=q$. Let

$$h\in L^{r'}(\mathbb{R}^{n}).\left(\frac{1}{r}+\frac{1}{r'}=1\right)$$

$$\left|\int f*g(\mathbf{x})\,h(\mathbf{x})\,dx\right|=\left|\int\int f(\mathbf{y})\,g(\mathbf{x}-\mathbf{y})\,h(\mathbf{x})\,dxdy\right|.$$

$$\leq\int\int|f(\mathbf{y})|\,|g(\mathbf{x}-\mathbf{y})|^{\theta}\,|g(\mathbf{x}-\mathbf{y})|^{1-\theta}\,|h(\mathbf{x})|\,dydx$$

$$\leq\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}\,|h(\mathbf{x})|\right)^{r'}dx\right)^{1/r'}\cdot$$

$$\left(\int\left(|f(\mathbf{y})|\,|g(\mathbf{x}-\mathbf{y})|^{\theta}\right)^{r}dx\right)^{1/r}dy$$

$$\leq\left[\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}\,|h(\mathbf{x})|\right)^{r'}dx\right)^{p'/r'}dy\right]^{1/p'}\cdot$$

$$\left[\int\left(\int\left(|f(\mathbf{y})|\,|g(\mathbf{x}-\mathbf{y})|^{\theta}\right)^{r}dx\right)^{p/r}dy\right]^{1/p}$$

$$\leq\left[\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}\,|h(\mathbf{x})|\right)^{p'}dy\right)^{r'/p'}dx\right]^{1/r'}\cdot$$

$$\left[\int|f(\mathbf{y})|^{p}\left(\int|g(\mathbf{x}-\mathbf{y})|^{\theta r}\,dx\right)^{p/r}dy\right]^{1/p}$$

$$= \left[\int |h(\mathbf{x})|^{r'} \left(\int |g(\mathbf{x}-\mathbf{y})|^{(1-\theta)p'} dy \right)^{r'/p'} dx \right]^{1/r'} ||g||_q^{q/r} ||f||_p$$

$$= ||g||_q^{q/r} ||g||_q^{q/p'} ||f||_p ||h||_{r'} = ||g||_q ||f||_p ||h||_{r'} . \tag{13.19}$$

Young's inequality says that

$$||f * g||_r \leq ||g||_q ||f||_p . \tag{13.20}$$

Therefore $||f * g||_r \leq ||g||_q ||f||_p$. How does this inequality follow from the above computation? Does 13.19 continue to hold if r, p, q are only assumed to be in $[1, \infty]$? Explain. Does 13.20 hold even if r, p, and q are only assumed to lie in $[1, \infty]$?

3. Suppose $(\Omega, \mu, \mathcal{S})$ is a finite measure space and that $\{f_n\}$ is a sequence of functions which converge weakly to 0 in $L^p(\Omega)$. Suppose also that $f_n(x) \to 0$ a.e. Show that then $f_n \to 0$ in $L^{p-\varepsilon}(\Omega)$ for every $p > \varepsilon > 0$.

4. Give an example of a sequence of functions in $L^\infty(-\pi, \pi)$ which converges weak $*$ to zero but which does not converge pointwise a.e. to zero. Convergence weak $*$ to 0 means that for every $g \in L^1(-\pi, \pi)$, $\int_{-\pi}^{\pi} g(t) f_n(t) dt \to 0$.

Integrals And Derivatives

14.1 The Fundamental Theorem Of Calculus

The version of the fundamental theorem of calculus found in Calculus has already been referred to frequently. It says that if f is a Riemann integrable function, the function

$$x \to \int_a^x f(t)\,dt,$$

has a derivative at every point where f is continuous. It is natural to ask what occurs for f in L^1. It is an amazing fact that the same result is obtained asside from a set of measure zero even though f, being only in L^1 may fail to be continuous anywhere. Proofs of this result are based on some form of the Vitali covering theorem presented above. In what follows, the measure space is $(\mathbb{R}^n, \mathcal{S}, m)$ where m is n-dimensional Lebesgue measure although the same theorems can be proved for arbitrary Radon measures [29]. To save notation, m is written in place of m_n.

By Lemma 8.7 on Page 162 and the completeness of m, the Lebesgue measurable sets are exactly those measurable in the sense of Caratheodory. Also, to save on notation m is also the name of the outer measure defined on all of $\mathcal{P}(\mathbb{R}^n)$ which is determined by m_n. Recall

$$B(\mathbf{p}, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{p}| < r\}. \tag{14.1}$$

Also define the following.

$$\text{If } B = B(\mathbf{p}, r), \text{ then } \widehat{B} = B(\mathbf{p}, 5r). \tag{14.2}$$

The first version of the Vitali covering theorem presented above will now be used to establish the fundamental theorem of calculus. The space of locally integrable functions is the most general one for which the maximal function defined below makes sense.

Definition 14.1 *$f \in L^1_{loc}(\mathbb{R}^n)$ means $f\mathcal{X}_{B(0,R)} \in L^1(\mathbb{R}^n)$ for all $R > 0$. For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy Littlewood Maximal Function, Mf, is defined by*

$$Mf(\mathbf{x}) \equiv \sup_{r>0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y})|dy.$$

Theorem 14.2 *If $f \in L^1(\mathbb{R}^n)$, then for $\alpha > 0$,*

$$\overline{m}([Mf > \alpha]) \leq \frac{5^n}{\alpha}||f||_1.$$

(Here and elsewhere, $[Mf > \alpha] \equiv \{\mathbf{x} \in \mathbb{R}^n : Mf(\mathbf{x}) > \alpha\}$ with other occurrences of $[\ ]$ being defined similarly.)

Proof: Let $S \equiv [Mf > \alpha]$. For $\mathbf{x} \in S$, choose $r_{\mathbf{x}} > 0$ with

$$\frac{1}{m(B(\mathbf{x}, r_{\mathbf{x}}))} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f|\, dm > \alpha.$$

The $r_{\mathbf{x}}$ are all bounded because

$$m(B(\mathbf{x}, r_{\mathbf{x}})) < \frac{1}{\alpha} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f|\, dm < \frac{1}{\alpha}||f||_1.$$

By the Vitali covering theorem, there are disjoint balls $B(\mathbf{x}_i, r_i)$ such that

$$S \subseteq \cup_{i=1}^{\infty} B(\mathbf{x}_i, 5r_i)$$

and

$$\frac{1}{m(B(\mathbf{x}_i, r_i))} \int_{B(\mathbf{x}_i, r_i)} |f|\, dm > \alpha.$$

Therefore

$$\begin{aligned} \overline{m}(S) &\leq \sum_{i=1}^{\infty} m(B(\mathbf{x}_i, 5r_i)) = 5^n \sum_{i=1}^{\infty} m(B(\mathbf{x}_i, r_i)) \\ &\leq \frac{5^n}{\alpha} \sum_{i=1}^{\infty} \int_{B(\mathbf{x}_i, r_i)} |f|\, dm \\ &\leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f|\, dm, \end{aligned}$$

the last inequality being valid because the balls $B(\mathbf{x}_i, r_i)$ are disjoint. This proves the theorem.

Note that at this point it is unknown whether S is measurable. This is why $\overline{m}(S)$ and not $m(S)$ is written.

The following is the fundamental theorem of calculus from elementary calculus.

Lemma 14.3 *Suppose g is a continuous function. Then for all $\mathbf{x}$,*

$$\lim_{r \to 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) dy = g(\mathbf{x}).$$

Proof: Note that

$$g(\mathbf{x}) = \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} g(\mathbf{x})\,dy$$

and so

$$\begin{aligned}
&\left| g(\mathbf{x}) - \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} g(\mathbf{y})dy \right| \\
= & \left| \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} (g(\mathbf{y}) - g(\mathbf{x}))\,dy \right| \\
\leq & \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\,dy.
\end{aligned}$$

Now by continuity of g at $\mathbf{x}$, there exists $r > 0$ such that if $|\mathbf{x}-\mathbf{y}| < r$, $|g(\mathbf{y}) - g(\mathbf{x})| < \varepsilon$. For such r, the last expression is less than

$$\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} \varepsilon dy < \varepsilon.$$

This proves the lemma.

Definition 14.4 *Let $f \in L^1(\mathbb{R}^k, m)$. A point, $\mathbf{x} \in \mathbb{R}^k$ is said to be a Lebesgue point if*

$$\limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\,dm = 0.$$

Note that if $\mathbf{x}$ is a Lebesgue point, then

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} f(\mathbf{y})\,dm = f(\mathbf{x}).$$

and so the symmetric derivative exists at all Lebesgue points.

Theorem 14.5 *(Fundamental Theorem of Calculus) Let $f \in L^1(\mathbb{R}^k)$. Then there exists a set of measure $0, N$, such that if $\mathbf{x} \notin N$, then*

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|dy = 0.$$

Proof: Let $\lambda > 0$ and let $\varepsilon > 0$. By density of $C_c(\mathbb{R}^k)$ in $L^1(\mathbb{R}^k, m)$ there exists $g \in C_c(\mathbb{R}^k)$ such that $||g - f||_{L^1(\mathbb{R}^k)} < \varepsilon$. Now since g is continuous,

$$\begin{aligned}
& \limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\,dm \\
= & \limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\,dm \\
& - \lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\,dm
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{r\to 0}\left(\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-f\left(\mathbf{x}\right)\right|-\left|g\left(\mathbf{y}\right)-g\left(\mathbf{x}\right)\right|dm\right) \\
&\leq \limsup_{r\to 0}\left(\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|\left|f\left(\mathbf{y}\right)-f\left(\mathbf{x}\right)\right|-\left|g\left(\mathbf{y}\right)-g\left(\mathbf{x}\right)\right|\right|dm\right) \\
&\leq \limsup_{r\to 0}\left(\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-g\left(\mathbf{y}\right)-\left(f\left(\mathbf{x}\right)-g\left(\mathbf{x}\right)\right)\right|dm\right) \\
&\leq \limsup_{r\to 0}\left(\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-g\left(\mathbf{y}\right)\right|dm\right)+\left|f\left(\mathbf{x}\right)-g\left(\mathbf{x}\right)\right| \\
&\leq M\left(\left[f-g\right]\right)\left(\mathbf{x}\right)+\left|f\left(\mathbf{x}\right)-g\left(\mathbf{x}\right)\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left[\limsup_{r\to 0}\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-f\left(\mathbf{x}\right)\right|dm>\lambda\right] \\
\subseteq\ &\left[M\left(\left[f-g\right]\right)>\frac{\lambda}{2}\right]\cup\left[\left|f-g\right|>\frac{\lambda}{2}\right]
\end{aligned}
$$

Now

$$
\begin{aligned}
\varepsilon &> \int\left|f-g\right|dm\geq\int_{\left[\left|f-g\right|>\frac{\lambda}{2}\right]}\left|f-g\right|dm \\
&\geq \frac{\lambda}{2}m\left(\left[\left|f-g\right|>\frac{\lambda}{2}\right]\right)
\end{aligned}
$$

This along with the weak estimate of Theorem 14.2 implies

$$
\begin{aligned}
&m\left(\left[\limsup_{r\to 0}\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-f\left(\mathbf{x}\right)\right|dm>\lambda\right]\right) \\
<\ &\left(\frac{2}{\lambda}5^k+\frac{2}{\lambda}\right)\left|\left|f-g\right|\right|_{L^1(\mathbb{R}^k)} \\
<\ &\left(\frac{2}{\lambda}5^k+\frac{2}{\lambda}\right)\varepsilon.
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, it follows

$$
m\left(\left[\limsup_{r\to 0}\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-f\left(\mathbf{x}\right)\right|dm>\lambda\right]\right)=0.
$$

Now let

$$
N=\left[\limsup_{r\to 0}\frac{1}{m\left(B\left(\mathbf{x},r\right)\right)}\int_{B(\mathbf{x},r)}\left|f\left(\mathbf{y}\right)-f\left(\mathbf{x}\right)\right|dm>0\right]
$$

and

$$N_n = \left[\limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm > \frac{1}{n}\right]$$

It was just shown that $m(N_n) = 0$. Also, $N = \cup_{n=1}^{\infty} N_n$. Therefore, $m(N) = 0$ also. It follows that for $\mathbf{x} \notin N$,

$$\limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm = 0$$

and this proves a.e. point is a Lebesgue point.

Of course it is sufficient to assume f is only in $L^1_{loc}(\mathbb{R}^k)$.

Corollary 14.6 *(Fundamental Theorem of Calculus) Let $f \in L^1_{loc}(\mathbb{R}^k)$. Then there exists a set of measure $0, N$, such that if $\mathbf{x} \notin N$, then*

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0.$$

Proof: Consider $B(\mathbf{0},n)$ where n is a positive integer. Then $f_n \equiv f\mathcal{X}_{B(\mathbf{0},n)} \in L^1(\mathbb{R}^k)$ and so there exists a set of measure 0, N_n such that if $\mathbf{x} \in B(\mathbf{0},n) \setminus N_n$, then

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f_n(\mathbf{y}) - f_n(\mathbf{x})| dy = \lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0.$$

Let $N = \cup_{n=1}^{\infty} N_n$. Then if $\mathbf{x} \notin N$, the above equation holds.

Corollary 14.7 *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(\mathbf{y}) dy = f(\mathbf{x}) \quad a.e.\ \mathbf{x}. \tag{14.3}$$

Proof:

$$\left| \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(\mathbf{y}) dy - f(\mathbf{x}) \right| \le \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dy$$

and the last integral converges to 0 a.e. $\mathbf{x}$.

Definition 14.8 *For N the set of Theorem 14.5 or Corollary 14.6, N^C is called the Lebesgue set or the set of Lebesgue points.*

The next corollary is a one dimensional version of what was just presented.

Corollary 14.9 *Let $f \in L^1(\mathbb{R})$ and let*

$$F(x) = \int_{-\infty}^{x} f(t) dt.$$

Then for a.e. x, $F'(x) = f(x)$.

Proof: For $h > 0$

$$\frac{1}{h}\int_{x}^{x+h}|f(y)-f(x)|dy \leq 2(\frac{1}{2h})\int_{x-h}^{x+h}|f(y)-f(x)|dy$$

By Theorem 14.5, this converges to 0 a.e. Similarly

$$\frac{1}{h}\int_{x-h}^{x}|f(y)-f(x)|dy$$

converges to 0 a.e. x.

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| \leq \frac{1}{h}\int_{x}^{x+h}|f(y)-f(x)|dy \tag{14.4}$$

and

$$\left|\frac{F(x)-F(x-h)}{h}-f(x)\right| \leq \frac{1}{h}\int_{x-h}^{x}|f(y)-f(x)|dy. \tag{14.5}$$

Now the expression on the right in 14.4 and 14.5 converges to zero for a.e. x. Therefore, by 14.4, for a.e. x the derivative from the right exists and equals $f(x)$ while from 14.5 the derivative from the left exists and equals $f(x)$ a.e. It follows

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x) \text{ a.e. } x$$

This proves the corollary.

14.2 Absolutely Continuous Functions

Definition 14.10 *Let $[a,b]$ be a closed and bounded interval and let $f:[a,b]\to\mathbb{R}$. Then f is said to be absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sum_{i=1}^{m}|y_i - x_i| < \delta$, then $\sum_{i=1}^{m}|f(y_i) - f(x_i)| < \varepsilon$.*

Definition 14.11 *A finite subset, P of $[a,b]$ is called a partition of $[x,y]\subseteq[a,b]$ if $P=\{x_0,x_1,\cdots,x_n\}$ where*

$$x = x_0 < x_1 < \cdots, < x_n = y.$$

For $f:[a,b]\to\mathbb{R}$ and $P=\{x_0,x_1,\cdots,x_n\}$ define

$$V_P[x,y] \equiv \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|.$$

Denoting by $\mathcal{P}[x,y]$ the set of all partitions of $[x,y]$ define

$$V[x,y] \equiv \sup_{P\in\mathcal{P}[x,y]} V_P[x,y].$$

For simplicity, $V[a,x]$ will be denoted by $V(x)$. It is called the total variation of the function, f.

There are some simple facts about the total variation of an absolutely continuous function, f which are contained in the next lemma.

Lemma 14.12 *Let f be an absolutely continuous function defined on $[a,b]$ and let V be its total variation function as described above. Then V is an increasing bounded function. Also if P and Q are two partitions of $[x,y]$ with $P\subseteq Q$, then $V_P[x,y]\leq V_Q[x,y]$ and if $[x,y]\subseteq[z,w]$,*

$$V[x,y]\leq V[z,w] \tag{14.6}$$

If $P=\{x_0,x_1,\cdots,x_n\}$ is a partition of $[x,y]$, then

$$V[x,y]=\sum_{i=1}^{n}V[x_i,x_{i-1}]. \tag{14.7}$$

Also if $y>x$,

$$V(y)-V(x)\geq|f(y)-f(x)| \tag{14.8}$$

and the function, $x\to V(x)-f(x)$ is increasing. The total variation function, V is absolutely continuous.

Proof: The claim that V is increasing is obvious as is the next claim about $P\subseteq Q$ leading to $V_P[x,y]\leq V_Q[x,y]$. To verify this, simply add in one point at a time and verify that from the triangle inequality, the sum involved gets no smaller. The claim that V is increasing consistent with set inclusion of intervals is also clearly true and follows directly from the definition.

Now let $t<V[x,y]$ where $P_0=\{x_0,x_1,\cdots,x_n\}$ is a partition of $[x,y]$. There exists a partition, P of $[x,y]$ such that $t<V_P[x,y]$. Without loss of generality it can be assumed that $\{x_0,x_1,\cdots,x_n\}\subseteq P$ since if not, you can simply add in the points of P_0 and the resulting sum for the total variation will get no smaller. Let P_i be those points of P which are contained in $[x_{i-1},x_i]$. Then

$$t<V_p[x,y]=\sum_{i=1}^{n}V_{P_i}[x_{i-1},x_i]\leq\sum_{i=1}^{n}V[x_{i-1},x_i].$$

Since $t<V[x,y]$ is arbitrary,

$$V[x,y]\leq\sum_{i=1}^{n}V[x_i,x_{i-1}] \tag{14.9}$$

Note that 14.9 does not depend on f being absolutely continuous. Suppose now that f is absolutely continuous. Let δ correspond to $\varepsilon=1$. Then if $[x,y]$ is an interval of length no larger than δ, the definition of absolute continuity implies

$$V[x,y]<1.$$

Then from 14.9

$$V[a, n\delta] \leq \sum_{i=1}^{n} V[a + (i-1)\delta, a + i\delta] < \sum_{i=1}^{n} 1 = n.$$

Thus V is bounded on $[a, b]$. Now let P_i be a partition of $[x_{i-1}, x_i]$ such that

$$V_{P_i}[x_{i-1}, x_i] > V[x_{i-1}, x_i] - \frac{\varepsilon}{n}$$

Then letting $P = \cup P_i$,

$$-\varepsilon + \sum_{i=1}^{n} V[x_{i-1}, x_i] < \sum_{i=1}^{n} V_{P_i}[x_{i-1}, x_i] = V_P[x, y] \leq V[x, y].$$

Since ε is arbitrary, 14.7 follows from this and 14.9.

Now let $x < y$

$$\begin{aligned} V(y) - f(y) - (V(x) - f(x)) &= V(y) - V(x) - (f(y) - f(x)) \\ &\geq V(y) - V(x) - |f(y) - f(x)| \geq 0. \end{aligned}$$

It only remains to verify that V is absolutely continuous.

Let $\varepsilon > 0$ be given and let δ correspond to $\varepsilon/2$ in the definition of absolute continuity applied to f. Suppose $\sum_{i=1}^{n} |y_i - x_i| < \delta$ and consider $\sum_{i=1}^{n} |V(y_i) - V(x_i)|$. By 14.9 this last equals $\sum_{i=1}^{n} V[x_i, y_i]$. Now let P_i be a partition of $[x_i, y_i]$ such that $V_{P_i}[x_i, y_i] + \frac{\varepsilon}{2n} > V[x_i, y_i]$. Then by the definition of absolute continuity,

$$\sum_{i=1}^{n} |V(y_i) - V(x_i)| = \sum_{i=1}^{n} V[x_i, y_i] \leq \sum_{i=1}^{n} V_{P_i}[x_i, y_i] + \eta < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

and shows V is absolutely continuous as claimed.

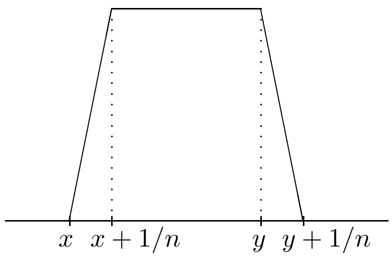
Lemma 14.13 *Suppose $f : [a, b] \to \mathbb{R}$ is absolutely continuous and increasing. Then f' exists a.e., is in $L^1([a, b])$, and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

Proof: Define L, a positive linear functional on $C([a, b])$ by

$$Lg \equiv \int_a^b g df$$

where this integral is the Riemann Stieltjes integral with respect to the integrating function, f. By the Riesz representation theorem for positive linear functionals, there exists a unique Radon measure, μ such that $Lg = \int g d\mu$. Now consider the following picture for $g_n \in C([a, b])$ in which g_n equals 1 for x between $x + 1/n$ and y.



Then $g_n(t) \to \mathcal{X}_{(x,y]}(t)$ pointwise. Therefore, by the dominated convergence theorem,

$$\mu((x,y]) = \lim_{n\to\infty} \int g_n d\mu.$$

However,

$$\begin{aligned}\left(f(y) - f\left(x + \frac{1}{n}\right)\right) &\leq \int g_n d\mu = \int_a^b g_n df \leq \left(f\left(y + \frac{1}{n}\right) - f(y)\right) \\ &+ \left(f(y) - f\left(x + \frac{1}{n}\right)\right) + \left(f\left(x + \frac{1}{n}\right) - f(x)\right)\end{aligned}$$

and so as $n \to \infty$ the continuity of f implies

$$\mu((x,y]) = f(y) - f(x).$$

Similarly, $\mu(x,y) = f(y) - f(y)$ and $\mu([x,y]) = f(y) - f(x)$, the argument used to establish this being very similar to the above. It follows in particular that

$$f(x) - f(a) = \int_{[a,x]} d\mu.$$

Note that up till now, no referrence has been made to the absolute continuity of f. Any increasing continuous function would be fine.

Now if E is a Borel set such that $m(E) = 0$, Then the outer regularity of m implies there exists an open set, V containing E such that $m(V) < \delta$ where δ corresponds to ε in the definition of absolute continuity of f. Then letting $\{I_k\}$ be the connected components of V it follows $E \subseteq \cup_{k=1}^\infty I_k$ with $\sum_k m(I_k) = m(V) < \delta$. Therefore, from absolute continuity of f, it follows that for $I_k = (a_k, b_k)$ and each n

$$\mu(\cup_{k=1}^n I_k) = \sum_{k=1}^n \mu(I_k) = \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$$

and so letting $n \to \infty$,

$$\mu(E) \leq \mu(V) = \sum_{k=1}^\infty |f(b_k) - f(a_k)| \leq \varepsilon.$$

Since ε is arbitrary, it follows $\mu(E) = 0$. Therefore, $\mu \ll m$ and so by the Radon Nikodym theorem there exists a unique $h \in L^1([a,b])$ such that

$$\mu(E) = \int_E h dm.$$

In particular,

$$\mu([a,x]) = f(x) - f(a) = \int_{[a,x]} h dm.$$

From the fundamental theorem of calculus $f'(x) = h(x)$ at every Lebesgue point of h. Therefore, writing in usual notation,

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

as claimed. This proves the lemma.

With the above lemmas, the following is the main theorem about absolutely continuous functions.

Theorem 14.14 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous if and only if $f'(x)$ exists a.e., $f' \in L^1([a,b])$ and*

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

Proof: Suppose first that f is absolutely continuous. By Lemma 14.12 the total variation function, V is absolutely continuous and $f(x) = V(x) - (V(x) - f(x))$ where both V and $V - f$ are increasing and absolutely continuous. By Lemma 14.13

$$\begin{aligned} f(x) - f(a) &= V(x) - V(a) - [(V(x) - f(x)) - (V(a) - f(a))] \\ &= \int_a^x V'(t)\,dt - \int_a^x (V-f)'(t)\,dt. \end{aligned}$$

Now f' exists and is in L^1 becasue $f = V - (V - f)$ and V and $V - f$ have derivatives in L^1. Therefore, $(V-f)' = V' - f'$ and so the above reduces to

$$f(x) - f(a) = \int_a^x f'(t)\,dt.$$

This proves one half of the theorem.

Now suppose $f' \in L^1$ and $f(x) = f(a) + \int_a^x f'(t)\,dt$. It is necessary to verify that f is absolutely continuous. But this follows easily from Lemma 7.45 on Page 151 which implies that a single function, f' is uniformly integrable. This lemma implies that if $\sum_i |y_i - x_i|$ is sufficiently small then

$$\sum_i \left| \int_{x_i}^{y_i} f'(t)\,dt \right| = \sum_i |f(y_i) - f(x_i)| < \varepsilon.$$

14.3 Differentiation Of Measures With Respect To Lebesgue Measure

Recall the Vitali covering theorem in Corollary 9.20 on Page 209.

Corollary 14.15 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Definition 14.16 *Let μ be a Radon mesure defined on $\mathbb{R}^n$. Then*

$$\frac{d\mu}{dm}(\mathbf{x}) \equiv \lim_{r\to 0} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}$$

whenever this limit exists.

It turns out this limit exists for m a.e. $\mathbf{x}$. To verify this here is another definition.

Definition 14.17 *Let $f(r)$ be a function having values in $[-\infty,\infty]$. Then*

$$\begin{aligned} \limsup_{r\to 0+} f(r) &\equiv \lim_{r\to 0} (\sup\{f(t) : t \in [0,r]\}) \\ \liminf_{r\to 0+} f(r) &\equiv \lim_{r\to 0} (\inf\{f(t) : t \in [0,r]\}) \end{aligned}$$

This is well defined because the function $r \to \inf\{f(t) : t \in [0,r]\}$ is increasing and $r \to \sup\{f(t) : t \in [0,r]\}$ is decreasing. Also note that $\lim_{r\to 0+} f(r)$ exists if and only if

$$\limsup_{r\to 0+} f(r) = \liminf_{r\to 0+} f(r)$$

and if this happens

$$\lim_{r\to 0+} f(r) = \liminf_{r\to 0+} f(r) = \limsup_{r\to 0+} f(r).$$

The claims made in the above definition follow immediately from the definition of what is meant by a limit in $[-\infty,\infty]$ and are left for the reader.

Theorem 14.18 *Let μ be a Borel measure on $\mathbb{R}^n$ then $\frac{d\mu}{dm}(\mathbf{x})$ exists in $[-\infty,\infty]$ m a.e.*

Proof: Let $p < q$ and let p, q be rational numbers. Define $N_{pq}(M)$ as

$$\left\{\mathbf{x} \in B(\mathbf{0},M) \text{ such that } \limsup_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))} > q > p > \liminf_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}\right\},$$

Also define N_{pq} as

$$\left\{\mathbf{x} \in \mathbb{R}^n \text{ such that } \limsup_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))} > q > p > \liminf_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}\right\},$$

and N as

$$\left\{\mathbf{x}\in\mathbb{R}^n \text{ such that } \limsup_{r\to 0+}\frac{\mu\left(B\left(\mathbf{x},r\right)\right)}{m\left(B\left(\mathbf{x},r\right)\right)}>\liminf_{r\to 0+}\frac{\mu\left(B\left(\mathbf{x},r\right)\right)}{m\left(B\left(\mathbf{x},r\right)\right)}\right\}.$$

I will show $\overline{m}\left(N_{pq}\left(M\right)\right)=0$. Use outer regularity to obtain an open set, V containing $N_{pq}\left(M\right)$ such that

$$\overline{m}\left(N_{pq}\left(M\right)\right)+\varepsilon>m\left(V\right).$$

From the definition of $N_{pq}\left(M\right)$, it follows that for each $\mathbf{x}\in N_{pq}\left(M\right)$ there exist arbitrarily small $r>0$ such that

$$\frac{\mu\left(B\left(\mathbf{x},r\right)\right)}{m\left(B\left(\mathbf{x},r\right)\right)}<p.$$

Only consider those r which are small enough to be contained in $B\left(\mathbf{0},M\right)$ so that the collection of such balls has bounded radii. This is a Vitali cover of $N_{pq}\left(M\right)$ and so by Corollary 14.15 there exists a sequence of disjoint balls of this sort, $\left\{B_i\right\}_{i=1}^{\infty}$ such that

$$\mu\left(B_i\right)<pm\left(B_i\right),\ \overline{m}\left(N_{pq}\left(M\right)\setminus\cup_{i=1}^{\infty}B_i\right)=0. \tag{14.10}$$

Now for $\mathbf{x}\in N_{pq}\left(M\right)\cap\left(\cup_{i=1}^{\infty}B_i\right)$ (most of $N_{pq}\left(M\right)$), there exist arbitrarily small balls, $B\left(\mathbf{x},r\right)$, such that $B\left(\mathbf{x},r\right)$ is contained in some set of $\left\{B_i\right\}_{i=1}^{\infty}$ and

$$\frac{\mu\left(B\left(\mathbf{x},r\right)\right)}{m\left(B\left(\mathbf{x},r\right)\right)}>q.$$

This is a Vitali cover of $N_{pq}\left(M\right)\cap\left(\cup_{i=1}^{\infty}B_i\right)$ and so there exists a sequence of disjoint balls of this sort, $\left\{B_j'\right\}_{j=1}^{\infty}$ such that

$$\overline{m}\left(\left(N_{pq}\left(M\right)\cap\left(\cup_{i=1}^{\infty}B_i\right)\right)\setminus\cup_{j=1}^{\infty}B_j'\right)=0,\ \mu\left(B_j'\right)>qm\left(B_j'\right). \tag{14.11}$$

It follows from 14.10 and 14.11 that

$$\overline{m}\left(N_{pq}\left(M\right)\right)\leq\overline{m}\left(\left(N_{pq}\left(M\right)\cap\left(\cup_{i=1}^{\infty}B_i\right)\right)\right)\leq m\left(\cup_{j=1}^{\infty}B_j'\right) \tag{14.12}$$

Therefore,

$$\begin{aligned}\sum_j\mu\left(B_j'\right) &> q\sum_j m\left(B_j'\right)\geq q\overline{m}\left(N_{pq}\left(M\right)\cap\left(\cup_iB_i\right)\right)=q\overline{m}\left(N_{pq}\left(M\right)\right)\\ &\geq p\overline{m}\left(N_{pq}\left(M\right)\right)\geq p\left(m\left(V\right)-\varepsilon\right)\geq p\sum_i m\left(B_i\right)-p\varepsilon\\ &\geq \sum_i\mu\left(B_i\right)-p\varepsilon\geq\sum_j\mu\left(B_j'\right)-p\varepsilon.\end{aligned}$$

It follows

$$p\varepsilon\geq\left(q-p\right)\overline{m}\left(N_{pq}\left(M\right)\right)$$

Since ε is arbitrary, $m(N_{pq}(M)) = 0$. Now $N_{pq} \subseteq \cup_{M=1}^{\infty} N_{pq}(M)$ and so $m(N_{pq}) = 0$. Now

$$N = \cup_{p.q \in \mathbb{Q}} N_{pq}$$

and since this is a countable union of sets of measure zero, $m(N) = 0$ also. This proves the theorem.

From Theorem 13.8 on Page 299 it follows that if μ is a complex measure then $|\mu|$ is a finite measure. This makes possible the following definition.

Definition 14.19 *Let μ be a real measure. Define the following measures. For E a measurable set,*

$$\begin{aligned} \mu^+(E) &\equiv \frac{1}{2}(|\mu| + \mu)(E), \\ \mu^-(E) &\equiv \frac{1}{2}(|\mu| - \mu)(E). \end{aligned}$$

These are measures thanks to Theorem 13.7 on Page 297 and $\mu^+ - \mu^- = \mu$. These measures have values in $[0, \infty)$. They are called the positive and negative parts of μ respectively. For μ a complex measure, define $\operatorname{Re}\mu$ *and* $\operatorname{Im}\mu$ *by*

$$\begin{aligned} \operatorname{Re}\mu(E) &\equiv \frac{1}{2}\left(\mu(E) + \overline{\mu(E)}\right) \\ \operatorname{Im}\mu(E) &\equiv \frac{1}{2i}\left(\mu(E) - \overline{\mu(E)}\right) \end{aligned}$$

Then $\operatorname{Re}\mu$ *and* $\operatorname{Im}\mu$ *are both real measures. Thus for μ a complex measure,*

$$\begin{aligned} \mu &= \operatorname{Re}\mu^+ - \operatorname{Re}\mu^- + i\left(\operatorname{Im}\mu^+ - \operatorname{Im}\mu^-\right) \\ &= \nu_1 - \nu_1 + i(\nu_3 - \nu_4) \end{aligned}$$

where each ν_i is a real measure having values in $[0, \infty)$.

Then there is an obvious corollary to Theorem 14.18.

Corollary 14.20 *Let μ be a complex Borel measure on $\mathbb{R}^n$. Then $\frac{d\mu}{dm}(\mathbf{x})$ exists a.e.*

Proof: Letting ν_i be defined in Definition 14.19. By Theorem 14.18, for m a.e. $\mathbf{x}$, $\frac{d\nu_i}{dm}(\mathbf{x})$ exists. This proves the corollary because μ is just a finite sum of these ν_i.

Theorem 13.2 on Page 291, the Radon Nikodym theorem, implies that if you have two finite measures, μ and λ, you can write λ as the sum of a measure absolutely continuous with respect to μ and one which is singular to μ in a unique way. The next topic is related to this. It has to do with the differentiation of a measure which is singular with respect to Lebesgue measure.

Theorem 14.21 *Let μ be a Radon measure on $\mathbb{R}^n$ and suppose there exists a μ measurable set, N such that for all Borel sets, E, $\mu(E) = \mu(E \cap N)$ where $\overline{m}(N) = 0$. Then*

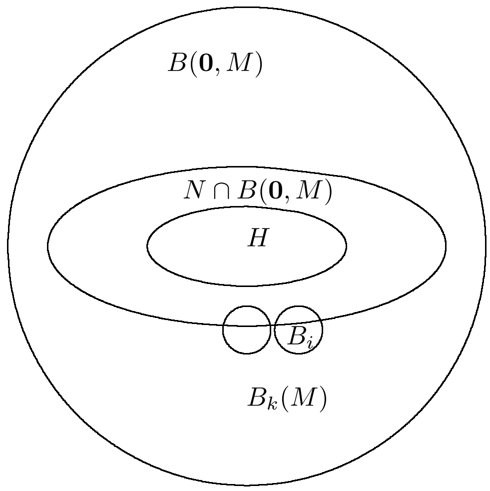
$$\frac{d\mu}{dm}(\mathbf{x}) = 0 \ m \ a.e.$$

Proof: For $k \in \mathbb{N}$, let

$$\begin{aligned} B_k(M) &\equiv \left\{ \mathbf{x} \in N^C : \limsup_{r \to 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k} \right\} \cap B(\mathbf{0}, M), \\ B_k &\equiv \left\{ \mathbf{x} \in N^C : \limsup_{r \to 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k} \right\}, \\ B &\equiv \left\{ \mathbf{x} \in N^C : \limsup_{r \to 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > 0 \right\}. \end{aligned}$$

Let $\varepsilon > 0$. Since μ is regular, there exists H, a compact set such that $H \subseteq N \cap B(\mathbf{0}, M)$ and

$$\mu(N \cap B(\mathbf{0}, M) \setminus H) < \varepsilon.$$



For each $\mathbf{x} \in B_k(M)$, there exist arbitrarily small $r > 0$ such that $B(\mathbf{x}, r) \subseteq B(\mathbf{0}, M) \setminus H$ and

$$\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k}. \tag{14.13}$$

Two such balls are illustrated in the above picture. This is a Vitali cover of $B_k(M)$ and so there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^{\infty}$ such that

$\overline{m}(B_k(M) \setminus \cup_i B_i) = 0$. Therefore,

$$\begin{aligned}\overline{m}(B_k(M)) &\leq \overline{m}(B_k(M) \cap (\cup_i B_i)) \leq \sum_i \overline{m}(B_i) \leq k \sum_i \mu(B_i) \\ &= k \sum_i \mu(B_i \cap N) = k \sum_i \mu(B_i \cap N \cap B(\mathbf{0}, M)) \\ &\leq k\mu(N \cap B(\mathbf{0}, M) \setminus H) < \varepsilon k\end{aligned}$$

Since ε was arbitrary, this shows $\overline{m}(B_k(M)) = 0$.

Therefore,

$$\overline{m}(B_k) \leq \sum_{M=1}^{\infty} \overline{m}(B_k(M)) = 0$$

and $\overline{m}(B) \leq \sum_k \overline{m}(B_k) = 0$. Since $\overline{m}(N) = 0$, this proves the theorem.

It is easy to obtain a different version of the above theorem. This is done with the aid of the following lemma.

Lemma 14.22 *Suppose μ is a Borel measure on $\mathbb{R}^n$ having values in $[0, \infty)$. Then there exists a Radon measure, μ_1 such that $\mu_1 = \mu$ on all Borel sets.*

Proof: By assumption, $\mu(\mathbb{R}^n) < \infty$ and so it is possible to define a positive linear functional, L on $C_c(\mathbb{R}^n)$ by

$$Lf \equiv \int f d\mu.$$

By the Riesz representation theorem for positive linear functionals of this sort, there exists a unique Radon measure, μ_1 such that for all $f \in C_c(\mathbb{R}^n)$,

$$\int f d\mu_1 = Lf = \int f d\mu.$$

Now let V be an open set and let $K_k \equiv \left\{\mathbf{x} \in V : \operatorname{dist}\left(\mathbf{x}, V^C\right) \leq 1/k\right\} \cap \overline{B(\mathbf{0},k)}$. Then $\{K_k\}$ is an incresing sequence of compact sets whose union is V. Let $K_k \prec f_k \prec V$. Then $f_k(\mathbf{x}) \to \mathcal{X}_V(\mathbf{x})$ for every $\mathbf{x}$. Therefore,

$$\mu_1(V) = \lim_{k\to\infty} \int f_k d\mu_1 = \lim_{k\to\infty} \int f_k d\mu = \mu(V)$$

and so $\mu = \mu_1$ on open sets. Now if K is a compact set, let

$$V_k \equiv \{\mathbf{x} \in \mathbb{R}^n : \operatorname{dist}(\mathbf{x}, K) < 1/k\}.$$

Then V_k is an open set and $\cap_k V_k = K$. Letting $K \prec f_k \prec V_k$, it follows that $f_k(\mathbf{x}) \to \mathcal{X}_K(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Therefore, by the dominated convergence theorem with a dominating function, $\mathcal{X}_{\mathbb{R}^n}$

$$\mu_1(K) = \lim_{k\to\infty} \int f_k d\mu_1 = \lim_{k\to\infty} \int f_k d\mu = \mu(K)$$

and so μ and μ_1 are equal on all compact sets. It follows $\mu = \mu_1$ on all countable unions of compact sets and countable intersections of open sets.

Now let E be a Borel set. By regularity of μ_1, there exist sets, H and G such that H is the countable union of an increasing sequence of compact sets, G is the countable intersection of a decreasing sequence of open sets, $H \subseteq E \subseteq G$, and $\mu_1(H) = \mu_1(G) = \mu_1(E)$. Therefore,

$$\mu_1(H) = \mu(H) \leq \mu(E) \leq \mu(G) = \mu_1(G) = \mu_1(E) = \mu_1(H).$$

therefore, $\mu(E) = \mu_1(E)$ and this proves the lemma.

Corollary 14.23 *Suppose μ is a complex Borel measure defined on $\mathbb{R}^n$ for which there exists a μ measurable set, N such that for all Borel sets, E, $\mu(E) = \mu(E \cap N)$ where $\overline{m}(N) = 0$. Then*

$$\frac{d\mu}{dm}(\mathbf{x}) = 0 \ m \ a.e.$$

Proof: Each of $\operatorname{Re}\mu^+, \operatorname{Re}\mu^-, \operatorname{Im}\mu^+$, and $\operatorname{Im}\mu^-$ are real measures having values in $[0,\infty)$ and so by Lemma 14.22 each is a Radon measure having the same property that μ has in terms of being supported on a set of m measure zero. Therefore, for ν equal to any of these, $\frac{d\nu}{dm}(\mathbf{x}) = 0$ m a.e. This proves the corollary.

14.4 Exercises

1. Let E be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if

$$\lim_{r\to 0} \frac{m_n(E \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} = 1.$$

Show that a.e. point of E is a point of density. **Hint:** The numerator of the above quotient is $\int_{B(\mathbf{x},r)} \mathcal{X}_E(\mathbf{x})\, dm$. Now consider the fundamental theorem of calculus.

2. Show that if $f \in L^1_{loc}(\mathbb{R}^n)$ and $\int f\phi dx = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, then $f(\mathbf{x}) = 0$ a.e.

3. $\uparrow$Now suppose that for $u \in L^1_{loc}(\mathbb{R}^n)$, $w \in L^1_{loc}(\mathbb{R}^n)$ is a weak partial derivative of u with respect to x_i if whenever $h_k \to 0$ it follows that for all $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{(u(\mathbf{x} + h_k \mathbf{e}_i) - u(\mathbf{x}))}{h_k} \phi(\mathbf{x})\, dx = \int_{\mathbb{R}^n} w(\mathbf{x})\, \phi(\mathbf{x})\, dx. \tag{14.14}$$

and in this case, write $w = u_{,i}$. Using Problem 2 show this is well defined.

4. $\uparrow$Show that $w \in L^1_{loc}(\mathbb{R}^n)$ is a weak partial derivative of u with respect to x_i in the sense of Problem 3 if and only if for all $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$-\int u(\mathbf{x})\, \phi_{,i}(\mathbf{x})\, dx = \int w(\mathbf{x})\, \phi(\mathbf{x})\, dx.$$

5. If $f \in L^1_{loc}(\mathbb{R}^n)$, the fundamental theorem of calculus says

$$\lim_{r\to 0} \frac{1}{m_n(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dy = 0$$

for a.e. $\mathbf{x}$. Suppose now that $\{E_k\}$ is a sequence of measurable sets and r_k is a sequence of positive numbers converging to zero such that $E_k \subseteq B(\mathbf{x}, r_k)$ and $m_n(E_k) \geq c m_n(B(\mathbf{x}, r_k))$ where c is some positive number. Then show

$$\lim_{k\to\infty} \frac{1}{m_n(E_k)} \int_{E_k} |f(\mathbf{y}) - f(\mathbf{x})|\, dy = 0$$

for a.e. $\mathbf{x}$. Such a sequence of sets is known as a regular family of sets [40] and is said to converge regularly to $\mathbf{x}$ in [27].

6. Let f be in $L^1_{loc}(\mathbb{R}^n)$. Show Mf is Borel measurable. **Hint:** First consider the function, $F_r(\mathbf{x}) \equiv \frac{1}{m_n(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{x})|\, dm_n$. Argue F_r is continuous. Then $Mf(\mathbf{x}) = \sup_{r>0} F_r(\mathbf{x})$.

7. If $f \in L^p, 1 < p < \infty$, show $Mf \in L^p$ and

$$||Mf||_p \leq A(p,n)||f||_p.$$

Hint: Let

$$f_1(\mathbf{x}) \equiv \begin{cases} f(\mathbf{x}) \text{ if } |f(\mathbf{x})| > \alpha/2, \\ 0 \text{ if } |f(\mathbf{x})| \leq \alpha/2. \end{cases}$$

Argue $[Mf(\mathbf{x}) > \alpha] \subseteq [Mf_1(\mathbf{x}) > \alpha/2]$. Then use the distribution function. Recall why

$$\int (Mf)^p dx = \int_0^\infty p\alpha^{p-1} m([Mf > \alpha]) d\alpha$$

$$\leq \int_0^\infty p\alpha^{p-1} m([Mf_1 > \alpha/2]) d\alpha.$$

Now use the fundamental estimate satisfied by the maximal function and Fubini's Theorem as needed.

8. Show $|f(\mathbf{x})| \leq Mf(\mathbf{x})$ at every Lebesgue point of f whenever $f \in L^1_{loc}(\mathbb{R}^n)$.

9. In the proof of the Vitali covering theorem, Theorem 9.11 on Page 202, there is nothing sacred about the constant $\frac{1}{2}$. Go through the proof replacing this constant with λ where $\lambda \in (0,1)$. Show that it follows that for every $\delta > 0$, the conclusion of the Vitali covering theorem can be obtained with 5 replaced by $(3+\delta)$ in the definition of $\widehat{B}$. In this context, see Rudin [36] who proves a different version of the Vitali covering theorem involving only finite covers and gets the constant 3. See also Problem 10.

10. Suppose A is covered by a finite collection of Balls, $\mathcal{F}$. Show that then there exists a disjoint collection of these balls, $\{B_i\}_{i=1}^p$, such that $A \subseteq \cup_{i=1}^p \widehat{B}_i$ where 5 can be replaced with 3 in the definition of $\widehat{B}$. **Hint:** Since the collection of balls is finite, they can be arranged in order of decreasing radius.

11. Suppose E is a Lebesgue measurable set which has positive measure and let B be an arbitrary open ball and let D be a set dense in $\mathbb{R}^n$. Establish the result of Smítal, [10]which says that under these conditions, $\overline{m_n}((E+D)\cap B) = m_n(B)$ where here $\overline{m_n}$ denotes the outer measure determined by m_n. Is this also true for X, an arbitrary possibly non measurable set replacing E in which $\overline{m_n}(X) > 0$? **Hint:** Let $\mathbf{x}$ be a point of density of E and let D' denote those elements of D, $\mathbf{d}$, such that $\mathbf{d}+\mathbf{x} \in B$. Thus D' is dense in B. Now use translation invariance of Lebesgue measure to verify there exists, $R > 0$ such that if $r < R$, we have the following holding for $\mathbf{d} \in D'$ and $r_{\mathbf{d}} < R$.
$$\overline{m_n}((E+D)\cap B(\mathbf{x}+\mathbf{d}, r_{\mathbf{d}})) \geq$$
$$m_n((E+\mathbf{d})\cap B(\mathbf{x}+\mathbf{d}, r_{\mathbf{d}})) \geq (1-\varepsilon)\, m_n(B(\mathbf{x}+\mathbf{d}, r_{\mathbf{d}})).$$
Argue the balls, $m_n(B(\mathbf{x}+\mathbf{d}, r_{\mathbf{d}}))$, form a Vitali cover of B.

12. Consider the construction employed to obtain the Cantor set, but instead of removing the middle third interval, remove only enough that the sum of the lengths of all the open intervals which are removed is less than one. That which remains is called a fat Cantor set. Show it is a compact set which has measure greater than zero which contains no interval and has the property that every point is a limit point of the set. Let P be such a fat Cantor set and consider
$$f(x) = \int_0^x \mathcal{X}_{P^C}(t)\, dt.$$
Show that f is a strictly increasing function which has the property that its derivative equals zero on a set of positive measure.

13. Let f be a function defined on an interval, (a,b). The Dini derivates are defined as
$$\begin{aligned} D_+ f(x) &\equiv \lim\inf_{h\to 0+} \frac{f(x+h)-f(x)}{h}, \\ D^+ f(x) &\equiv \lim\sup_{h\to 0+} \frac{f(x+h)-f(x)}{h} \end{aligned}$$
$$\begin{aligned} D_- f(x) &\equiv \lim\inf_{h\to 0+} \frac{f(x)-f(x-h)}{h}, \\ D^- f(x) &\equiv \lim\sup_{h\to 0+} \frac{f(x)-f(x-h)}{h}. \end{aligned}$$

Suppose f is continuous on (a,b) and for all $x \in (a,b)$, $D_+ f(x) \geq 0$. Show that then f is increasing on (a,b). **Hint:** Consider the function, $H(x) \equiv f(x)(d-c) - x(f(d) - f(c))$ where $a < c < d < b$. Thus $H(c) = H(d)$. Also it is easy to see that H cannot be constant if $f(d) < f(c)$ due to the assumption that $D_+ f(x) \geq 0$. If there exists $x_1 \in (a,b)$ where $H(x_1) > H(c)$, then let $x_0 \in (c,d)$ be the point where the maximum of f occurs. Consider $D_+ f(x_0)$. If, on the other hand, $H(x) < H(c)$ for all $x \in (c,d)$, then consider $D_+ H(c)$.

14. $\uparrow$ Suppose in the situation of the above problem we only know

$$D_+ f(x) \geq 0 \text{ a.e.}$$

Does the conclusion still follow? What if we only know $D_+ f(x) \geq 0$ for every x outside a countable set? **Hint:** In the case of $D_+ f(x) \geq 0$,consider the bad function in the exercises for the chapter on the construction of measures which was based on the Cantor set. In the case where $D_+ f(x) \geq 0$ for all but countably many x, by replacing $f(x)$ with $\widetilde{f}(x) \equiv f(x) + \varepsilon x$, consider the situation where $D_+ \widetilde{f}(x) > 0$ for all but countably many x. If in this situation, $\widetilde{f}(c) > \widetilde{f}(d)$ for some $c < d$, and $y \in \left(\widetilde{f}(d), \widetilde{f}(c)\right)$,let

$$z \equiv \sup\left\{x \in [c,d] : \widetilde{f}(x) > y_0\right\}.$$

Show that $\widetilde{f}(z) = y_0$ and $D_+ \widetilde{f}(z) \leq 0$. Conclude that if $\widetilde{f}$ fails to be increasing, then $D_+ \widetilde{f}(z) \leq 0$ for uncountably many points, z. Now draw a conclusion about f.

15. $\uparrow$ Let $f : [a,b] \to \mathbb{R}$ be increasing. Show

$$m\left(\overbrace{\left[D^+ f(x) > q > p > D_+ f(x)\right]}^{N_{pq}}\right) = 0 \tag{14.15}$$

and conclude that aside from a set of measure zero, $D^+ f(x) = D_+ f(x)$. Similar reasoning will show $D^- f(x) = D_- f(x)$ a.e. and $D^+ f(x) = D_- f(x)$ a.e. and so off some set of measure zero, we have

$$D_- f(x) = D^- f(x) = D^+ f(x) = D_+ f(x)$$

which implies the derivative exists and equals this common value. **Hint**: To show 14.15, let U be an open set containing N_{pq} such that $\overline{m}(N_{pq}) + \varepsilon > m(U)$. For each $x \in N_{pq}$ there exist $y > x$ arbitrarily close to x such that

$$f(y) - f(x) < p(y - x).$$

Thus the set of such intervals, $\{[x, y]\}$ which are contained in U constitutes a Vitali cover of N_{pq}. Let $\{[x_i, y_i]\}$ be disjoint and

$$\overline{m}\left(N_{pq} \setminus \cup_i [x_i, y_i]\right) = 0.$$

Now let $V \equiv \cup_i (x_i, y_i)$. Then also we have

$$\overline{m}\left(N_{pq} \setminus \overbrace{\cup_i (x_i, y_i)}^{=V}\right) = 0.$$

and so $\overline{m}(N_{pq} \cap V) = \overline{m}(N_{pq})$. For each $x \in N_{pq} \cap V$, there exist $y > x$ arbitrarily close to x such that

$$f(y) - f(x) > q(y - x).$$

Thus the set of such intervals, $\{[x', y']\}$ which are contained in V is a Vitali cover of $N_{pq} \cap V$. Let $\{[x_i', y_i']\}$ be disjoint and

$$\overline{m}\left(N_{pq} \cap V \setminus \cup_i [x_i', y_i']\right) = 0.$$

Then verify the following:

$$\begin{aligned}
\sum_i f(y_i') - f(x_i') &> q \sum_i (y_i' - x_i') \geq q\overline{m}(N_{pq} \cap V) = q\overline{m}(N_{pq}) \\
&\geq p\overline{m}(N_{pq}) > p(m(U) - \varepsilon) \geq p \sum_i (y_i - x_i) - p\varepsilon \\
&\geq \sum_i (f(y_i) - f(x_i)) - p\varepsilon \geq \sum_i f(y_i') - f(x_i') - p\varepsilon
\end{aligned}$$

and therefore, $(q - p)\overline{m}(N_{pq}) \leq p\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves that there is a right derivative a.e. A similar argument does the other cases.

16. Suppose $|f(x) - f(y)| \leq K|x - y|$. Show there exists $g \in L^\infty(\mathbb{R})$, $||g||_\infty \leq K$, and
$$f(y) - f(x) = \int_x^y g(t)dt.$$
Hint: Let $F(x) = Kx + f(x)$ and let λ be the measure representing $\int f dF$. Show $\lambda \ll m$.

17. We $P = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < \cdots < x_n = b$. Define
$$\sum_P |f(x_i) - f(x_{i-1})| \equiv \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$
A function, $f : [a, b] \to \mathbb{R}$ is said to be of bounded variation if
$$\sup_P \left\{\sum_P |f(x_i) - f(x_{i-1})|\right\} < \infty.$$

Show that whenever f is of bounded variation it can be written as the difference of two increasing functions. Explain why such bounded variation functions have derivatives a.e.

Fourier Transforms

15.1 An Algebra Of Special Functions

First recall the following definition of a polynomial.

Definition 15.1 *$\alpha = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in \mathbb{R}^n$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

and f a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

A polynomial in n variables of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} a_\alpha \mathbf{x}^\alpha.$$

Here α is a multi-index as just described and $a_\alpha \in \mathbb{C}$. Also define for $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index

$$D^\alpha f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Definition 15.2 *Define $\mathcal{G}_1$ to be the functions of the form $p(\mathbf{x}) e^{-a|\mathbf{x}|^2}$ where $a > 0$ and $p(\mathbf{x})$ is a polynomial. Let $\mathcal{G}$ be all finite sums of functions in $\mathcal{G}_1$. Thus $\mathcal{G}$ is an algebra of functions which has the property that if $f \in \mathcal{G}$ then $\overline{f} \in \mathcal{G}$.*

It is always assumed, unless stated otherwise that the measure will be Lebesgue measure.

Lemma 15.3 *$\mathcal{G}$ is dense in $C_0(\mathbb{R}^n)$ with respect to the norm,*

$$||f||_\infty \equiv \sup \{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\}$$

Proof: By the Weierstrass approximation theorem, it suffices to show $\mathcal{G}$ separates the points and annihilates no point. It was already observed in the above definition that $\overline{f} \in \mathcal{G}$ whenever $f \in \mathcal{G}$. If $\mathbf{y}_1 \neq \mathbf{y}_2$ suppose first that $|\mathbf{y}_1| \neq |\mathbf{y}_2|$. Then in this case, you can let $f(\mathbf{x}) \equiv e^{-|\mathbf{x}|^2}$ and $f \in \mathcal{G}$ and $f(\mathbf{y}_1) \neq f(\mathbf{y}_2)$. If $|\mathbf{y}_1| = |\mathbf{y}_2|$, then suppose $y_{1k} \neq y_{2k}$. This must happen for some k because $\mathbf{y}_1 \neq \mathbf{y}_2$. Then let $f(\mathbf{x}) \equiv x_k e^{-|\mathbf{x}|^2}$. Thus $\mathcal{G}$ separates points. Now $e^{-|\mathbf{x}|^2}$ is never equal to zero and so $\mathcal{G}$ annihilates no point of $\mathbb{R}^n$. This proves the lemma.

These functions are clearly quite specialized. Therefore, the following theorem is somewhat surprising.

Theorem 15.4 *For each $p \geq 1, p < \infty, \mathcal{G}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof: Let $f \in L^p(\mathbb{R}^n)$. Then there exists $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_p < \varepsilon$. Now let $b > 0$ be large enough that

$$\int_{\mathbb{R}^n} \left(e^{-b|\mathbf{x}|^2}\right)^p dx < \varepsilon^p.$$

Then $\mathbf{x} \to g(\mathbf{x}) e^{b|\mathbf{x}|^2}$ is in $C_c(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$. Therefore, from Lemma 15.3 there exists $\psi \in \mathcal{G}$ such that

$$\left|\left|ge^{b|\cdot|^2} - \psi\right|\right|_\infty < 1$$

Therefore, letting $\phi(\mathbf{x}) \equiv e^{-b|\mathbf{x}|^2}\psi(\mathbf{x})$ it follows that $\phi \in \mathcal{G}$ and for all $\mathbf{x} \in \mathbb{R}^n$,

$$|g(\mathbf{x}) - \phi(\mathbf{x})| < e^{-b|\mathbf{x}|^2}$$

Therefore,

$$\left(\int_{\mathbb{R}^n} |g(\mathbf{x}) - \phi(\mathbf{x})|^p dx\right)^{1/p} \leq \left(\int_{\mathbb{R}^n} \left(e^{-b|\mathbf{x}|^2}\right)^p dx\right)^{1/p} < \varepsilon.$$

It follows

$$||f - \phi||_p \leq ||f - g||_p + ||g - \phi||_p < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem.

The following lemma is also interesting even if it is obvious.

Lemma 15.5 *For $\psi \in \mathcal{G}$, p a polynomial, and α, β multiindices, $D^\alpha\psi \in \mathcal{G}$ and $p\psi \in \mathcal{G}$. Also*

$$\sup\{|\mathbf{x}^\beta D^\alpha \psi(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty$$

15.2 Fourier Transforms Of Functions In $\mathcal{G}$

Definition 15.6 *For $\psi \in \mathcal{G}$ Define the Fourier transform, F and the inverse Fourier transform, F^{-1} by*

$$F\psi(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})dx,$$

$$F^{-1}\psi(\mathbf{t}) \equiv (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})dx.$$

where $\mathbf{t}\cdot\mathbf{x} \equiv \sum_{i=1}^n t_i x_i$*.Note there is no problem with this definition because* ψ *is in* $L^1(\mathbb{R}^n)$ *and therefore,*

$$\left|e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})\right| \leq |\psi(\mathbf{x})|,$$

an integrable function.

One reason for using the functions, $\mathcal{G}$ is that it is very easy to compute the Fourier transform of these functions. The first thing to do is to verify F and F^{-1} map $\mathcal{G}$ to $\mathcal{G}$ and that $F^{-1}\circ F(\psi) = \psi$.

Lemma 15.7 *The following formulas are true*

$$\int_{\mathbb{R}} e^{-c(x+it)^2}dx = \int_{\mathbb{R}} e^{-c(x-it)^2}dx = \frac{\sqrt{\pi}}{\sqrt{c}}, \tag{15.1}$$

$$\int_{\mathbb{R}^n} e^{-c(\mathbf{x}+i\mathbf{t})\cdot(\mathbf{x}+i\mathbf{t})}dx = \int_{\mathbb{R}^n} e^{-c(\mathbf{x}-i\mathbf{t})\cdot(\mathbf{x}-i\mathbf{t})}dx = \left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^n, \tag{15.2}$$

$$\int_{\mathbb{R}} e^{-ct^2}e^{-ist}dt = \int_{\mathbb{R}} e^{-ct^2}e^{ist}dt = e^{-\frac{s^2}{4c}}\frac{\sqrt{\pi}}{\sqrt{c}}, \tag{15.3}$$

$$\int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2}e^{-i\mathbf{s}\cdot\mathbf{t}}dt = \int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2}e^{i\mathbf{s}\cdot\mathbf{t}}dt = e^{-\frac{|\mathbf{s}|^2}{4c}}\left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^n. \tag{15.4}$$

Proof: Consider the first one. Simple manipulations yield

$$H(t) \equiv \int_{\mathbb{R}} e^{-c(x+it)^2}dx = e^{ct^2}\int_{\mathbb{R}} e^{-cx^2}\cos(2cxt)\,dx.$$

Now using the dominated convergence theorem to justify passing derivatives inside the integral where necessary and using integration by parts,

$$\begin{aligned} H'(t) &= 2cte^{ct^2}\int_{\mathbb{R}} e^{-cx^2}\cos(2cxt)\,dx - e^{ct^2}\int_{\mathbb{R}} e^{-cx^2}\sin(2cxt)\,2xcdx \\ &= 2ctH(t) - e^{ct^2}2ct\int_{\mathbb{R}} e^{-cx^2}\cos(2cxt)\,dx = 2ct(H(t)-H(t)) = 0 \end{aligned}$$

and so $H(t) = H(0) = \int_{\mathbb{R}} e^{-cx^2}dx \equiv I$. Thus

$$I^2 = \int_{\mathbb{R}^2} e^{-c\left(x^2+y^2\right)}dxdy = \int_0^\infty \int_0^{2\pi} e^{-cr^2}rd\theta dr = \frac{\pi}{c}.$$

Therefore, $I = \sqrt{\pi}/\sqrt{c}$. Since the sign of t is unimportant, this proves 15.1. This also proves 15.2 after writing as iterated integrals.

Consider 15.3.

$$\begin{aligned}
\int_{\mathbb{R}} e^{-ct^2} e^{ist} dt &= \int_{\mathbb{R}} e^{-c\left(t^2 - \frac{ist}{c} + \left(\frac{is}{2c}\right)^2\right)} e^{-\frac{s^2}{4c}} dt \\
&= e^{-\frac{s^2}{4c}} \int_{\mathbb{R}} e^{-c\left(t - \frac{is}{2c}\right)^2} dt = e^{-\frac{s^2}{4c}} \frac{\sqrt{\pi}}{\sqrt{c}}.
\end{aligned}$$

Changing the variable $t \to -t$ gives the other part of 15.3.

Finally 15.4 follows from using iterated integrals.

With these formulas, it is easy to verify F, F^{-1} map $\mathcal{G}$ to $\mathcal{G}$ and $F \circ F^{-1} = F^{-1} \circ F = id$.

Theorem 15.8 *Each of F and F^{-1} map $\mathcal{G}$ to $\mathcal{G}$. Also $F^{-1} \circ F(\psi) = \psi$ and $F \circ F^{-1}(\psi) = \psi$.*

Proof: The first claim will be shown if it is shown that $F\psi \in \mathcal{G}$ for $\psi(\mathbf{x}) = \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2}$ because an arbitrary function of $\mathcal{G}$ is a finite sum of scalar multiples of functions such as ψ. Using Lemma 15.7,

$$\begin{aligned}
F\psi(\mathbf{t}) &\equiv \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2} dx \\
&= \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D_t^\alpha \left(\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} e^{-b|\mathbf{x}|^2} dx\right) \\
&= \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{2b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right)
\end{aligned}$$

and this is clearly in $\mathcal{G}$ because it equals a polynomial times $e^{-\frac{|\mathbf{t}|^2}{2b}}$. It remains to verify the other assertion. As in the first case, it suffices to consider $\psi(\mathbf{x}) = \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2}$. Using Lemma 15.7 and ordinary integration by parts on the iterated integrals, $\int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2} e^{i\mathbf{s}\cdot\mathbf{t}} dt = e^{-\frac{|\mathbf{s}|^2}{2c}} \left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^n$

$$
\begin{aligned}
& F^{-1} \circ F(\psi)(\mathbf{s}) \\
\equiv \;& \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2} dx dt \\
= \;& \left(\frac{1}{2\pi}\right)^{n} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} (-i)^{-|\alpha|} D_t^\alpha \left(\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} e^{-b|\mathbf{x}|^2} dx dt\right) \\
= \;& \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} \left(\frac{1}{2\pi}\right)^{n/2} (-i)^{-|\alpha|} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) dt \\
= \;& \left(\frac{1}{2\pi}\right)^{n} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n (-i)^{-|\alpha|} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{4b}}\right) dt \\
= \;& \left(\frac{1}{2\pi}\right)^{n} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n (-i)^{-|\alpha|} (-1)^{|\alpha|} \mathbf{s}^\alpha (i)^{|\alpha|} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} e^{-\frac{|\mathbf{t}|^2}{4b}} dt \\
= \;& \left(\frac{1}{2\pi}\right)^{n} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \mathbf{s}^\alpha \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} e^{-\frac{|\mathbf{t}|^2}{4b}} dt \\
= \;& \left(\frac{1}{2\pi}\right)^{n} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \mathbf{s}^\alpha e^{-\frac{|\mathbf{s}|^2}{4(1/(4b))}} \left(\frac{\sqrt{\pi}}{\sqrt{1/(4b)}}\right)^n \\
= \;& \left(\frac{1}{2\pi}\right)^{n} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \mathbf{s}^\alpha e^{-b|\mathbf{s}|^2} \left(\sqrt{\pi} 2\sqrt{b}\right)^n = \mathbf{s}^\alpha e^{-b|\mathbf{s}|^2} = \psi(\mathbf{s}).
\end{aligned}
$$

This little computation proves the theorem. The other case is entirely similar.

15.3 Fourier Transforms Of Just About Anything

Definition 15.9 *Let $\mathcal{G}^*$ denote the vector space of linear functions defined on $\mathcal{G}$ which have values in $\mathbb{C}$. Thus $T \in \mathcal{G}^*$ means $T : \mathcal{G} \to \mathbb{C}$ and T is linear,*

$$T(a\psi + b\phi) = aT(\psi) + bT(\phi) \text{ for all } a, b \in \mathbb{C}, \quad \psi, \phi \in \mathcal{G}$$

Let $\psi \in \mathcal{G}$. Then define $T_\psi \in \mathcal{G}^*$ by

$$T_\psi(\phi) \equiv \int_{\mathbb{R}^n} \psi(\mathbf{x}) \phi(\mathbf{x}) dx$$

Lemma 15.10 *The following is obtained for all $\phi, \psi \in \mathcal{G}$.*

$$T_{F\psi}(\phi) = T_\psi(F\phi), \; T_{F^{-1}\psi}(\phi) = T_\psi\left(F^{-1}\phi\right)$$

Also if $\psi \in \mathcal{G}$ and $T_\psi = 0$, then $\psi = 0$.

Proof:

$$
\begin{aligned}
T_{F\psi}(\phi) &\equiv \int_{\mathbb{R}^n} F\psi(\mathbf{t})\,\phi(\mathbf{t})\,dt \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})dx\phi(\mathbf{t})\,dt \\
&= \int_{\mathbb{R}^n} \psi(\mathbf{x}) \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\phi(\mathbf{t})\,dtdx \\
&= \int_{\mathbb{R}^n} \psi(\mathbf{x})F\phi(\mathbf{x})\,dx \equiv T_\psi(F\phi)
\end{aligned}
$$

The other claim is similar.

Suppose now $T_\psi = 0$. Then

$$\int_{\mathbb{R}^n} \psi\phi dx = 0$$

for all $\phi \in \mathcal{G}$. Therefore, this is true for $\phi = \psi$ and so $\psi = 0$. This proves the lemma.

From now on regard $\mathcal{G} \subseteq \mathcal{G}^*$ and for $\psi \in \mathcal{G}$ write $\psi(\phi)$ instead of $T_\psi(\phi)$. It was just shown that with this interpretation[1],

$$F\psi(\phi) = \psi(F(\phi)),\ F^{-1}\psi(\phi) = \psi\left(F^{-1}\phi\right).$$

This lemma suggests a way to define the Fourier transform of something in $\mathcal{G}^*$.

Definition 15.11 *For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by*

$$FT(\phi) \equiv T(F\phi),\ F^{-1}T(\phi) \equiv T\left(F^{-1}\phi\right)$$

Lemma 15.12 *F and F^{-1} are both one to one, onto, and are inverses of each other.*

Proof: First note F and F^{-1} are both linear. This follows directly from the definition. Suppose now $FT = 0$. Then $FT(\phi) = T(F\phi) = 0$ for all $\phi \in \mathcal{G}$. But F and F^{-1} map $\mathcal{G}$ onto $\mathcal{G}$ because if $\psi \in \mathcal{G}$, then $\psi = F\left(F^{-1}(\psi)\right)$. Therefore, $T = 0$ and so F is one to one. Similarly F^{-1} is one to one. Now

$$F^{-1}(FT)(\phi) \equiv (FT)\left(F^{-1}\phi\right) \equiv T\left(F\left(F^{-1}(\phi)\right)\right) = T\phi.$$

Therefore, $F^{-1} \circ F(T) = T$. Similarly, $F \circ F^{-1}(T) = T$. Thus both F and F^{-1} are one to one and onto and are inverses of each other as suggested by the notation. This proves the lemma.

Probably the most interesting things in $\mathcal{G}^*$ are functions of various kinds. The following lemma has to do with this situation.

[1]This is not all that different from what was done with the derivative. Remember when you consider the derivative of a function of one variable, in elementary courses you think of it as a number but thinking of it as a linear transformation acting on $\mathbb{R}$ is better because this leads to the concept of a derivative which generalizes to functions of many variables. So it is here. You can think of $\psi \in \mathcal{G}$ as simply an element of $\mathcal{G}$ but it is better to think of it as an element of $\mathcal{G}^*$ as just described.

Lemma 15.13 *If* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} f\phi dx = 0$ *for all* $\phi \in C_c(\mathbb{R}^n)$, *then* $f = 0$ *a.e.*

Proof: First suppose $f \geq 0$. Let

$$E \equiv \{\mathbf{x} : f(\mathbf{x}) \geq r\},\ E_R \equiv E \cap B(\mathbf{0},R).$$

Let K_m be an increasing sequence of compact sets and let V_m be a decreasing sequence of open sets satisfying

$$K_m \subseteq E_R \subseteq V_m,\ m_n(V_m) \leq m_n(K_m) + 2^{-m}, V_1 \subseteq B(\mathbf{0},R).$$

Therefore,

$$m_n(V_m \setminus K_m) \leq 2^{-m}.$$

Let

$$\phi_m \in C_c(V_m),\ K_m \prec \phi_m \prec V_m.$$

Then $\phi_m(\mathbf{x}) \to \mathcal{X}_{E_R}(\mathbf{x})$ a.e. because the set where $\phi_m(\mathbf{x})$ fails to converge to this set is contained in the set of all $\mathbf{x}$ which are in infinitely many of the sets $V_m \setminus K_m$. This set has measure zero because

$$\sum_{m=1}^{\infty} m_n(V_m \setminus K_m) < \infty$$

and so, by the dominated convergence theorem,

$$0 = \lim_{m\to\infty} \int_{\mathbb{R}^n} f\phi_m dx = \lim_{m\to\infty} \int_{V_1} f\phi_m dx = \int_{E_R} f dx \geq rm(E_R).$$

Thus, $m_n(E_R) = 0$ and therefore $m_n(E) = \lim_{R\to\infty} m_n(E_R) = 0$. Since $r > 0$ is arbitrary, it follows

$$m_n([f > 0]) = \cup_{k=1}^{\infty} m_n\left(\left[f > k^{-1}\right]\right) = 0.$$

Now suppose f has values in $\mathbb{R}$. Let $E_+ = [f \geq 0]$ and $E_- = [f < 0]$. Thus these are two measurable sets. As in the first part, let K_m and V_m be sequences of compact and open sets such that $K_m \subseteq E_+ \cap B(\mathbf{0}, R) \subseteq V_m \subseteq B(\mathbf{0}, R)$ and let $K_m \prec \phi_m \prec V_m$ with $m_n(V_m \setminus K_m) < 2^{-m}$. Thus $\phi_m \in C_c(\mathbb{R}^n)$ and the sequence converges pointwise to $\mathcal{X}_{E_+\cap B(\mathbf{0},R)}$. Then by the dominated convergence theorem, if ψ is any function in $C_c(\mathbb{R}^n)$

$$0 = \int f\phi_m \psi dm_n \to \int f\psi \mathcal{X}_{E_+\cap B(\mathbf{0},R)} dm_n.$$

Hence, letting $R \to \infty$,

$$\int f\psi \mathcal{X}_{E_+} dm_n = \int f_+ \psi dm_n = 0$$

Since ψ is arbitrary, the first part of the argument applies to f_+ and implies $f_+ = 0$. Similarly $f_- = 0$. Finally, if f is complcx valued, the assumptions mean

$$\int \operatorname{Re}(f)\,\phi dm_n = 0, \int \operatorname{Im}(f)\,\phi dm_n = 0$$

for all $\phi \in C_c(\mathbb{R}^n)$ and so both $\operatorname{Re}(f), \operatorname{Im}(f)$ equal zero a.e. This proves the lemma.

Corollary 15.14 *Let $f \in L^1(\mathbb{R}^n)$ and suppose*

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\,dx = 0$$

for all $\phi \in \mathcal{G}$. Then $f = 0$ a.e.

Proof: Let $\psi \in C_c(\mathbb{R}^n)$. Then by the Stone Weierstrass approximation theorem, there exists a sequence of functions, $\{\phi_k\} \subseteq \mathcal{G}$ such that $\phi_k \to \psi$ uniformly. Then by the dominated convergence theorem,

$$\int f\psi dx = \lim_{k\to\infty}\int f\phi_k dx = 0.$$

By Lemma 15.13 $f = 0$.

The next theorem is the main result of this sort.

Theorem 15.15 *Let $f \in L^p(\mathbb{R}^n), p \geq 1$, or suppose f is measurable and has polynomial growth,*

$$|f(\mathbf{x})| \leq K\left(1 + |\mathbf{x}|^2\right)^m$$

for some $m \in \mathbb{N}$. Then if

$$\int f\psi dx = 0$$

for all $\psi \in \mathcal{G}$ then it follows $f = 0$.

Proof: The case where $f \in L^1(\mathbb{R}^n)$ was dealt with in Corollary 15.14. Suppose $f \in L^p(\mathbb{R}^n)$ for $p > 1$. Then by Holder's inequality and the density of $\mathcal{G}$ in $L^{p'}(\mathbb{R}^n)$, it follows that $\int fgdx = 0$ for all $g \in L^{p'}(\mathbb{R}^n)$. By the Riesz representation theorem, $f = 0$.

It remains to consider the case where f has polynomial growth. Thus $\mathbf{x} \to f(\mathbf{x})\,e^{-|\mathbf{x}|^2} \in L^1(\mathbb{R}^n)$. Therefore, for all $\psi \in \mathcal{G}$,

$$0 = \int f(\mathbf{x})\,e^{-|\mathbf{x}|^2}\psi(\mathbf{x})\,dx$$

because $e^{-|\mathbf{x}|^2}\psi(\mathbf{x}) \in \mathcal{G}$. Therefore, by the first part, $f(\mathbf{x})\,e^{-|\mathbf{x}|^2} = 0$ a.e.

The following theorem shows that you can consider most functions you are likely to encounter as elements of $\mathcal{G}^*$.

Theorem 15.16 *Let f be a measurable function with polynomial growth,*

$$|f(\mathbf{x})| \leq C\left(1+|\mathbf{x}|^2\right)^N \quad \text{for some } N,$$

or let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1,\infty]$. Then $f \in \mathcal{G}^*$ if

$$f(\phi) \equiv \int f\phi dx.$$

Proof: Let f have polynomial growth first. Then the above integral is clearly well defined and so in this case, $f \in \mathcal{G}^*$.

Next suppose $f \in L^p(\mathbb{R}^n)$ with $\infty > p \geq 1$. Then it is clear again that the above integral is well defined because of the fact that ϕ is a sum of polynomials times exponentials of the form $e^{-c|\mathbf{x}|^2}$ and these are in $L^{p'}(\mathbb{R}^n)$. Also $\phi \rightarrow f(\phi)$ is clearly linear in both cases. This proves the theorem.

This has shown that for nearly any reasonable function, you can define its Fourier transform as described above. Also you should note that $\mathcal{G}^*$ includes $C_0(\mathbb{R}^n)'$, the space of complex measures whose total variation are Radon measures. It is especially interesting when the Fourier transform yields another function of some sort.

15.3.1 Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$

First suppose $f \in L^1(\mathbb{R}^n)$.

Theorem 15.17 *Let $f \in L^1(\mathbb{R}^n)$. Then $Ff(\phi) = \int_{\mathbb{R}^n} g\phi dt$ where*

$$g(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\, dx$$

and $F^{-1}f(\phi) = \int_{\mathbb{R}^n} g\phi dt$ where $g(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\, dx$. In short,

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx,$$

$$F^{-1}f(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx.$$

Proof: From the definition and Fubini's theorem,

$$\begin{aligned} Ff(\phi) &\equiv \int_{\mathbb{R}^n} f(\mathbf{t})\, F\phi(\mathbf{t})\, dt = \int_{\mathbb{R}^n} f(\mathbf{t}) \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \phi(\mathbf{x})\, dx dt \\ &= \int_{\mathbb{R}^n} \left(\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{t})\, e^{-i\mathbf{t}\cdot\mathbf{x}} dt\right) \phi(\mathbf{x})\, dx. \end{aligned}$$

Since $\phi \in \mathcal{G}$ is arbitrary, it follows from Theorem 15.15 that $Ff(\mathbf{x})$ is given by the claimed formula. The case of F^{-1} is identical.

Here are interesting properties of these Fourier transforms of functions in L^1.

Theorem 15.18 *If $f \in L^1(\mathbb{R}^n)$ and $||f_k - f||_1 \to 0$, then Ff_k and $F^{-1}f_k$ converge uniformly to Ff and $F^{-1}f$ respectively. If $f \in L^1(\mathbb{R}^n)$, then $F^{-1}f$ and Ff are both continuous and bounded. Also,*

$$\lim_{|\mathbf{x}|\to\infty} F^{-1}f(\mathbf{x}) = \lim_{|\mathbf{x}|\to\infty} Ff(\mathbf{x}) = 0. \tag{15.5}$$

Furthermore, for $f \in L^1(\mathbb{R}^n)$ both Ff and $F^{-1}f$ are uniformly continuous.

Proof: The first claim follows from the following inequality.

$$\begin{aligned}
|Ff_k(\mathbf{t}) - Ff(\mathbf{t})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left|e^{-i\mathbf{t}\cdot\mathbf{x}} f_k(\mathbf{x}) - e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\right| dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f_k(\mathbf{x}) - f(\mathbf{x})| \, dx \\
&= (2\pi)^{-n/2} ||f - f_k||_1 .
\end{aligned}$$

which a similar argument holding for F^{-1}.

Now consider the second claim of the theorem.

$$|Ff(\mathbf{t}) - Ff(\mathbf{t}')| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left|e^{-i\mathbf{t}\cdot\mathbf{x}} - e^{-i\mathbf{t}'\cdot\mathbf{x}}\right| |f(\mathbf{x})| \, dx$$

The integrand is bounded by $2|f(\mathbf{x})|$, a function in $L^1(\mathbb{R}^n)$ and converges to 0 as $\mathbf{t}' \to \mathbf{t}$ and so the dominated convergence theorem implies Ff is continuous. To see $Ff(\mathbf{t})$ is uniformly bounded,

$$|Ff(\mathbf{t})| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(\mathbf{x})| \, dx < \infty.$$

A similar argument gives the same conclusions for F^{-1}.

It remains to verify 15.5 and the claim that Ff and $F^{-1}f$ are uniformly continuous.

$$|Ff(\mathbf{t})| \leq \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx\right|$$

Now let $\varepsilon > 0$ be given and let $g \in C_c^\infty(\mathbb{R}^n)$ such that $(2\pi)^{-n/2} ||g - f||_1 < \varepsilon/2$. Then

$$\begin{aligned}
|Ff(\mathbf{t})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(\mathbf{x}) - g(\mathbf{x})| \, dx \\
&\quad + \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx\right| \\
&\leq \varepsilon/2 + \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx\right| .
\end{aligned}$$

Now integrating by parts, it follows that for $||\mathbf{t}||_\infty \equiv \max\{|t_j| : j = 1, \cdots, n\} > 0$

$$|Ff(\mathbf{t})| \leq \varepsilon/2 + (2\pi)^{-n/2} \left|\frac{1}{||\mathbf{t}||_\infty} \int_{\mathbb{R}^n} \sum_{j=1}^n \left|\frac{\partial g(\mathbf{x})}{\partial x_j}\right| dx\right| \tag{15.6}$$

and this last expression converges to zero as $||\mathbf{t}||_\infty \to \infty$. The reason for this is that if $t_j \neq 0$, integration by parts with respect to x_j gives

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx = (2\pi)^{-n/2} \frac{1}{-it_j} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \frac{\partial g(\mathbf{x})}{\partial x_j} dx.$$

Therefore, choose the j for which $||\mathbf{t}||_\infty = |t_j|$ and the result of 15.6 holds. Therefore, from 15.6, if $||\mathbf{t}||_\infty$ is large enough, $|Ff(\mathbf{t})| < \varepsilon$. Similarly, $\lim_{||\mathbf{t}|| \to \infty} F^{-1}(\mathbf{t}) = 0$. Consider the claim about uniform continuity. Let $\varepsilon > 0$ be given. Then there exists R such that if $||\mathbf{t}||_\infty > R$, then $|Ff(\mathbf{t})| < \frac{\varepsilon}{2}$. Since Ff is continuous, it is uniformly continuous on the compact set, $[-R-1, R+1]^n$. Therefore, there exists δ_1 such that if $||\mathbf{t} - \mathbf{t}'||_\infty < \delta_1$ for $\mathbf{t}', \mathbf{t} \in [-R-1, R+1]^n$, then

$$|Ff(\mathbf{t}) - Ff(\mathbf{t}')| < \varepsilon/2. \tag{15.7}$$

Now let $0 < \delta < \min(\delta_1, 1)$ and suppose $||\mathbf{t} - \mathbf{t}'||_\infty < \delta$. If both $\mathbf{t}, \mathbf{t}'$ are contained in $[-R, R]^n$, then 15.7 holds. If $\mathbf{t} \in [-R, R]^n$ and $\mathbf{t}' \notin [-R, R]^n$, then both are contained in $[-R-1, R+1]^n$ and so this verifies 15.7 in this case. The other case is that neither point is in $[-R, R]^n$ and in this case,

$$\begin{aligned} |Ff(\mathbf{t}) - Ff(\mathbf{t}')| &\leq |Ff(\mathbf{t})| + |Ff(\mathbf{t}')| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves the theorem.

There is a very interesting relation between the Fourier transform and convolutions.

Theorem 15.19 *Let $f, g \in L^1(\mathbb{R}^n)$. Then $f*g \in L^1$ and $F(f*g) = (2\pi)^{n/2} FfFg$.*

Proof: Consider

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| \, dy dx.$$

The function, $(\mathbf{x}, \mathbf{y}) \to |f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})|$ is Lebesgue measurable and so by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| \, dy dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| \, dx dy \\ &= ||f||_1 \, ||g||_1 < \infty. \end{aligned}$$

It follows that for a.e. $\mathbf{x}$, $\int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| \, dy < \infty$ and for each of these values of $\mathbf{x}$, it follows that $\int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) \, dy$ exists and equals a function of $\mathbf{x}$ which is in $L^1(\mathbb{R}^n)$, $f * g(\mathbf{x})$. Now

$$\begin{aligned}
& F(f*g)(\mathbf{t}) \\
\equiv\ & (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f*g(\mathbf{x})\,dx \\
=\ & (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\,g(\mathbf{y})\,dydx \\
=\ & (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{y}} g(\mathbf{y})\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot(\mathbf{x}-\mathbf{y})} f(\mathbf{x}-\mathbf{y})\,dxdy \\
=\ & (2\pi)^{n/2}\,Ff(\mathbf{t})\,Fg(\mathbf{t}).
\end{aligned}$$

There are many other considerations involving Fourier transforms of functions in $L^1(\mathbb{R}^n)$.

15.3.2 Fourier Transforms Of Functions In $L^2(\mathbb{R}^n)$

Consider Ff and $F^{-1}f$ for $f \in L^2(\mathbb{R}^n)$. First note that the formula given for Ff and $F^{-1}f$ when $f \in L^1(\mathbb{R}^n)$ will not work for $f \in L^2(\mathbb{R}^n)$ unless f is also in $L^1(\mathbb{R}^n)$. Recall that $\overline{a+ib} = a - ib$.

Theorem 15.20 *For $\phi \in \mathcal{G}$, $||F\phi||_2 = ||F^{-1}\phi||_2 = ||\phi||_2$.*

Proof: First note that for $\psi \in \mathcal{G}$,

$$F(\overline{\psi}) = \overline{F^{-1}(\psi)}\,,\ F^{-1}(\overline{\psi}) = \overline{F(\psi)}. \tag{15.8}$$

This follows from the definition. For example,

$$\begin{aligned}
F\overline{\psi}(\mathbf{t}) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\overline{\psi}(\mathbf{x})\,dx \\
&= \overline{(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})\,dx}
\end{aligned}$$

Let $\phi, \psi \in \mathcal{G}$. It was shown above that

$$\int_{\mathbb{R}^n}(F\phi)\psi(\mathbf{t})dt = \int_{\mathbb{R}^n}\phi(F\psi)dx.$$

Similarly,

$$\int_{\mathbb{R}^n}\phi(F^{-1}\psi)dx = \int_{\mathbb{R}^n}(F^{-1}\phi)\psi dt. \tag{15.9}$$

Now, 15.8 - 15.9 imply

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi|^2 dx &= \int_{\mathbb{R}^n} \phi\overline{F^{-1}(F\phi)}dx \\ &= \int_{\mathbb{R}^n} \phi F(\overline{F\phi})dx \\ &= \int_{\mathbb{R}^n} F\phi(\overline{F\phi})dx \\ &= \int_{\mathbb{R}^n} |F\phi|^2 dx.\end{aligned}$$

Similarly

$$||\phi||_2 = ||F^{-1}\phi||_2.$$

This proves the theorem.

Lemma 15.21 *Let $f \in L^2(\mathbb{R}^n)$ and let $\phi_k \to f$ in $L^2(\mathbb{R}^n)$ where $\phi_k \in \mathcal{G}$. (Such a sequence exists because of density of $\mathcal{G}$ in $L^2(\mathbb{R}^n)$.) Then Ff and $F^{-1}f$ are both in $L^2(\mathbb{R}^n)$ and the following limits take place in L^2.*

$$\lim_{k\to\infty} F(\phi_k) = F(f), \lim_{k\to\infty} F^{-1}(\phi_k) = F^{-1}(f).$$

Proof: Let $\psi \in \mathcal{G}$ be given. Then

$$\begin{aligned}Ff(\psi) &\equiv f(F\psi) \equiv \int_{\mathbb{R}^n} f(\mathbf{x}) F\psi(\mathbf{x}) dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \phi_k(\mathbf{x}) F\psi(\mathbf{x}) dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} F\phi_k(\mathbf{x}) \psi(\mathbf{x}) dx.\end{aligned}$$

Also by Theorem 15.20 $\{F\phi_k\}_{k=1}^{\infty}$ is Cauchy in $L^2(\mathbb{R}^n)$ and so it converges to some $h \in L^2(\mathbb{R}^n)$. Therefore, from the above,

$$Ff(\psi) = \int_{\mathbb{R}^n} h(\mathbf{x}) \psi(\mathbf{x})$$

which shows that $F(f) \in L^2(\mathbb{R}^n)$ and $h = F(f)$. The case of F^{-1} is entirely similar. This proves the lemma.

Since Ff and $F^{-1}f$ are in $L^2(\mathbb{R}^n)$, this also proves the following theorem.

Theorem 15.22 *If $f \in L^2(\mathbb{R}^n)$, Ff and $F^{-1}f$ are the unique elements of $L^2(\mathbb{R}^n)$ such that for all $\phi \in \mathcal{G}$,*

$$\int_{\mathbb{R}^n} Ff(\mathbf{x})\phi(\mathbf{x})dx = \int_{\mathbb{R}^n} f(\mathbf{x})F\phi(\mathbf{x})dx, \tag{15.10}$$

$$\int_{\mathbb{R}^n} F^{-1}f(\mathbf{x})\phi(\mathbf{x})dx = \int_{\mathbb{R}^n} f(\mathbf{x})F^{-1}\phi(\mathbf{x})dx. \tag{15.11}$$

Theorem 15.23 *(Plancherel)*

$$||f||_2 = ||Ff||_2 = ||F^{-1}f||_2. \tag{15.12}$$

Proof: Use the density of $\mathcal{G}$ in $L^2(\mathbb{R}^n)$ to obtain a sequence, $\{\phi_k\}$ converging to f in $L^2(\mathbb{R}^n)$. Then by Lemma 15.21

$$||Ff||_2 = \lim_{k\to\infty} ||F\phi_k||_2 = \lim_{k\to\infty} ||\phi_k||_2 = ||f||_2 .$$

Similarly,

$$||f||_2 = ||F^{-1}f||_2.$$

This proves the theorem.

The following corollary is a simple generalization of this. To prove this corollary, use the following simple lemma which comes as a consequence of the Cauchy Schwarz inequality.

Lemma 15.24 *Suppose $f_k \to f$ in $L^2(\mathbb{R}^n)$ and $g_k \to g$ in $L^2(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k g_k dx = \int_{\mathbb{R}^n} fg dx$$

Proof:

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} fg dx\right| &\leq \left|\int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} f_k g dx\right| + \\ &\quad \left|\int_{\mathbb{R}^n} f_k g dx - \int_{\mathbb{R}^n} fg dx\right| \\ &\leq ||f_k||_2 \, ||g-g_k||_2 + ||g||_2 \, ||f_k - f||_2 .\end{aligned}$$

Now $||f_k||_2$ is a Cauchy sequence and so it is bounded independent of k. Therefore, the above expression is smaller than ε whenever k is large enough. This proves the lemma.

Corollary 15.25 *For $f, g \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} f\overline{g} dx = \int_{\mathbb{R}^n} Ff \, \overline{Fg} dx = \int_{\mathbb{R}^n} F^{-1}f \, \overline{F^{-1}g} dx.$$

Proof: First note the above formula is obvious if $f, g \in \mathcal{G}$. To see this, note

$$\begin{aligned}\int_{\mathbb{R}^n} Ff \, \overline{Fg} dx &= \int_{\mathbb{R}^n} Ff(x) \overline{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\mathbf{t}} g(t) \, dt} dx \\ &= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\mathbf{t}} Ff(x) \, dx \overline{g(t)} dt \\ &= \int_{\mathbb{R}^n} \left(F^{-1} \circ F\right) f(t) \overline{g(t)} dt \\ &= \int_{\mathbb{R}^n} f(t) \overline{g(t)} dt.\end{aligned}$$

The formula with F^{-1} is exactly similar.

Now to verify the corollary, let $\phi_k \to f$ in $L^2(\mathbb{R}^n)$ and let $\psi_k \to g$ in $L^2(\mathbb{R}^n)$. Then by Lemma 15.21

$$\begin{aligned}\int_{\mathbb{R}^n} Ff\,\overline{Fg}dx &= \lim_{k\to\infty}\int_{\mathbb{R}^n} F\phi_k\,\overline{F\psi_k}dx \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} \phi_k\overline{\psi_k}dx \\ &= \int_{\mathbb{R}^n} f\overline{g}dx\end{aligned}$$

A similar argument holds for F^{-1}.This proves the corollary.

How does one compute Ff and $F^{-1}f$?

Theorem 15.26 *For $f \in L^2(\mathbb{R}^n)$, let $f_r = f\mathcal{X}_{E_r}$ where E_r is a bounded measurable set with $E_r \uparrow \mathbb{R}^n$. Then the following limits hold in $L^2(\mathbb{R}^n)$.*

$$Ff = \lim_{r\to\infty} Ff_r\ ,\ F^{-1}f = \lim_{r\to\infty} F^{-1}f_r.$$

Proof: $||f - f_r||_2 \to 0$ and so $||Ff - Ff_r||_2 \to 0$ and $||F^{-1}f - F^{-1}f_r||_2 \to 0$ by Plancherel's Theorem. This proves the theorem.

What are Ff_r and $F^{-1}f_r$? Let $\phi \in \mathcal{G}$

$$\begin{aligned}\int_{\mathbb{R}^n} Ff_r\phi dx &= \int_{\mathbb{R}^n} f_r F\phi dx \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}\phi(\mathbf{y})dydx \\ &= \int_{\mathbb{R}^n}[(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}dx]\phi(\mathbf{y})dy.\end{aligned}$$

Since this holds for all $\phi \in \mathcal{G}$, a dense subset of $L^2(\mathbb{R}^n)$, it follows that

$$Ff_r(\mathbf{y}) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}dx.$$

Similarly

$$F^{-1}f_r(\mathbf{y}) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f_r(\mathbf{x})e^{i\mathbf{x}\cdot\mathbf{y}}dx.$$

This shows that to take the Fourier transform of a function in $L^2(\mathbb{R}^n)$, it suffices to take the limit as $r \to \infty$ in $L^2(\mathbb{R}^n)$ of $(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}dx$. A similar procedure works for the inverse Fourier transform.

Note this reduces to the earlier definition in case $f \in L^1(\mathbb{R}^n)$. Now consider the convolution of a function in L^2 with one in L^1.

Theorem 15.27 *Let $h \in L^2(\mathbb{R}^n)$ and let $f \in L^1(\mathbb{R}^n)$. Then $h * f \in L^2(\mathbb{R}^n)$,*

$$F^{-1}(h * f) = (2\pi)^{n/2} F^{-1}hF^{-1}f,$$

$$F(h * f) = (2\pi)^{n/2} FhFf,$$

and

$$||h * f||_2 \leq ||h||_2 \, ||f||_1 \, . \tag{15.13}$$

Proof: An application of Minkowski's inequality yields

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |h(\mathbf{x} - \mathbf{y})| \, |f(\mathbf{y})| \, dy \right)^2 dx \right)^{1/2} \leq ||f||_1 \, ||h||_2 \, . \tag{15.14}$$

Hence $\int |h(\mathbf{x} - \mathbf{y})| \, |f(\mathbf{y})| \, dy < \infty$ a.e. $\mathbf{x}$ and

$$\mathbf{x} \to \int h(\mathbf{x} - \mathbf{y}) \, f(\mathbf{y}) \, dy$$

is in $L^2(\mathbb{R}^n)$. Let $E_r \uparrow \mathbb{R}^n$, $m(E_r) < \infty$. Thus,

$$h_r \equiv \mathcal{X}_{E_r} h \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

and letting $\phi \in \mathcal{G}$,

$$\int F(h_r * f)(\phi) \, dx$$

$$\begin{aligned}
&\equiv \int (h_r * f)(F\phi) \, dx \\
&= (2\pi)^{-n/2} \int\int\int h_r(\mathbf{x} - \mathbf{y}) \, f(\mathbf{y}) \, e^{-i\mathbf{x}\cdot\mathbf{t}} \phi(\mathbf{t}) \, dt dy dx \\
&= (2\pi)^{-n/2} \int\int \left(\int h_r(\mathbf{x} - \mathbf{y}) \, e^{-i(\mathbf{x}-\mathbf{y})\cdot\mathbf{t}} dx \right) f(\mathbf{y}) \, e^{-i\mathbf{y}\cdot\mathbf{t}} dy \phi(\mathbf{t}) \, dt \\
&= \int (2\pi)^{n/2} Fh_r(\mathbf{t}) \, Ff(\mathbf{t}) \, \phi(\mathbf{t}) \, dt.
\end{aligned}$$

Since ϕ is arbitrary and $\mathcal{G}$ is dense in $L^2(\mathbb{R}^n)$,

$$F(h_r * f) = (2\pi)^{n/2} Fh_r Ff.$$

Now by Minkowski's Inequality, $h_r * f \to h * f$ in $L^2(\mathbb{R}^n)$ and also it is clear that $h_r \to h$ in $L^2(\mathbb{R}^n)$; so, by Plancherel's theorem, you may take the limit in the above and conclude

$$F(h * f) = (2\pi)^{n/2} FhFf.$$

The assertion for F^{-1} is similar and 15.13 follows from 15.14.

15.3.3 The Schwartz Class

The problem with $\mathcal{G}$ is that it does not contain $C_c^\infty(\mathbb{R}^n)$. I have used it in presenting the Fourier transform because the functions in $\mathcal{G}$ have a very specific form which made some technical details work out easier than in any other approach I have seen. The Schwartz class is a larger class of functions which does contain $C_c^\infty(\mathbb{R}^n)$ and also has the same nice properties as $\mathcal{G}$. The functions in the Schwartz class are infinitely differentiable and they vanish very rapidly as $|\mathbf{x}| \to \infty$ along with all their partial derivatives. This is the description of these functions, not a specific form involving polynomials times $e^{-\alpha|\mathbf{x}|^2}$. To describe this precisely requires some notation.

Definition 15.28 *$f \in \mathfrak{S}$, the Schwartz class, if $f \in C^\infty(\mathbb{R}^n)$ and for all positive integers N,*

$$\rho_N(f) < \infty$$

where

$$\rho_N(f) = \sup\{(1 + |\mathbf{x}|^2)^N |D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n \ , \ |\alpha| \leq N\}.$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and

$$\sup\{|\mathbf{x}^\beta D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty \tag{15.15}$$

for all multi indices α and β.

Also note that if $f \in \mathfrak{S}$, then $p(f) \in \mathfrak{S}$ for any polynomial, p with $p(0) = 0$ and that

$$\mathfrak{S} \subseteq L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

for any $p \geq 1$. To see this assertion about the $p(f)$, it suffices to consider the case of the product of two elements of the Schwartz class. If $f, g \in \mathfrak{S}$, then $D^\alpha(fg)$ is a finite sum of derivatives of f times derivatives of g. Therefore, $\rho_N(fg) < \infty$ for all N. You may wonder about examples of things in $\mathfrak{S}$. Clearly any function in $C_c^\infty(\mathbb{R}^n)$ is in $\mathfrak{S}$. However there are other functions in $\mathfrak{S}$. For example $e^{-|\mathbf{x}|^2}$ is in $\mathfrak{S}$ as you can verify for yourself and so is any function from $\mathcal{G}$. Note also that the density of $C_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ shows that $\mathfrak{S}$ is dense in $L^p(\mathbb{R}^n)$ for every p.

Recall the Fourier transform of a function in $L^1(\mathbb{R}^n)$ is given by

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx.$$

Therefore, this gives the Fourier transform for $f \in \mathfrak{S}$. The nice property which $\mathfrak{S}$ has in common with $\mathcal{G}$ is that the Fourier transform and its inverse map $\mathfrak{S}$ one to one onto $\mathfrak{S}$. This means I could have presented the whole of the above theory in terms of $\mathfrak{S}$ rather than in terms of $\mathcal{G}$. However, it is more technical.

Theorem 15.29 *If $f \in \mathfrak{S}$, then Ff and $F^{-1}f$ are also in $\mathfrak{S}$.*

Proof: To begin with, let $\alpha = \mathbf{e}_j = (0, 0, \cdots, 1, 0, \cdots, 0)$, the 1 in the j^{th} slot.

$$\frac{F^{-1}f(\mathbf{t}+h\mathbf{e}_j) - F^{-1}f(\mathbf{t})}{h} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}f(\mathbf{x})(\frac{e^{ihx_j}-1}{h})dx. \tag{15.16}$$

Consider the integrand in 15.16.

$$\begin{aligned}\left|e^{i\mathbf{t}\cdot\mathbf{x}}f(\mathbf{x})(\frac{e^{ihx_j}-1}{h})\right| &= |f(\mathbf{x})|\left|(\frac{e^{i(h/2)x_j}-e^{-i(h/2)x_j}}{h})\right| \\ &= |f(\mathbf{x})|\left|\frac{i\sin((h/2)x_j)}{(h/2)}\right| \\ &\leq |f(\mathbf{x})||x_j|\end{aligned}$$

and this is a function in $L^1(\mathbb{R}^n)$ because $f \in \mathfrak{S}$. Therefore by the Dominated Convergence Theorem,

$$\begin{aligned}\frac{\partial F^{-1}f(\mathbf{t})}{\partial t_j} &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}ix_jf(\mathbf{x})dx \\ &= i(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\mathbf{x}^{\mathbf{e}_j}f(\mathbf{x})dx.\end{aligned}$$

Now $\mathbf{x}^{\mathbf{e}_j}f(\mathbf{x}) \in \mathfrak{S}$ and so one can continue in this way and take derivatives indefinitely. Thus $F^{-1}f \in C^\infty(\mathbb{R}^n)$ and from the above argument,

$$D^\alpha F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}(i\mathbf{x})^\alpha f(\mathbf{x})dx.$$

To complete showing $F^{-1}f \in \mathfrak{S}$,

$$\mathbf{t}^\beta D^\alpha F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\mathbf{t}^\beta(i\mathbf{x})^a f(\mathbf{x})dx.$$

Integrate this integral by parts to get

$$\mathbf{t}^\beta D^\alpha F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} i^{|\beta|}e^{i\mathbf{t}\cdot\mathbf{x}}D^\beta((i\mathbf{x})^a f(\mathbf{x}))dx. \tag{15.17}$$

Here is how this is done.

$$\begin{aligned}\int_{\mathbb{R}} e^{it_jx_j}t_j^{\beta_j}(i\mathbf{x})^\alpha f(\mathbf{x})dx_j &= \frac{e^{it_jx_j}}{it_j}t_j^{\beta_j}(i\mathbf{x})^\alpha f(\mathbf{x})\mid_{-\infty}^{\infty} + \\ &\quad i\int_{\mathbb{R}} e^{it_jx_j}t_j^{\beta_j-1}D^{\mathbf{e}_j}((i\mathbf{x})^\alpha f(\mathbf{x}))dx_j\end{aligned}$$

where the boundary term vanishes because $f \in \mathfrak{S}$. Returning to 15.17, use the fact that $|e^{ia}| = 1$ to conclude

$$|\mathbf{t}^\beta D^\alpha F^{-1}f(\mathbf{t})| \leq C\int_{\mathbb{R}^n}|D^\beta((i\mathbf{x})^a f(\mathbf{x}))|dx < \infty.$$

It follows $F^{-1}f \in \mathfrak{S}$. Similarly $Ff \in \mathfrak{S}$ whenever $f \in \mathfrak{S}$.

Theorem 15.30 *Let $\psi \in \mathfrak{S}$. Then $(F \circ F^{-1})(\psi) = \psi$ and $(F^{-1} \circ F)(\psi) = \psi$ whenever $\psi \in \mathfrak{S}$. Also F and F^{-1} map $\mathfrak{S}$ one to one and onto $\mathfrak{S}$.*

Proof: The first claim follows from the fact that F and F^{-1} are inverses of each other which was established above. For the second, let $\psi \in \mathfrak{S}$. Then $\psi = F\left(F^{-1}\psi\right)$. Thus F maps $\mathfrak{S}$ onto $\mathfrak{S}$. If $F\psi = 0$, then do F^{-1} to both sides to conclude $\psi = 0$. Thus F is one to one and onto. Similarly, F^{-1} is one to one and onto.

15.3.4 Convolution

To begin with it is necessary to discuss the meaning of ϕf where $f \in \mathcal{G}^*$ and $\phi \in \mathcal{G}$. What should it mean? First suppose $f \in L^p(\mathbb{R}^n)$ or measurable with polynomial growth. Then ϕf also has these properties. Hence, it should be the case that $\phi f(\psi) = \int_{\mathbb{R}^n} \phi f \psi dx = \int_{\mathbb{R}^n} f(\phi\psi)\, dx$. This motivates the following definition.

Definition 15.31 *Let $T \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then $\phi T \equiv T\phi \in \mathcal{G}^*$ will be defined by*

$$\phi T(\psi) \equiv T(\phi\psi).$$

The next topic is that of convolution. It was just shown that

$$F(f * \phi) = (2\pi)^{n/2} F\phi F f,\ F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1} f$$

whenever $f \in L^2(\mathbb{R}^n)$ and $\phi \in \mathcal{G}$ so the same definition is retained in the general case because it makes perfect sense and agrees with the earlier definition.

Definition 15.32 *Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then define the convolution of f with an element of $\mathcal{G}$ as follows.*

$$f * \phi \equiv (2\pi)^{n/2} F^{-1}(F\phi F f) \in \mathcal{G}^*$$

There is an obvious question. With this definition, is it true that $F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1} f$ as it was earlier?

Theorem 15.33 *Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$.*

$$F(f * \phi) = (2\pi)^{n/2} F\phi F f, \tag{15.18}$$

$$F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1} f. \tag{15.19}$$

Proof: Note that 15.18 follows from Definition 15.32 and both assertions hold for $f \in \mathcal{G}$. Consider 15.19. Here is a simple formula involving a pair of functions in $\mathcal{G}$.

$$\left(\psi * F^{-1}F^{-1}\phi\right)(\mathbf{x})$$

$$\begin{aligned}
&= \left(\int\int\int \psi(\mathbf{x}-\mathbf{y}) e^{i\mathbf{y}\cdot\mathbf{y}_1} e^{i\mathbf{y}_1\cdot\mathbf{z}} \phi(\mathbf{z})\, dz dy_1 dy\right)(2\pi)^n \\
&= \left(\int\int\int \psi(\mathbf{x}-\mathbf{y}) e^{-i\mathbf{y}\cdot\tilde{\mathbf{y}}_1} e^{-i\tilde{\mathbf{y}}_1\cdot\mathbf{z}} \phi(\mathbf{z})\, dz d\tilde{y}_1 dy\right)(2\pi)^n \\
&= (\psi * FF\phi)(\mathbf{x}).
\end{aligned}$$

Now for $\psi \in \mathcal{G}$,

$$(2\pi)^{n/2} F\left(F^{-1}\phi F^{-1} f\right)(\psi) \equiv (2\pi)^{n/2}\left(F^{-1}\phi F^{-1} f\right)(F\psi) \equiv$$

$$(2\pi)^{n/2} F^{-1} f\left(F^{-1}\phi F\psi\right) \equiv (2\pi)^{n/2} f\left(F^{-1}\left(F^{-1}\phi F\psi\right)\right) =$$

$$f\left((2\pi)^{n/2} F^{-1}\left(\left(FF^{-1}F^{-1}\phi\right)(F\psi)\right)\right) \equiv$$

$$f\left(\psi * F^{-1}F^{-1}\phi\right) = f(\psi * FF\phi) \tag{15.20}$$

Also

$$(2\pi)^{n/2} F^{-1}(F\phi F f)(\psi) \equiv (2\pi)^{n/2}(F\phi F f)\left(F^{-1}\psi\right) \equiv$$

$$(2\pi)^{n/2} F f\left(F\phi F^{-1}\psi\right) \equiv (2\pi)^{n/2} f\left(F\left(F\phi F^{-1}\psi\right)\right) =$$

$$= f\left(F\left((2\pi)^{n/2}\left(F\phi F^{-1}\psi\right)\right)\right)$$

$$= f\left(F\left((2\pi)^{n/2}\left(F^{-1}FF\phi F^{-1}\psi\right)\right)\right) = f\left(F\left(F^{-1}(FF\phi * \psi)\right)\right)$$

$$f(FF\phi * \psi) = f(\psi * FF\phi). \tag{15.21}$$

The last line follows from the following.

$$\begin{aligned}
\int FF\phi(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, dy &= \int F\phi(\mathbf{x}-\mathbf{y}) F\psi(\mathbf{y})\, dy \\
&= \int F\psi(\mathbf{x}-\mathbf{y}) F\phi(\mathbf{y})\, dy \\
&= \int \psi(\mathbf{x}-\mathbf{y}) FF\phi(y)\, dy.
\end{aligned}$$

From 15.21 and 15.20 , since ψ was arbitrary,

$$(2\pi)^{n/2} F\left(F^{-1}\phi F^{-1} f\right) = (2\pi)^{n/2} F^{-1}(F\phi F f) \equiv f * \phi$$

which shows 15.19.

15.4 Exercises

1. For $f \in L^1(\mathbb{R}^n)$, show that if $F^{-1}f \in L^1$ or $Ff \in L^1$, then f equals a continuous bounded function a.e.

2. Suppose $f, g \in L^1(\mathbb{R})$ and $Ff = Fg$. Show $f = g$ a.e.

3. Show that if $f \in L^1(\mathbb{R}^n)$, then $\lim_{|\mathbf{x}|\to\infty} Ff(\mathbf{x}) = 0$.

4. $\uparrow$ Suppose $f * f = f$ or $f * f = 0$ and $f \in L^1(\mathbb{R})$. Show $f = 0$.

5. For this problem define $\int_a^\infty f(t)\,dt \equiv \lim_{r\to\infty} \int_a^r f(t)\,dt$. Note this coincides with the Lebesgue integral when $f \in L^1(a,\infty)$. Show

   (a) $\int_0^\infty \frac{\sin(u)}{u} du = \frac{\pi}{2}$

   (b) $\lim_{r\to\infty} \int_\delta^\infty \frac{\sin(ru)}{u} du = 0$ whenever $\delta > 0$.

   (c) If $f \in L^1(\mathbb{R})$, then $\lim_{r\to\infty} \int_{\mathbb{R}} \sin(ru) f(u)\,du = 0$.

   Hint: For the first two, use $\frac{1}{u} = \int_0^\infty e^{-ut} dt$ and apply Fubini's theorem to $\int_0^R \sin u \int_{\mathbb{R}} e^{-ut} dt du$. For the last part, first establish it for $f \in C_c^\infty(\mathbb{R})$ and then use the density of this set in $L^1(\mathbb{R})$ to obtain the result. This is sometimes called the Riemann Lebesgue lemma.

6. $\uparrow$Suppose that $g \in L^1(\mathbb{R})$ and that at some $x > 0$, g is locally Holder continuous from the right and from the left. This means

$$\lim_{r\to 0+} g(x+r) \equiv g(x+)$$

   exists,

$$\lim_{r\to 0+} g(x-r) \equiv g(x-)$$

   exists and there exist constants $K, \delta > 0$ and $r \in (0,1]$ such that for $|x-y| < \delta$,

$$|g(x+) - g(y)| < K|x-y|^r$$

   for $y > x$ and

$$|g(x-) - g(y)| < K|x-y|^r$$

   for $y < x$. Show that under these conditions,

$$\lim_{r\to\infty} \frac{2}{\pi} \int_0^\infty \frac{\sin(ur)}{u} \left(\frac{g(x-u) + g(x+u)}{2} \right) du$$

$$= \frac{g(x+) + g(x-)}{2}.$$

7. ↑ Let $g \in L^1(\mathbb{R})$ and suppose g is locally Holder continuous from the right and from the left at x. Show that then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ixt} \int_{-\infty}^{\infty} e^{-ity} g(y)\, dydt = \frac{g(x+) + g(x-)}{2}.$$

This is very interesting. If $g \in L^2(\mathbb{R})$, this shows $F^{-1}(Fg)(x) = \frac{g(x+)+g(x-)}{2}$, the midpoint of the jump in g at the point, x. In particular, if $g \in \mathcal{G}$, $F^{-1}(Fg) = g$. **Hint:** Show the left side of the above equation reduces to

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin(ur)}{u} \left(\frac{g(x-u) + g(x+u)}{2} \right) du$$

and then use Problem 6 to obtain the result.

8. ↑ A measurable function g defined on $(0, \infty)$ has exponential growth if $|g(t)| \leq Ce^{\eta t}$ for some η. For $\operatorname{Re}(s) > \eta$, define the Laplace Transform by

$$Lg(s) \equiv \int_0^{\infty} e^{-su} g(u)\, du.$$

Assume that g has exponential growth as above and is Holder continuous from the right and from the left at t. Pick $\gamma > \eta$. Show that

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{\gamma t} e^{iyt} Lg(\gamma + iy)\, dy = \frac{g(t+) + g(t-)}{2}.$$

This formula is sometimes written in the form

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} Lg(s)\, ds$$

and is called the complex inversion integral for Laplace transforms. It can be used to find inverse Laplace transforms. **Hint:**

$$\frac{1}{2\pi} \int_{-R}^{R} e^{\gamma t} e^{iyt} Lg(\gamma + iy)\, dy =$$

$$\frac{1}{2\pi} \int_{-R}^{R} e^{\gamma t} e^{iyt} \int_0^{\infty} e^{-(\gamma + iy)u} g(u)\, dudy.$$

Now use Fubini's theorem and do the integral from $-R$ to R to get this equal to

$$\frac{e^{\gamma t}}{\pi} \int_{-\infty}^{\infty} e^{-\gamma u} \overline{g}(u) \frac{\sin(R(t-u))}{t-u} du$$

where $\overline{g}$ is the zero extension of g off $[0, \infty)$. Then this equals

$$\frac{e^{\gamma t}}{\pi} \int_{-\infty}^{\infty} e^{-\gamma(t-u)} \overline{g}(t-u) \frac{\sin(Ru)}{u} du$$

which equals

$$\frac{2e^{\gamma t}}{\pi}\int_0^\infty \frac{\overline{g}\,(t-u)\,e^{-\gamma(t-u)}+\overline{g}\,(t+u)\,e^{-\gamma(t+u)}}{2}\frac{\sin(Ru)}{u}du$$

and then apply the result of Problem 6.

9. Suppose $f\in\mathfrak{S}$. Show $F(f_{x_j})(\mathbf{t}) = it_jFf(\mathbf{t})$.

10. Let $f\in\mathfrak{S}$ and let k be a positive integer.

$$||f||_{k,2}\equiv(||f||_2^2+\sum_{|\alpha|\le k}||D^\alpha f||_2^2)^{1/2}.$$

One could also define

$$|||f|||_{k,2}\equiv(\int_{R^n}|Ff(\mathbf{x})|^2(1+|\mathbf{x}|^2)^k dx)^{1/2}.$$

Show both $||\ ||_{k,2}$ and $|||\ |||_{k,2}$ are norms on $\mathfrak{S}$ and that they are equivalent. These are Sobolev space norms. For which values of k does the second norm make sense? How about the first norm?

11. $\uparrow$ Define $H^k(\mathbb{R}^n), k\ge 0$ by $f\in L^2(\mathbb{R}^n)$ such that

$$(\int|Ff(\mathbf{x})|^2(1+|\mathbf{x}|^2)^k dx)^{\frac{1}{2}}<\infty,$$

$$|||f|||_{k,2}\equiv(\int|Ff(\mathbf{x})|^2(1+|\mathbf{x}|^2)^k dx)^{\frac{1}{2}}.$$

Show $H^k(\mathbb{R}^n)$ is a Banach space, and that if k is a positive integer, $H^k(\mathbb{R}^n)$ $=\{\ f\in L^2(\mathbb{R}^n):$ there exists $\{u_j\}\subseteq\mathcal{G}$ with $||u_j-f||_2\to 0$ and $\{u_j\}$ is a Cauchy sequence in $||\ ||_{k,2}$ of Problem 10$\}$. This is one way to define Sobolev Spaces. **Hint:** One way to do the second part of this is to define a new measure, μ by

$$\mu(E)\equiv\int_E\left(1+|\mathbf{x}|^2\right)^k dx.$$

Then show μ is a Radon measure and show there exists $\{g_m\}$ such that $g_m\in\mathcal{G}$ and $g_m\to Ff$ in $L^2(\mu)$. Thus $g_m = Ff_m,\ f_m\in\mathcal{G}$ because F maps $\mathcal{G}$ onto $\mathcal{G}$. Then by Problem 10, $\{f_m\}$ is Cauchy in the norm $||\ ||_{k,2}$.

12. $\uparrow$ If $2k>n$, show that if $f\in H^k(\mathbb{R}^n)$, then f equals a bounded continuous function a.e. **Hint:** Show that for k this large, $Ff\in L^1(\mathbb{R}^n)$, and then use Problem 1. To do this, write

$$|Ff(\mathbf{x})| = |Ff(\mathbf{x})|(1+|\mathbf{x}|^2)^{\frac{k}{2}}(1+|\mathbf{x}|^2)^{\frac{-k}{2}},$$

So

$$\int|Ff(\mathbf{x})|dx=\int|Ff(\mathbf{x})|(1+|\mathbf{x}|^2)^{\frac{k}{2}}(1+|\mathbf{x}|^2)^{\frac{-k}{2}}dx.$$

Use the Cauchy Schwarz inequality. This is an example of a Sobolev imbedding Theorem.

13. Let $u \in \mathcal{G}$. Then $Fu \in \mathcal{G}$ and so, in particular, it makes sense to form the integral,
$$\int_{\mathbb{R}} Fu\left(\mathbf{x}', x_n\right) dx_n$$
where $\left(\mathbf{x}', x_n\right) = \mathbf{x} \in \mathbb{R}^n$. For $u \in \mathcal{G}$, define $\gamma u\left(\mathbf{x}'\right) \equiv u\left(\mathbf{x}', 0\right)$. Find a constant such that $F\left(\gamma u\right)\left(\mathbf{x}'\right)$ equals this constant times the above integral. **Hint:** By the dominated convergence theorem
$$\int_{\mathbb{R}} Fu\left(\mathbf{x}', x_n\right) dx_n = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} Fu\left(\mathbf{x}', x_n\right) dx_n.$$
Now use the definition of the Fourier transform and Fubini's theorem as required in order to obtain the desired relationship.

14. Recall the Fourier series of a function in $L^2(-\pi, \pi)$ converges to the function in $L^2(-\pi, \pi)$. Prove a similar theorem with $L^2(-\pi, \pi)$ replaced by $L^2(-m\pi, m\pi)$ and the functions
$$\left\{ (2\pi)^{-(1/2)} e^{inx} \right\}_{n \in \mathbb{Z}}$$
used in the Fourier series replaced with
$$\left\{ (2m\pi)^{-(1/2)} e^{i \frac{n}{m} x} \right\}_{n \in \mathbb{Z}}$$
Now suppose f is a function in $L^2(\mathbb{R})$ satisfying $Ff(t) = 0$ if $|t| > m\pi$. Show that if this is so, then
$$f(x) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} f\left(\frac{-n}{m}\right) \frac{\sin\left(\pi\left(mx + n\right)\right)}{mx + n}.$$
Here m is a positive integer. This is sometimes called the Shannon sampling theorem.**Hint:** First note that since $Ff \in L^2$ and is zero off a finite interval, it follows $Ff \in L^1$. Also
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-m\pi}^{m\pi} e^{itx} Ff(x)\, dx$$
and you can conclude from this that f has all derivatives and they are all bounded. Thus f is a very nice function. You can replace Ff with its Fourier series. Then consider carefully the Fourier coefficient of Ff. Argue it equals $f\left(\frac{-n}{m}\right)$ or at least an appropriate constant times this. When you get this the rest will fall quickly into place if you use Ff is zero off $[-m\pi, m\pi]$.

Part III

Complex Analysis

The Complex Numbers

The reader is presumed familiar with the algebraic properties of complex numbers, including the operation of conjugation. Here a short review of the distance in $\mathbb{C}$ is presented.

The length of a complex number, referred to as the modulus of z and denoted by $|z|$ is given by

$$|z| \equiv \left(x^2 + y^2\right)^{1/2} = (z\overline{z})^{1/2},$$

Then $\mathbb{C}$ is a metric space with the distance between two complex numbers, z and w defined as

$$d(z, w) \equiv |z - w|.$$

This metric on $\mathbb{C}$ is the same as the usual metric of $\mathbb{R}^2$. A sequence, $z_n \to z$ if and only if $x_n \to x$ in $\mathbb{R}$ and $y_n \to y$ in $\mathbb{R}$ where $z = x + iy$ and $z_n = x_n + iy_n$. For example if $z_n = \frac{n}{n+1} + i\frac{1}{n}$, then $z_n \to 1 + 0i = 1$.

Definition 16.1 *A sequence of complex numbers, $\{z_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that $n, m > N$ implies $|z_n - z_m| < \varepsilon$.*

This is the usual definition of Cauchy sequence. There are no new ideas here.

Proposition 16.2 *The complex numbers with the norm just mentioned forms a complete normed linear space.*

Proof: Let $\{z_n\}$ be a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers and so they converge to real numbers, x and y respectively. Thus $z_n = x_n + iy_n \to x + iy$. $\mathbb{C}$ is a linear space with the field of scalars equal to $\mathbb{C}$. It only remains to verify that $|\ |$ satisfies the axioms of a norm which are:

$$|z + w| \leq |z| + |w|$$

$$|z| \geq 0 \text{ for all } z$$

$$|z| = 0 \text{ if and only if } z = 0$$

$$|\alpha z| = |\alpha|\,|z|.$$

The only one of these axioms of a norm which is not completely obvious is the first one, the triangle inequality. Let $z = x + iy$ and $w = u + iv$

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}) \\ &\leq |z|^2 + |w|^2 + 2|(z\overline{w})| = (|z| + |w|)^2 \end{aligned}$$

and this verifies the triangle inequality.

Definition 16.3 *An infinite sum of complex numbers is defined as the limit of the sequence of partial sums. Thus,*

$$\sum_{k=1}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Just as in the case of sums of real numbers, an infinite sum converges if and only if the sequence of partial sums is a Cauchy sequence.

From now on, when f is a function of a complex variable, it will be assumed that f has values in X, a complex Banach space. Usually in complex analysis courses, f has values in $\mathbb{C}$ but there are many important theorems which don't require this so I will leave it fairly general for a while. Later the functions will have values in $\mathbb{C}$. If you are only interested in this case, think $\mathbb{C}$ whenever you see X.

Definition 16.4 *A sequence of functions of a complex variable, $\{f_n\}$ converges uniformly to a function, g for $z \in S$ if for every $\varepsilon > 0$ there exists N_ε such that if $n > N_\varepsilon$, then*

$$||f_n(z) - g(z)|| < \varepsilon$$

for all $z \in S$. The infinite sum $\sum_{k=1}^{\infty} f_n$ converges uniformly on S if the partial sums converge uniformly on S. Here $||\cdot||$ refers to the norm in X, the Banach space in which f has its values.

The following proposition is also a routine application of the above definition. Neither the definition nor this proposition say anything new.

Proposition 16.5 *A sequence of functions, $\{f_n\}$ defined on a set S, converges uniformly to some function, g if and only if for all $\varepsilon > 0$ there exists N_ε such that whenever $m, n > N_\varepsilon$,*

$$||f_n - f_m||_\infty < \varepsilon.$$

Here $||f||_\infty \equiv \sup\{||f(z)|| : z \in S\}$.

Just as in the case of functions of a real variable, one of the important theorems is the Weierstrass M test. Again, there is nothing new here. It is just a review of earlier material.

Theorem 16.6 *Let $\{f_n\}$ be a sequence of complex valued functions defined on $S \subseteq \mathbb{C}$. Suppose there exists M_n such that $||f_n||_\infty < M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly on S.*

Proof: Let $z \in S$. Then letting $m < n$

$$\left|\left|\sum_{k=1}^{n} f_k(z) - \sum_{k=1}^{m} f_k(z)\right|\right| \leq \sum_{k=m+1}^{n} ||f_k(z)|| \leq \sum_{k=m+1}^{\infty} M_k < \varepsilon$$

whenever m is large enough. Therefore, the sequence of partial sums is uniformly Cauchy on S and therefore, converges uniformly to $\sum_{k=1}^{\infty} f_k(z)$ on S.

16.1 The Extended Complex Plane

The set of complex numbers has already been considered along with the topology of $\mathbb{C}$ which is nothing but the topology of $\mathbb{R}^2$. Thus, for $z_n = x_n + iy_n$, $z_n \to z \equiv x + iy$ if and only if $x_n \to x$ and $y_n \to y$. The norm in $\mathbb{C}$ is given by

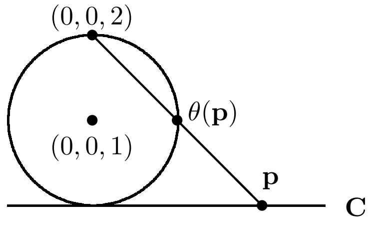
$$|x + iy| \equiv ((x + iy)(x - iy))^{1/2} = \left(x^2 + y^2\right)^{1/2}$$

which is just the usual norm in $\mathbb{R}^2$ identifying (x, y) with $x + iy$. Therefore, $\mathbb{C}$ is a complete metric space topologically like $\mathbb{R}^2$ and so the Heine Borel theorem that compact sets are those which are closed and bounded is valid. Thus, as far as topology is concerned, there is nothing new about $\mathbb{C}$.

The extended complex plane, denoted by $\widehat{\mathbb{C}}$, consists of the complex plane, $\mathbb{C}$ along with another point not in $\mathbb{C}$ known as ∞. For example, ∞ could be any point in $\mathbb{R}^3$. A sequence of complex numbers, z_n, converges to ∞ if, whenever K is a compact set in $\mathbb{C}$, there exists a number, N such that for all $n > N$, $z_n \notin K$. Since compact sets in $\mathbb{C}$ are closed and bounded, this is equivalent to saying that for all $R > 0$, there exists N such that if $n > N$, then $z_n \notin B(0, R)$ which is the same as saying $\lim_{n\to\infty} |z_n| = \infty$ where this last symbol has the same meaning as it does in calculus.

A geometric way of understanding this in terms of more familiar objects involves a concept known as the Riemann sphere.

Consider the unit sphere, S^2 given by $(z - 1)^2 + y^2 + x^2 = 1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, p in the complex plane to the point $(0, 0, 2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty) \equiv (0, 0, 2)$.



Then θ^{-1} is sometimes called sterographic projection. The mapping θ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, z_n converges to ∞ if and only if θz_n converges to $(0,0,2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta(z_n) \to \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 16.7 *Let $z, w \in \widehat{\mathbb{C}}$ including possibly $w = \infty$. Then let $d(x,w) \equiv |\theta(z) - \theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^3$.*

Theorem 16.8 $\left(\widehat{\mathbb{C}}, d\right)$ *is a compact, hence complete metric space.*

Proof: Suppose $\{z_n\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\{\theta(z_n)\}$ is a sequence in S^2 which is compact. Therefore, there exists a subsequence, $\{\theta z_{n_k}\}$ and a point, $z \in S^2$ such that $\theta z_{n_k} \to \theta z$ in S^2 which implies immediately that $d(z_{n_k}, z) \to 0$. A compact metric space must be complete.

16.2 Exercises

1. Prove the root test for series of complex numbers. If $a_k \in \mathbb{C}$ and $r \equiv \limsup_{n\to\infty} |a_n|^{1/n}$ then
$$\sum_{k=0}^{\infty} a_k \begin{cases} \text{converges absolutely if } r < 1 \\ \text{diverges if } r > 1 \\ \text{test fails if } r = 1. \end{cases}$$

2. Does $\lim_{n\to\infty} n\left(\frac{2+i}{3}\right)^n$ exist? Tell why and find the limit if it does exist.

3. Let $A_0 = 0$ and let $A_n \equiv \sum_{k=1}^{n} a_k$ if $n > 0$. Prove the partial summation formula,
$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$
Now using this formula, suppose $\{b_n\}$ is a sequence of real numbers which converges to 0 and is decreasing. Determine those values of ω such that $|\omega| = 1$ and $\sum_{k=1}^{\infty} b_k \omega^k$ converges.

4. Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be given by $f(x+iy) = u(x,y) + iv(x,y)$. Show f is continuous on U if and only if $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are both continuous.

Riemann Stieltjes Integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. I will base this on the notion of Riemann Stieltjes integrals as in [11], [31], and [23]. The Riemann Stieltjes integral is a generalization of the usual Riemann integral and requires the concept of a function of bounded variation.

Definition 17.1 *Let $\gamma : [a,b] \to \mathbb{C}$ be a function. Then γ is of bounded variation if*

$$\sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < \cdots < t_n = b \right\} \equiv V(\gamma, [a,b]) < \infty$$

where the sums are taken over all possible lists, $\{a = t_0 < \cdots < t_n = b\}$.

The idea is that it makes sense to talk of the length of the curve $\gamma([a,b])$, defined as $V(\gamma,[a,b])$. For this reason, in the case that γ is continuous, such an image of a bounded variation function is called a rectifiable curve.

Definition 17.2 *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to X$. Letting $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, define*

$$||\mathcal{P}|| \equiv \max \{|t_j - t_{j-1}| : j = 1, \cdots, n\}$$

and the Riemann Steiltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^{n} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, τ_j used. It is understood that this point is arbitrary.) Define $\int_\gamma f d\gamma$ as the unique number which satisfies the following condition. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||\mathcal{P}|| \leq \delta$, then

$$\left| \int_\gamma f d\gamma - S(\mathcal{P}) \right| < \varepsilon.$$

Sometimes this is written as

$$\int_{\gamma} f d\gamma \equiv \lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}).$$

The set of points in the curve, $\gamma([a, b])$ *will be denoted sometimes by* γ^*.

Then γ^* is a set of points in $\mathbb{C}$ and as t moves from a to b, $\gamma(t)$ moves from $\gamma(a)$ to $\gamma(b)$. Thus γ^* has a first point and a last point. If $\phi : [c, d] \to [a, b]$ is a continuous nondecreasing function, then $\gamma \circ \phi : [c, d] \to \mathbb{C}$ is also of bounded variation and yields the same set of points in $\mathbb{C}$ with the same first and last points.

Theorem 17.3 *Let* ϕ *and* γ *be as just described. Then assuming that*

$$\int_{\gamma} f d\gamma$$

exists, so does

$$\int_{\gamma \circ \phi} f d(\gamma \circ \phi)$$

and

$$\int_{\gamma} f d\gamma = \int_{\gamma \circ \phi} f d(\gamma \circ \phi). \tag{17.1}$$

Proof: There exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ such that $||\mathcal{P}|| < \delta$, then

$$\left| \int_{\gamma} f d\gamma - S(\mathcal{P}) \right| < \varepsilon.$$

By continuity of ϕ, there exists $\sigma > 0$ such that if $\mathcal{Q}$ is a partition of $[c, d]$ with $||\mathcal{Q}|| < \sigma, \mathcal{Q} = \{s_0, \cdots, s_n\}$, then $|\phi(s_j) - \phi(s_{j-1})| < \delta$. Thus letting $\mathcal{P}$ denote the points in $[a, b]$ given by $\phi(s_j)$ for $s_j \in \mathcal{Q}$, it follows that $||\mathcal{P}|| < \delta$ and so

$$\left| \int_{\gamma} f d\gamma - \sum_{j=1}^{n} f(\gamma(\phi(\tau_j)))(\gamma(\phi(s_j)) - \gamma(\phi(s_{j-1}))) \right| < \varepsilon$$

where $\tau_j \in [s_{j-1}, s_j]$. Therefore, from the definition 17.1 holds and

$$\int_{\gamma \circ \phi} f d(\gamma \circ \phi)$$

exists.

This theorem shows that $\int_{\gamma} f d\gamma$ is independent of the particular γ used in its computation to the extent that if ϕ is any nondecreasing function from another interval, $[c, d]$, mapping to $[a, b]$, then the same value is obtained by replacing γ with $\gamma \circ \phi$.

The fundamental result in this subject is the following theorem.

Theorem 17.4 *Let $f : \gamma^* \to X$ be continuous and let $\gamma : [a,b] \to \mathbb{C}$ be continuous and of bounded variation. Then $\int_\gamma f d\gamma$ exists. Also letting $\delta_m > 0$ be such that $|t-s| < \delta_m$ implies $||f(\gamma(t)) - f(\gamma(s))|| < \frac{1}{m}$,*

$$\left| \int_\gamma f d\gamma - S(\mathcal{P}) \right| \leq \frac{2V(\gamma,[a,b])}{m}$$

whenever $||\mathcal{P}|| < \delta_m$.

Proof: The function, $f \circ \gamma$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\{\delta_m\}$ such that if $|s-t| < \delta_m$, then

$$|f(\gamma(t)) - f(\gamma(s))| < \frac{1}{m}.$$

Let

$$F_m \equiv \overline{\{S(\mathcal{P}) : ||\mathcal{P}|| < \delta_m\}}.$$

Thus F_m is a closed set. (The symbol, $S(\mathcal{P})$ in the above definition, means to include all sums corresponding to $\mathcal{P}$ for any choice of τ_j.) It is shown that

$$\operatorname{diam}(F_m) \leq \frac{2V(\gamma,[a,b])}{m} \tag{17.2}$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^\infty F_m$. This is because X is complete. It will then follow $I = \int_\gamma f(t)\, d\gamma(t)$. To verify 17.2, it suffices to verify that whenever $\mathcal{P}$ and $\mathcal{Q}$ are partitions satisfying $||\mathcal{P}|| < \delta_m$ and $||\mathcal{Q}|| < \delta_m$,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq \frac{2}{m} V(\gamma,[a,b]). \tag{17.3}$$

Suppose $||\mathcal{P}|| < \delta_m$ and $\mathcal{Q} \supseteq \mathcal{P}$. Then also $||\mathcal{Q}|| < \delta_m$. To begin with, suppose that $\mathcal{P} \equiv \{t_0, \cdots, t_p, \cdots, t_n\}$ and $\mathcal{Q} \equiv \{t_0, \cdots, t_{p-1}, t^*, t_p, \cdots, t_n\}$. Thus $\mathcal{Q}$ contains only one more point than $\mathcal{P}$. Letting $S(\mathcal{Q})$ and $S(\mathcal{P})$ be Riemann Steiltjes sums,

$$S(\mathcal{Q}) \equiv \sum_{j=1}^{p-1} f(\gamma(\sigma_j))(\gamma(t_j) - \gamma(t_{j-1})) + f(\gamma(\sigma_*))(\gamma(t^*) - \gamma(t_{p-1}))$$

$$+ f(\gamma(\sigma^*))(\gamma(t_p) - \gamma(t^*)) + \sum_{j=p+1}^{n} f(\gamma(\sigma_j))(\gamma(t_j) - \gamma(t_{j-1})),$$

$$S(\mathcal{P}) \equiv \sum_{j=1}^{p-1} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})) +$$

$$\overbrace{f(\gamma(\tau_p))(\gamma(t^*) - \gamma(t_{p-1})) + f(\gamma(\tau_p))(\gamma(t_p) - \gamma(t^*))}^{=f(\gamma(\tau_p))(\gamma(t_p)-\gamma(t_{p-1}))}$$

$$+\sum_{j=p+1}^{n} f(\gamma(\tau_j))(\gamma(t_j)-\gamma(t_{j-1})).$$

Therefore,

$$|S(\mathcal{P})-S(\mathcal{Q})| \leq \sum_{j=1}^{p-1} \frac{1}{m}|\gamma(t_j)-\gamma(t_{j-1})| + \frac{1}{m}|\gamma(t^*)-\gamma(t_{p-1})| +$$

$$\frac{1}{m}|\gamma(t_p)-\gamma(t^*)| + \sum_{j=p+1}^{n} \frac{1}{m}|\gamma(t_j)-\gamma(t_{j-1})| \leq \frac{1}{m} V(\gamma,[a,b]). \tag{17.4}$$

Clearly the extreme inequalities would be valid in 17.4 if $\mathcal{Q}$ had more than one extra point. You simply do the above trick more than one time. Let $S(\mathcal{P})$ and $S(\mathcal{Q})$ be Riemann Steiltjes sums for which $||\mathcal{P}||$ and $||\mathcal{Q}||$ are less than δ_m and let $\mathcal{R} \equiv \mathcal{P} \cup \mathcal{Q}$. Then from what was just observed,

$$|S(\mathcal{P})-S(\mathcal{Q})| \leq |S(\mathcal{P})-S(\mathcal{R})| + |S(\mathcal{R})-S(\mathcal{Q})| \leq \frac{2}{m} V(\gamma,[a,b]).$$

and this shows 17.3 which proves 17.2. Therefore, there exists a unique complex number, $I \in \cap_{m=1}^{\infty} F_m$ which satisfies the definition of $\int_\gamma f d\gamma$. This proves the theorem.

The following theorem follows easily from the above definitions and theorem.

Theorem 17.5 *Let $f \in C(\gamma^*)$ and let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and continuous. Let*

$$M \geq \max\{||f \circ \gamma(t)|| : t \in [a,b]\}. \tag{17.5}$$

Then

$$\left|\left|\int_\gamma f d\gamma\right|\right| \leq M V(\gamma,[a,b]). \tag{17.6}$$

Also if $\{f_n\}$ is a sequence of functions of $C(\gamma^*)$ which is converging uniformly to the function, f on γ^*, then

$$\lim_{n\to\infty} \int_\gamma f_n d\gamma = \int_\gamma f d\gamma. \tag{17.7}$$

Proof: Let 17.5 hold. From the proof of the above theorem, when $||\mathcal{P}|| < \delta_m$,

$$\left|\left|\int_\gamma f d\gamma - S(\mathcal{P})\right|\right| \leq \frac{2}{m} V(\gamma,[a,b])$$

and so

$$\left|\left|\int_\gamma f d\gamma\right|\right| \leq ||S(\mathcal{P})|| + \frac{2}{m} V(\gamma,[a,b])$$

$$\begin{aligned} &\leq \sum_{j=1}^{n} M\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{2}{m} V(\gamma,[a, b]) \\ &\leq M V(\gamma,[a, b])+\frac{2}{m} V(\gamma,[a, b]) . \end{aligned}$$

This proves 17.6 since m is arbitrary. To verify 17.7 use the above inequality to write

$$\left|\left|\int_{\gamma} f d\gamma-\int_{\gamma} f_{n} d\gamma\right|\right|=\left|\left|\int_{\gamma}\left(f-f_{n}\right) d\gamma(t)\right|\right|$$
$$\leq \max \left\{\left|\left|f \circ \gamma(t)-f_{n} \circ \gamma(t)\right|\right|: t \in[a, b]\right\} V(\gamma,[a, b]) .$$

Since the convergence is assumed to be uniform, this proves 17.7.

It turns out to be much easier to evaluate such integrals in the case where γ is also $C^{1}([a, b])$. The following theorem about approximation will be very useful but first here is an easy lemma.

Lemma 17.6 *Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be in $C^{1}([a, b])$. Then $V(\gamma,[a, b])<\infty$ so γ is of bounded variation.*

Proof: This follows from the following

$$\begin{aligned} \sum_{j=1}^{n}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| &= \sum_{j=1}^{n}\left|\int_{t_{j-1}}^{t_{j}} \gamma^{\prime}(s) ds\right| \\ &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}}\left|\gamma^{\prime}(s)\right| ds \\ &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}}\left|\left|\gamma^{\prime}\right|\right|_{\infty} ds \\ &= \left|\left|\gamma^{\prime}\right|\right|_{\infty}(b-a) . \end{aligned}$$

Therefore it follows $V(\gamma,[a, b]) \leq\left|\left|\gamma^{\prime}\right|\right|_{\infty}(b-a)$. Here $||\gamma||_{\infty}=\max \{|\gamma(t)|: t \in[a, b]\}$.

Theorem 17.7 *Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Let Ω be an open set containing γ^{*} and let $f: \Omega \times K \rightarrow X$ be continuous for K a compact set in $\mathbb{C}$, and let $\varepsilon>0$ be given. Then there exists $\eta:[a, b] \rightarrow \mathbb{C}$ such that $\eta(a)=\gamma(a), \gamma(b)=\eta(b), \eta \in C^{1}([a, b])$, and*

$$||\gamma-\eta||<\varepsilon, \tag{17.8}$$

$$\left|\int_{\gamma} f(\cdot, z) d\gamma-\int_{\eta} f(\cdot, z) d\eta\right|<\varepsilon, \tag{17.9}$$

$$V(\eta,[a, b]) \leq V(\gamma,[a, b]), \tag{17.10}$$

where $||\gamma-\eta|| \equiv \max \{|\gamma(t)-\eta(t)|: t \in[a, b]\}$.

Proof: Extend γ to be defined on all $\mathbb{R}$ according to $\gamma(t) = \gamma(a)$ if $t < a$ and $\gamma(t) = \gamma(b)$ if $t > b$. Now define

$$\gamma_h(t) \equiv \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \gamma(s)\, ds.$$

where the integral is defined in the obvious way. That is,

$$\int_a^b \alpha(t) + i\beta(t)\, dt \equiv \int_a^b \alpha(t)\, dt + i \int_a^b \beta(t)\, dt.$$

Therefore,

$$\gamma_h(b) = \frac{1}{2h} \int_b^{b+2h} \gamma(s)\, ds = \gamma(b),$$
$$\gamma_h(a) = \frac{1}{2h} \int_{a-2h}^{a} \gamma(s)\, ds = \gamma(a).$$

Also, because of continuity of γ and the fundamental theorem of calculus,

$$\gamma_h'(t) = \frac{1}{2h} \left\{ \gamma\left(t + \frac{2h}{b-a}(t-a)\right)\left(1 + \frac{2h}{b-a}\right) - \right.$$
$$\left. \gamma\left(-2h + t + \frac{2h}{b-a}(t-a)\right)\left(1 + \frac{2h}{b-a}\right)\right\}$$

and so $\gamma_h \in C^1([a,b])$. The following lemma is significant.

Lemma 17.8 $V(\gamma_h, [a,b]) \leq V(\gamma, [a,b])$.

Proof: Let $a = t_0 < t_1 < \cdots < t_n = b$. Then using the definition of γ_h and changing the variables to make all integrals over $[0, 2h]$,

$$\sum_{j=1}^{n} |\gamma_h(t_j) - \gamma_h(t_{j-1})| =$$

$$\sum_{j=1}^{n} \left| \frac{1}{2h} \int_0^{2h} \left[\gamma\left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a)\right) - \right.\right.$$
$$\left.\left. \gamma\left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a)\right)\right]\right|$$
$$\leq \frac{1}{2h} \int_0^{2h} \sum_{j=1}^{n} \left| \gamma\left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a)\right) - \right.$$
$$\left. \gamma\left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a)\right)\right| ds.$$

For a given $s \in [0, 2h]$, the points, $s - 2h + t_j + \frac{2h}{b-a}(t_j - a)$ for $j = 1, \cdots, n$ form an increasing list of points in the interval $[a - 2h, b + 2h]$ and so the integrand is bounded above by $V(\gamma, [a - 2h, b + 2h]) = V(\gamma, [a, b])$. It follows

$$\sum_{j=1}^{n} |\gamma_h(t_j) - \gamma_h(t_{j-1})| \leq V(\gamma, [a, b])$$

which proves the lemma.

With this lemma the proof of the theorem can be completed without too much trouble. Let H be an open set containing γ^* such that $\overline{H}$ is a compact subset of Ω. Let $0 < \varepsilon < \text{dist}\left(\gamma^*, H^C\right)$. Then there exists δ_1 such that if $h < \delta_1$, then for all t,

$$\begin{aligned} |\gamma(t) - \gamma_h(t)| &\leq \frac{1}{2h}\int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} |\gamma(s) - \gamma(t)|\, ds \\ &< \frac{1}{2h}\int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \varepsilon ds = \varepsilon \end{aligned} \tag{17.11}$$

due to the uniform continuity of γ. This proves 17.8.

From 17.2 and the above lemma, there exists δ_2 such that if $||\mathcal{P}|| < \delta_2$, then for all $z \in K$,

$$\left|\left|\int_{\gamma} f(\cdot, z)\, d\gamma(t) - S(\mathcal{P})\right|\right| < \frac{\varepsilon}{3}, \left|\left|\int_{\gamma_h} f(\cdot, z)\, d\gamma_h(t) - S_h(\mathcal{P})\right|\right| < \frac{\varepsilon}{3}$$

for all h. Here $S(\mathcal{P})$ is a Riemann Steiltjes sum of the form

$$\sum_{i=1}^{n} f(\gamma(\tau_i), z)(\gamma(t_i) - \gamma(t_{i-1}))$$

and $S_h(\mathcal{P})$ is a similar Riemann Steiltjes sum taken with respect to γ_h instead of γ. Because of 17.11 $\gamma_h(t)$ has values in $H \subseteq \Omega$. Therefore, fix the partition, $\mathcal{P}$, and choose h small enough that in addition to this, the following inequality is valid for all $z \in K$.

$$|S(\mathcal{P}) - S_h(\mathcal{P})| < \frac{\varepsilon}{3}$$

This is possible because of 17.11 and the uniform continuity of f on $\overline{H} \times K$. It follows

$$\left|\left|\int_{\gamma} f(\cdot, z)\, d\gamma(t) - \int_{\gamma_h} f(\cdot, z)\, d\gamma_h(t)\right|\right| \leq$$

$$\left|\left|\int_{\gamma} f(\cdot, z)\, d\gamma(t) - S(\mathcal{P})\right|\right| + ||S(\mathcal{P}) - S_h(\mathcal{P})||$$

$$+ \left|\left|S_h(\mathcal{P}) - \int_{\gamma_h} f(\cdot, z)\, d\gamma_h(t)\right|\right| < \varepsilon.$$

Formula 17.10 follows from the lemma. This proves the theorem.

Of course the same result is obtained without the explicit dependence of f on z.

This is a very useful theorem because if γ is $C^1([a,b])$, it is easy to calculate $\int_\gamma f d\gamma$ and the above theorem allows a reduction to the case where γ is C^1. The next theorem shows how easy it is to compute these integrals in the case where γ is C^1. First note that if f is continuous and $\gamma \in C^1([a,b])$, then by Lemma 17.6 and the fundamental existence theorem, Theorem 17.4, $\int_\gamma f d\gamma$ exists.

Theorem 17.9 *If $f : \gamma^* \to X$ is continuous and $\gamma : [a,b] \to \mathbb{C}$ is in $C^1([a,b])$, then*

$$\int_\gamma f d\gamma = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{17.12}$$

Proof: Let $\mathcal{P}$ be a partition of $[a,b]$, $\mathcal{P} = \{t_0, \cdots, t_n\}$ and $||\mathcal{P}||$ is small enough that whenever $|t-s| < ||\mathcal{P}||$,

$$|f(\gamma(t)) - f(\gamma(s))| < \varepsilon \tag{17.13}$$

and

$$\left|\left|\int_\gamma f d\gamma - \sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))\right|\right| < \varepsilon.$$

Now

$$\sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})) = \int_a^b \sum_{j=1}^n f(\gamma(\tau_j))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds$$

where here

$$\mathcal{X}_{[a,b]}(s) \equiv \begin{cases} 1 \text{ if } s \in [a,b] \\ 0 \text{ if } s \notin [a,b] \end{cases}.$$

Also,

$$\int_a^b f(\gamma(s))\gamma'(s)\,ds = \int_a^b \sum_{j=1}^n f(\gamma(s))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds$$

and thanks to 17.13,

$$\left|\left|\overbrace{\int_a^b \sum_{j=1}^n f(\gamma(\tau_j))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds}^{=\sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j)-\gamma(t_{j-1}))} - \overbrace{\int_a^b \sum_{j=1}^n f(\gamma(s))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds}^{=\int_a^b f(\gamma(s))\gamma'(s)ds}\right|\right|$$

$$\begin{aligned}
&\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||f(\gamma(\tau_j)) - f(\gamma(s))||\,|\gamma'(s)|\,ds \leq ||\gamma'||_\infty \sum_j \varepsilon(t_j - t_{j-1}) \\
&= \varepsilon\,||\gamma'||_\infty (b-a).
\end{aligned}$$

It follows that

$$\left|\left|\int_\gamma fd\gamma-\int_a^b f\left(\gamma\left(s\right)\right)\gamma'\left(s\right)ds\right|\right|\leq\left|\left|\int_\gamma fd\gamma-\sum_{j=1}^n f\left(\gamma\left(\tau_j\right)\right)\left(\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right)\right|\right|$$

$$+\left|\left|\sum_{j=1}^n f\left(\gamma\left(\tau_j\right)\right)\left(\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right)-\int_a^b f\left(\gamma\left(s\right)\right)\gamma'\left(s\right)ds\right|\right|\leq\varepsilon\left|\left|\gamma'\right|\right|_\infty\left(b-a\right)+\varepsilon.$$

Since ε is arbitrary, this verifies 17.12.

Definition 17.10 *Let Ω be an open subset of $\mathbb{C}$ and let $\gamma:[a,b]\to\Omega$ be a continuous function with bounded variation $f:\Omega\to X$ be a continuous function. Then the following notation is more customary.*

$$\int_\gamma f\left(z\right)dz\equiv\int_\gamma fd\gamma.$$

The expression, $\int_\gamma f\left(z\right)dz$, is called a contour integral and γ is referred to as the contour. A function $f:\Omega\to X$ for Ω an open set in $\mathbb{C}$ has a primitive if there exists a function, F, the primitive, such that $F'\left(z\right)=f\left(z\right)$. Thus F is just an antiderivative. Also if $\gamma_k:[a_k,b_k]\to\mathbb{C}$ is continuous and of bounded variation, for $k=1,\cdots,m$ and $\gamma_k\left(b_k\right)=\gamma_{k+1}\left(a_k\right)$, define

$$\int_{\sum_{k=1}^m\gamma_k}f\left(z\right)dz\equiv\sum_{k=1}^m\int_{\gamma_k}f\left(z\right)dz. \tag{17.14}$$

In addition to this, for $\gamma:[a,b]\to\mathbb{C}$, define $-\gamma:[a,b]\to\mathbb{C}$ by $-\gamma\left(t\right)\equiv\gamma\left(b+a-t\right)$. Thus γ simply traces out the points of γ^* in the opposite order.

The following lemma is useful and follows quickly from Theorem 17.3.

Lemma 17.11 *In the above definition, there exists a continuous bounded variation function, γ defined on some closed interval, $[c,d]$, such that $\gamma\left([c,d]\right)=\cup_{k=1}^m\gamma_k\left([a_k,b_k]\right)$ and $\gamma\left(c\right)=\gamma_1\left(a_1\right)$ while $\gamma\left(d\right)=\gamma_m\left(b_m\right)$. Furthermore,*

$$\int_\gamma f\left(z\right)dz=\sum_{k=1}^m\int_{\gamma_k}f\left(z\right)dz.$$

If $\gamma:[a,b]\to\mathbb{C}$ is of bounded variation and continuous, then

$$\int_\gamma f\left(z\right)dz=-\int_{-\gamma}f\left(z\right)dz.$$

Re stating Theorem 17.7 with the new notation in the above definition,

Theorem 17.12 *Let K be a compact set in $\mathbb{C}$ and let $f : \Omega \times K \to X$ be continuous for Ω an open set in $\mathbb{C}$. Also let $\gamma : [a,b] \to \Omega$ be continuous with bounded variation. Then if $r > 0$ is given, there exists $\eta : [a,b] \to \Omega$ such that $\eta(a) = \gamma(a)$, $\eta(b) = \gamma(b)$, η is $C^1([a,b])$, and*

$$\left| \int_\gamma f(z,w)\,dz - \int_\eta f(z,w)\,dz \right| < r, \ ||\eta - \gamma|| < r.$$

It will be very important to consider which functions have primitives. It turns out, it is not enough for f to be continuous in order to possess a primitive. This is in stark contrast to the situation for functions of a real variable in which the fundamental theorem of calculus will deliver a primitive for any continuous function. The reason for the interest in such functions is the following theorem and its corollary.

Theorem 17.13 *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and of bounded variation. Also suppose $F'(z) = f(z)$ for all $z \in \Omega$, an open set containing γ^* and f is continuous on Ω. Then*

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof: By Theorem 17.12 there exists $\eta \in C^1([a,b])$ such that $\gamma(a) = \eta(a)$, and $\gamma(b) = \eta(b)$ such that

$$\left|\left| \int_\gamma f(z)\,dz - \int_\eta f(z)\,dz \right|\right| < \varepsilon.$$

Then since η is in $C^1([a,b])$,

$$\begin{aligned} \int_\eta f(z)\,dz &= \int_a^b f(\eta(t))\,\eta'(t)\,dt = \int_a^b \frac{dF(\eta(t))}{dt}dt \\ &= F(\eta(b)) - F(\eta(a)) = F(\gamma(b)) - F(\gamma(a)). \end{aligned}$$

Therefore,

$$\left|\left| (F(\gamma(b)) - F(\gamma(a))) - \int_\gamma f(z)\,dz \right|\right| < \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

Corollary 17.14 *If $\gamma : [a,b] \to \mathbb{C}$ is continuous, has bounded variation, is a closed curve, $\gamma(a) = \gamma(b)$, and $\gamma^* \subseteq \Omega$ where Ω is an open set on which $F'(z) = f(z)$, then*

$$\int_\gamma f(z)\,dz = 0.$$

17.1 Exercises

1. Let $\gamma : [a, b] \to \mathbb{R}$ be increasing. Show $V(\gamma, [a, b]) = \gamma(b) - \gamma(a)$.

2. Suppose $\gamma : [a, b] \to \mathbb{C}$ satisfies a Lipschitz condition, $|\gamma(t) - \gamma(s)| \leq K|s - t|$. Show γ is of bounded variation and that $V(\gamma, [a, b]) \leq K|b - a|$.

3. $\gamma : [c_0, c_m] \to \mathbb{C}$ is piecewise smooth if there exist numbers, $c_k, k = 1, \cdots, m$ such that $c_0 < c_1 < \cdots < c_{m-1} < c_m$ such that γ is continuous and $\gamma : [c_k, c_{k+1}] \to \mathbb{C}$ is C^1. Show that such piecewise smooth functions are of bounded variation and give an estimate for $V(\gamma, [c_0, c_m])$.

4. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = r(\cos mt + i \sin mt)$ for m an integer. Find $\int_\gamma \frac{dz}{z}$.

5. Show that if $\gamma : [a, b] \to \mathbb{C}$ then there exists an increasing function $h : [0, 1] \to [a, b]$ such that $\gamma \circ h([0, 1]) = \gamma^*$.

6. Let $\gamma : [a, b] \to \mathbb{C}$ be an arbitrary continuous curve having bounded variation and let f, g have continuous derivatives on some open set containing γ^*. Prove the usual integration by parts formula.
$$\int_\gamma f g' dz = f(\gamma(b)) g(\gamma(b)) - f(\gamma(a)) g(\gamma(a)) - \int_\gamma f' g dz.$$

7. Let $f(z) \equiv |z|^{-(1/2)} e^{-i\frac{\theta}{2}}$ where $z = |z| e^{i\theta}$. This function is called the principle branch of $z^{-(1/2)}$. Find $\int_\gamma f(z)\, dz$ where γ is the semicircle in the upper half plane which goes from $(1, 0)$ to $(-1, 0)$ in the counter clockwise direction. Next do the integral in which γ goes in the clockwise direction along the semicircle in the lower half plane.

8. Prove an open set, U is connected if and only if for every two points in U, there exists a C^1 curve having values in U which joins them.

9. Let $\mathcal{P}, \mathcal{Q}$ be two partitions of $[a, b]$ with $\mathcal{P} \subseteq \mathcal{Q}$. Each of these partitions can be used to form an approximation to $V(\gamma, [a, b])$ as described above. Recall the total variation was the supremum of sums of a certain form determined by a partition. How is the sum associated with $\mathcal{P}$ related to the sum associated with $\mathcal{Q}$? Explain.

10. Consider the curve,
$$\gamma(t) = \begin{cases} t + it^2 \sin\left(\frac{1}{t}\right) \text{ if } t \in (0, 1] \\ 0 \text{ if } t = 0 \end{cases}.$$
Is γ a continuous curve having bounded variation? What if the t^2 is replaced with t? Is the resulting curve continuous? Is it a bounded variation curve?

11. Suppose $\gamma : [a, b] \to \mathbb{R}$ is given by $\gamma(t) = t$. What is $\int_\gamma f(t)\, d\gamma$? Explain.

Fundamentals Of Complex Analysis

18.1 Analytic Functions

Definition 18.1 *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to X$. Then f is analytic on Ω if for every $z \in \Omega$,*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \equiv f'(z)$$

exists and is a continuous function of $z \in \Omega$. Here $h \in \mathbb{C}$.

Note that if f is analytic, it must be the case that f is continuous. It is more common to not include the requirement that f' is continuous but it is shown later that the continuity of f' follows.

What are some examples of analytic functions? In the case where $X = \mathbb{C}$, the simplest example is any polynomial. Thus

$$p(z) \equiv \sum_{k=0}^{n} a_k z^k$$

is an analytic function and

$$p'(z) = \sum_{k=1}^{n} a_k k z^{k-1}.$$

More generally, power series are analytic. This will be shown soon but first here is an important definition and a convergence theorem called the root test.

Definition 18.2 *Let $\{a_k\}$ be a sequence in X. Then $\sum_{k=1}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ whenever this limit exists. When the limit exists, the series is said to converge.*

Theorem 18.3 *Consider* $\sum_{k=1}^{\infty} a_k$ *and let* $\rho \equiv \limsup_{k\to\infty} ||a_k||^{1/k}$ *. Then if* $\rho < 1$*, the series converges absolutely and if* $\rho > 1$ *the series diverges spectacularly in the sense that* $\lim_{k\to\infty} a_k \neq 0$*. If* $\rho = 1$ *the test fails. Also* $\sum_{k=1}^{\infty} a_k (z-a)^k$ *converges on some disk* $B(a,R)$ *. It converges absolutely if* $|z-a| < R$ *and uniformly on* $B(a,r_1)$ *whenever* $r_1 < R$*. The function* $f(z) = \sum_{k=1}^{\infty} a_k (z-a)^k$ *is continuous on* $B(a,R)$ *.*

Proof: Suppose $\rho < 1$. Then there exists $r \in (\rho, 1)$. Therefore, $||a_k|| \leq r^k$ for all k large enough and so by a comparison test, $\sum_k ||a_k||$ converges because the partial sums are bounded above. Therefore, the partial sums of the original series form a Cauchy sequence in X and so they also converge due to completeness of X.

Now suppose $\rho > 1$. Then letting $\rho > r > 1$, it follows $||a_k||^{1/k} \geq r$ infinitely often. Thus $||a_k|| \geq r^k$ infinitely often. Thus there exists a subsequence for which $||a_{n_k}||$ converges to ∞. Therefore, the series cannot converge.

Now consider $\sum_{k=1}^{\infty} a_k (z-a)^k$. This series converges absolutely if

$$\limsup_{k\to\infty} ||a_k||^{1/k} |z-a| < 1$$

which is the same as saying $|z-a| < 1/\rho$ where $\rho \equiv \limsup_{k\to\infty} ||a_k||^{1/k}$. Let $R = 1/\rho$.

Now suppose $r_1 < R$. Consider $|z-a| \leq r_1$. Then for such z,

$$||a_k|| \, |z-a|^k \leq ||a_k|| \, r_1^k$$

and

$$\limsup_{k\to\infty} \left(||a_k|| \, r_1^k\right)^{1/k} = \limsup_{k\to\infty} ||a_k||^{1/k} r_1 = \frac{r_1}{R} < 1$$

so $\sum_k ||a_k|| \, r_1^k$ converges. By the Weierstrass M test, $\sum_{k=1}^{\infty} a_k (z-a)^k$ converges uniformly for $|z-a| \leq r_1$. Therefore, f is continuous on $B(a,R)$ as claimed because it is the uniform limit of continuous functions, the partial sums of the infinite series.

What if $\rho = 0$? In this case,

$$\limsup_{k\to\infty} ||a_k||^{1/k} |z-a| = 0 \cdot |z-a| = 0$$

and so $R = \infty$ and the series, $\sum ||a_k|| \, |z-a|^k$ converges everywhere.

What if $\rho = \infty$? Then in this case, the series converges only at $z = a$ because if $z \neq a$,

$$\limsup_{k\to\infty} ||a_k||^{1/k} |z-a| = \infty.$$

Theorem 18.4 *Let* $f(z) \equiv \sum_{k=1}^{\infty} a_k (z-a)^k$ *be given in Theorem 18.3 where* $R > 0$*. Then* f *is analytic on* $B(a,R)$ *. So are all its derivatives.*

Proof: Consider $g(z) = \sum_{k=2}^{\infty} a_k k (z-a)^{k-1}$ on $B(a,R)$ where $R = \rho^{-1}$ as above. Let $r_1 < r < R$. Then letting $|z-a| < r_1$ and $h < r - r_1$,

$$
\begin{aligned}
&\left|\left|\frac{f(z+h) - f(z)}{h} - g(z)\right|\right| \\
\leq\ & \sum_{k=2}^{\infty} ||a_k|| \left| \frac{(z+h-a)^k - (z-a)^k}{h} - k(z-a)^{k-1} \right| \\
\leq\ & \sum_{k=2}^{\infty} ||a_k|| \left| \frac{1}{h} \left(\sum_{i=0}^{k} \binom{k}{i} (z-a)^{k-i} h^i - (z-a)^k \right) - k(z-a)^{k-1} \right| \\
=\ & \sum_{k=2}^{\infty} ||a_k|| \left| \frac{1}{h} \left(\sum_{i=1}^{k} \binom{k}{i} (z-a)^{k-i} h^i \right) - k(z-a)^{k-1} \right| \\
\leq\ & \sum_{k=2}^{\infty} ||a_k|| \left| \left(\sum_{i=2}^{k} \binom{k}{i} (z-a)^{k-i} h^{i-1} \right) \right| \\
\leq\ & |h| \sum_{k=2}^{\infty} ||a_k|| \left(\sum_{i=0}^{k-2} \binom{k}{i+2} |z-a|^{k-2-i} |h|^i \right) \\
=\ & |h| \sum_{k=2}^{\infty} ||a_k|| \left(\sum_{i=0}^{k-2} \binom{k-2}{i} \frac{k(k-1)}{(i+2)(i+1)} |z-a|^{k-2-i} |h|^i \right) \\
\leq\ & |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} \left(\sum_{i=0}^{k-2} \binom{k-2}{i} |z-a|^{k-2-i} |h|^i \right) \\
=\ & |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} (|z-a| + |h|)^{k-2} < |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} r^{k-2}.
\end{aligned}
$$

Then

$$
\limsup_{k \to \infty} \left(||a_k|| \frac{k(k-1)}{2} r^{k-2} \right)^{1/k} = \rho r < 1
$$

and so

$$
\left|\left| \frac{f(z+h) - f(z)}{h} - g(z) \right|\right| \leq C |h|.
$$

therefore, $g(z) = f'(z)$. Now by 18.3 it also follows that f' is continuous. Since $r_1 < R$ was arbitrary, this shows that $f'(z)$ is given by the differentiated series above for $|z-a| < R$. Now a repeat of the argument shows all the derivatives of f exist and are continuous on $B(a,R)$.

18.1.1 Cauchy Riemann Equations

Next consider the very important Cauchy Riemann equations which give conditions under which complex valued functions of a complex variable are analytic.

Theorem 18.5 *Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a function, such that for $z = x + iy \in \Omega$,*

$$f(z) = u(x, y) + iv(x, y).$$

Then f is analytic if and only if u, v are $C^1(\Omega)$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore,

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

Proof: Suppose f is analytic first. Then letting $t \in \mathbb{R}$,

$$f'(z) = \lim_{t \to 0} \frac{f(z+t) - f(z)}{t} =$$

$$\lim_{t \to 0} \left(\frac{u(x+t, y) + iv(x+t, y)}{t} - \frac{u(x, y) + iv(x, y)}{t} \right)$$

$$= \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}.$$

But also

$$f'(z) = \lim_{t \to 0} \frac{f(z+it) - f(z)}{it} =$$

$$\lim_{t \to 0} \left(\frac{u(x, y+t) + iv(x, y+t)}{it} - \frac{u(x, y) + iv(x, y)}{it} \right)$$

$$\frac{1}{i}\left(\frac{\partial u(x, y)}{\partial y} + i\frac{\partial v(x, y)}{\partial y} \right)$$

$$= \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y}.$$

This verifies the Cauchy Riemann equations. We are assuming that $z \to f'(z)$ is continuous. Therefore, the partial derivatives of u and v are also continuous. To see this, note that from the formulas for $f'(z)$ given above, and letting $z_1 = x_1 + iy_1$

$$\left| \frac{\partial v(x, y)}{\partial y} - \frac{\partial v(x_1, y_1)}{\partial y} \right| \leq |f'(z) - f'(z_1)|,$$

showing that $(x, y) \to \frac{\partial v(x,y)}{\partial y}$ is continuous since $(x_1, y_1) \to (x, y)$ if and only if $z_1 \to z$. The other cases are similar.

Now suppose the Cauchy Riemann equations hold and the functions, u and v are $C^1(\Omega)$. Then letting $h = h_1 + ih_2$,

$$f(z+h) - f(z) = u(x + h_1, y + h_2)$$

$$+iv(x+h_1,y+h_2)-(u(x,y)+iv(x,y))$$

We know u and v are both differentiable and so

$$f(z+h)-f(z)=\frac{\partial u}{\partial x}(x,y)h_1+\frac{\partial u}{\partial y}(x,y)h_2+$$

$$i\left(\frac{\partial v}{\partial x}(x,y)h_1+\frac{\partial v}{\partial y}(x,y)h_2\right)+o(h).$$

Dividing by h and using the Cauchy Riemann equations,

$$\frac{f(z+h)-f(z)}{h}=\frac{\frac{\partial u}{\partial x}(x,y)h_1+i\frac{\partial v}{\partial y}(x,y)h_2}{h}+$$

$$\frac{i\frac{\partial v}{\partial x}(x,y)h_1+\frac{\partial u}{\partial y}(x,y)h_2}{h}+\frac{o(h)}{h}$$

$$=\frac{\partial u}{\partial x}(x,y)\frac{h_1+ih_2}{h}+i\frac{\partial v}{\partial x}(x,y)\frac{h_1+ih_2}{h}+\frac{o(h)}{h}$$

Taking the limit as $h\to 0$,

$$f'(z)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y).$$

It follows from this formula and the assumption that u,v are $C^1(\Omega)$ that f' is continuous.

It is routine to verify that all the usual rules of derivatives hold for analytic functions. In particular, the product rule, the chain rule, and quotient rule.

18.1.2 An Important Example

An important example of an analytic function is $e^z\equiv\exp(z)\equiv e^x(\cos y+i\sin y)$ where $z=x+iy$. You can verify that this function satisfies the Cauchy Riemann equations and that all the partial derivatives are continuous. Also from the above discussion, $(e^z)'=e^x\cos(y)+ie^x\sin y=e^z$. Later I will show that e^z is given by the usual power series. An important property of this function is that it can be used to parameterize the circle centered at z_0 having radius r.

Lemma 18.6 *Let γ denote the closed curve which is a circle of radius r centered at z_0. Then a parameterization this curve is $\gamma(t)=z_0+re^{it}$ where $t\in[0,2\pi]$.*

Proof: $|\gamma(t)-z_0|^2=\left|re^{it}re^{-it}\right|=r^2$. Also, you can see from the definition of the sine and cosine that the point described in this way moves counter clockwise over this circle.

18.2 Exercises

1. Verify all the usual rules of differentiation including the product and chain rules.

2. Suppose f and $f' : U \to \mathbb{C}$ are analytic and $f(z) = u(x,y) + iv(x,y)$. Verify $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace's equation. We say these functions satisfying Laplace's equation are harmonic functions. If u is a harmonic function defined on $B(0,r)$ show that $v(x,y) \equiv \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt$ is such that $u + iv$ is analytic.

3. Let $f : U \to \mathbb{C}$ be analytic and $f(z) = u(x,y) + iv(x,y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.

4. Define a function $f(z) \equiv \overline{z} \equiv x - iy$ where $z = x + iy$. Is f analytic?

5. If $f(z) = u(x,y) + iv(x,y)$ and f is analytic, verify that
$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'(z)|^2 .$$

6. Show that if $u(x,y) + iv(x,y) = f(z)$ is analytic, then $\nabla u \cdot \nabla v = 0$. Recall
$$\nabla u(x,y) = \langle u_x(x,y), u_y(x,y)\rangle.$$

7. Show that every polynomial is analytic.

8. If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U, an open set of $\mathbb{C}$, and if $f : U \to \mathbb{C}$ is analytic, we can consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, $\langle x, y\rangle$. Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \to \mathbb{C}$ is analytic. Show that the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$ is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. . **Hint:** To make this easy to show, first observe that $\langle x, y\rangle \cdot \langle a, b\rangle = \frac{1}{2}(z\overline{w} + \overline{z}w)$ where $z = x + iy$ and $w = a + ib$.

9. Analytic functions are even better than what is described in Problem 8. In addition to preserving angles, they also preserve orientation. To verify this show that if $z = x + iy$ and $w = a + ib$ are two complex numbers, then $\langle x, y, 0\rangle$ and $\langle a, b, 0\rangle$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $\langle x, y, 0\rangle \times \langle a, b, 0\rangle$, yields a vector normal to the two given vectors such that the triple, $\langle x, y, 0\rangle$, $\langle a, b, 0\rangle$, and $\langle x, y, 0\rangle \times \langle a, b, 0\rangle$ satisfies the right hand rule

and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product either points in the direction of the positive z axis or in the direction of the negative z axis. Thus, either the vectors $\langle x, y, 0\rangle, \langle a, b, 0\rangle, \mathbf{k}$ form a right handed system or the vectors $\langle a, b, 0\rangle, \langle x, y, 0\rangle, \mathbf{k}$ form a right handed system. These are the two possible orientations. Show that in the situation of Problem 8 the orientation of $\gamma'(t_0), \eta'(s_0), \mathbf{k}$ is the same as the orientation of the vectors $(f\circ\gamma)'(t_0), (f\circ\eta)'(s_0), \mathbf{k}$. Such mappings are called conformal. If f is analytic and $f'(z) \neq 0$, then we know from this problem and the above that f is a conformal map. **Hint:** You can do this by verifying that $(f\circ\gamma)'(t_0) \times (f\circ\eta)'(s_0) = |f'(\gamma(t_0))|^2 \gamma'(t_0) \times \eta'(s_0)$. To make the verification easier, you might first establish the following simple formula for the cross product where here $x + iy = z$ and $a + ib = w$.

$$(x, y, 0) \times (a, b, 0) = \operatorname{Re}(zi\overline{w})\,\mathbf{k}.$$

10. Write the Cauchy Riemann equations in terms of polar coordinates. Recall the polar coordinates are given by

$$x = r\cos\theta,\ y = r\sin\theta.$$

This means, letting $u(x, y) = u(r, \theta), v(x, y) = v(r, \theta)$, write the Cauchy Riemann equations in terms of r and θ. You should eventually show the Cauchy Riemann equations are equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

11. Show that a real valued analytic function must be constant.

18.3 Cauchy's Formula For A Disk

The Cauchy integral formula is the most important theorem in complex analysis. It will be established for a disk in this chapter and later will be generalized to much more general situations but the version given here will suffice to prove many interesting theorems needed in the later development of the theory. The following are some advanced calculus results.

Lemma 18.7 *Let $f : [a, b] \to \mathbb{C}$. Then $f'(t)$ exists if and only if $\operatorname{Re} f'(t)$ and $\operatorname{Im} f'(t)$ exist. Furthermore,*

$$f'(t) = \operatorname{Re} f'(t) + i \operatorname{Im} f'(t).$$

Proof: The if part of the equivalence is obvious.

Now suppose $f'(t)$ exists. Let both t and $t + h$ be contained in $[a, b]$

$$\left|\frac{\operatorname{Re} f(t+h) - \operatorname{Re} f(t)}{h} - \operatorname{Re}(f'(t))\right| \leq \left|\frac{f(t+h) - f(t)}{h} - f'(t)\right|$$

and this converges to zero as $h \to 0$. Therefore, $\operatorname{Re} f'(t) = \operatorname{Re}(f'(t))$. Similarly, $\operatorname{Im} f'(t) = \operatorname{Im}(f'(t))$.

Lemma 18.8 *If $g : [a,b] \to \mathbb{C}$ and g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: From the above lemma, you can apply the mean value theorem to the real and imaginary parts of g.

Applying the above lemma to the components yields the following lemma.

Lemma 18.9 *If $g : [a,b] \to \mathbb{C}^n = X$ and g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) = 0$, then $g(t)$ is a constant.*

If you want to have X be a complex Banach space, the result is still true.

Lemma 18.10 *If $g : [a,b] \to X$ and g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: Let $\Lambda \in X'$. Then $\Lambda g : [a,b] \to \mathbb{C}$. Therefore, from Lemma 18.8, for each $\Lambda \in X'$, $\Lambda g(s) = \Lambda g(t)$ and since X' separates the points, it follows $g(s) = g(t)$ so g is constant.

Lemma 18.11 *Let $\phi : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\, ds. \tag{18.1}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{18.2}$$

Proof: The first claim follows from the uniform continuity of ϕ on $[a,b] \times [c,d]$, which uniform continuity results from the set being compact. To establish 18.2, let t and $t+h$ be contained in $[c,d]$ and form, using the mean value theorem,

$$\begin{aligned} \frac{g(t+h) - g(t)}{h} &= \frac{1}{h} \int_a^b [\phi(s,t+h) - \phi(s,t)]\, ds \\ &= \frac{1}{h} \int_a^b \frac{\partial \phi(s,t+\theta h)}{\partial t} h ds \\ &= \int_a^b \frac{\partial \phi(s,t+\theta h)}{\partial t} ds, \end{aligned}$$

where θ may depend on s but is some number between 0 and 1. Then by the uniform continuity of $\frac{\partial \phi}{\partial t}$, it follows that 18.2 holds.

Corollary 18.12 *Let $\phi : [a,b] \times [c,d] \to \mathbb{C}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\,ds. \tag{18.3}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{18.4}$$

Proof: Apply Lemma 18.11 to the real and imaginary parts of ϕ.

Applying the above corollary to the components, you can also have the same result for ϕ having values in $\mathbb{C}^n$.

Corollary 18.13 *Let $\phi : [a,b] \times [c,d] \to \mathbb{C}^n$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\,ds. \tag{18.5}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{18.6}$$

If you want to consider ϕ having values in X, a complex Banach space a similar result holds.

Corollary 18.14 *Let $\phi : [a,b] \times [c,d] \to X$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\,ds. \tag{18.7}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{18.8}$$

Proof: Let $\Lambda \in X'$. Then $\Lambda\phi : [a,b] \times [c,d] \to \mathbb{C}$ is continuous and $\frac{\partial \Lambda\phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$. Therefore, from 18.8,

$$\Lambda(g'(t)) = (\Lambda g)'(t) = \int_a^b \frac{\partial \Lambda\phi(s,t)}{\partial t} ds = \Lambda \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds$$

and since X' separates the points, it follows 18.8 holds.

The following is Cauchy's integral formula for a disk.

Theorem 18.15 *Let $f : \Omega \to X$ be analytic on the open set, Ω and let*

$$\overline{B(z_0, r)} \subseteq \Omega.$$

Let $\gamma(t) \equiv z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then if $z \in B(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw. \tag{18.9}$$

Proof: Consider for $\alpha \in [0, 1]$,

$$g(\alpha) \equiv \int_0^{2\pi} \frac{f\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)}{re^{it} + z_0 - z} rie^{it} dt.$$

If α equals one, this reduces to the integral in 18.9. The idea is to show g is a constant and that $g(0) = f(z) 2\pi i$. First consider the claim about $g(0)$.

$$\begin{aligned} g(0) &= \left(\int_0^{2\pi} \frac{re^{it}}{re^{it} + z_0 - z} dt\right) if(z) \\ &= if(z) \left(\int_0^{2\pi} \frac{1}{1 - \frac{z - z_0}{re^{it}}} dt\right) \\ &= if(z) \int_0^{2\pi} \sum_{n=0}^{\infty} r^{-n} e^{-int} (z - z_0)^n dt \end{aligned}$$

because $\left|\frac{z - z_0}{re^{it}}\right| < 1$. Since this sum converges uniformly you can interchange the sum and the integral to obtain

$$\begin{aligned} g(0) &= if(z) \sum_{n=0}^{\infty} r^{-n} (z - z_0)^n \int_0^{2\pi} e^{-int} dt \\ &= 2\pi i f(z) \end{aligned}$$

because $\int_0^{2\pi} e^{-int} dt = 0$ if $n > 0$.

Next consider the claim that g is constant. By Corollary 18.13, for $\alpha \in (0, 1)$,

$$\begin{aligned} g'(\alpha) &= \int_0^{2\pi} \frac{f'\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)\left(re^{it} + z_0 - z\right)}{re^{it} + z_0 - z} rie^{it} dt \\ &= \int_0^{2\pi} f'\left(z + \alpha\left(z_0 + re^{it} - z\right)\right) rie^{it} dt \\ &= \int_0^{2\pi} \frac{d}{dt}\left(f\left(z + \alpha\left(z_0 + re^{it} - z\right)\right) \frac{1}{\alpha}\right) dt \\ &= f\left(z + \alpha\left(z_0 + re^{i2\pi} - z\right)\right) \frac{1}{\alpha} - f\left(z + \alpha\left(z_0 + re^{0} - z\right)\right) \frac{1}{\alpha} = 0. \end{aligned}$$

Now g is continuous on $[0, 1]$ and $g'(t) = 0$ on $(0, 1)$ so by Lemma 18.9, g equals a constant. This constant can only be $g(0) = 2\pi i f(z)$. Thus,

$$g(1) = \int_\gamma \frac{f(w)}{w - z} dw = g(0) = 2\pi i f(z).$$

This proves the theorem.

This is a very significant theorem. A few applications are given next.

Theorem 18.16 *Let $f : \Omega \to X$ be analytic where Ω is an open set in $\mathbb{C}$. Then f has infinitely many derivatives on Ω. Furthermore, for all $z \in B(z_0, r)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw \tag{18.10}$$

where $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$ for r small enough that $B(z_0, r) \subseteq \Omega$.

Proof: Let $z \in B(z_0, r) \subseteq \Omega$ and let $\overline{B(z_0, r)} \subseteq \Omega$. Then, letting $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$, and h small enough,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw,\ f(z+h) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z-h}dw$$

Now

$$\frac{1}{w-z-h} - \frac{1}{w-z} = \frac{h}{(-w+z+h)(-w+z)}$$

and so

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{2\pi hi}\int_\gamma \frac{hf(w)}{(-w+z+h)(-w+z)}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(-w+z+h)(-w+z)}dw.\end{aligned}$$

Now for all h sufficiently small, there exists a constant C independent of such h such that

$$\begin{aligned}&\left|\frac{1}{(-w+z+h)(-w+z)} - \frac{1}{(-w+z)(-w+z)}\right| \\ = &\left|\frac{h}{(w-z-h)(w-z)^2}\right| \leq C|h|\end{aligned}$$

and so, the integrand converges uniformly as $h \to 0$ to

$$= \frac{f(w)}{(w-z)^2}$$

Therefore, the limit as $h \to 0$ may be taken inside the integral to obtain

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}dw.$$

Continuing in this way, yields 18.10.

This is a very remarkable result. It shows the existence of one continuous derivative implies the existence of all derivatives, in contrast to the theory of functions of a real variable. Actually, more than what is stated in the theorem was shown. The above proof establishes the following corollary.

Corollary 18.17 *Suppose f is continuous on $\partial B(z_0,r)$ and suppose that for all $z \in B(z_0,r)$,*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw,$$

where $\gamma(t) \equiv z_0 + re^{it}, t \in [0,2\pi]$. Then f is analytic on $B(z_0,r)$ and in fact has infinitely many derivatives on $B(z_0,r)$.

Another application is the following lemma.

Lemma 18.18 *Let $\gamma(t) = z_0 + re^{it}$, for $t \in [0,2\pi]$, suppose $f_n \to f$ uniformly on $\overline{B(z_0,r)}$, and suppose*

$$f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w)}{w-z}dw \tag{18.11}$$

for $z \in B(z_0,r)$. Then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw, \tag{18.12}$$

implying that f is analytic on $B(z_0,r)$.

Proof: From 18.11 and the uniform convergence of f_n to f on $\gamma([0,2\pi])$, the integrals in 18.11 converge to

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw.$$

Therefore, the formula 18.12 follows.

Uniform convergence on a closed disk of the analytic functions implies the target function is also analytic. This is amazing. Think of the Weierstrass approximation theorem for polynomials. You can obtain a continuous nowhere differentiable function as the uniform limit of polynomials.

The conclusions of the following proposition have all been obtained earlier in Theorem 18.4 but they can be obtained more easily if you use the above theorem and lemmas.

Proposition 18.19 *Let $\{a_n\}$ denote a sequence in X. Then there exists $R \in [0,\infty]$ such that*

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k$$

converges absolutely if $|z-z_0| < R$, diverges if $|z-z_0| > R$ and converges uniformly on $B(z_0,r)$ for all $r < R$. Furthermore, if $R > 0$, the function,

$$f(z) \equiv \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

is analytic on $B(z_0,R)$.

Proof: The assertions about absolute convergence are routine from the root test if

$$R \equiv \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$$

with $R = \infty$ if the quantity in parenthesis equals zero. The root test can be used to verify absolute convergence which then implies convergence by completeness of X.

The assertion about uniform convergence follows from the Weierstrass M test and $M_n \equiv |a_n| r^n$. ($\sum_{n=0}^{\infty} |a_n| r^n < \infty$ by the root test). It only remains to verify the assertion about $f(z)$ being analytic in the case where $R > 0$.

Let $0 < r < R$ and define $f_n(z) \equiv \sum_{k=0}^{n} a_k (z - z_0)^k$. Then f_n is a polynomial and so it is analytic. Thus, by the Cauchy integral formula above,

$$f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w)}{w-z}dw$$

where $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$. By Lemma 18.18 and the first part of this proposition involving uniform convergence,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw.$$

Therefore, f is analytic on $B(z_0, r)$ by Corollary 18.17. Since $r < R$ is arbitrary, this shows f is analytic on $B(z_0, R)$.

This proposition shows that all functions having values in X which are given as power series are analytic on their circle of convergence, the set of complex numbers, z, such that $|z - z_0| < R$. In fact, every analytic function can be realized as a power series.

Theorem 18.20 *If $f : \Omega \to X$ is analytic and if $B(z_0, r) \subseteq \Omega$, then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{18.13}$$

for all $|z - z_0| < r$. Furthermore,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{18.14}$$

Proof: Consider $|z - z_0| < r$ and let $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then for $w \in \gamma([0, 2\pi])$,

$$\left|\frac{z - z_0}{w - z_0}\right| < 1$$

and so, by the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)\left(1-\frac{z-z_0}{w-z_0}\right)}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw. \end{aligned}$$

Since the series converges uniformly, you can interchange the integral and the sum to obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\right)(z-z_0)^n \\ &\equiv \sum_{n=0}^{\infty} a_n (z-z_0)^n \end{aligned}$$

By Theorem 18.16, 18.14 holds.

Note that this also implies that if a function is analytic on an open set, then all of its derivatives are also analytic. This follows from Theorem 18.4 which says that a function given by a power series has all derivatives on the disk of convergence.

18.4 Exercises

1. Show that if $|e_k| \leq \varepsilon$, then $\left|\sum_{k=m}^{\infty} e_k\left(r^k - r^{k+1}\right)\right| < \varepsilon$ if $0 \leq r < 1$. **Hint:** Let $|\theta| = 1$ and verify that

$$\theta\sum_{k=m}^{\infty} e_k\left(r^k - r^{k+1}\right) = \left|\sum_{k=m}^{\infty} e_k\left(r^k - r^{k+1}\right)\right| = \sum_{k=m}^{\infty}\operatorname{Re}(\theta e_k)\left(r^k - r^{k+1}\right)$$

   where $-\varepsilon < \operatorname{Re}(\theta e_k) < \varepsilon$.

2. Abel's theorem says that if $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence equal to 1 and if $A = \sum_{n=0}^{\infty} a_n$, then $\lim_{r\to 1-}\sum_{n=0}^{\infty} a_n r^n = A$. **Hint:** Show $\sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} A_k\left(r^k - r^{k+1}\right)$ where A_k denotes the k^{th} partial sum of $\sum a_j$. Thus

$$\sum_{k=0}^{\infty} a_k r^k = \sum_{k=m+1}^{\infty} A_k\left(r^k - r^{k+1}\right) + \sum_{k=0}^{m} A_k\left(r^k - r^{k+1}\right),$$

   where $|A_k - A| < \varepsilon$ for all $k \geq m$. In the first sum, write $A_k = A + e_k$ and use Problem 1. Use this theorem to verify that $\arctan(1) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}$.

3. Find the integrals using the Cauchy integral formula.

   (a) $\int_\gamma \frac{\sin z}{z-i} dz$ where $\gamma(t) = 2e^{it} : t \in [0, 2\pi]$.

   (b) $\int_\gamma \frac{1}{z-a} dz$ where $\gamma(t) = a + re^{it} : t \in [0, 2\pi]$

   (c) $\int_\gamma \frac{\cos z}{z^2} dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$

   (d) $\int_\gamma \frac{\log(z)}{z^n} dz$ where $\gamma(t) = 1 + \frac{1}{2}e^{it} : t \in [0, 2\pi]$ and $n = 0, 1, 2$. In this problem, $\log(z) \equiv \ln|z| + i\arg(z)$ where $\arg(z) \in (-\pi, \pi)$ and $z = |z| e^{i\arg(z)}$. Thus $e^{\log(z)} = z$ and $\log(z)' = \frac{1}{z}$.

4. Let $\gamma(t) = 4e^{it} : t \in [0, 2\pi]$ and find $\int_\gamma \frac{z^2+4}{z(z^2+1)} dz$.

5. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < R$. Show that then

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

   for all $r \in [0, R)$. **Hint:** Let

$$f_n(z) \equiv \sum_{k=0}^{n} a_k z^k,$$

   show

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f_n\left(re^{i\theta}\right) \right|^2 d\theta = \sum_{k=0}^{n} |a_k|^2 r^{2k}$$

   and then take limits as $n \to \infty$ using uniform convergence.

6. The Cauchy integral formula, marvelous as it is, can actually be improved upon. The Cauchy integral formula involves representing f by the values of f on the boundary of the disk, $B(a, r)$. It is possible to represent f by using only the values of $\operatorname{Re} f$ on the boundary. This leads to the Schwarz formula . Supply the details in the following outline.

   Suppose f is analytic on $|z| < R$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{18.15}$$

   with the series converging uniformly on $|z| = R$. Then letting $|w| = R$,

$$2u(w) = f(w) + \overline{f(w)}$$

   and so

$$2u(w) = \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} \overline{a_k} \left(\overline{w}\right)^k. \tag{18.16}$$

Now letting $\gamma(t) = Re^{it},\ t \in [0, 2\pi]$

$$\begin{aligned}\int_\gamma \frac{2u(w)}{w} dw &= (a_0 + \overline{a_0}) \int_\gamma \frac{1}{w} dw \\ &= 2\pi i (a_0 + \overline{a_0}).\end{aligned}$$

Thus, multiplying 18.16 by w^{-1},

$$\frac{1}{\pi i} \int_\gamma \frac{u(w)}{w} dw = a_0 + \overline{a_0}.$$

Now multiply 18.16 by $w^{-(n+1)}$ and integrate again to obtain

$$a_n = \frac{1}{\pi i} \int_\gamma \frac{u(w)}{w^{n+1}} dw.$$

Using these formulas for a_n in 18.15, we can interchange the sum and the integral (Why can we do this?) to write the following for $|z| < R$.

$$\begin{aligned}f(z) &= \frac{1}{\pi i} \int_\gamma \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^{k+1} u(w)\, dw - \overline{a_0} \\ &= \frac{1}{\pi i} \int_\gamma \frac{u(w)}{w - z} dw - \overline{a_0},\end{aligned}$$

which is the Schwarz formula. Now $\operatorname{Re} a_0 = \frac{1}{2\pi i} \int_\gamma \frac{u(w)}{w} dw$ and $\overline{a_0} = \operatorname{Re} a_0 - i \operatorname{Im} a_0$. Therefore, we can also write the Schwarz formula as

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{u(w)(w + z)}{(w - z) w} dw + i \operatorname{Im} a_0. \tag{18.17}$$

7. Take the real parts of the second form of the Schwarz formula to derive the Poisson formula for a disk,

$$u\left(re^{i\alpha}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u\left(Re^{i\theta}\right)\left(R^2 - r^2\right)}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} d\theta. \tag{18.18}$$

8. Suppose that $u(w)$ is a given real continuous function defined on $\partial B(0, R)$ and define $f(z)$ for $|z| < R$ by 18.17. Show that f, so defined is analytic. Explain why u given in 18.18 is harmonic. Show that

$$\lim_{r \to R-} u\left(re^{i\alpha}\right) = u\left(Re^{i\alpha}\right).$$

Thus u is a harmonic function which approaches a given function on the boundary and is therefore, a solution to the Dirichlet problem.

9. Suppose $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for all $|z - z_0| < R$. Show that $f'(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1}$ for all $|z - z_0| < R$. **Hint:** Let $f_n(z)$ be a partial sum of f. Show that f_n' converges uniformly to some function, g on $|z - z_0| \leq r$ for any $r < R$. Now use the Cauchy integral formula for a function and its derivative to identify g with f'.

10. Use Problem 9 to find the exact value of $\sum_{k=0}^{\infty} k^2 \left(\frac{1}{3}\right)^k$.

11. Prove the binomial formula,
$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$
where
$$\binom{\alpha}{n} \equiv \frac{\alpha \cdots (\alpha - n + 1)}{n!}.$$
Can this be used to give a proof of the binomial formula,
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k?$$
Explain.

12. Suppose f is analytic on $B(z_0, r)$ and continuous on $\overline{B(z_0, r)}$ and $|f(z)| \leq M$ on $\overline{B(z_0, r)}$. Show that then $\left|f^{(n)}(a)\right| \leq \frac{Mn!}{r^n}$.

18.5 Zeros Of An Analytic Function

In this section we give a very surprising property of analytic functions which is in stark contrast to what takes place for functions of a real variable.

Definition 18.21 *A region is a connected open set.*

It turns out the zeros of an analytic function which is not constant on some region cannot have a limit point. This is also a good time to define the order of a zero.

Definition 18.22 *Suppose f is an analytic function defined near a point, α where $f(\alpha) = 0$. Thus α is a zero of the function, f. The zero is of order m if $f(z) = (z - \alpha)^m g(z)$ where g is an analytic function which is not equal to zero at α.*

Theorem 18.23 *Let Ω be a connected open set (region) and let $f : \Omega \to X$ be analytic. Then the following are equivalent.*

1. *$f(z) = 0$ for all $z \in \Omega$*
2. *There exists $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all n.*

3. There exists $z_0 \in \Omega$ which is a limit point of the set,

$$Z \equiv \{z \in \Omega : f(z) = 0\}.$$

Proof: It is clear the first condition implies the second two. Suppose the third holds. Then for z near z_0

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where $k \geq 1$ since z_0 is a zero of f. Suppose $k < \infty$. Then,

$$f(z) = (z - z_0)^k g(z)$$

where $g(z_0) \neq 0$. Letting $z_n \to z_0$ where $z_n \in Z, z_n \neq z_0$, it follows

$$0 = (z_n - z_0)^k g(z_n)$$

which implies $g(z_n) = 0$. Then by continuity of g, we see that $g(z_0) = 0$ also, contrary to the choice of k. Therefore, k cannot be less than ∞ and so z_0 is a point satisfying the second condition.

Now suppose the second condition and let

$$S \equiv \left\{z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n\right\}.$$

It is clear that S is a closed set which by assumption is nonempty. However, this set is also open. To see this, let $z \in S$. Then for all w close enough to z,

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} (w - z)^k = 0.$$

Thus f is identically equal to zero near $z \in S$. Therefore, all points near z are contained in S also, showing that S is an open set. Now $\Omega = S \cup (\Omega \setminus S)$, the union of two disjoint open sets, S being nonempty. It follows the other open set, $\Omega \setminus S$, must be empty because Ω is connected. Therefore, the first condition is verified. This proves the theorem. (See the following diagram.)

$$\begin{tikzcd} & 1.) \arrow[dl, shift right] \arrow[dr] & \\ 2.) \arrow[ur, shift right] & & 3.) \arrow[ll] \end{tikzcd}$$

Note how radically different this is from the theory of functions of a real variable. Consider, for example the function

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

which has a derivative for all $x \in \mathbb{R}$ and for which 0 is a limit point of the set, Z, even though f is not identically equal to zero.

Here is a very important application called Euler's formula. Recall that

$$e^z \equiv e^x (\cos (y) + i \sin (y)) \tag{18.19}$$

Is it also true that $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$?

Theorem 18.24 *(Euler's Formula) Let $z = x + iy$. Then*

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Proof: It was already observed that e^z given by 18.19 is analytic. So is $\exp (z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!}$. In fact the power series converges for all $z \in \mathbb{C}$. Furthermore the two functions, e^z and $\exp (z)$ agree on the real line which is a set which contains a limit point. Therefore, they agree for all values of $z \in \mathbb{C}$.

This formula shows the famous two identities,

$$e^{i\pi} = -1 \text{ and } e^{2\pi i} = 1.$$

18.6 Liouville's Theorem

The following theorem pertains to functions which are analytic on all of $\mathbb{C}$, "entire" functions.

Definition 18.25 *A function, $f : \mathbb{C} \to \mathbb{C}$ or more generally, $f : \mathbb{C} \to X$ is entire means it is analytic on $\mathbb{C}$.*

Theorem 18.26 *(Liouville's theorem) If f is a bounded entire function having values in X, then f is a constant.*

Proof: Since f is entire, pick any $z \in \mathbb{C}$ and write

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w - z)^2} dw$$

where $\gamma_R (t) = z + Re^{it}$ for $t \in [0, 2\pi]$. Therefore,

$$||f'(z)|| \leq C \frac{1}{R}$$

where C is some constant depending on the assumed bound on f. Since R is arbitrary, let $R \to \infty$ to obtain $f'(z) = 0$ for any $z \in \mathbb{C}$. It follows from this that f is constant for if z_j $j = 1, 2$ are two complex numbers, let $h(t) = f(z_1 + t(z_2 - z_1))$ for $t \in [0, 1]$. Then $h'(t) = f'(z_1 + t(z_2 - z_1))(z_2 - z_1) = 0$. By Lemmas 18.8 - 18.10 h is a constant on $[0, 1]$ which implies $f(z_1) = f(z_2)$.

With Liouville's theorem it becomes possible to give an easy proof of the fundamental theorem of algebra. It is ironic that all the best proofs of this theorem in algebra come from the subjects of analysis or topology. Out of all the proofs that have been given of this very important theorem, the following one based on Liouville's theorem is the easiest.

Theorem 18.27 *(Fundamental theorem of Algebra) Let*

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial where $n \geq 1$ and each coefficient is a complex number. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Suppose not. Then $p(z)^{-1}$ is an entire function. Also

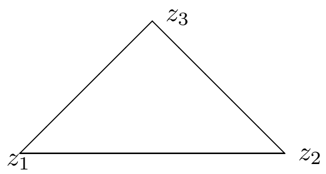
$$|p(z)| \geq |z|^n - \left(|a_{n-1}| |z|^{n-1} + \cdots + |a_1| |z| + |a_0|\right)$$

and so $\lim_{|z|\to\infty} |p(z)| = \infty$ which implies $\lim_{|z|\to\infty} \left|p(z)^{-1}\right| = 0$. It follows that, since $p(z)^{-1}$ is bounded for z in any bounded set, we must have that $p(z)^{-1}$ is a bounded entire function. But then it must be constant. However since $p(z)^{-1} \to 0$ as $|z| \to \infty$, this constant can only be 0. However, $\frac{1}{p(z)}$ is never equal to zero. This proves the theorem.

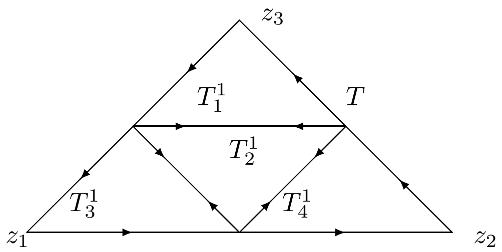
18.7 The General Cauchy Integral Formula

18.7.1 The Cauchy Goursat Theorem

This section gives a fundamental theorem which is essential to the development which follows and is closely related to the question of when a function has a primitive. First of all, if you have two points in $\mathbb{C}$, z_1 and z_2, you can consider $\gamma(t) \equiv z_1 + t(z_2 - z_1)$ for $t \in [0,1]$ to obtain a continuous bounded variation curve from z_1 to z_2. More generally, if $z_1, \cdots, z_m$ are points in $\mathbb{C}$ you can obtain a continuous bounded variation curve from z_1 to z_m which consists of first going from z_1 to z_2 and then from z_2 to z_3 and so on, till in the end one goes from z_{m-1} to z_m. We denote this piecewise linear curve as $\gamma(z_1, \cdots, z_m)$. Now let T be a triangle with vertices z_1, z_2 and z_3 encountered in the counter clockwise direction as shown.



Denote by $\int_{\partial T} f(z)\, dz$, the expression, $\int_{\gamma(z_1,z_2,z_3,z_1)} f(z)\, dz$. Consider the following picture.



By Lemma 17.11

$$\int_{\partial T} f(z)\, dz = \sum_{k=1}^{4} \int_{\partial T_k^1} f(z)\, dz. \tag{18.20}$$

On the "inside lines" the integrals cancel as claimed in Lemma 17.11 because there are two integrals going in opposite directions for each of these inside lines.

Theorem 18.28 *(Cauchy Goursat) Let $f : \Omega \to X$ have the property that $f'(z)$ exists for all $z \in \Omega$ and let T be a triangle contained in Ω. Then*

$$\int_{\partial T} f(w)\, dw = 0.$$

Proof: Suppose not. Then

$$\left\| \int_{\partial T} f(w)\, dw \right\| = \alpha \neq 0.$$

From 18.20 it follows

$$\alpha \leq \sum_{k=1}^{4} \left\| \int_{\partial T_k^1} f(w)\, dw \right\|$$

and so for at least one of these T_k^1, denoted from now on as T_1,

$$\left\| \int_{\partial T_1} f(w)\, dw \right\| \geq \frac{\alpha}{4}.$$

Now let T_1 play the same role as T, subdivide as in the above picture, and obtain T_2 such that

$$\left\| \int_{\partial T_2} f(w)\, dw \right\| \geq \frac{\alpha}{4^2}.$$

Continue in this way, obtaining a sequence of triangles,

$$T_k \supseteq T_{k+1}, \operatorname{diam}(T_k) \leq \operatorname{diam}(T)\, 2^{-k},$$

and

$$\left\| \int_{\partial T_k} f(w)\, dw \right\| \geq \frac{\alpha}{4^k}.$$

Then let $z \in \cap_{k=1}^{\infty} T_k$ and note that by assumption, $f'(z)$ exists. Therefore, for all k large enough,

$$\int_{\partial T_k} f(w)\, dw = \int_{\partial T_k} f(z) + f'(z)(w-z) + g(w)\, dw$$

where $||g(w)|| < \varepsilon |w-z|$. Now observe that $w \to f(z) + f'(z)(w-z)$ has a primitive, namely,

$$F(w) = f(z) w + f'(z)(w-z)^2 /2.$$

Therefore, by Corollary 17.14.

$$\int_{\partial T_k} f(w)\, dw = \int_{\partial T_k} g(w)\, dw.$$

From the definition, of the integral,

$$\begin{aligned} \frac{\alpha}{4^k} &\leq \left|\left|\int_{\partial T_k} g(w)\, dw\right|\right| \leq \varepsilon \operatorname{diam}(T_k) (\text{length of } \partial T_k) \\ &\leq \varepsilon 2^{-k} (\text{length of } T) \operatorname{diam}(T) 2^{-k}, \end{aligned}$$

and so

$$\alpha \leq \varepsilon (\text{length of } T) \operatorname{diam}(T).$$

Since ε is arbitrary, this shows $\alpha = 0$, a contradiction. Thus $\int_{\partial T} f(w)\, dw = 0$ as claimed.

This fundamental result yields the following important theorem.

Theorem 18.29 *(Morera*[1]*) Let Ω be an open set and let $f'(z)$ exist for all $z \in \Omega$. Let $D \equiv \overline{B(z_0, r)} \subseteq \Omega$. Then there exists $\varepsilon > 0$ such that f has a primitive on $B(z_0, r+\varepsilon)$.*

Proof: Choose $\varepsilon > 0$ small enough that $B(z_0, r+\varepsilon) \subseteq \Omega$. Then for $w \in B(z_0, r+\varepsilon)$, define

$$F(w) \equiv \int_{\gamma(z_0, w)} f(u)\, du.$$

Then by the Cauchy Goursat theorem, and $w \in B(z_0, r+\varepsilon)$, it follows that for $|h|$ small enough,

$$\begin{aligned} \frac{F(w+h) - F(w)}{h} &= \frac{1}{h} \int_{\gamma(w, w+h)} f(u)\, du \\ = \frac{1}{h} \int_0^1 f(w+th)\, h dt &= \int_0^1 f(w+th)\, dt \end{aligned}$$

which converges to $f(w)$ due to the continuity of f at w. This proves the theorem.

The following is a slight generalization of the above theorem which is also referred to as Morera's theorem.

[1] Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886

Corollary 18.30 *Let Ω be an open set and suppose that whenever*

$$\gamma\left(z_1, z_2, z_3, z_1\right)$$

is a closed curve bounding a triangle T, which is contained in Ω, and f is a continuous function defined on Ω, it follows that

$$\int_{\gamma(z_1,z_2,z_3,z_1)} f\left(z\right) dz = 0,$$

then f is analytic on Ω.

Proof: As in the proof of Morera's theorem, let $\overline{B\left(z_0, r\right)} \subseteq \Omega$ and use the given condition to construct a primitive, F for f on $B\left(z_0, r\right)$. Then F is analytic and so by Theorem 18.16, it follows that F and hence f have infinitely many derivatives, implying that f is analytic on $B\left(z_0, r\right)$. Since z_0 is arbitrary, this shows f is analytic on Ω.

18.7.2 A Redundant Assumption

Earlier in the definition of analytic, it was assumed the derivative is continuous. This assumption is **redundant.**

Theorem 18.31 *Let Ω be an open set in $\mathbb{C}$ and suppose $f : \Omega \to X$ has the property that $f'\left(z\right)$ exists for each $z \in \Omega$. Then f is analytic on Ω.*

Proof: Let $z_0 \in \Omega$ and let $B\left(z_0, r\right) \subseteq \Omega$. By Morera's theorem f has a primitive, F on $B\left(z_0, r\right)$. It follows that F is analytic because it has a derivative, f, and this derivative is continuous. Therefore, by Theorem 18.16 F has infinitely many derivatives on $B\left(z_0, r\right)$ implying that f also has infinitely many derivatives on $B\left(z_0, r\right)$. Thus f is analytic as claimed.

It follows a function is analytic on an open set, Ω if and only if $f'\left(z\right)$ exists for $z \in \Omega$. This is because it was just shown the derivative, if it exists, is automatically continuous.

The same proof used to prove Theorem 18.29 implies the following corollary.

Corollary 18.32 *Let Ω be a convex open set and suppose that $f'\left(z\right)$ exists for all $z \in \Omega$. Then f has a primitive on Ω.*

Note that this implies that if Ω is a convex open set on which $f'\left(z\right)$ exists and if $\gamma : [a, b] \to \Omega$ is a closed, continuous curve having bounded variation, then letting F be a primitive of f Theorem 17.13 implies

$$\int_\gamma f\left(z\right) dz = F\left(\gamma\left(b\right)\right) - F\left(\gamma\left(a\right)\right) = 0.$$

Notice how different this is from the situation of a function of a real variable! It is possible for a function of a real variable to have a derivative everywhere and yet the derivative can be discontinuous. A simple example is the following.

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$

Then $f'(x)$ exists for all $x \in \mathbb{R}$. Indeed, if $x \neq 0$, the derivative equals $2x \sin \frac{1}{x} - \cos \frac{1}{x}$ which has no limit as $x \to 0$. However, from the definition of the derivative of a function of one variable, $f'(0) = 0$.

18.7.3 Classification Of Isolated Singularities

First some notation.

Definition 18.33 *Let $B'(a,r) \equiv \{z \in \mathbb{C} \text{ such that } 0 < |z-a| < r\}$. Thus this is the usual ball without the center. A function is said to have an isolated singularity at the point $a \in \mathbb{C}$ if f is analytic on $B'(a,r)$ for some $r > 0$.*

It turns out isolated singularities can be neatly classified into three types, removable singularities, poles, and essential singularities. The next theorem deals with the case of a removable singularity.

Definition 18.34 *An isolated singularity of f is said to be removable if there exists an analytic function, g analytic at a and near a such that $f = g$ at all points near a.*

Theorem 18.35 *Let $f : B'(a,r) \to X$ be analytic. Thus f has an isolated singularity at a. Suppose also that*

$$\lim_{z \to a} f(z)(z-a) = 0.$$

Then there exists a unique analytic function, $g : B(a,r) \to X$ such that $g = f$ on $B'(a,r)$. Thus the singularity at a is removable.

Proof: Let $h(z) \equiv (z-a)^2 f(z), h(a) \equiv 0$. Then h is analytic on $B(a,r)$ because it is easy to see that $h'(a) = 0$. It follows h is given by a power series,

$$h(z) = \sum_{k=2}^{\infty} a_k (z-a)^k$$

where $a_0 = a_1 = 0$ because of the observation above that $h'(a) = h(a) = 0$. It follows that for $|z-a| > 0$

$$f(z) = \sum_{k=2}^{\infty} a_k (z-a)^{k-2} \equiv g(z).$$

This proves the theorem.

What of the other case where the singularity is not removable? This situation is dealt with by the amazing Casorati Weierstrass theorem.

Theorem 18.36 *(Casorati Weierstrass) Let a be an isolated singularity and suppose for some $r > 0, f(B'(a,r))$ is not dense in $\mathbb{C}$. Then either a is a removable singularity or there exist finitely many $b_1, \cdots, b_M$ for some finite number, M such that for z near a,*

$$f(z) = g(z) + \sum_{k=1}^{M} \frac{b_k}{(z-a)^k} \tag{18.21}$$

where $g(z)$ is analytic near a.

Proof: Suppose $B(z_0, \delta)$ has no points of $f(B'(a,r))$. Such a ball must exist if $f(B'(a,r))$ is not dense. Then for $z \in B'(a,r), |f(z) - z_0| \geq \delta > 0$. It follows from Theorem 18.35 that $\frac{1}{f(z)-z_0}$ has a removable singularity at a. Hence, there exists h an analytic function such that for z near a,

$$h(z) = \frac{1}{f(z) - z_0}. \tag{18.22}$$

There are two cases. First suppose $h(a) = 0$. Then $\sum_{k=1}^{\infty} a_k (z-a)^k = \frac{1}{f(z)-z_0}$ for z near a. If all the $a_k = 0$, this would be a contradiction because then the left side would equal zero for z near a but the right side could not equal zero. Therefore, there is a first m such that $a_m \neq 0$. Hence there exists an analytic function, $k(z)$ which is not equal to zero in some ball, $B(a, \varepsilon)$ such that

$$k(z)(z-a)^m = \frac{1}{f(z) - z_0}.$$

Hence, taking both sides to the -1 power,

$$f(z) - z_0 = \frac{1}{(z-a)^m} \sum_{k=0}^{\infty} b_k (z-a)^k$$

and so 18.21 holds.

The other case is that $h(a) \neq 0$. In this case, raise both sides of 18.22 to the -1 power and obtain

$$f(z) - z_0 = h(z)^{-1},$$

a function analytic near a. Therefore, the singularity is removable. This proves the theorem.

This theorem is the basis for the following definition which classifies isolated singularities.

Definition 18.37 *Let a be an isolated singularity of a complex valued function, f. When 18.21 holds for z near a, then a is called a pole. The order of the pole in 18.21 is M. If for every $r > 0, f(B'(a,r))$ is dense in $\mathbb{C}$ then a is called an essential singularity.*

In terms of the above definition, isolated singularities are either removable, a pole, or essential. There are no other possibilities.

Theorem 18.38 *Suppose $f : \Omega \to \mathbb{C}$ has an isolated singularity at $a \in \Omega$. Then a is a pole if and only if*

$$\lim_{z \to a} d\left(f\left(z\right), \infty\right) = 0$$

in $\widehat{\mathbb{C}}$.

Proof: Suppose first f has a pole at a. Then by definition, $f\left(z\right) = g\left(z\right) + \sum_{k=1}^{M} \frac{b_k}{(z-a)^k}$ for z near a where g is analytic. Then

$$\begin{aligned} \left|f\left(z\right)\right| &\geq \frac{\left|b_M\right|}{\left|z-a\right|^M} - \left|g\left(z\right)\right| - \sum_{k=1}^{M-1} \frac{\left|b_k\right|}{\left|z-a\right|^k} \\ &= \frac{1}{\left|z-a\right|^M} \left(\left|b_M\right| - \left(\left|g\left(z\right)\right| \left|z-a\right|^M + \sum_{k=1}^{M-1} \left|b_k\right| \left|z-a\right|^{M-k} \right) \right). \end{aligned}$$

Now $\lim_{z \to a} \left(\left|g\left(z\right)\right| \left|z-a\right|^M + \sum_{k=1}^{M-1} \left|b_k\right| \left|z-a\right|^{M-k} \right) = 0$ and so the above inequality proves $\lim_{z \to a} \left|f\left(z\right)\right| = \infty$. Referring to the diagram on Page 372, you see this is the same as saying

$$\lim_{z \to a} \left|\theta f\left(z\right) - \left(0,0,2\right)\right| = \lim_{z \to a} \left|\theta f\left(z\right) - \theta\left(\infty\right)\right| = \lim_{z \to a} d\left(f\left(z\right), \infty\right) = 0$$

Conversely, suppose $\lim_{z \to a} d\left(f\left(z\right), \infty\right) = 0$. Then from the diagram on Page 372, it follows $\lim_{z \to a} \left|f\left(z\right)\right| = \infty$ and in particular, a cannot be either removable or an essential singularity by the Casorati Weierstrass theorem, Theorem 18.36. The only case remaining is that a is a pole. This proves the theorem.

Definition 18.39 *Let $f : \Omega \to \mathbb{C}$ where Ω is an open subset of $\mathbb{C}$. Then f is called meromorphic if all singularities are isolated and are either poles or removable and this set of singularities has no limit point. It is convenient to regard meromorphic functions as having values in $\widehat{\mathbb{C}}$ where if a is a pole, $f\left(a\right) \equiv \infty$. From now on, this will be assumed when a meromorphic function is being considered.*

The usefulness of the above convention about $f\left(a\right) \equiv \infty$ at a pole is made clear in the following theorem.

Theorem 18.40 *Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \widehat{\mathbb{C}}$ be meromorphic. Then f is continuous with respect to the metric, d on $\widehat{\mathbb{C}}$.*

Proof: Let $z_n \to z$ where $z \in \Omega$. Then if z is a pole, it follows from Theorem 18.38 that

$$d\left(f\left(z_n\right), \infty\right) \equiv d\left(f\left(z_n\right), f\left(z\right)\right) \to 0.$$

If z is not a pole, then $f\left(z_n\right) \to f\left(z\right)$ in $\mathbb{C}$ which implies $\left|\theta\left(f\left(z_n\right)\right) - \theta\left(f\left(z\right)\right)\right| = d\left(f\left(z_n\right), f\left(z\right)\right) \to 0$. Recall that θ is continuous on $\mathbb{C}$.

18.7.4 The Cauchy Integral Formula

This section presents the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number also called the index, defined in the following theorem. This winding number, along with the earlier results, especially Liouville's theorem, yields an extremely general Cauchy integral formula.

Definition 18.41 *Let $\gamma : [a, b] \to \mathbb{C}$ and suppose $z \notin \gamma^*$. The winding number, $n(\gamma, z)$ is defined by*

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

The main interest is in the case where γ is closed curve. However, the same notation will be used for any such curve.

Theorem 18.42 *Let $\gamma : [a, b] \to \mathbb{C}$ be continuous and have bounded variation with $\gamma(a) = \gamma(b)$. Also suppose that $z \notin \gamma^*$. Define*

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}. \tag{18.23}$$

Then $n(\gamma, \cdot)$ is continuous and integer valued. Furthermore, there exists a sequence, $\eta_k : [a, b] \to \mathbb{C}$ such that η_k is $C^1([a, b])$,

$$||\eta_k - \gamma|| < \frac{1}{k}, \eta_k(a) = \eta_k(b) = \gamma(a) = \gamma(b),$$

and $n(\eta_k, z) = n(\gamma, z)$ for all k large enough. Also $n(\gamma, \cdot)$ is constant on every connected component of $\mathbb{C} \setminus \gamma^*$ and equals zero on the unbounded component of $\mathbb{C} \setminus \gamma^*$.

Proof: First consider the assertion about continuity.

$$\begin{aligned} |n(\gamma, z) - n(\gamma, z_1)| &\leq C \left| \int_\gamma \left(\frac{1}{w - z} - \frac{1}{w - z_1} \right) dw \right| \\ &\leq \widetilde{C} \,(\text{Length of } \gamma)\, |z_1 - z| \end{aligned}$$

whenever z_1 is close enough to z. This proves the continuity assertion. Note this did not depend on γ being closed.

Next it is shown that for a closed curve the winding number equals an integer. To do so, use Theorem 17.12 to obtain η_k, a function in $C^1([a, b])$ such that $z \notin \eta_k([a, b])$ for all k large enough, $\eta_k(x) = \gamma(x)$ for $x = a, b$, and

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \right| < \frac{1}{k},\ ||\eta_k - \gamma|| < \frac{1}{k}.$$

It is shown that each of $\frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z}$ is an integer. To simplify the notation, write η instead of η_k.

$$\int_\eta \frac{dw}{w - z} = \int_a^b \frac{\eta'(s)\, ds}{\eta(s) - z}.$$

Define

$$g(t) \equiv \int_a^t \frac{\eta'(s)\,ds}{\eta(s) - z}. \tag{18.24}$$

Then

$$\begin{aligned}\left(e^{-g(t)}(\eta(t) - z)\right)' &= e^{-g(t)}\eta'(t) - e^{-g(t)}g'(t)(\eta(t) - z) \\ &= e^{-g(t)}\eta'(t) - e^{-g(t)}\eta'(t) = 0.\end{aligned}$$

It follows that $e^{-g(t)}(\eta(t) - z)$ equals a constant. In particular, using the fact that $\eta(a) = \eta(b)$,

$$e^{-g(b)}(\eta(b) - z) = e^{-g(a)}(\eta(a) - z) = (\eta(a) - z) = (\eta(b) - z)$$

and so $e^{-g(b)} = 1$. This happens if and only if $-g(b) = 2m\pi i$ for some integer m. Therefore, 18.24 implies

$$2m\pi i = \int_a^b \frac{\eta'(s)\,ds}{\eta(s) - z} = \int_\eta \frac{dw}{w - z}.$$

Therefore, $\frac{1}{2\pi i}\int_{\eta_k} \frac{dw}{w-z}$ is a sequence of integers converging to $\frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z} \equiv n(\gamma, z)$ and so $n(\gamma, z)$ must also be an integer and $n(\eta_k, z) = n(\gamma, z)$ for all k large enough.

Since $n(\gamma, \cdot)$ is continuous and integer valued, it follows from Corollary 5.58 on Page 112 that it must be constant on every connected component of $\mathbb{C} \setminus \gamma^*$. It is clear that $n(\gamma, z)$ equals zero on the unbounded component because from the formula,

$$\lim_{z\to\infty} |n(\gamma, z)| \leq \lim_{z\to\infty} V(\gamma, [a, b]) \left(\frac{1}{|z| - c}\right)$$

where $c \geq \max\{|w| : w \in \gamma^*\}$.This proves the theorem.

Corollary 18.43 *Suppose $\gamma : [a, b] \to \mathbb{C}$ is a continuous bounded variation curve and $n(\gamma, z)$ is an integer where $z \notin \gamma^*$. Then $\gamma(a) = \gamma(b)$. Also $z \to n(\gamma, z)$ for $z \notin \gamma^*$ is continuous.*

Proof: Letting η be a C^1 curve for which $\eta(a) = \gamma(a)$ and $\eta(b) = \gamma(b)$ and which is close enough to γ that $n(\eta, z) = n(\gamma, z)$, the argument is similar to the above. Let

$$g(t) \equiv \int_a^t \frac{\eta'(s)\,ds}{\eta(s) - z}. \tag{18.25}$$

Then

$$\begin{aligned}\left(e^{-g(t)}(\eta(t) - z)\right)' &= e^{-g(t)}\eta'(t) - e^{-g(t)}g'(t)(\eta(t) - z) \\ &= e^{-g(t)}\eta'(t) - e^{-g(t)}\eta'(t) = 0.\end{aligned}$$

Hence

$$e^{-g(t)}(\eta(t) - z) = c \neq 0. \tag{18.26}$$

By assumption

$$g(b) = \int_\eta \frac{1}{w-z} dw = 2\pi i m$$

for some integer, m. Therefore, from 18.26

$$1 = e^{2\pi m i} = \frac{\eta(b) - z}{c}.$$

Thus $c = \eta(b) - z$ and letting $t = a$ in 18.26,

$$1 = \frac{\eta(a) - z}{\eta(b) - z}$$

which shows $\eta(a) = \eta(b)$. This proves the corollary since the assertion about continuity was already observed.

It is a good idea to consider a simple case to get an idea of what the winding number is measuring. To do so, consider $\gamma : [a, b] \to \mathbb{C}$ such that γ is continuous, closed and bounded variation. Suppose also that γ is one to one on (a, b). Such a curve is called a simple closed curve. It can be shown that such a simple closed curve divides the plane into exactly two components, an "inside" bounded component and an "outside" unbounded component. This is called the Jordan Curve theorem or the Jordan separation theorem. This is a difficult theorem which requires some very hard topology such as homology theory or degree theory. It won't be used here beyond making reference to it. For now, it suffices to simply assume that γ is such that this result holds. This will usually be obvious anyway. Also suppose that it is possible to change the parameter to be in $[0, 2\pi]$, in such a way that $\gamma(t) + \lambda\left(z + re^{it} - \gamma(t)\right) - z \neq 0$ for all $t \in [0, 2\pi]$ and $\lambda \in [0, 1]$. (As t goes from 0 to 2π the point $\gamma(t)$ traces the curve $\gamma([0, 2\pi])$ in the counter clockwise direction.) Suppose $z \in D$, the inside of the simple closed curve and consider the curve $\delta(t) = z + re^{it}$ for $t \in [0, 2\pi]$ where r is chosen small enough that $\overline{B(z, r)} \subseteq D$. Then it happens that $n(\delta, z) = n(\gamma, z)$.

Proposition 18.44 *Under the above conditions,*

$$n(\delta, z) = n(\gamma, z)$$

and $n(\delta, z) = 1$.

Proof: By changing the parameter, assume that $[a, b] = [0, 2\pi]$. From Theorem 18.42 it suffices to assume also that γ is C^1. Define $h_\lambda(t) \equiv \gamma(t) + \lambda\left(z + re^{it} - \gamma(t)\right)$ for $\lambda \in [0, 1]$. (This function is called a homotopy of the curves γ and δ.) Note that for each $\lambda \in [0, 1]$, $t \to h_\lambda(t)$ is a closed C^1 curve. Also,

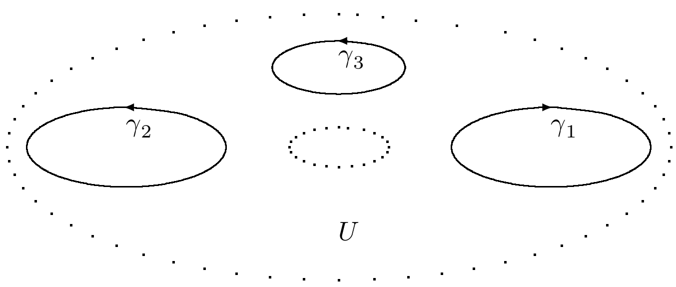
$$\frac{1}{2\pi i} \int_{h_\lambda} \frac{1}{w-z} dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t) + \lambda\left(rie^{it} - \gamma'(t)\right)}{\gamma(t) + \lambda\left(z + re^{it} - \gamma(t)\right) - z} dt.$$

This number is an integer and it is routine to verify that it is a continuous function of λ. When $\lambda = 0$ it equals $n(\gamma, z)$ and when $\lambda = 1$ it equals $n(\delta, z)$. Therefore, $n(\delta, z) = n(\gamma, z)$. It only remains to compute $n(\delta, z)$.

$$n(\delta, z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{re^{it}} dt = 1.$$

This proves the proposition.

Now if γ was not one to one but caused the point, $\gamma(t)$ to travel around γ^* twice, you could modify the above argument to have the parameter interval, $[0, 4\pi]$ and still find $n(\delta, z) = n(\gamma, z)$ only this time, $n(\delta, z) = 2$. Thus the winding number is just what its name suggests. It measures the number of times the curve winds around the point. One might ask why bother with the winding number if this is all it does. The reason is that the notion of counting the number of times a curve winds around a point is rather vague. The winding number is precise. It is also the natural thing to consider in the general Cauchy integral formula presented below. Consider a situation typified by the following picture in which Ω is the open set between the dotted curves and γ_j are closed rectifiable curves in Ω.



The following theorem is the general Cauchy integral formula.

Definition 18.45 *Let $\{\gamma_k\}_{k=1}^n$ be continuous oriented curves having bounded variation. Then this is called a cycle if whenever, $z \notin \cup_{k=1}^n \gamma_k^*$, $\sum_{k=1}^n n(\gamma_k, z)$ is an integer.*

By Theorem 18.42 if each γ_k is a closed curve, then $\{\gamma_k\}_{k=1}^n$ is a cycle.

Theorem 18.46 *Let Ω be an open subset of the plane and let $f : \Omega \to X$ be analytic. If $\gamma_k : [a_k, b_k] \to \Omega,\ k = 1, \cdots, m$ are continuous curves having bounded variation such that for all $z \notin \cup_{k=1}^m \gamma_k([a_k, b_k])$*

$$\sum_{k=1}^m n(\gamma_k, z) \text{ equals an integer}$$

and for all $z \notin \Omega$,

$$\sum_{k=1}^m n(\gamma_k, z) = 0.$$

Then for all $z \in \Omega \setminus \cup_{k=1}^{m} \gamma_k([a_k, b_k])$,

$$f(z) \sum_{k=1}^{m} n(\gamma_k, z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw.$$

Proof: Let ϕ be defined on $\Omega \times \Omega$ by

$$\phi(z, w) \equiv \begin{cases} \frac{f(w) - f(z)}{w - z} \text{ if } w \neq z \\ f'(z) \text{ if } w = z \end{cases}.$$

Then ϕ is analytic as a function of both z and w and is continuous in $\Omega \times \Omega$. This is easily seen using Theorem 18.35. Consider the case of $w \to \phi(z, w)$.

$$\lim_{w \to z} (w - z)(\phi(z, w) - \phi(z, z)) = \lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} - f'(z) \right) = 0.$$

Thus $w \to \phi(z, w)$ has a removable singularity at z. The case of $z \to \phi(z, w)$ is similar.

Define

$$h(z) \equiv \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w) \, dw.$$

Is h is analytic on Ω? To show this is the case, verify

$$\int_{\partial T} h(z) \, dz = 0$$

for every triangle, T, contained in Ω and apply Corollary 18.30. To do this, use Theorem 17.12 to obtain for each k, a sequence of functions, $\eta_{kn} \in C^1([a_k, b_k])$ such that

$$\eta_{kn}(x) = \gamma_k(x) \text{ for } x \in \{a_k, b_k\}$$

and

$$\eta_{kn}([a_k, b_k]) \subseteq \Omega, \ \ ||\eta_{kn} - \gamma_k|| < \frac{1}{n},$$

$$\left|\left| \int_{\eta_{kn}} \phi(z, w) \, dw - \int_{\gamma_k} \phi(z, w) \, dw \right|\right| < \frac{1}{n}, \tag{18.27}$$

for all $z \in T$. Then applying Fubini's theorem,

$$\int_{\partial T} \int_{\eta_{kn}} \phi(z, w) \, dw dz = \int_{\eta_{kn}} \int_{\partial T} \phi(z, w) \, dz dw = 0$$

because ϕ is given to be analytic. By 18.27,

$$\int_{\partial T} \int_{\gamma_k} \phi(z, w) \, dw dz = \lim_{n \to \infty} \int_{\partial T} \int_{\eta_{kn}} \phi(z, w) \, dw dz = 0$$

and so h is analytic on Ω as claimed.

Now let H denote the set,

$$H \equiv \left\{ z \in \mathbb{C} \setminus \cup_{k=1}^{m} \gamma_k\left([a_k, b_k]\right) : \sum_{k=1}^{m} n\left(\gamma_k, z\right) = 0 \right\}.$$

H is an open set because $z \to \sum_{k=1}^{m} n\left(\gamma_k, z\right)$ is integer valued by assumption and continuous. Define

$$g(z) \equiv \begin{cases} h(z) \text{ if } z \in \Omega \\ \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(w)}{w-z}dw \text{ if } z \in H \end{cases}. \tag{18.28}$$

Why is $g(z)$ well defined? For $z \in \Omega \cap H$, $z \notin \cup_{k=1}^{m}\gamma_k\left([a_k, b_k]\right)$ and so

$$\begin{aligned} g(z) &= \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\phi(z,w)\,dw = \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(w)-f(z)}{w-z}dw \\ &= \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(z)}{w-z}dw \\ &= \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(w)}{w-z}dw \end{aligned}$$

because $z \in H$. This shows $g(z)$ is well defined. Also, g is analytic on Ω because it equals h there. It is routine to verify that g is analytic on H also because of the second line of 18.28. By assumption, $\Omega^C \subseteq H$ because it is assumed that $\sum_k n\left(\gamma_k, z\right) = 0$ for $z \notin \Omega$ and so $\Omega \cup H = \mathbb{C}$ showing that g is an entire function.

Now note that $\sum_{k=1}^{m} n\left(\gamma_k, z\right) = 0$ for all z contained in the unbounded component of $\mathbb{C} \setminus \cup_{k=1}^{m} \gamma_k\left([a_k, b_k]\right)$ which component contains $B(0,r)^C$ for r large enough. It follows that for $|z| > r$, it must be the case that $z \in H$ and so for such z, the bottom description of $g(z)$ found in 18.28 is valid. Therefore, it follows

$$\lim_{|z|\to\infty} ||g(z)|| = 0$$

and so g is bounded and entire. By Liouville's theorem, g is a constant. Hence, from the above equation, the constant can only equal zero.

For $z \in \Omega \setminus \cup_{k=1}^{m}\gamma_k\left([a_k, b_k]\right)$,

$$0 = h(z) = \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\phi(z,w)\,dw = \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(w)-f(z)}{w-z}dw =$$

$$\frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(w)}{w-z}dw - f(z)\sum_{k=1}^{m} n\left(\gamma_k, z\right).$$

This proves the theorem.

Corollary 18.47 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \mathbb{C}$ is analytic,

$$\sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = 0.$$

Proof: This follows from Theorem 18.46 as follows. Let

$$g(w) = f(w)(w - z)$$

where $z \in \Omega \setminus \cup_{k=1}^{m} \gamma_k([a_k, b_k])$. Then by this theorem,

$$0 = 0 \sum_{k=1}^{m} n(\gamma_k, z) = g(z) \sum_{k=1}^{m} n(\gamma_k, z) =$$

$$\sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw.$$

Another simple corollary to the above theorem is Cauchy's theorem for a simply connected region.

Definition 18.48 *An open set, $\Omega \subseteq \mathbb{C}$ is a region if it is open and connected. A region, Ω is simply connected if $\widehat{\mathbb{C}} \setminus \Omega$ is connected where $\widehat{\mathbb{C}}$ is the extended complex plane. In the future, the term simply connected open set will be an open set which is connected and $\widehat{\mathbb{C}} \setminus \Omega$ is connected .*

Corollary 18.49 *Let $\gamma : [a, b] \to \Omega$ be a continuous closed curve of bounded variation where Ω is a simply connected region in $\mathbb{C}$ and let $f : \Omega \to X$ be analytic. Then*

$$\int_{\gamma} f(w)\, dw = 0.$$

Proof: Let D denote the unbounded component of $\widehat{\mathbb{C}} \setminus \gamma^*$. Thus $\infty \in \widehat{\mathbb{C}} \setminus \gamma^*$. Then the connected set, $\widehat{\mathbb{C}} \setminus \Omega$ is contained in D since every point of $\widehat{\mathbb{C}} \setminus \Omega$ must be in some component of $\widehat{\mathbb{C}} \setminus \gamma^*$ and ∞ is contained in both $\widehat{\mathbb{C}} \setminus \Omega$ and D. Thus D must be the component that contains $\widehat{\mathbb{C}} \setminus \Omega$. It follows that $n(\gamma, \cdot)$ must be constant on $\widehat{\mathbb{C}} \setminus \Omega$, its value being its value on D. However, for $z \in D$,

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} dw$$

and so $\lim_{|z|\to\infty} n(\gamma, z) = 0$ showing $n(\gamma, z) = 0$ on D. Therefore this verifies the hypothesis of Theorem 18.46. Let $z \in \Omega \cap D$ and define

$$g(w) \equiv f(w)(w-z).$$

Thus g is analytic on Ω and by Theorem 18.46,

$$0 = n(z,\gamma) g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w-z} dw = \frac{1}{2\pi i} \int_\gamma f(w) dw.$$

This proves the corollary.

The following is a very significant result which will be used later.

Corollary 18.50 *Suppose Ω is a simply connected open set and $f : \Omega \to X$ is analytic. Then f has a primitive, F, on Ω. Recall this means there exists F such that $F'(z) = f(z)$ for all $z \in \Omega$.*

Proof: Pick a point, $z_0 \in \Omega$ and let V denote those points, z of Ω for which there exists a curve, $\gamma : [a,b] \to \Omega$ such that γ is continuous, of bounded variation, $\gamma(a) = z_0$, and $\gamma(b) = z$. Then it is easy to verify that V is both open and closed in Ω and therefore, $V = \Omega$ because Ω is connected. Denote by $\gamma_{z_0,z}$ such a curve from z_0 to z and define

$$F(z) \equiv \int_{\gamma_{z_0,z}} f(w) dw.$$

Then F is well defined because if $\gamma_j, j = 1,2$ are two such curves, it follows from Corollary 18.49 that

$$\int_{\gamma_1} f(w) dw + \int_{-\gamma_2} f(w) dw = 0,$$

implying that

$$\int_{\gamma_1} f(w) dw = \int_{\gamma_2} f(w) dw.$$

Now this function, F is a primitive because, thanks to Corollary 18.49

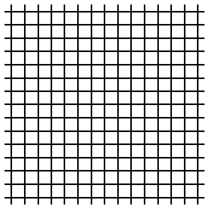
$$\begin{aligned}(F(z+h) - F(z)) h^{-1} &= \frac{1}{h} \int_{\gamma_{z,z+h}} f(w) dw \\ &= \frac{1}{h} \int_0^1 f(z+th) h dt\end{aligned}$$

and so, taking the limit as $h \to 0$, $F'(z) = f(z)$.

18.7.5 An Example Of A Cycle

The next theorem deals with the existence of a cycle with nice properties. Basically, you go around the compact subset of an open set with suitable contours while staying in the open set. The method involves the following simple concept.

Definition 18.51 *A tiling of $\mathbb{R}^2 = \mathbb{C}$ is the union of infinitely many equally spaced vertical and horizontal lines. You can think of the small squares which result as tiles. To tile the plane or $\mathbb{R}^2 = \mathbb{C}$ means to consider such a union of horizontal and vertical lines. It is like graph paper. See the picture below for a representation of part of a tiling of $\mathbb{C}$.*



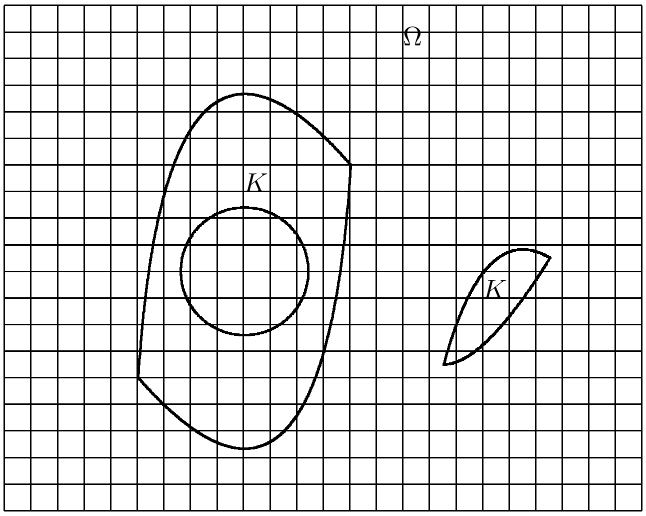
Theorem 18.52 *Let K be a compact subset of an open set, Ω. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each j, $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^{m} n(\Gamma_k, p) = 1.$$

while for all $z \notin \Omega$

$$\sum_{k=1}^{m} n(\Gamma_k, z) = 0.$$

Proof: Let $\delta = \text{dist}\left(K, \Omega^C\right)$. Since K is compact, $\delta > 0$. Now tile the plane with squares, each of which has diameter less than $\delta/2$.

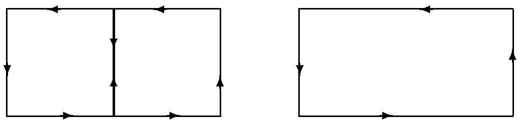


Let S denote the set of all the closed squares in this tiling which have nonempty intersection with K.Thus, all the squares of S are contained in Ω. First suppose p is a point of K which is in the interior of one of these squares in the tiling. Denote by ∂S_k the boundary of S_k one of the squares in S, oriented in the counter clockwise direction and S_m denote the square of S which contains the point, p in its interior. Let the edges of the square, S_j be $\left\{\gamma_k^j\right\}_{k=1}^4$. Thus a short computation shows $n\left(\partial S_m, p\right)=1$ but $n\left(\partial S_j, p\right)=0$ for all $j \neq m$. The reason for this is that for z in S_j, the values $\left\{z-p: z \in S_j\right\}$ lie in an open square, Q which is located at a positive distance from 0. Then $\widehat{\mathbb{C}} \setminus Q$ is connected and $1/(z-p)$ is analytic on Q. It follows from Corollary 18.50 that this function has a primitive on Q and so

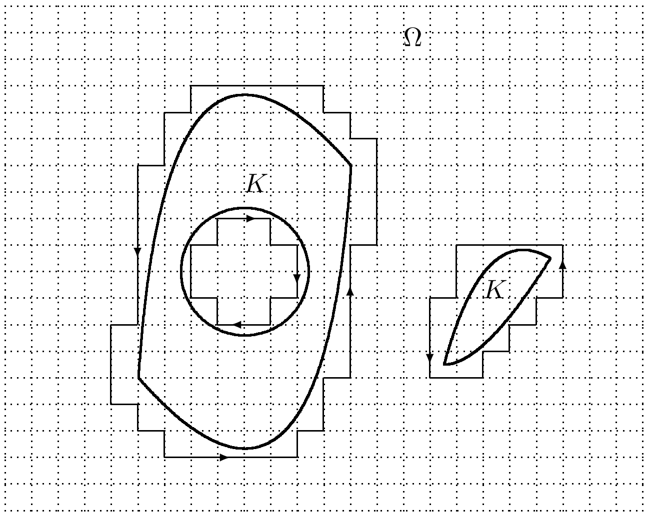
$$\int_{\partial S_j} \frac{1}{z-p} dz=0.$$

Similarly, if $z \notin \Omega, n\left(\partial S_j, z\right)=0$. On the other hand, a direct computation will verify that $n\left(p, \partial S_m\right)=1$. Thus $1=\sum_{j, k} n\left(p, \gamma_k^j\right)=\sum_{S_j \in S} n\left(p, \partial S_j\right)$ and if $z \notin \Omega$, $0=\sum_{j, k} n\left(z, \gamma_k^j\right)=\sum_{S_j \in S} n\left(z, \partial S_j\right)$.

If γ_k^{j*} coincides with γ_l^{l*}, then the contour integrals taken over this edge are taken in opposite directions and so the edge the two squares have in common can be deleted without changing $\sum_{j, k} n\left(z, \gamma_k^j\right)$ for any z not on any of the lines in the tiling. For example, see the picture,



From the construction, if any of the γ_k^{j*} contains a point of K then this point is on one of the four edges of S_j and at this point, there is at least one edge of some S_l which also contains this point. As just discussed, this shared edge can be deleted without changing $\sum_{i, j} n\left(z, \gamma_k^j\right)$. Delete the edges of the S_k which intersect K but not the endpoints of these edges. That is, delete the open edges. When this is done, delete all isolated points. Let the resulting oriented curves be denoted by $\{\gamma_k\}_{k=1}^m$. Note that you might have $\gamma_k^*=\gamma_l^*$. The construction is illustrated in the following picture.



Then as explained above, $\sum_{k=1}^{m} n(p, \gamma_k) = 1$. It remains to prove the claim about the closed curves.

Each orientation on an edge corresponds to a direction of motion over that edge. Call such a motion over the edge a route. Initially, every vertex, (corner of a square in S) has the property there are the same number of routes to and from that vertex. When an open edge whose closure contains a point of K is deleted, every vertex either remains unchanged as to the number of routes to and from that vertex or it loses both a route away and a route to. Thus the property of having the same number of routes to and from each vertex is preserved by deleting these open edges.. The isolated points which result lose all routes to and from. It follows that upon removing the isolated points you can begin at any of the remaining vertices and follow the routes leading out from this and successive vertices according to orientation and eventually return to that end. Otherwise, there would be a vertex which would have only one route leading to it which does not happen. Now if you have used all the routes out of this vertex, pick another vertex and do the same process. Otherwise, pick an unused route out of the vertex and follow it to return. Continue this way till all routes are used exactly once, resulting in closed oriented curves, Γ_k. Then

$$\sum_k n(\Gamma_k, p) = \sum_j n(\gamma_k, p) = 1.$$

In case $p \in K$ is on some line of the tiling, it is not on any of the Γ_k because $\Gamma_k^* \cap K = \emptyset$ and so the continuity of $z \to n(\Gamma_k, z)$ yields the desired result in this case also. This proves the lemma.

18.8 Exercises

1. If U is simply connected, f is analytic on U and f has no zeros in U, show there exists an analytic function, F, defined on U such that $e^F = f$.

2. Let f be defined and analytic near the point $a \in \mathbb{C}$. Show that then $f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ whenever $|z-a| < R$ where R is the distance between a and the nearest point where f fails to have a derivative. The number R, is called the radius of convergence and the power series is said to be expanded about a.

3. Find the radius of convergence of the function $\frac{1}{1+z^2}$ expanded about $a = 2$. Note there is nothing wrong with the function, $\frac{1}{1+x^2}$ when considered as a function of a real variable, x for any value of x. However, if you insist on using power series, you find there is a limitation on the values of x for which the power series converges due to the presence in the complex plane of a point, i, where the function fails to have a derivative.

4. Suppose f is analytic on all of $\mathbb{C}$ and satisfies $|f(z)| < A + B|z|^{1/2}$. Show f is constant.

5. What if you defined an open set, U to be simply connected if $\mathbb{C} \setminus U$ is connected. Would it amount to the same thing? **Hint:** Consider the outside of $B(0,1)$.

6. Let $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Find $\int_\gamma \frac{1}{z^n} dz$ for $n = 1, 2, \cdots$.

7. Show $i \int_0^{2\pi} (2\cos\theta)^{2n} d\theta = \int_\gamma \left(z + \frac{1}{z}\right)^{2n} \left(\frac{1}{z}\right) dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Then evaluate this integral using the binomial theorem and the previous problem.

8. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.

9. Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic for $z \in U$, an open set. Let $g(z) = u^*(x,y) + iv^*(x,y)$ where
$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = Q \begin{pmatrix} u \\ v \end{pmatrix}$$
where Q is a unitary matrix. That is $QQ^* = Q^*Q = I$. When will g be analytic?

10. Suppose f is analytic on an open set, U, except for $\gamma^* \subset U$ where γ is a one to one continuous function having bounded variation, but it is known that f is continuous on γ^*. Show that in fact f is analytic on γ^* also. **Hint:** Pick a point on γ^*, say $\gamma(t_0)$ and suppose for now that $t_0 \in (a,b)$. Pick $r > 0$ such that $B = B(\gamma(t_0), r) \subseteq U$. Then show there exists $t_1 < t_0$ and $t_2 > t_0$ such

that $\gamma([t_1,t_2]) \subseteq \overline{B}$ and $\gamma(t_i) \notin B$. Thus $\gamma([t_1,t_2])$ is a path across B going through the center of B which divides B into two open sets, B_1, and B_2 along with γ^*. Let the boundary of B_k consist of $\gamma([t_1,t_2])$ and a circular arc, C_k. Now letting $z \in B_k$, the line integral of $\frac{f(w)}{w-z}$ over γ^* in two different directions cancels. Therefore, if $z \in B_k$, you can argue that $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$. By continuity, this continues to hold for $z \in \gamma((t_1,t_2))$. Therefore, f must be analytic on $\gamma((t_1,t_1))$ also. This shows that f must be analytic on $\gamma((a,b))$. To get the endpoints, simply extend γ to have the same properties but defined on $[a-\varepsilon, b+\varepsilon]$ and repeat the above argument or else do this at the beginning and note that you get $[a,b] \subseteq (a-\varepsilon, b+\varepsilon)$.

11. Let U be an open set contained in the upper half plane and suppose that there are finitely many line segments on the x axis which are contained in the boundary of U. Now suppose that f is defined, real, and continuous on these line segments and is defined and analytic on U. Now let $\widetilde{U}$ denote the reflection of U across the x axis. Show that it is possible to extend f to a function, g defined on all of

$$W \equiv \widetilde{U} \cup U \cup \{\text{the line segments mentioned earlier}\}$$

such that g is analytic in W. **Hint:** For $z \in \widetilde{U}$, the reflection of U across the x axis, let $g(z) \equiv \overline{f(\overline{z})}$. Show that g is analytic on $\widetilde{U} \cup U$ and continuous on the line segments. Then use Problem 10 or Morera's theorem to argue that g is analytic on the line segments also. The result of this problem is know as the Schwarz reflection principle.

12. Show that rotations and translations of analytic functions yield analytic functions and use this observation to generalize the Schwarz reflection principle to situations in which the line segments are part of a line which is not the x axis. Thus, give a version which involves reflection about an arbitrary line.

The Open Mapping Theorem

19.1 A Local Representation

The open mapping theorem, is an even more surprising result than the theorem about the zeros of an analytic function. The following proof of this important theorem uses an interesting local representation of the analytic function.

Theorem 19.1 *(Open mapping theorem) Let Ω be a region in $\mathbb{C}$ and suppose $f : \Omega \to \mathbb{C}$ is analytic. Then $f(\Omega)$ is either a point or a region. In the case where $f(\Omega)$ is a region, it follows that for each $z_0 \in \Omega$, there exists an open set, V containing z_0 and $m \in \mathbb{N}$ such that for all $z \in V$,*

$$f(z) = f(z_0) + \phi(z)^m \tag{19.1}$$

where $\phi : V \to B(0,\delta)$ is one to one, analytic and onto, $\phi(z_0) = 0$, $\phi'(z) \neq 0$ on V and ϕ^{-1} analytic on $B(0,\delta)$. If f is one to one then $m = 1$ for each z_0 and $f^{-1} : f(\Omega) \to \Omega$ is analytic.

Proof: Suppose $f(\Omega)$ is not a point. Then if $z_0 \in \Omega$ it follows there exists $r > 0$ such that $f(z) \neq f(z_0)$ for all $z \in B(z_0, r) \setminus \{z_0\}$. Otherwise, z_0 would be a limit point of the set,

$$\{z \in \Omega : f(z) - f(z_0) = 0\}$$

which would imply from Theorem 18.23 that $f(z) = f(z_0)$ for all $z \in \Omega$. Therefore, making r smaller if necessary and using the power series of f,

$$f(z) = f(z_0) + (z - z_0)^m g(z) \ \left(\overset{?}{=} \left((z - z_0) g(z)^{1/m}\right)^m\right)$$

for all $z \in B(z_0, r)$, where $g(z) \neq 0$ on $B(z_0, r)$. As implied in the above formula, one wonders if you can take the m^{th} root of $g(z)$.

$\frac{g'}{g}$ is an analytic function on $B(z_0, r)$ and so by Corollary 18.32 it has a primitive on $B(z_0, r)$, h. Therefore by the product rule and the chain rule, $(ge^{-h})' = 0$ and so there exists a constant, $C = e^{a+ib}$ such that on $B(z_0, r)$,

$$ge^{-h} = e^{a+ib}.$$

Therefore,

$$g(z) = e^{h(z)+a+ib}$$

and so, modifying h by adding in the constant, $a+ib$, $g(z) = e^{h(z)}$ where $h'(z) = \frac{g'(z)}{g(z)}$ on $B(z_0, r)$. Letting

$$\phi(z) = (z - z_0) e^{\frac{h(z)}{m}}$$

implies formula 19.1 is valid on $B(z_0, r)$. Now

$$\phi'(z_0) = e^{\frac{h(z_0)}{m}} \neq 0.$$

Shrinking r if necessary you can assume $\phi'(z) \neq 0$ on $B(z_0, r)$. Is there an open set, V contained in $B(z_0, r)$ such that ϕ maps V onto $B(0, \delta)$ for some $\delta > 0$?

Let $\phi(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Consider the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

where u, v are C^1 because ϕ is given to be analytic. The Jacobian of this map at $(x, y) \in B(z_0, r)$ is

$$\left| \begin{array}{cc} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{array} \right| = \left| \begin{array}{cc} u_x(x, y) & -v_x(x, y) \\ v_x(x, y) & u_x(x, y) \end{array} \right|$$

$$= u_x(x, y)^2 + v_x(x, y)^2 = \left|\phi'(z)\right|^2 \neq 0.$$

This follows from a use of the Cauchy Riemann equations. Also

$$\begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore, by the inverse function theorem there exists an open set, V, containing z_0 and $\delta > 0$ such that $(u, v)^T$ maps V one to one onto $B(0, \delta)$. Thus ϕ is one to one onto $B(0, \delta)$ as claimed. Applying the same argument to other points, z of V and using the fact that $\phi'(z) \neq 0$ at these points, it follows ϕ maps open sets to open sets. In other words, ϕ^{-1} is continuous.

It also follows that ϕ^m maps V onto $B(0, \delta^m)$. Therefore, the formula 19.1 implies that f maps the open set, V, containing z_0 to an open set. This shows $f(\Omega)$ is an open set because z_0 was arbitrary. It is connected because f is continuous and Ω is connected. Thus $f(\Omega)$ is a region. It remains to verify that ϕ^{-1} is analytic on $B(0, \delta)$. Since ϕ^{-1} is continuous,

$$\lim_{\phi(z_1) \rightarrow \phi(z)} \frac{\phi^{-1}(\phi(z_1)) - \phi^{-1}(\phi(z))}{\phi(z_1) - \phi(z)} = \lim_{z_1 \rightarrow z} \frac{z_1 - z}{\phi(z_1) - \phi(z)} = \frac{1}{\phi'(z)}.$$

Therefore, ϕ^{-1} is analytic as claimed.

It only remains to verify the assertion about the case where f is one to one. If $m>1$, then $e^{\frac{2\pi i}{m}}\neq 1$ and so for $z_1\in V$,

$$e^{\frac{2\pi i}{m}}\phi(z_1)\neq\phi(z_1). \tag{19.2}$$

But $e^{\frac{2\pi i}{m}}\phi(z_1)\in B(0,\delta)$ and so there exists $z_2\neq z_1$(since ϕ is one to one) such that $\phi(z_2)=e^{\frac{2\pi i}{m}}\phi(z_1)$. But then

$$\phi(z_2)^m=\left(e^{\frac{2\pi i}{m}}\phi(z_1)\right)^m=\phi(z_1)^m$$

implying $f(z_2)=f(z_1)$ contradicting the assumption that f is one to one. Thus $m=1$ and $f'(z)=\phi'(z)\neq 0$ on V. Since f maps open sets to open sets, it follows that f^{-1} is continuous and so

$$\begin{aligned}\left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1)\to f(z)}\frac{f^{-1}(f(z_1))-f^{-1}(f(z))}{f(z_1)-f(z)}\\ &= \lim_{z_1\to z}\frac{z_1-z}{f(z_1)-f(z)}=\frac{1}{f'(z)}.\end{aligned}$$

This proves the theorem.

One does not have to look very far to find that this sort of thing does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. Take for example, the function $f(x)=x^2$. Then $f(\mathbb{R})$ is neither a point nor a region. In fact $f(\mathbb{R})$ fails to be open.

Corollary 19.2 *Suppose in the situation of Theorem 19.1 $m>1$ for the local representation of f given in this theorem. Then there exists $\delta>0$ such that if $w\in B(f(z_0),\delta)=f(V)$ for V an open set containing z_0, then $f^{-1}(w)$ consists of m distinct points in V. (f is m to one on V)*

Proof: Let $w\in B(f(z_0),\delta)$. Then $w=f(\widehat{z})$ where $\widehat{z}\in V$. Thus $f(\widehat{z})=f(z_0)+\phi(\widehat{z})^m$. Consider the m distinct numbers, $\left\{e^{\frac{2k\pi i}{m}}\phi(\widehat{z})\right\}_{k=1}^m$. Then each of these numbers is in $B(0,\delta)$ and so since ϕ maps V one to one onto $B(0,\delta)$, there are m distinct numbers in V, $\{z_k\}_{k=1}^m$ such that $\phi(z_k)=e^{\frac{2k\pi i}{m}}\phi(\widehat{z})$. Then

$$\begin{aligned}f(z_k) &= f(z_0)+\phi(z_k)^m=f(z_0)+\left(e^{\frac{2k\pi i}{m}}\phi(\widehat{z})\right)^m\\ &= f(z_0)+e^{2k\pi i}\phi(\widehat{z})^m=f(z_0)+\phi(\widehat{z})^m=f(\widehat{z})=w\end{aligned}$$

This proves the corollary.

19.1.1 Branches Of The Logarithm

The argument used in to prove the next theorem was used in the proof of the open mapping theorem. It is a very important result and deserves to be stated as a theorem.

Theorem 19.3 *Let Ω be a simply connected region and suppose $f : \Omega \to \mathbb{C}$ is analytic and nonzero on Ω. Then there exists an analytic function, g such that $e^{g(z)} = f(z)$ for all $z \in \Omega$.*

Proof: The function, f'/f is analytic on Ω and so by Corollary 18.50 there is a primitive for f'/f, denoted as g_1. Then

$$\left(e^{-g_1} f\right)' = -\frac{f'}{f} e^{-g_1} f + e^{-g_1} f' = 0$$

and so since Ω is connected, it follows $e^{-g_1} f$ equals a constant, e^{a+ib}. Therefore, $f(z) = e^{g_1(z)+a+ib}$. Define $g(z) \equiv g_1(z) + a + ib$.

The function, g in the above theorem is called a branch of the logarithm of f and is written as $\log(f(z))$.

Definition 19.4 *Let ρ be a ray starting at 0. Thus ρ is a straight line of infinite length extending in one direction with its initial point at 0.*

A special case of the above theorem is the following.

Theorem 19.5 *Let ρ be a ray starting at 0. Then there exists an analytic function, $L(z)$ defined on $\mathbb{C} \setminus \rho$ such that*

$$e^{L(z)} = z.$$

This function, L is called a branch of the logarithm. This branch of the logarithm satisfies the usual formula for logarithms, $L(zw) = L(z) + L(w)$ provided $zw \notin \rho$.

Proof: $\mathbb{C} \setminus \rho$ is a simply connected region because its complement with respect to $\widehat{\mathbb{C}}$ is connected. Furthermore, the function, $f(z) = z$ is not equal to zero on $\mathbb{C} \setminus \rho$. Therefore, by Theorem 19.3 there exists an analytic function $L(z)$ such that $e^{L(z)} = f(z) = z$. Now consider the problem of finding a description of $L(z)$. Each $z \in \mathbb{C} \setminus \rho$ can be written in a unique way in the form

$$z = |z| \, e^{i \arg_\theta(z)}$$

where $\arg_\theta(z)$ is the angle in $(\theta, \theta + 2\pi)$ associated with z. (You could of course have considered this to be the angle in $(\theta - 2\pi, \theta)$ associated with z or in infinitely many other open intervals of length 2π. The description of the log is not unique.) Then letting $L(z) = a + ib$

$$z = |z| \, e^{i \arg_\theta(z)} = e^{L(z)} = e^a e^{ib}$$

and so you can let $L(z) = \ln |z| + i \arg_\theta(z)$.

Does $L(z)$ satisfy the usual properties of the logarithm? That is, for $z, w \in \mathbb{C} \setminus \rho$, is $L(zw) = L(z) + L(w)$? This follows from the usual rules of exponents. You know $e^{z+w} = e^z e^w$. (You can verify this directly or you can reduce to the case where z, w are real. If z is a fixed real number, then the equation holds for all real w. Therefore, it must also hold for all complex w because the real line contains a limit point. Now for this fixed w, the equation holds for all z real. Therefore, by similar reasoning, it holds for all complex z.)

Now suppose $z, w \in \mathbb{C} \setminus \rho$ and $zw \notin \rho$. Then

$$e^{L(zw)} = zw,\ e^{L(z)+L(w)} = e^{L(z)}e^{L(w)} = zw$$

and so $L(zw) = L(z) + L(w)$ as claimed. This proves the theorem.

In the case where the ray is the negative real axis, it is called the principal branch of the logarithm. Thus $\arg(z)$ is a number between $-\pi$ and π.

Definition 19.6 *Let* log *denote the branch of the logarithm which corresponds to the ray for $\theta = \pi$. That is, the ray is the negative real axis. Sometimes this is called the principal branch of the logarithm.*

19.2 Maximum Modulus Theorem

Here is another very significant theorem known as the maximum modulus theorem which follows immediately from the open mapping theorem.

Theorem 19.7 *(maximum modulus theorem) Let Ω be a bounded region and let $f : \Omega \to \mathbb{C}$ be analytic and $f : \overline{\Omega} \to \mathbb{C}$ continuous. Then if $z \in \Omega$,*

$$|f(z)| \leq \max\{|f(w)| : w \in \partial\Omega\}. \tag{19.3}$$

If equality is achieved for any $z \in \Omega$, then f is a constant.

Proof: Suppose f is not a constant. Then $f(\Omega)$ is a region and so if $z \in \Omega$, there exists $r > 0$ such that $B(f(z), r) \subseteq f(\Omega)$. It follows there exists $z_1 \in \Omega$ with $|f(z_1)| > |f(z)|$. Hence $\max\left\{|f(w)| : w \in \overline{\Omega}\right\}$ is not achieved at any interior point of Ω. Therefore, the point at which the maximum is achieved must lie on the boundary of Ω and so

$$\max\{|f(w)| : w \in \partial\Omega\} = \max\left\{|f(w)| : w \in \overline{\Omega}\right\} > |f(z)|$$

for all $z \in \Omega$ or else f is a constant. This proves the theorem.

You can remove the assumption that Ω is bounded and give a slightly different version.

Theorem 19.8 *Let $f : \Omega \to \mathbb{C}$ be analytic on a region, Ω and suppose $\overline{B(a, r)} \subseteq \Omega$. Then*

$$|f(a)| \leq \max\left\{\left|f\left(a + re^{i\theta}\right)\right| : \theta \in [0, 2\pi]\right\}.$$

Equality occurs for some $r > 0$ and $a \in \Omega$ if and only if f is constant in Ω hence equality occurs for all such a, r.

Proof: The claimed inequality holds by Theorem 19.7. Suppose equality in the above is achieved for some $\overline{B(a,r)} \subseteq \Omega$. Then by Theorem 19.7 f is equal to a constant, w on $B(a,r)$. Therefore, the function, $f(\cdot) - w$ has a zero set which has a limit point in Ω and so by Theorem 18.23 $f(z) = w$ for all $z \in \Omega$.

Conversely, if f is constant, then the equality in the above inequality is achieved for all $\overline{B(a,r)} \subseteq \Omega$.

Next is yet another version of the maximum modulus principle which is in Conway [11]. Let Ω be an open set.

Definition 19.9 *Define $\partial_\infty \Omega$ to equal $\partial\Omega$ in the case where Ω is bounded and $\partial\Omega \cup \{\infty\}$ in the case where Ω is not bounded.*

Definition 19.10 *Let f be a complex valued function defined on a set $S \subseteq \mathbb{C}$ and let a be a limit point of S.*

$$\limsup_{z\to a} |f(z)| \equiv \lim_{r\to 0} \{\sup |f(w)| : w \in B'(a,r) \cap S\}.$$

The limit exists because $\{\sup |f(w)| : w \in B'(a,r) \cap S\}$ is decreasing in r. In case $a = \infty$,

$$\limsup_{z\to\infty} |f(z)| \equiv \lim_{r\to\infty} \{\sup |f(w)| : |w| > r, w \in S\}$$

Note that if $\limsup_{z\to a} |f(z)| \leq M$ and $\delta > 0$, then there exists $r > 0$ such that if $z \in B'(a,r) \cap S$, then $|f(z)| < M + \delta$. If $a = \infty$, there exists $r > 0$ such that if $|z| > r$ and $z \in S$, then $|f(z)| < M + \delta$.

Theorem 19.11 *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be analytic. Suppose also that for every $a \in \partial_\infty \Omega$,*

$$\limsup_{z\to a} |f(z)| \leq M < \infty.$$

Then in fact $|f(z)| \leq M$ for all $z \in \Omega$.

Proof: Let $\delta > 0$ and let $H \equiv \{z \in \Omega : |f(z)| > M + \delta\}$. Suppose $H \neq \emptyset$. Then H is an open subset of Ω. I claim that H is actually bounded. If Ω is bounded, there is nothing to show so assume Ω is unbounded. Then the condition involving the lim sup implies there exists $r > 0$ such that if $|z| > r$ and $z \in \Omega$, then $|f(z)| \leq M + \delta/2$. It follows H is contained in $\overline{B(0,r)}$ and so it is bounded. Now consider the components of Ω. One of these components contains points from H. Let this component be denoted as V and let $H_V \equiv H \cap V$. Thus H_V is a bounded open subset of V. Let U be a component of H_V. First suppose $\overline{U} \subseteq V$. In this case, it follows that on $\partial U, |f(z)| = M + \delta$ and so by Theorem 19.7 $|f(z)| \leq M + \delta$ for all $z \in U$ contradicting the definition of H. Next suppose ∂U contains a point of $\partial V, a$. Then in this case, a violates the condition on lim sup. Either way you get a contradiction. Hence $H = \emptyset$ as claimed. Since $\delta > 0$ is arbitrary, this shows $|f(z)| \leq M$.

19.3 Extensions Of Maximum Modulus Theorem

19.3.1 Phragmên Lindelöf Theorem

This theorem is an extension of Theorem 19.11. It uses a growth condition near the extended boundary to conclude that f is bounded. I will present the version found in Conway [11]. It seems to be more of a method than an actual theorem. There are several versions of it.

Theorem 19.12 *Let Ω be a simply connected region in $\mathbb{C}$ and suppose f is analytic on Ω. Also suppose there exists a function, ϕ which is nonzero and uniformly bounded on Ω. Let M be a positive number. Now suppose $\partial_\infty \Omega = A \cup B$ such that for every $a \in A$, $\limsup_{z \to a} |f(z)| \leq M$ and for every $b \in B$, and $\eta > 0$, $\limsup_{z \to b} |f(z)| |\phi(z)|^\eta \leq M$. Then $|f(z)| \leq M$ for all $z \in \Omega$.*

Proof: By Theorem 19.3 there exists $\log(\phi(z))$ analytic on Ω. Now define $g(z) \equiv \exp(\eta \log(\phi(z)))$ so that $g(z) = \phi(z)^\eta$. Now also

$$|g(z)| = |\exp(\eta \log(\phi(z)))| = |\exp(\eta \ln |\phi(z)|)| = |\phi(z)|^\eta.$$

Let $m \geq |\phi(z)|$ for all $z \in \Omega$. Define $F(z) \equiv f(z) g(z) m^{-\eta}$. Thus F is analytic and for $b \in B$,

$$\limsup_{z \to b} |F(z)| = \limsup_{z \to b} |f(z)| |\phi(z)|^\eta m^{-\eta} \leq M m^{-\eta}$$

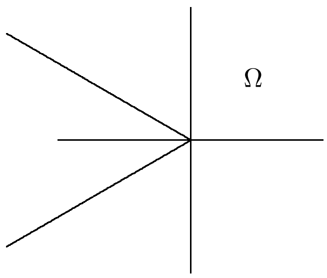
while for $a \in A$,

$$\limsup_{z \to a} |F(z)| \leq M.$$

Therefore, for $\alpha \in \partial_\infty \Omega$, $\limsup_{z \to \alpha} |F(z)| \leq \max(M, M\eta^{-\eta})$ and so by Theorem 19.11, $|f(z)| \leq \left(\frac{m^\eta}{|\phi(z)|^\eta}\right) \max(M, M\eta^{-\eta})$. Now let $\eta \to 0$ to obtain $|f(z)| \leq M$.

In applications, it is often the case that $B = \{\infty\}$.

Now here is an interesting case of this theorem. It involves a particular form for Ω, in this case $\Omega = \left\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\right\}$ where $a \geq \frac{1}{2}$.



Then $\partial\Omega$ equals the two slanted lines. Also on Ω you can define a logarithm, $\log(z) = \ln|z| + i \arg(z)$ where $\arg(z)$ is the angle associated with z between $-\pi$

and π. Therefore, if c is a real number you can define z^c for such z in the usual way:

$$\begin{aligned} z^c &\equiv \exp(c\log(z)) = \exp(c[\ln|z| + i\arg(z)]) \\ &= |z|^c \exp(ic\arg(z)) = |z|^c(\cos(c\arg(z)) + i\sin(c\arg(z))). \end{aligned}$$

If $|c| < a$, then $|c\arg(z)| < \frac{\pi}{2}$ and so $\cos(c\arg(z)) > 0$. Therefore, for such c,

$$\begin{aligned} |\exp(-(z^c))| &= |\exp(-|z|^c(\cos(c\arg(z)) + i\sin(c\arg(z))))| \\ &= |\exp(-|z|^c(\cos(c\arg(z))))| \end{aligned}$$

which is bounded since $\cos(c\arg(z)) > 0$.

Corollary 19.13 *Let $\Omega = \left\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\right\}$ where $a \geq \frac{1}{2}$ and suppose f is analytic on Ω and satisfies $\limsup_{z\to a}|f(z)| \leq M$ on $\partial\Omega$ and suppose there are positive constants, P, b where $b < a$ and*

$$|f(z)| \leq P\exp\left(|z|^b\right)$$

for all $|z|$ large enough. Then $|f(z)| \leq M$ for all $z \in \Omega$.

Proof: Let $b < c < a$ and let $\phi(z) \equiv \exp(-(z^c))$. Then as discussed above, $\phi(z) \neq 0$ on Ω and $|\phi(z)|$ is bounded on Ω. Now

$$|\phi(z)|^\eta = |\exp(-|z|^c\eta(\cos(c\arg(z))))|$$

$$\limsup_{z\to\infty}|f(z)||\phi(z)|^\eta = \limsup_{z\to\infty}\frac{P\exp\left(|z|^b\right)}{|\exp(|z|^c\eta(\cos(c\arg(z))))|} = 0 \leq M$$

and so by Theorem 19.12 $|f(z)| \leq M$.

The following is another interesting case. This case is presented in Rudin [36]

Corollary 19.14 *Let Ω be the open set consisting of $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ and suppose f is analytic on Ω , continuous on $\overline{\Omega}$, and bounded on Ω. Suppose also that $f(z) \leq 1$ on the two lines $\operatorname{Re} z = a$ and $\operatorname{Re} z = b$. Then $|f(z)| \leq 1$ for all $z \in \Omega$.*

Proof: This time let $\phi(z) = \frac{1}{1+z-a}$. Thus $|\phi(z)| \leq 1$ because $\operatorname{Re}(z-a) > 0$ and $\phi(z) \neq 0$ for all $z \in \Omega$. Also, $\limsup_{z\to\infty}|\phi(z)|^\eta = 0$ for every $\eta > 0$. Therefore, if a is a point of the sides of Ω, $\limsup_{z\to a}|f(z)| \leq 1$ while $\limsup_{z\to\infty}|f(z)||\phi(z)|^\eta = 0 \leq 1$ and so by Theorem 19.12, $|f(z)| \leq 1$ on Ω.

This corollary yields an interesting conclusion.

Corollary 19.15 *Let Ω be the open set consisting of $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ and suppose f is analytic on Ω , continuous on $\overline{\Omega}$, and bounded on Ω. Define*

$$M(x) \equiv \sup\{|f(z)| : \operatorname{Re} z = x\}$$

Then for $x \in (a, b)$.

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}.$$

Proof: Let $\varepsilon > 0$ and define

$$g(z) \equiv (M(a)+\varepsilon)^{\frac{b-z}{b-a}} (M(b)+\varepsilon)^{\frac{z-a}{b-a}}$$

where for $M > 0$ and $z \in \mathbb{C}$, $M^z \equiv \exp(z \ln(M))$. Thus $g \neq 0$ and so f/g is analytic on Ω and continuous on $\overline{\Omega}$. Also on the left side,

$$\left| \frac{f(a+iy)}{g(a+iy)} \right| = \left| \frac{f(a+iy)}{(M(a)+\varepsilon)^{\frac{b-a-iy}{b-a}}} \right| = \left| \frac{f(a+iy)}{(M(a)+\varepsilon)^{\frac{b-a}{b-a}}} \right| \leq 1$$

while on the right side a similar computation shows $\left|\frac{f}{g}\right| \leq 1$ also. Therefore, by Corollary 19.14 $|f/g| \leq 1$ on Ω. Therefore, letting $x + iy = z$,

$$|f(z)| \leq \left| (M(a)+\varepsilon)^{\frac{b-z}{b-a}} (M(b)+\varepsilon)^{\frac{z-a}{b-a}} \right| = \left| (M(a)+\varepsilon)^{\frac{b-x}{b-a}} (M(b)+\varepsilon)^{\frac{x-a}{b-a}} \right|$$

and so

$$M(x) \leq (M(a)+\varepsilon)^{\frac{b-x}{b-a}} (M(b)+\varepsilon)^{\frac{x-a}{b-a}}.$$

Since $\varepsilon > 0$ is arbitrary, it yields the conclusion of the corollary.

Another way of saying this is that $x \to \ln(M(x))$ is a convex function.

This corollary has an interesting application known as the Hadamard three circles theorem.

19.3.2 Hadamard Three Circles Theorem

Let $0 < R_1 < R_2$ and suppose f is analytic on $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$. Then letting $R_1 < a < b < R_2$, note that $g(z) \equiv \exp(z)$ maps the strip $\{z \in \mathbb{C} : \ln a < \operatorname{Re} z < b\}$ onto $\{z \in \mathbb{C} : a < |z| < b\}$ and that in fact, g maps the line $\ln r + iy$ onto the circle $re^{i\theta}$. Now let $M(x)$ be defined as above and m be defined by

$$m(r) \equiv \max_{\theta} \left| f\left(re^{i\theta}\right) \right|.$$

Then for $a < r < b$, Corollary 19.15 implies

$$\begin{aligned} m(r) &= \sup_y \left| f\left(e^{\ln r + iy}\right) \right| = M(\ln r) \leq M(\ln a)^{\frac{\ln b - \ln r}{\ln b - \ln a}} M(\ln b)^{\frac{\ln r - \ln a}{\ln b - \ln a}} \\ &= m(a)^{\ln(b/r)/\ln(b/a)} m(b)^{\ln(r/a)/\ln(b/a)} \end{aligned}$$

and so

$$m(r)^{\ln(b/a)} \leq m(a)^{\ln(b/r)} m(b)^{\ln(r/a)}.$$

Taking logarithms, this yields

$$\ln\left(\frac{b}{a}\right) \ln(m(r)) \leq \ln\left(\frac{b}{r}\right) \ln(m(a)) + \ln\left(\frac{r}{a}\right) \ln(m(b))$$

which says the same as $r \to \ln(m(r))$ is a convex function of $\ln r$.

The next example, also in Rudin [36] is very dramatic. An unbelievably weak assumption is made on the growth of the function and still you get a uniform bound in the conclusion.

Corollary 19.16 *Let* $\Omega = \left\{z \in \mathbb{C} : |\mathrm{Im}\,(z)| < \frac{\pi}{2}\right\}$. *Suppose* f *is analytic on* Ω, *continuous on* $\overline{\Omega}$, *and there exist constants,* $\alpha < 1$ *and* $A < \infty$ *such that*

$$|f(z)| \leq \exp(A \exp(\alpha |x|)) \text{ for } z = x + iy$$

and

$$\left| f\left(x \pm i\frac{\pi}{2}\right) \right| \leq 1$$

for all $x \in \mathbb{R}$. *Then* $|f(z)| \leq 1$ *on* Ω.

Proof: This time let $\phi(z) = [\exp(A\exp(\beta z))\exp(A\exp(-\beta z))]^{-1}$ where $\alpha < \beta < 1$. Then $\phi(z) \neq 0$ on Ω and for $\eta > 0$

$$|\phi(z)|^{\eta} = \frac{1}{|\exp(\eta A \exp(\beta z))\exp(\eta A \exp(-\beta z))|}$$

Now

$$\begin{aligned} & \exp(\eta A \exp(\beta z))\exp(\eta A \exp(-\beta z)) \\ = \; & \exp(\eta A (\exp(\beta z) + \exp(-\beta z))) \\ = \; & \exp\left[\eta A \left(\cos(\beta y)\left(e^{\beta x} + e^{-\beta x}\right) + i \sin(\beta y)\left(e^{\beta x} - e^{-\beta x}\right)\right)\right] \end{aligned}$$

and so

$$|\phi(z)|^{\eta} = \frac{1}{\exp[\eta A (\cos(\beta y)(e^{\beta x} + e^{-\beta x}))]}$$

Now $\cos \beta y > 0$ because $\beta < 1$ and $|y| < \frac{\pi}{2}$. Therefore,

$$\limsup_{z \to \infty} |f(z)| |\phi(z)|^{\eta} \leq 0 \leq 1$$

and so by Theorem 19.12, $|f(z)| \leq 1$.

19.3.3 Schwarz's Lemma

This interesting lemma comes from the maximum modulus theorem. It will be used later as part of the proof of the Riemann mapping theorem.

Lemma 19.17 *Suppose* $F : B(0,1) \to B(0,1)$, F *is analytic, and* $F(0) = 0$. *Then for all* $z \in B(0,1)$,

$$|F(z)| \leq |z|, \tag{19.4}$$

and

$$|F'(0)| \leq 1. \tag{19.5}$$

If equality holds in 19.5 then there exists $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and*

$$F(z) = \lambda z. \tag{19.6}$$

Proof: First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,

$$\left|\frac{F(z)}{z}\right| \leq \max_{t\in[0,2\pi]} \frac{\left|F\left(re^{it}\right)\right|}{r} \leq \frac{1}{r}.$$

Then letting $r \to 1$,

$$\left|\frac{F(z)}{z}\right| \leq 1$$

this shows 19.4 and it also verifies 19.5 on taking the limit as $z \to 0$. If equality holds in 19.5, then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, λ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$.

Rudin [36] gives a memorable description of what this lemma says. It says that if an analytic function maps the unit ball to itself, keeping 0 fixed, then it must do one of two things, either be a rotation or move all points closer to 0. (This second part follows in case $|F'(0)| < 1$ because in this case, you must have $|F(z)| \neq |z|$ and so by 19.4, $|F(z)| < |z|$)

19.3.4 One To One Analytic Maps On The Unit Ball

The transformation in the next lemma is of fundamental importance.

Lemma 19.18 *Let $\alpha \in B(0,1)$ and define*

$$\phi_\alpha(z) \equiv \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Then $\phi_\alpha : B(0,1) \to B(0,1)$, $\phi_\alpha : \partial B(0,1) \to \partial B(0,1)$, and is one to one and onto. Also $\phi_{-\alpha} = \phi_\alpha^{-1}$. Also

$$\phi_\alpha'(0) = 1 - |\alpha|^2,\ \phi'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Proof: First of all, for $|z| < 1/|\alpha|$,

$$\phi_\alpha \circ \phi_{-\alpha}(z) \equiv \frac{\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right) - \alpha}{1 - \overline{\alpha}\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right)} = z$$

after a few computations. If I show that ϕ_α maps $B(0,1)$ to $B(0,1)$ for all $|\alpha| < 1$, this will have shown that ϕ_α is one to one and onto $B(0,1)$.

Consider $\left|\phi_\alpha\left(e^{i\theta}\right)\right|$. This yields

$$\left|\frac{e^{i\theta}-\alpha}{1-\overline{\alpha}e^{i\theta}}\right| = \left|\frac{1-\alpha e^{-i\theta}}{1-\overline{\alpha}e^{i\theta}}\right| = 1$$

where the first equality is obtained by multiplying by $\left|e^{-i\theta}\right| = 1$. Therefore, ϕ_α maps $\partial B(0,1)$ one to one and onto $\partial B(0,1)$. Now notice that ϕ_α is analytic on $B(0,1)$ because the only singularity, a pole is at $z = 1/\overline{\alpha}$. By the maximum modulus theorem, it follows

$$|\phi_\alpha(z)| < 1$$

whenever $|z| < 1$. The same is true of $\phi_{-\alpha}$.

It only remains to verify the assertions about the derivatives. Long division gives $\phi_\alpha(z) = (-\overline{\alpha})^{-1} + \left(\frac{-\alpha+(\overline{\alpha})^{-1}}{1-\overline{\alpha}z}\right)$ and so

$$\begin{aligned} \phi_\alpha'(z) &= (-1)(1-\overline{\alpha}z)^{-2}\left(-\alpha+(\overline{\alpha})^{-1}\right)(-\overline{\alpha}) \\ &= \overline{\alpha}(1-\overline{\alpha}z)^{-2}\left(-\alpha+(\overline{\alpha})^{-1}\right) \\ &= (1-\overline{\alpha}z)^{-2}\left(-|\alpha|^2+1\right) \end{aligned}$$

Hence the two formulas follow. This proves the lemma.

One reason these mappings are so important is the following theorem.

Theorem 19.19 *Suppose f is an analytic function defined on $B(0,1)$ and f maps $B(0,1)$ one to one and onto $B(0,1)$. Then there exists θ such that*

$$f(z) = e^{i\theta}\phi_\alpha(z)$$

for some $\alpha \in B(0,1)$.

Proof: Let $f(\alpha) = 0$. Then $h(z) \equiv f \circ \phi_{-\alpha}(z)$ maps $B(0,1)$ one to one and onto $B(0,1)$ and has the property that $h(0) = 0$. Therefore, by the Schwarz lemma,

$$|h(z)| \leq |z|.$$

but it is also the case that $h^{-1}(0) = 0$ and h^{-1} maps $B(0,1)$ to $B(0,1)$. Therefore, the same inequality holds for h^{-1}. Therefore,

$$|z| = \left|h^{-1}(h(z))\right| \leq |h(z)|$$

and so $|h(z)| = |z|$. By the Schwarz lemma again, $h(z) \equiv f\left(\phi_{-\alpha}(z)\right) = e^{i\theta}z$. Letting $z = \phi_\alpha$, you get $f(z) = e^{i\theta}\phi_\alpha(z)$.

19.4 Exercises

1. Consider the function, $g(z) = \frac{z-i}{z+i}$. Show this is analytic on the upper half plane, $P+$ and maps the upper half plane one to one and onto $B(0,1)$. **Hint:** First show g maps the real axis to $\partial B(0,1)$. This is really easy because you end up looking at a complex number divided by its conjugate. Thus $|g(z)| = 1$ for z on $\partial(P+)$. Now show that $\limsup_{z\to\infty}|g(z)| = 1$. Then apply a version of the maximum modulus theorem. You might note that $g(z) = 1 + \frac{-2i}{z+i}$. This will show $|g(z)| \leq 1$. Next pick $w \in B(0,1)$ and solve $g(z) = w$. You just have to show there exists a unique solution and its imaginary part is positive.

2. Does there exist an entire function f which maps $\mathbb{C}$ onto the upper half plane?

3. Letting g be the function of Problem 1 show that $\left(g^{-1}\right)'(0) = 2$. Also note that $g^{-1}(0) = i$. Now suppose f is an analytic function defined on the upper half plane which has the property that $|f(z)| \leq 1$ and $f(i) = \beta$ where $|\beta| < 1$. Find an upper bound to $|f'(i)|$. Also find all functions, f which satisfy the condition, $f(i) = \beta, |f(z)| \leq 1$, and achieve this maximum value. **Hint:** You could consider the function, $h(z) \equiv \phi_\beta \circ f \circ g^{-1}(z)$ and check the conditions for the Schwarz lemma for this function, h.

4. This and the next two problems follow a presentation of an interesting topic in Rudin [36]. Let ϕ_α be given in Lemma 19.18. Suppose f is an analytic function defined on $B(0,1)$ which satisfies $|f(z)| \leq 1$. Suppose also there are $\alpha, \beta \in B(0,1)$ and it is required $f(\alpha) = \beta$. If f is such a function, show that $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** To show this consider $g = \phi_\beta \circ f \circ \phi_{-\alpha}$. Show $g(0) = 0$ and $|g(z)| \leq 1$ on $B(0,1)$. Now use Lemma 19.17.

5. In Problem 4 show there exists a function, f analytic on $B(0,1)$ such that $f(\alpha) = \beta$, $|f(z)| \leq 0$, and $|f'(\alpha)| = \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** You do this by choosing g in the above problem such that equality holds in Lemma 19.17. Thus you need $g(z) = \lambda z$ where $|\lambda| = 1$ and solve $g = \phi_\beta \circ f \circ \phi_{-\alpha}$ for f.

6. Suppose that $f : B(0,1) \to B(0,1)$ and that f is analytic, one to one, and onto with $f(\alpha) = 0$. Show there exists $\lambda, |\lambda| = 1$ such that $f(z) = \lambda \phi_\alpha(z)$. This gives a different way to look at Theorem 19.19. **Hint:** Let $g = f^{-1}$. Then $g'(0) f'(\alpha) = 1$. However, $f(\alpha) = 0$ and $g(0) = \alpha$. From Problem 4 with $\beta = 0$, you can conclude an inequality for $|f'(\alpha)|$ and another one for $|g'(0)|$. Then use the fact that the product of these two equals 1 which comes from the chain rule to conclude that equality must take place. Now use Problem 5 to obtain the form of f.

7. In Corollary 19.16 show that it is essential that $\alpha < 1$. That is, show there exists an example where the conclusion is not satisfied with a slightly weaker growth condition. **Hint:** Consider $\exp(\exp(z))$.

8. Suppose $\{f_n\}$ is a sequence of functions which are analytic on Ω, a bounded region such that each f_n is also continuous on $\overline{\Omega}$. Suppose that $\{f_n\}$ converges uniformly on $\partial\Omega$. Show that then $\{f_n\}$ converges uniformly on $\overline{\Omega}$ and that the function to which the sequence converges is analytic on Ω and continuous on $\overline{\Omega}$.

9. Suppose Ω is a bounded region and there exists a point $z_0 \in \Omega$ such that $|f(z_0)| = \min\left\{|f(z)| : z \in \overline{\Omega}\right\}$. Can you conclude f must equal a constant?

10. Suppose f is continuous on $\overline{B(a,r)}$ and analytic on $B(a,r)$ and that f is not constant. Suppose also $|f(z)| = C \neq 0$ for all $|z - a| = r$. Show that there exists $\alpha \in B(a,r)$ such that $f(\alpha) = 0$. **Hint:** If not, consider f/C and C/f. Both would be analytic on $B(a,r)$ and are equal to 1 on the boundary.

11. Suppose f is analytic on $B(0,1)$ but for every $a \in \partial B(0,1)$, $\lim_{z\to a} |f(z)| = \infty$. Show there exists a sequence, $\{z_n\} \subseteq B(0,1)$ such that $\lim_{n\to\infty} |z_n| = 1$ and $f(z_n) = 0$.

19.5 Counting Zeros

The above proof of the open mapping theorem relies on the very important inverse function theorem from real analysis. There are other approaches to this important theorem which do not rely on the big theorems from real analysis and are more oriented toward the use of the Cauchy integral formula and specialized techniques from complex analysis. One of these approaches is given next which involves the notion of "counting zeros". The next theorem is the one about counting zeros. It will also be used later in the proof of the Riemann mapping theorem.

Theorem 19.20 *Let Ω be an open set in $\mathbb{C}$ and let $\gamma : [a,b] \to \Omega$ be closed, continuous, bounded variation, and $n(\gamma, z) = 0$ for all $z \notin \Omega$. Suppose also that f is analytic on Ω having zeros $a_1, \cdots, a_m$ where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on γ^*. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: Let $f(z) = \prod_{j=1}^m (z - a_j) g(z)$ where $g(z) \neq 0$ on Ω. Hence

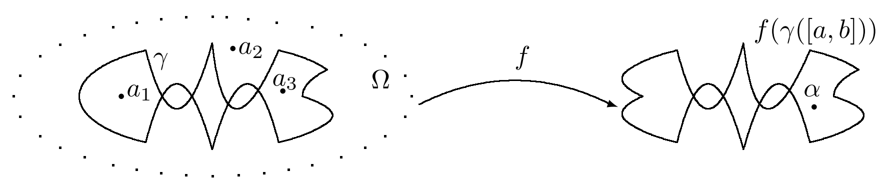
$$\frac{f'(z)}{f(z)} = \sum_{j=1}^m \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}$$

and so

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{j=1}^m n(\gamma, a_j) + \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz.$$

But the function, $z \to \frac{g'(z)}{g(z)}$ is analytic and so by Corollary 18.47, the last integral in the above expression equals 0. Therefore, this proves the theorem.

The following picture is descriptive of the situation described in the next theorem.



Theorem 19.21 *Let Ω be a region, let $\gamma : [a,b] \to \Omega$ be closed continuous, and bounded variation such that $n(\gamma, z) = 0$ for all $z \notin \Omega$. Also suppose $f : \Omega \to \mathbb{C}$*

is analytic and that $\alpha \notin f(\gamma^*)$. *Then* $f \circ \gamma : [a,b] \to \mathbb{C}$ *is continuous, closed, and bounded variation. Also suppose* $\{a_1, \cdots, a_m\} = f^{-1}(\alpha)$ *where these points are counted according to their multiplicities as zeros of the function* $f - \alpha$ *Then*

$$n(f \circ \gamma, \alpha) = \sum_{k=1}^{m} n(\gamma, a_k).$$

Proof: It is clear that $f \circ \gamma$ is continuous. It only remains to verify that it is of bounded variation. Suppose first that $\gamma^* \subseteq B \subseteq \overline{B} \subseteq \Omega$ where B is a ball. Then

$$|f(\gamma(t)) - f(\gamma(s))| =$$

$$\begin{aligned} & \left| \int_0^1 f'(\gamma(s) + \lambda(\gamma(t) - \gamma(s)))(\gamma(t) - \gamma(s))\, d\lambda \right| \\ \leq \quad & C|\gamma(t) - \gamma(s)| \end{aligned}$$

where $C \geq \max\left\{|f'(z)| : z \in \overline{B}\right\}$. Hence, in this case,

$$V(f \circ \gamma, [a,b]) \leq CV(\gamma, [a,b]).$$

Now let ε denote the distance between γ^* and $\mathbb{C} \setminus \Omega$. Since γ^* is compact, $\varepsilon > 0$. By uniform continuity there exists $\delta = \frac{b-a}{p}$ for p a positive integer such that if $|s-t| < \delta$, then $|\gamma(s) - \gamma(t)| < \frac{\varepsilon}{2}$. Then

$$\gamma([t, t+\delta]) \subseteq \overline{B\left(\gamma(t), \frac{\varepsilon}{2}\right)} \subseteq \Omega.$$

Let $C \geq \max\left\{|f'(z)| : z \in \cup_{j=1}^{p} \overline{B\left(\gamma(t_j), \frac{\varepsilon}{2}\right)}\right\}$ where $t_j \equiv \frac{j}{p}(b-a) + a$. Then from what was just shown,

$$\begin{aligned} V(f \circ \gamma, [a,b]) & \leq \sum_{j=0}^{p-1} V(f \circ \gamma, [t_j, t_{j+1}]) \\ & \leq C \sum_{j=0}^{p-1} V(\gamma, [t_j, t_{j+1}]) < \infty \end{aligned}$$

showing that $f \circ \gamma$ is bounded variation as claimed. Now from Theorem 18.42 there exists $\eta \in C^1([a,b])$ such that

$$\eta(a) = \gamma(a) = \gamma(b) = \eta(b),\ \eta([a,b]) \subseteq \Omega,$$

and

$$n(\eta, a_k) = n(\gamma, a_k),\ n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha) \tag{19.7}$$

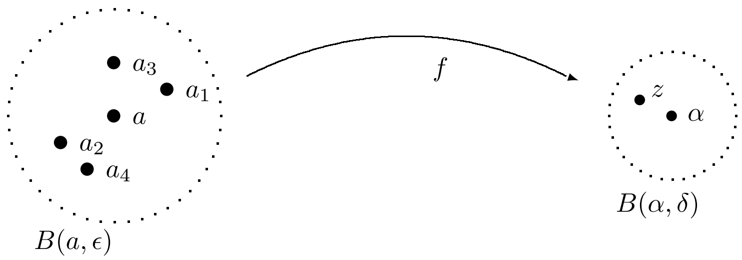
for $k = 1, \cdots, m$. Then

$$n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha)$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{f\circ\eta}\frac{dw}{w-\alpha} \\
&= \frac{1}{2\pi i}\int_a^b \frac{f'(\eta(t))}{f(\eta(t))-\alpha}\eta'(t)\,dt \\
&= \frac{1}{2\pi i}\int_\eta \frac{f'(z)}{f(z)-\alpha}dz \\
&= \sum_{k=1}^m n(\eta,a_k)
\end{aligned}$$

By Theorem 19.20. By 19.7, this equals $\sum_{k=1}^m n(\gamma,a_k)$ which proves the theorem.

The next theorem is incredible and is very interesting for its own sake. The following picture is descriptive of the situation of this theorem.



Theorem 19.22 *Let $f : B(a,R) \to \mathbb{C}$ be analytic and let*

$$f(z)-\alpha = (z-a)^m g(z),\ \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a,R)$. ($f(z)-\alpha$ has a zero of order m at $z=a$.) Then there exist $\varepsilon,\delta > 0$ with the property that for each z satisfying $0 < |z-\alpha| < \delta$, there exist points,

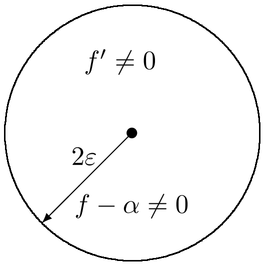
$$\{a_1,\cdots,a_m\} \subseteq B(a,\varepsilon),$$

such that

$$f^{-1}(z)\cap B(a,\varepsilon) = \{a_1,\cdots,a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot)-z$.

Proof: By Theorem 18.23 f is not constant on $B(a,R)$ because it has a zero of order m. Therefore, using this theorem again, there exists $\varepsilon > 0$ such that $\overline{B(a,2\varepsilon)} \subseteq B(a,R)$ and there are no solutions to the equation $f(z)-\alpha = 0$ for $z \in \overline{B(a,2\varepsilon)}$ except a. Also assume ε is small enough that for $0 < |z-a| \leq 2\varepsilon$, $f'(z) \neq 0$. This can be done since otherwise, a would be a limit point of a sequence of points, z_n, having $f'(z_n) = 0$ which would imply, by Theorem 18.23 that $f' = 0$ on $B(a,R)$, contradicting the assumption that $f-\alpha$ has a zero of order m and is therefore not constant. Thus the situation is described by the following picture.



Now pick $\gamma(t) = a + \varepsilon e^{it}, t \in [0, 2\pi]$. Then $\alpha \notin f(\gamma^*)$ so there exists $\delta > 0$ with

$$B(\alpha, \delta) \cap f(\gamma^*) = \emptyset. \tag{19.8}$$

Therefore, $B(\alpha, \delta)$ is contained on one component of $\mathbb{C} \setminus f(\gamma([0, 2\pi]))$. Therefore, $n(f \circ \gamma, \alpha) = n(f \circ \gamma, z)$ for all $z \in B(\alpha, \delta)$. Now consider f restricted to $\overline{B(a, 2\varepsilon)}$. For $z \in B(\alpha, \delta)$, $f^{-1}(z)$ must consist of a finite set of points because $f'(w) \neq 0$ for all w in $\overline{B(a, 2\varepsilon) \setminus \{a\}}$ implying that the zeros of $f(\cdot) - z$ in $\overline{B(a, 2\varepsilon)}$ have no limit point. Since $\overline{B(a, 2\varepsilon)}$ is compact, this means there are only finitely many. By Theorem 19.21,

$$n(f \circ \gamma, z) = \sum_{k=1}^{p} n(\gamma, a_k) \tag{19.9}$$

where $\{a_1, \cdots, a_p\} = f^{-1}(z)$. Each point, a_k of $f^{-1}(z)$ is either inside the circle traced out by γ, yielding $n(\gamma, a_k) = 1$, or it is outside this circle yielding $n(\gamma, a_k) = 0$ because of 19.8. It follows the sum in 19.9 reduces to the number of points of $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$. Thus, letting those points in $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$ be denoted by $\{a_1, \cdots, a_r\}$

$$n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r.$$

Also, by Theorem 19.20, $m = n(f \circ \gamma, \alpha)$ because a is a zero of $f - \alpha$ of order m. Therefore, for $z \in B(\alpha, \delta)$

$$m = n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r$$

Therefore, $r = m$. Each of these a_k is a zero of order 1 of the function $f(\cdot) - z$ because $f'(a_k) \neq 0$. This proves the theorem.

This is a very fascinating result partly because it implies that for values of f near a value, α, at which $f(\cdot) - \alpha$ has a zero of order m for $m > 1$, the inverse image of these values includes at least m points, not just one. Thus the topological properties of the inverse image changes radically. This theorem also shows that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$.

Theorem 19.23 *(open mapping theorem) Let Ω be a region and $f : \Omega \to \mathbb{C}$ be analytic. Then $f(\Omega)$ is either a point or a region. If f is one to one, then $f^{-1} : f(\Omega) \to \Omega$ is analytic.*

Proof: If f is not constant, then for every $\alpha \in f(\Omega)$, it follows from Theorem 18.23 that $f(\cdot) - \alpha$ has a zero of order $m < \infty$ and so from Theorem 19.22 for each $a \in \Omega$ there exist $\varepsilon, \delta > 0$ such that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$ which clearly implies that f maps open sets to open sets. Therefore, $f(\Omega)$ is open, connected because f is continuous. If f is one to one, Theorem 19.22 implies that for every $\alpha \in f(\Omega)$ the zero of $f(\cdot) - \alpha$ is of order 1. Otherwise, that theorem implies that for z near α, there are m points which f maps to z contradicting the assumption that f is one to one. Therefore, $f'(z) \neq 0$ and since f^{-1} is continuous, due to f being an open map, it follows

$$\begin{aligned}\left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1) \to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\ &= \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}.\end{aligned}$$

This proves the theorem.

19.6 An Application To Linear Algebra

Gerschgorin's theorem gives a convenient way to estimate eigenvalues of a matrix from easy to obtain information. For A an $n \times n$ matrix, denote by $\sigma(A)$ the collection of all eigenvalues of A.

Theorem 19.24 *Let A be an $n \times n$ matrix. Consider the n Gerschgorin discs defined as*

$$D_i \equiv \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

Then every eigenvalue is contained in some Gerschgorin disc.

This theorem says to add up the absolute values of the entries of the i^{th} row which are off the main diagonal and form the disc centered at a_{ii} having this radius. The union of these discs contains $\sigma(A)$.

Proof: Suppose $A\mathbf{x} = \lambda \mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. Then for $A = (a_{ij})$

$$\sum_{j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i.$$

Therefore, if we pick k such that $|x_k| \geq |x_j|$ for all x_j, it follows that $|x_k| \neq 0$ since $|\mathbf{x}| \neq 0$ and

$$|x_k| \sum_{j \neq k} |a_{kj}| \geq \sum_{j \neq k} |a_{kj}| |x_j| \geq |\lambda - a_{kk}| |x_k|.$$

Now dividing by $|x_k|$ we see that λ is contained in the k^{th} Gerschgorin disc.

More can be said using the theory about counting zeros. To begin with the distance between two $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$ as follows.

$$||A - B||^2 \equiv \sum_{ij} |a_{ij} - b_{ij}|^2 .$$

Thus two matrices are close if and only if their corresponding entries are close.

Let A be an $n \times n$ matrix. Recall the eigenvalues of A are given by the zeros of the polynomial, $p_A(z) = \det(zI - A)$ where I is the $n \times n$ identity. Then small changes in A will produce small changes in $p_A(z)$ and $p_A'(z)$. Let γ_k denote a very small closed circle which winds around z_k, one of the eigenvalues of A, in the counter clockwise direction so that $n(\gamma_k, z_k) = 1$. This circle is to enclose only z_k and is to have no other eigenvalue on it. Then apply Theorem 19.20. According to this theorem

$$\frac{1}{2\pi i} \int_\gamma \frac{p_A'(z)}{p_A(z)} dz$$

is always an integer equal to the multiplicity of z_k as a root of $p_A(t)$. Therefore, small changes in A result in no change to the above contour integral because it must be an integer and small changes in A result in small changes in the integral. Therefore whenever every entry of the matrix B is close enough to the corresponding entry of the matrix A, the two matrices have the same number of zeros inside γ_k under the usual convention that zeros are to be counted according to multiplicity. By making the radius of the small circle equal to ε where ε is less than the minimum distance between any two distinct eigenvalues of A, this shows that if B is close enough to A, every eigenvalue of B is closer than ε to some eigenvalue of A. The next theorem is about continuous dependence of eigenvalues.

Theorem 19.25 *If λ is an eigenvalue of A, then if $||B - A||$ is small enough, some eigenvalue of B will be within ε of λ.*

Consider the situation that $A(t)$ is an $n \times n$ matrix and that $t \to A(t)$ is continuous for $t \in [0, 1]$.

Lemma 19.26 *Let $\lambda(t) \in \sigma(A(t))$ for $t < 1$ and let $\Sigma_t = \cup_{s \geq t} \sigma(A(s))$. Also let K_t be the connected component of $\lambda(t)$ in Σ_t. Then there exists $\eta > 0$ such that $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t + \eta]$.*

Proof: Denote by $D(\lambda(t), \delta)$ the disc centered at $\lambda(t)$ having radius $\delta > 0$, with other occurrences of this notation being defined similarly. Thus

$$D(\lambda(t), \delta) \equiv \{z \in \mathbb{C} : |\lambda(t) - z| \leq \delta\}.$$

Suppose $\delta > 0$ is small enough that $\lambda(t)$ is the only element of $\sigma(A(t))$ contained in $D(\lambda(t), \delta)$ and that $p_{A(t)}$ has no zeroes on the boundary of this disc. Then by continuity, and the above discussion and theorem, there exists $\eta > 0, t + \eta < 1$, such that for $s \in [t, t + \eta]$, $p_{A(s)}$ also has no zeroes on the boundary of this disc and that

$A(s)$ has the same number of eigenvalues, counted according to multiplicity, in the disc as $A(t)$. Thus $\sigma(A(s)) \cap D(\lambda(t), \delta) \neq \emptyset$ for all $s \in [t, t+\eta]$. Now let

$$H = \bigcup_{s \in [t,t+\eta]} \sigma(A(s)) \cap D(\lambda(t), \delta).$$

I will show H is connected. Suppose not. Then $H = P \cup Q$ where P, Q are separated and $\lambda(t) \in P$. Let

$$s_0 \equiv \inf\{s : \lambda(s) \in Q \text{ for some } \lambda(s) \in \sigma(A(s))\}.$$

There exists $\lambda(s_0) \in \sigma(A(s_0)) \cap D(\lambda(t), \delta)$. If $\lambda(s_0) \notin Q$, then from the above discussion there are

$$\lambda(s) \in \sigma(A(s)) \cap Q$$

for $s > s_0$ arbitrarily close to $\lambda(s_0)$. Therefore, $\lambda(s_0) \in Q$ which shows that $s_0 > t$ because $\lambda(t)$ is the only element of $\sigma(A(t))$ in $D(\lambda(t), \delta)$ and $\lambda(t) \in P$. Now let $s_n \uparrow s_0$. Then $\lambda(s_n) \in P$ for any

$$\lambda(s_n) \in \sigma(A(s_n)) \cap D(\lambda(t), \delta)$$

and from the above discussion, for some choice of $s_n \to s_0$, $\lambda(s_n) \to \lambda(s_0)$ which contradicts P and Q separated and nonempty. Since P is nonempty, this shows $Q = \emptyset$. Therefore, H is connected as claimed. But $K_t \supseteq H$ and so $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$. This proves the lemma.

The following is the necessary theorem.

Theorem 19.27 *Suppose $A(t)$ is an $n \times n$ matrix and that $t \to A(t)$ is continuous for $t \in [0,1]$. Let $\lambda(0) \in \sigma(A(0))$ and define $\Sigma \equiv \cup_{t \in [0,1]} \sigma(A(t))$. Let $K_{\lambda(0)} = K_0$ denote the connected component of $\lambda(0)$ in Σ. Then $K_0 \cap \sigma(A(t)) \neq \emptyset$ for all $t \in [0,1]$.*

Proof: Let $S \equiv \{t \in [0,1] : K_0 \cap \sigma(A(s)) \neq \emptyset \text{ for all } s \in [0,t]\}$. Then $0 \in S$. Let $t_0 = \sup(S)$. Say $\sigma(A(t_0)) = \lambda_1(t_0), \cdots, \lambda_r(t_0)$. I claim at least one of these is a limit point of K_0 and consequently must be in K_0 which will show that S has a last point. Why is this claim true? Let $s_n \uparrow t_0$ so $s_n \in S$. Now let the discs, $D(\lambda_i(t_0), \delta), i = 1, \cdots, r$ be disjoint with $p_{A(t_0)}$ having no zeroes on γ_i the boundary of $D(\lambda_i(t_0), \delta)$. Then for n large enough it follows from Theorem 19.20 and the discussion following it that $\sigma(A(s_n))$ is contained in $\cup_{i=1}^r D(\lambda_i(t_0), \delta)$. Therefore, $K_0 \cap (\sigma(A(t_0)) + D(0, \delta)) \neq \emptyset$ for all δ small enough. This requires at least one of the $\lambda_i(t_0)$ to be in $\overline{K_0}$. Therefore, $t_0 \in S$ and S has a last point.

Now by Lemma 19.26, if $t_0 < 1$, then $K_0 \cup K_t$ would be a strictly larger connected set containing $\lambda(0)$. (The reason this would be strictly larger is that $K_0 \cap \sigma(A(s)) = \emptyset$ for some $s \in (t, t+\eta)$ while $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$.) Therefore, $t_0 = 1$ and this proves the theorem.

The following is an interesting corollary of the Gerschgorin theorem.

Corollary 19.28 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains an eigenvalue of A. Also, if there are n disjoint Gerschgorin discs, then each one contains an eigenvalue of A.*

Proof: Denote by $A(t)$ the matrix $\left(a_{ij}^t\right)$ where if $i \neq j$, $a_{ij}^t = ta_{ij}$ and $a_{ii}^t = a_{ii}$. Thus to get $A(t)$ we multiply all non diagonal terms by t. Let $t \in [0,1]$. Then $A(0) = diag(a_{11}, \cdots, a_{nn})$ and $A(1) = A$. Furthermore, the map, $t \to A(t)$ is continuous. Denote by D_j^t the Gerschgorin disc obtained from the j^{th} row for the matrix, $A(t)$. Then it is clear that $D_j^t \subseteq D_j$ the j^{th} Gerschgorin disc for A. Then a_{ii} is the eigenvalue for $A(0)$ which is contained in the disc, consisting of the single point a_{ii} which is contained in D_i. Letting K be the connected component in Σ for Σ defined in Theorem 19.27 which is determined by a_{ii}, it follows by Gerschgorin's theorem that $K \cap \sigma(A(t)) \subseteq \cup_{j=1}^n D_j^t \subseteq \cup_{j=1}^n D_j = D_i \cup (\cup_{j\neq i} D_j)$ and also, since K is connected, there are no points of K in both D_i and $(\cup_{j\neq i} D_j)$. Since at least one point of K is in D_i,(a_{ii}) it follows all of K must be contained in D_i. Now by Theorem 19.27 this shows there are points of $K \cap \sigma(A)$ in D_i. The last assertion follows immediately.

Actually, this can be improved slightly. It involves the following lemma.

Lemma 19.29 *In the situation of Theorem 19.27 suppose $\lambda(0) = K_0 \cap \sigma(A(0))$ and that $\lambda(0)$ is a simple root of the characteristic equation of $A(0)$. Then for all $t \in [0,1]$,*

$$\sigma(A(t)) \cap K_0 = \lambda(t)$$

where $\lambda(t)$ is a simple root of the characteristic equation of $A(t)$.

Proof: Let $S \equiv$

$$\{t \in [0,1] : K_0 \cap \sigma(A(s)) = \lambda(s), \text{ a simple eigenvalue for all } s \in [0,t]\}.$$

Then $0 \in S$ so it is nonempty. Let $t_0 = \sup(S)$ and suppose $\lambda_1 \neq \lambda_2$ are two elements of $\sigma(A(t_0)) \cap K_0$. Then choosing $\eta > 0$ small enough, and letting D_i be disjoint discs containing λ_i respectively, similar arguments to those of Lemma 19.26 imply

$$H_i \equiv \cup_{s\in[t_0-\eta,t_0]} \sigma(A(s)) \cap D_i$$

is a connected and nonempty set for $i = 1, 2$ which would require that $H_i \subseteq K_0$. But then there would be two different eigenvalues of $A(s)$ contained in K_0, contrary to the definition of t_0. Therefore, there is at most one eigenvalue, $\lambda(t_0) \in K_0 \cap \sigma(A(t_0))$. The possibility that it could be a repeated root of the characteristic equation must be ruled out. Suppose then that $\lambda(t_0)$ is a repeated root of the characteristic equation. As before, choose a small disc, D centered at $\lambda(t_0)$ and η small enough that

$$H \equiv \cup_{s\in[t_0-\eta,t_0]} \sigma(A(s)) \cap D$$

is a nonempty connected set containing either multiple eigenvalues of $A(s)$ or else a single repeated root to the characteristic equation of $A(s)$. But since H is connected and contains $\lambda(t_0)$ it must be contained in K_0 which contradicts the condition for

$s \in S$ for all these $s \in [t_0 - \eta, t_0]$. Therefore, $t_0 \in S$ as hoped. If $t_0 < 1$, there exists a small disc centered at $\lambda(t_0)$ and $\eta > 0$ such that for all $s \in [t_0, t_0 + \eta]$, $A(s)$ has only simple eigenvalues in D and the only eigenvalues of $A(s)$ which could be in K_0 are in D. (This last assertion follows from noting that $\lambda(t_0)$ is the only eigenvalue of $A(t_0)$ in K_0 and so the others are at a positive distance from K_0. For s close enough to t_0, the eigenvalues of $A(s)$ are either close to these eigenvalues of $A(t_0)$ at a positive distance from K_0 or they are close to the eigenvalue, $\lambda(t_0)$ in which case it can be assumed they are in D.) But this shows that t_0 is not really an upper bound to S. Therefore, $t_0 = 1$ and the lemma is proved.

With this lemma, the conclusion of the above corollary can be improved.

Corollary 19.30 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains exactly one eigenvalue of A and this eigenvalue is a simple root to the characteristic polynomial of A.*

Proof: In the proof of Corollary 19.28, first note that a_{ii} is a simple root of $A(0)$ since otherwise the i^{th} Gerschgorin disc would not be disjoint from the others. Also, K, the connected component determined by a_{ii} must be contained in D_i because it is connected and by Gerschgorin's theorem above, $K \cap \sigma(A(t))$ must be contained in the union of the Gerschgorin discs. Since all the other eigenvalues of $A(0)$, the a_{jj}, are outside D_i, it follows that $K \cap \sigma(A(0)) = a_{ii}$. Therefore, by Lemma 19.29, $K \cap \sigma(A(1)) = K \cap \sigma(A)$ consists of a single simple eigenvalue. This proves the corollary.

Example 19.31 *Consider the matrix,*

$$\begin{pmatrix} 5 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The Gerschgorin discs are $D(5,1), D(1,2)$, and $D(0,1)$. Then $D(5,1)$ is disjoint from the other discs. Therefore, there should be an eigenvalue in $D(5,1)$. The actual eigenvalues are not easy to find. They are the roots of the characteristic equation, $t^3 - 6t^2 + 3t + 5 = 0$. The numerical values of these are $-.669\,66, 1.423\,1$, and $5.246\,55$, verifying the predictions of Gerschgorin's theorem.

19.7 Exercises

1. Use Theorem 19.20 to give an alternate proof of the fundamental theorem of algebra. **Hint:** Take a contour of the form $\gamma_r = re^{it}$ where $t \in [0, 2\pi]$. Consider $\int_{\gamma_r} \frac{p'(z)}{p(z)} dz$ and consider the limit as $r \to \infty$.

2. Let M be an $n \times n$ matrix. Recall that the eigenvalues of M are given by the zeros of the polynomial, $p_M(z) = \det(M - zI)$ where I is the $n \times n$ identity. Formulate a theorem which describes how the eigenvalues depend on small

changes in M. **Hint:** You could define a norm on the space of $n \times n$ matrices as $||M|| \equiv tr\,(MM^*)^{1/2}$ where M^* is the conjugate transpose of M. Thus

$$||M|| = \left(\sum_{j,k} |M_{jk}|^2 \right)^{1/2}.$$

Argue that small changes will produce small changes in $p_M(z)$. Then apply Theorem 19.20 using γ_k a very small circle surrounding z_k, the k^{th} eigenvalue.

3. Suppose that two analytic functions defined on a region are equal on some set, S which contains a limit point. (Recall p is a limit point of S if every open set which contains p, also contains infinitely many points of S.) Show the two functions coincide. We defined $e^z \equiv e^x(\cos y + i \sin y)$ earlier and we showed that e^z, defined this way was analytic on $\mathbb{C}$. Is there any other way to define e^z on all of $\mathbb{C}$ such that the function coincides with e^x on the real axis?

4. You know various identities for real valued functions. For example $\cosh^2 x - \sinh^2 x = 1$. If you define $\cosh z \equiv \frac{e^z + e^{-z}}{2}$ and $\sinh z \equiv \frac{e^z - e^{-z}}{2}$, does it follow that
$$\cosh^2 z - \sinh^2 z = 1$$
for all $z \in \mathbb{C}$? What about
$$\sin(z+w) = \sin z \cos w + \cos z \sin w?$$
Can you verify these sorts of identities just from your knowledge about what happens for real arguments?

5. Was it necessary that U be a region in Theorem 18.23? Would the same conclusion hold if U were only assumed to be an open set? Why? What about the open mapping theorem? Would it hold if U were not a region?

6. Let $f : U \to \mathbb{C}$ be analytic and one to one. Show that $f'(z) \neq 0$ for all $z \in U$. Does this hold for a function of a real variable?

7. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq m$ on U. **Hint:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

8. If u is harmonic on some region, U, show that u coincides locally with the real part of an analytic function and that therefore, u has infinitely many

derivatives on U. **Hint:** Consider the case where $0 \in U$. You can always reduce to this case by a suitable translation. Now let $B(0,r) \subseteq U$ and use the Schwarz formula to obtain an analytic function whose real part coincides with u on $\partial B(0,r)$. Then use Problem 7.

9. Show the solution to the Dirichlet problem of Problem 8 on Page 400 is unique. You need to formulate this precisely and then prove uniqueness.

Residues

Definition 20.1 *The residue of f at an isolated singularity α which is a pole, written* $\operatorname{res}(f,\alpha)$ *is the coefficient of* $(z-\alpha)^{-1}$ *where*

$$f(z) = g(z) + \sum_{k=1}^{m} \frac{b_k}{(z-\alpha)^k}.$$

Thus $\operatorname{res}(f,\alpha) = b_1$ *in the above.*

At this point, recall Corollary 18.47 which is stated here for convenience.

Corollary 20.2 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \mathbb{C}$ is analytic,

$$\sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = 0.$$

The following theorem is called the residue theorem. Note the resemblance to Corollary 18.47.

Theorem 20.3 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \widehat{\mathbb{C}}$ is meromorphic with no pole of f contained in any γ_k^*,

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = \sum_{\alpha \in A} \operatorname{res}(f,\alpha) \sum_{k=1}^{m} n(\gamma_k, \alpha) \tag{20.1}$$

where here A denotes the set of poles of f in Ω. The sum on the right is a finite sum.

Proof: First note that there are at most finitely many α which are not in the unbounded component of $\mathbb{C} \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$. Thus there exists a finite set, $\{\alpha_1, \cdots, \alpha_N\} \subseteq A$ such that these are the only possibilities for which $\sum_{k=1}^{n} n(\gamma_k, \alpha)$ might not equal zero. Therefore, 20.1 reduces to

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = \sum_{j=1}^{N} \operatorname{res}(f, \alpha_j) \sum_{k=1}^{n} n(\gamma_k, \alpha_j)$$

and it is this last equation which is established. Near α_j,

$$f(z) = g_j(z) + \sum_{r=1}^{m_j} \frac{b_r^j}{(z - \alpha_j)^r} \equiv g_j(z) + Q_j(z).$$

where g_j is analytic at and near α_j. Now define

$$G(z) \equiv f(z) - \sum_{j=1}^{N} Q_j(z).$$

It follows that $G(z)$ has a removable singularity at each α_j. Therefore, by Corollary 18.47,

$$0 = \sum_{k=1}^{m} \int_{\gamma_k} G(z)\, dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z)\, dz - \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_k} Q_j(z)\, dz.$$

Now

$$\begin{aligned}
\sum_{k=1}^{m} \int_{\gamma_k} Q_j(z)\, dz &= \sum_{k=1}^{m} \int_{\gamma_k} \left(\frac{b_1^j}{(z - \alpha_j)} + \sum_{r=2}^{m_j} \frac{b_r^j}{(z - \alpha_j)^r} \right) dz \\
&= \sum_{k=1}^{m} \int_{\gamma_k} \frac{b_1^j}{(z - \alpha_j)} dz \equiv \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \operatorname{res}(f, \alpha_j)(2\pi i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{m} \int_{\gamma_k} f(z)\, dz &= \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_k} Q_j(z)\, dz \\
&= \sum_{j=1}^{N} \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \operatorname{res}(f, \alpha_j)(2\pi i) \\
&= 2\pi i \sum_{j=1}^{N} \operatorname{res}(f, \alpha_j) \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \\
&= (2\pi i) \sum_{\alpha \in A} \operatorname{res}(f, \alpha) \sum_{k=1}^{m} n(\gamma_k, \alpha)
\end{aligned}$$

which proves the theorem.

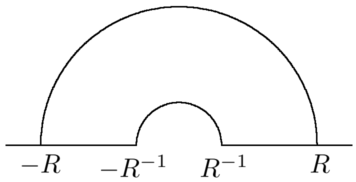
The following is an important example. This example can also be done by real variable methods and there are some who think that real variable methods are always to be preferred to complex variable methods. However, I will use the above theorem to work this example.

Example 20.4 *Find* $\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(x)}{x}dx$

Things are easier if you write it as

$$\lim_{R\to\infty}\frac{1}{i}\left(\int_{-R}^{-R^{-1}}\frac{e^{ix}}{x}dx+\int_{R^{-1}}^{R}\frac{e^{ix}}{x}dx\right).$$

This gives the same answer because $\cos(x)/x$ is odd. Consider the following contour in which the orientation involves counterclockwise motion exactly once around.



Denote by $\gamma_{R^{-1}}$ the little circle and γ_R the big one. Then on the inside of this contour there are no singularities of e^{iz}/z and it is contained in an open set with the property that the winding number with respect to this contour about any point not in the open set equals zero. By Theorem 18.22

$$\frac{1}{i}\left(\int_{-R}^{-R^{-1}}\frac{e^{ix}}{x}dx+\int_{\gamma_{R^{-1}}}\frac{e^{iz}}{z}dz+\int_{R^{-1}}^{R}\frac{e^{ix}}{x}dx+\int_{\gamma_R}\frac{e^{iz}}{z}dz\right)=0 \tag{20.2}$$

Now

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z}dz\right|=\left|\int_0^{\pi}e^{R(i\cos\theta-\sin\theta)}id\theta\right|\leq\int_0^{\pi}e^{-R\sin\theta}d\theta$$

and this last integral converges to 0 by the dominated convergence theorem. Now consider the other circle. By the dominated convergence theorem again,

$$\int_{\gamma_{R^{-1}}}\frac{e^{iz}}{z}dz=\int_{\pi}^{0}e^{R^{-1}(i\cos\theta-\sin\theta)}id\theta\to -i\pi$$

as $R \to \infty$. Then passing to the limit in 20.2,

$$\begin{aligned}
& \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \frac{1}{i} \left(\int_{-R}^{-R^{-1}} \frac{e^{ix}}{x} dx + \int_{R^{-1}}^{R} \frac{e^{ix}}{x} dx \right) \\
= & \lim_{R\to\infty} \frac{1}{i} \left(-\int_{\gamma_{R^{-1}}} \frac{e^{iz}}{z} dz - \int_{\gamma_R} \frac{e^{iz}}{z} dz \right) = \frac{-1}{i} (-i\pi) = \pi .
\end{aligned}$$

Example 20.5 *Find* $\lim_{R\to\infty} \int_{-R}^{R} e^{ixt} \frac{\sin x}{x} dx$. *Note this is essentially finding the inverse Fourier transform of the function,* $\sin(x)/x$.

This equals

$$\begin{aligned}
& \lim_{R\to\infty} \int_{-R}^{R} (\cos(xt) + i \sin(xt)) \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \int_{-R}^{R} \cos(xt) \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \int_{-R}^{R} \cos(xt) \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \frac{1}{2} \int_{-R}^{R} \frac{\sin(x(t+1)) + \sin(x(1-t))}{x} dx
\end{aligned}$$

Let $t \neq 1, -1$. Then changing variables yields

$$\lim_{R\to\infty} \left(\frac{1}{2} \int_{-R(1+t)}^{R(1+t)} \frac{\sin(u)}{u} du + \frac{1}{2} \int_{-R(1-t)}^{R(1-t)} \frac{\sin(u)}{u} du \right).$$

In case $|t| < 1$ Example 20.4 implies this limit is π. However, if $t > 1$ the limit equals 0 and this is also the case if $t < -1$. Summarizing,

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ixt} \frac{\sin x}{x} dx = \begin{cases} \pi \text{ if } |t| < 1 \\ 0 \text{ if } |t| > 1 \end{cases} .$$

20.1 Rouche's Theorem And The Argument Principle

20.1.1 Argument Principle

A simple closed curve is just one which is homeomorphic to the unit circle. The Jordan Curve theorem states that every simple closed curve in the plane divides the plane into exactly two connected components, one bounded and the other unbounded. This is a very hard theorem to prove. However, in most applications the conclusion is obvious. Nevertheless, to avoid using this big topological result and to attain some extra generality, I will state the following theorem in terms of the winding number to avoid using it. This theorem is called the argument principle. First recall that f has a zero of order m at α if $f(z) = g(z)(z-\alpha)^m$ where g is an analytic function which is not equal to zero at α. This is equivalent to having $f(z) = \sum_{k=m}^{\infty} a_k (z-\alpha)^k$ for z near α where $a_m \neq 0$. Also recall that f has a pole of order m at α if for z near α, $f(z)$ is of the form

$$f(z) = h(z) + \sum_{k=1}^{m} \frac{b_k}{(z-\alpha)^k} \tag{20.3}$$

where $b_m \neq 0$ and h is a function analytic near α.

Theorem 20.6 *(argument principle) Let f be meromorphic in Ω. Also suppose γ^* is a closed bounded variation curve containing none of the poles or zeros of f with the property that for all $z \notin \Omega, n(\gamma, z) = 0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the winding number of γ about these points equals 1. Let z_k be a zero of order r_k and let p_k be a pole of order l_k. Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k$$

Proof: This theorem follows from computing the residues of f'/f. It has residues at poles and zeros. I will do this now. First suppose f has a pole of order p at α. Then f has the form given in 20.3. Therefore,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{h'(z) - \sum_{k=1}^{p} \frac{kb_k}{(z-\alpha)^{k+1}}}{h(z) + \sum_{k=1}^{p} \frac{b_k}{(z-\alpha)^k}} \\ &= \frac{h'(z)(z-\alpha)^p - \sum_{k=1}^{p-1} kb_k (z-\alpha)^{-k-1+p} - \frac{pb_p}{(z-\alpha)}}{h(z)(z-\alpha)^p + \sum_{k=1}^{p-1} b_k (z-\alpha)^{p-k} + b_p} \end{aligned}$$

This is of the form

$$= \frac{b_p}{s(z) + b_p} \frac{r(z) - \frac{pb_p}{(z-\alpha)}}{b_p} = \frac{b_p}{s(z) + b_p} \left(\frac{r(z)}{b_p} - \frac{p}{(z-\alpha)} \right)$$

where $s(\alpha) = r(\alpha) = 0$. From this, it is clear $\operatorname{res}(f'/f, \alpha) = -p$, the order of the pole.

Next suppose f has a zero of order p at α. Then

$$\frac{f'(z)}{f(z)} = \frac{\sum_{k=p}^{\infty} a_k k (z-\alpha)^{k-1}}{\sum_{k=p}^{\infty} a_k (z-\alpha)^k} = \frac{\sum_{k=p}^{\infty} a_k k (z-\alpha)^{k-1-p}}{\sum_{k=p}^{\infty} a_k (z-\alpha)^{k-p}}$$

and from this it is clear $\operatorname{res}(f'/f) = p$, the order of the zero. The conclusion of this theorem now follows from Theorem 20.3.

One can also generalize the theorem to the case where there are many closed curves involved. This is proved in the same way as the above.

Theorem 20.7 *(argument principle) Let f be meromorphic in Ω and let γ_k : $[a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

and for all $z \notin \Omega$ and for $z \in \Omega$, $\sum_{k=1}^{m} n(\gamma_k, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the above sum of winding numbers equals 1. Let z_k be a zero of order r_k and let p_k be a pole of order l_k. Then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k$$

There is also a simple extension of this important principle which I found in [23].

Theorem 20.8 *(argument principle) Let f be meromorphic in Ω. Also suppose γ^* is a closed bounded variation curve with the property that for all $z \notin \Omega, n(\gamma, z) = 0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the winding number of γ about these points equals 1 listed according to multiplicity. Thus if there is a pole of order m there will be this value repeated m times in the list for the poles. Also let $g(z)$ be an analytic function. Then*

$$\frac{1}{2\pi i}\int_{\gamma} g(z) \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} g(z_k) - \sum_{k=1}^{m} g(p_k)$$

Proof: This theorem follows from computing the residues of $g(f'/f)$. It has residues at poles and zeros. I will do this now. First suppose f has a pole of order m at α. Then f has the form given in 20.3. Therefore,

$$\begin{aligned} & g(z) \frac{f'(z)}{f(z)} \\ = \quad & \frac{g(z)\left(h'(z) - \sum_{k=1}^{m} \frac{kb_k}{(z-\alpha)^{k+1}}\right)}{h(z) + \sum_{k=1}^{m} \frac{b_k}{(z-\alpha)^k}} \\ = \quad & g(z) \frac{h'(z)(z-\alpha)^m - \sum_{k=1}^{m-1} kb_k (z-\alpha)^{-k-1+m} - \frac{mb_m}{(z-\alpha)}}{h(z)(z-\alpha)^m + \sum_{k=1}^{m-1} b_k (z-\alpha)^{m-k} + b_m} \end{aligned}$$

From this, it is clear $\operatorname{res}(g(f'/f), \alpha) = -mg(\alpha)$, where m is the order of the pole. Thus α would have been listed m times in the list of poles. Hence the residue at this point is equivalent to adding $-g(\alpha)$ m times.

Next suppose f has a zero of order m at α. Then

$$g(z)\frac{f'(z)}{f(z)} = g(z)\frac{\sum_{k=m}^{\infty} a_k k(z-\alpha)^{k-1}}{\sum_{k=m}^{\infty} a_k (z-\alpha)^k} = g(z)\frac{\sum_{k=m}^{\infty} a_k k(z-\alpha)^{k-1-m}}{\sum_{k=m}^{\infty} a_k (z-\alpha)^{k-m}}$$

and from this it is clear $\operatorname{res}(g(f'/f)) = g(\alpha)m$, where m is the order of the zero. The conclusion of this theorem now follows from the residue theorem, Theorem 20.3.

The way people usually apply these theorems is to suppose γ^* is a simple closed bounded variation curve, often a circle. Thus it has an inside and an outside, the outside being the unbounded component of $\mathbb{C}\setminus\gamma^*$. The orientation of the curve is such that you go around it once in the counterclockwise direction. Then letting r_k and l_k be as described, the conclusion of the theorem follows. In applications, this is likely the way it will be.

20.1.2 Rouche's Theorem

With the argument principle, it is possible to prove Rouche's theorem . In the argument principle, denote by Z_f the quantity $\sum_{k=1}^{m} r_k$ and by P_f the quantity $\sum_{k=1}^{n} l_k$. Thus Z_f is the number of zeros of f counted according to the order of the zero with a similar definition holding for P_f. Thus the conclusion of the argument principle is.

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = Z_f - P_f$$

Rouche's theorem allows the comparison of $Z_h - P_h$ for $h = f, g$. It is a wonderful and amazing result.

Theorem 20.9 *(Rouche's theorem)Let f, g be meromorphic in an open set Ω. Also suppose γ^* is a closed bounded variation curve with the property that for all $z \notin \Omega, n(\gamma, z) = 0$, no zeros or poles are on γ^*, and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Let Z_f and P_f denote respectively the numbers of zeros and poles of f, which have the property that the winding number equals 1, counted according to order, with Z_g and P_g being defined similarly. Also suppose that for $z \in \gamma^*$*

$$|f(z) + g(z)| < |f(z)| + |g(z)|. \tag{20.4}$$

Then

$$Z_f - P_f = Z_g - P_g.$$

Proof: From the hypotheses,

$$\left|1 + \frac{f(z)}{g(z)}\right| < 1 + \left|\frac{f(z)}{g(z)}\right|$$

which shows that for all $z \in \gamma^*$,

$$\frac{f(z)}{g(z)} \in \mathbb{C} \setminus [0, \infty).$$

Letting l denote a branch of the logarithm defined on $\mathbb{C} \setminus [0, \infty)$, it follows that $l\left(\frac{f(z)}{g(z)}\right)$ is a primitive for the function,

$$\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}.$$

Therefore, by the argument principle,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_\gamma \frac{(f/g)'}{(f/g)} dz = \frac{1}{2\pi i} \int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right) dz \\ &= Z_f - P_f - (Z_g - P_g). \end{aligned}$$

This proves the theorem.

Often another condition other than 20.4 is used.

Corollary 20.10 *In the situation of Theorem 20.9 change 20.4 to the condition,*

$$|f(z) - g(z)| < |f(z)|$$

for $z \in \gamma^*$. Then the conclusion is the same.

Proof: The new condition implies $\left|1 - \frac{g}{f}(z)\right| < \left|\frac{g(z)}{f(z)}\right|$ on γ^*. Therefore, $\frac{g(z)}{f(z)} \notin (-\infty, 0]$ and so you can do the same argument with a branch of the logarithm.

20.1.3 A Different Formulation

In [38] I found this modification for Rouche's theorem concerned with the counting of zeros of analytic functions. This is a very useful form of Rouche's theorem because it makes no mention of a contour.

Theorem 20.11 *Let Ω be a bounded open set and suppose f, g are continuous on $\overline{\Omega}$ and analytic on Ω. Also suppose $|f(z)| < |g(z)|$ on $\partial\Omega$. Then g and $f + g$ have the same number of zeros in Ω provided each zero is counted according to multiplicity.*

Proof: Let $K = \left\{z \in \overline{\Omega} : |f(z)| \geq |g(z)|\right\}$. Then letting $\lambda \in [0, 1]$, if $z \notin K$, then $|f(z)| < |g(z)|$ and so

$$0 < |g(z)| - |f(z)| \leq |g(z)| - \lambda |f(z)| \leq |g(z) + \lambda f(z)|$$

which shows that all zeros of $g + \lambda f$ are contained in K which must be a compact subset of Ω due to the assumption that $|f(z)| < |g(z)|$ on $\partial\Omega$. By Theorem 18.52 on Page 419 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k^* \subseteq \Omega \setminus K$, $\sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then as above, it follows from the residue theorem or more directly, Theorem 20.7,

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda f'(z) + g'(z)}{\lambda f(z) + g(z)} dz = \sum_{j=1}^p m_j$$

where m_j is the order of the j^{th} zero of $\lambda f + g$ in K, hence in Ω. However,

$$\lambda \to \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda f'(z) + g'(z)}{\lambda f(z) + g(z)} dz$$

is integer valued and continuous so it gives the same value when $\lambda = 0$ as when $\lambda = 1$. When $\lambda = 0$ this gives the number of zeros of g in Ω and when $\lambda = 1$ it is the number of zeros of $f + g$. This proves the theorem.

Here is another formulation of this theorem.

Corollary 20.12 *Let Ω be a bounded open set and suppose f, g are continuous on $\overline{\Omega}$ and analytic on Ω. Also suppose $|f(z) - g(z)| < |g(z)|$ on $\partial\Omega$. Then f and g have the same number of zeros in Ω provided each zero is counted according to multiplicity.*

Proof: You let $f - g$ play the role of f in Theorem 20.11. Thus $f - g + g = f$ and g have the same number of zeros. Alternatively, you can give a proof of this directly as follows.

Let $K = \{z \in \Omega : |f(z) - g(z)| \geq |g(z)|\}$. Then if $g(z) + \lambda (f(z) - g(z)) = 0$ it follows

$$\begin{aligned} 0 &= |g(z) + \lambda (f(z) - g(z))| \geq |g(z)| - \lambda |f(z) - g(z)| \\ &\geq |g(z)| - |f(z) - g(z)| \end{aligned}$$

and so $z \in K$. Thus all zeros of $g(z) + \lambda (f(z) - g(z))$ are contained in K. By Theorem 18.52 on Page 419 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k^* \subseteq \Omega \setminus K, \sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then by Theorem 20.7,

$$\sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda (f'(z) - g'(z)) + g'(z)}{\lambda (f(z) - g(z)) + g(z)} dz = \sum_{j=1}^{p} m_j$$

where m_j is the order of the j^{th} zero of $\lambda (f - g) + g$ in K, hence in Ω. The left side is continuous as a function of λ and so the number of zeros of g corresponding to $\lambda = 0$ equals the number of zeros of f corresponding to $\lambda = 1$. This proves the corollary.

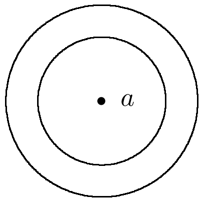
20.2 Singularities And The Laurent Series

20.2.1 What Is An Annulus?

In general, when you consider singularities, isolated or not, the fundamental tool is the Laurent series. This series is important for many other reasons also. In particular, it is fundamental to the spectral theory of various operators in functional analysis and is one way to obtain relationships between algebraic and analytical conditions essential in various convergence theorems. A Laurent series lives on an annulus. In all this f has values in X where X is a complex Banach space. If you like, let $X = \mathbb{C}$.

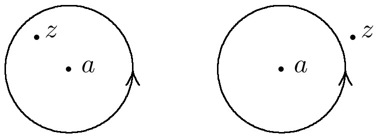
Definition 20.13 *Define* $\operatorname{ann}(a, R_1, R_2) \equiv \{z : R_1 < |z-a| < R_2\}$.

Thus $\operatorname{ann}(a, 0, R)$ would denote the punctured ball, $B(a, R) \setminus \{0\}$ and when $R_1 > 0$, the annulus looks like the following.



The annulus is the stuff between the two circles.

Here is an important lemma which is concerned with the situation described in the following picture.



Lemma 20.14 *Let* $\gamma_r(t) \equiv a + re^{it}$ *for* $t \in [0, 2\pi]$ *and let* $|z-a| < r$. *Then* $n(\gamma_r, z) = 1$. *If* $|z-a| > r$, *then* $n(\gamma_r, z) = 0$.

Proof: For the first claim, consider for $t \in [0,1]$,

$$f(t) \equiv n(\gamma_r, a + t(z-a)).$$

Then from properties of the winding number derived earlier, $f(t) \in \mathbb{Z}$, f is continuous, and $f(0) = 1$. Therefore, $f(t) = 1$ for all $t \in [0,1]$. This proves the first claim because $f(1) = n(\gamma_r, z)$.

For the second claim,

$$\begin{aligned}
n(\gamma_r, z) &= \frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{w-z}\,dw \\
&= \frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{w-a-(z-a)}\,dw \\
&= \frac{1}{2\pi i}\frac{-1}{z-a}\int_{\gamma_r} \frac{1}{1-\left(\frac{w-a}{z-a}\right)}\,dw \\
&= \frac{-1}{2\pi i(z-a)}\int_{\gamma_r} \sum_{k=0}^{\infty}\left(\frac{w-a}{z-a}\right)^k dw.
\end{aligned}$$

The series converges uniformly for $w \in \gamma_r$ because

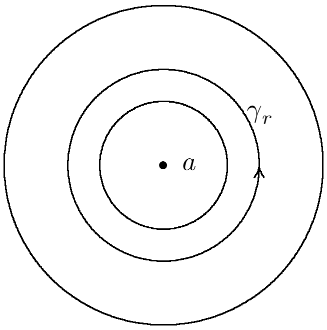
$$\left| \frac{w-a}{z-a} \right| = \frac{r}{r+c}$$

for some $c > 0$ due to the assumption that $|z-a| > r$. Therefore, the sum and the integral can be interchanged to give

$$n(\gamma_r, z) = \frac{-1}{2\pi i (z-a)} \sum_{k=0}^{\infty} \int_{\gamma_r} \left(\frac{w-a}{z-a} \right)^k dw = 0$$

because $w \to \left(\frac{w-a}{z-a} \right)^k$ has an antiderivative. This proves the lemma.

Now consider the following picture which pertains to the next lemma.



Lemma 20.15 *Let g be analytic on* $\operatorname{ann}(a, R_1, R_2)$. *Then if* $\gamma_r(t) \equiv a + re^{it}$ *for* $t \in [0, 2\pi]$ *and* $r \in (R_1, R_2)$, *then* $\int_{\gamma_r} g(z)\, dz$ *is independent of* r.

Proof: Let $R_1 < r_1 < r_2 < R_2$ and denote by $-\gamma_r(t)$ the curve, $-\gamma_r(t) \equiv a + re^{i(2\pi - t)}$ for $t \in [0, 2\pi]$. Then if $z \in \overline{B(a, R_1)}$, Lemma 20.14 implies both $n(\gamma_{r_2}, z)$ and $n(\gamma_{r_1}, z) = 1$ and so

$$n(-\gamma_{r_1}, z) + n(\gamma_{r_2}, z) = -1 + 1 = 0.$$

Also if $z \notin B(a, R_2)$, then Lemma 20.14 implies $n\left(\gamma_{r_j}, z\right) = 0$ for $j = 1, 2$. Therefore, whenever $z \notin \operatorname{ann}(a, R_1, R_2)$, the sum of the winding numbers equals zero. Therefore, by Theorem 18.46 applied to the function, $f(w) = g(z)(w - z)$ and $z \in \operatorname{ann}(a, R_1, R_2) \setminus \cup_{j=1}^{2} \gamma_{r_j}([0, 2\pi])$,

$$\begin{aligned} f(z)\left(n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)\right) &= 0\left(n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)\right) = \\ \frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{g(w)(w-z)}{w-z} dw &- \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{g(w)(w-z)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{\gamma_{r_2}} g(w)\, dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} g(w)\, dw \end{aligned}$$

which proves the desired result.

20.2.2 The Laurent Series

The Laurent series is like a power series except it allows for negative exponents. First here is a definition of what is meant by the convergence of such a series.

Definition 20.16 *$\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ converges if both the series,*

$$\sum_{n=0}^{\infty} a_n (z-a)^n \text{ and } \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

converge. When this is the case, the symbol, $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is defined as

$$\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}.$$

Lemma 20.17 *Suppose*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

for all $|z-a| \in (R_1, R_2)$. Then both $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converge absolutely and uniformly on $\{z : r_1 \leq |z-a| \leq r_2\}$ for any $r_1 < r_2$ satisfying $R_1 < r_1 < r_2 < R_2$.

Proof: Let $R_1 < |w-a| = r_1 - \delta < r_1$. Then $\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}$ converges and so

$$\lim_{n\to\infty} |a_{-n}| |w-a|^{-n} = \lim_{n\to\infty} |a_{-n}| (r_1-\delta)^{-n} = 0$$

which implies that for all n sufficiently large,

$$|a_{-n}| (r_1-\delta)^{-n} < 1.$$

Therefore,

$$\sum_{n=1}^{\infty} |a_{-n}| |z-a|^{-n} = \sum_{n=1}^{\infty} |a_{-n}| (r_1-\delta)^{-n} (r_1-\delta)^n |z-a|^{-n}.$$

Now for $|z-a| \geq r_1$,

$$|z-a|^{-n} \leq \frac{1}{r_1^n}$$

and so for all sufficiently large n

$$|a_{-n}| |z-a|^{-n} \leq \frac{(r_1-\delta)^n}{r_1^n}.$$

Therefore, by the Weierstrass M test, the series, $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converges absolutely and uniformly on the set

$$\{z \in \mathbb{C} : |z-a| \geq r_1\}.$$

Similar reasoning shows the series, $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges uniformly on the set

$$\{z \in \mathbb{C} : |z-a| \leq r_2\}.$$

This proves the Lemma.

Theorem 20.18 *Let f be analytic on $\operatorname{ann}(a, R_1, R_2)$. Then there exist numbers, $a_n \in \mathbb{C}$ such that for all $z \in \operatorname{ann}(a, R_1, R_2)$,*

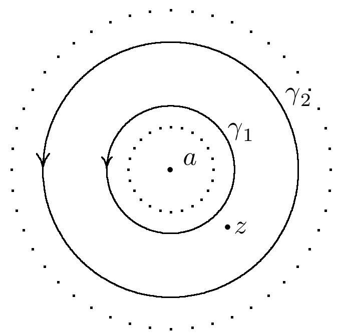
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \tag{20.5}$$

where the series converges absolutely and uniformly on $\overline{\operatorname{ann}(a, r_1, r_2)}$ whenever $R_1 < r_1 < r_2 < R_2$. Also

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw \tag{20.6}$$

where $\gamma(t) = a + re^{it}, t \in [0, 2\pi]$ for any $r \in (R_1, R_2)$. Furthermore the series is unique in the sense that if 20.5 holds for $z \in \operatorname{ann}(a, R_1, R_2)$, then a_n is given in 20.6.

Proof: Let $R_1 < r_1 < r_2 < R_2$ and define $\gamma_1(t) \equiv a + (r_1 - \varepsilon) e^{it}$ and $\gamma_2(t) \equiv a + (r_2 + \varepsilon) e^{it}$ for $t \in [0, 2\pi]$ and ε chosen small enough that $R_1 < r_1 - \varepsilon < r_2 + \varepsilon < R_2$.



Then using Lemma 20.14, if $z \notin \operatorname{ann}(a, R_1, R_2)$ then

$$n(-\gamma_1, z) + n(\gamma_2, z) = 0$$

and if $z \in \operatorname{ann}(a, r_1, r_2)$,

$$n(-\gamma_1, z) + n(\gamma_2, z) = 1.$$

Therefore, by Theorem 18.46, for $z \in \operatorname{ann}(a, r_1, r_2)$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left[\int_{-\gamma_1} \frac{f(w)}{w-z} dw + \int_{\gamma_2} \frac{f(w)}{w-z} dw\right] \\ &= \frac{1}{2\pi i}\left[\int_{\gamma_1} \frac{f(w)}{(z-a)\left[1-\frac{w-a}{z-a}\right]} dw + \int_{\gamma_2} \frac{f(w)}{(w-a)\left[1-\frac{z-a}{w-a}\right]} dw\right] \\ &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-a} \sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n dw + \\ &\quad \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(z-a)} \sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n dw. \end{aligned} \tag{20.7}$$

From the formula 20.7, it follows that for $z \in \overline{\operatorname{ann}(a, r_1, r_2)}$, the terms in the first sum are bounded by an expression of the form $C\left(\frac{r_2}{r_2+\varepsilon}\right)^n$ while those in the second are bounded by one of the form $C\left(\frac{r_1-\varepsilon}{r_1}\right)^n$ and so by the Weierstrass M test, the convergence is uniform and so the integrals and the sums in the above formula may be interchanged and after renaming the variable of summation, this yields

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw\right)(z-a)^n + \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}}\right)(z-a)^n. \tag{20.8}$$

Therefore, by Lemma 20.15, for any $r \in (R_1, R_2)$,

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} dw\right)(z-a)^n + \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}}\right)(z-a)^n. \tag{20.9}$$

and so

$$f(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} dw\right)(z-a)^n.$$

where $r \in (R_1, R_2)$ is arbitrary. This proves the existence part of the theorem. It remains to characterize a_n.

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ on $\operatorname{ann}(a, R_1, R_2)$ let

$$f_n(z) \equiv \sum_{k=-n}^{n} a_k (z-a)^k. \tag{20.10}$$

This function is analytic in $\text{ann}(a, R_1, R_2)$ and so from the above argument,

$$f_n(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k . \tag{20.11}$$

Also if $k > n$ or if $k < -n$,

$$\left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) = 0.$$

and so

$$f_n(z) = \sum_{k=-n}^{n} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k$$

which implies from 20.10 that for each $k \in [-n, n]$,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw = a_k$$

However, from the uniform convergence of the series,

$$\sum_{n=0}^{\infty} a_n (w-a)^n$$

and

$$\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}$$

ensured by Lemma 20.17 which allows the interchange of sums and integrals, if $k \in [-n, n]$,

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw \\
= \quad & \frac{1}{2\pi i} \int_{\gamma_r} \frac{\sum_{m=0}^{\infty} a_m (w-a)^m + \sum_{m=1}^{\infty} a_{-m} (w-a)^{-m}}{(w-a)^{k+1}} dw \\
= \quad & \sum_{m=0}^{\infty} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{m-(k+1)} dw \\
& + \sum_{m=1}^{\infty} a_{-m} \int_{\gamma_r} (w-a)^{-m-(k+1)} dw \\
= \quad & \sum_{m=0}^{n} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{m-(k+1)} dw \\
& + \sum_{m=1}^{n} a_{-m} \int_{\gamma_r} (w-a)^{-m-(k+1)} dw \\
= \quad & \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw
\end{aligned}$$

because if $l > n$ or $l < -n$,

$$\int_{\gamma_r} \frac{a_l (w-a)^l}{(w-a)^{k+1}} dw = 0$$

for all $k \in [-n, n]$. Therefore,

$$a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw$$

and so this establishes uniqueness. This proves the theorem.

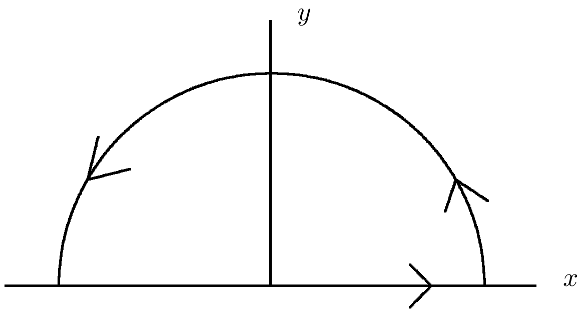
20.2.3 Contour Integrals And Evaluation Of Integrals

Here are some examples of hard integrals which can be evaluated by using residues. This will be done by integrating over various closed curves having bounded variation.

Example 20.19 *The first example we consider is the following integral.*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$$

One could imagine evaluating this integral by the method of partial fractions and it should work out by that method. However, we will consider the evaluation of this integral by the method of residues instead. To do so, consider the following picture.



Let $\gamma_r(t) = re^{it}, t \in [0, \pi]$ and let $\sigma_r(t) = t : t \in [-r, r]$. Thus γ_r parameterizes the top curve and σ_r parameterizes the straight line from $-r$ to r along the x axis. Denoting by Γ_r the closed curve traced out by these two, we see from simple estimates that

$$\lim_{r \to \infty} \int_{\gamma_r} \frac{1}{1+z^4} dz = 0.$$

This follows from the following estimate.

$$\left| \int_{\gamma_r} \frac{1}{1+z^4} dz \right| \leq \frac{1}{r^4 - 1} \pi r.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \lim_{r \to \infty} \int_{\Gamma_r} \frac{1}{1+z^4} dz.$$

We compute $\int_{\Gamma_r} \frac{1}{1+z^4} dz$ using the method of residues. The only residues of the integrand are located at points, z where $1 + z^4 = 0$. These points are

$$\begin{aligned} z &= -\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}, z = \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}, \\ z &= \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}, z = -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2} \end{aligned}$$

and it is only the last two which are found in the inside of Γ_r. Therefore, we need to calculate the residues at these points. Clearly this function has a pole of order one at each of these points and so we may calculate the residue at α in this list by evaluating

$$\lim_{z \to \alpha} (z - \alpha) \frac{1}{1+z^4}$$

Thus

$$\begin{aligned} & \operatorname{Res}\left(f, \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) \\ = & \lim_{z \to \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}} \left(z - \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)\right) \frac{1}{1+z^4} \\ = & -\frac{1}{8}\sqrt{2} - \frac{1}{8}i\sqrt{2} \end{aligned}$$

Similarly we may find the other residue in the same way

$$\begin{aligned} & \operatorname{Res}\left(f, -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) \\ = & \lim_{z \to -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}} \left(z - \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)\right) \frac{1}{1+z^4} \\ = & -\frac{1}{8}i\sqrt{2} + \frac{1}{8}\sqrt{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Gamma_r} \frac{1}{1+z^4} dz &= 2\pi i \left(-\frac{1}{8}i\sqrt{2} + \frac{1}{8}\sqrt{2} + \left(-\frac{1}{8}\sqrt{2} - \frac{1}{8}i\sqrt{2}\right)\right) \\ &= \frac{1}{2}\pi\sqrt{2}. \end{aligned}$$

Thus, taking the limit we obtain $\frac{1}{2}\pi\sqrt{2} = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$.

Obviously many different variations of this are possible. The main idea being that the integral over the semicircle converges to zero as $r \to \infty$.

Sometimes we don't blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

Example 20.20 *The integral is*

$$\int_0^\pi \frac{\cos\theta}{2+\cos\theta} d\theta$$

This integrand is even and so it equals

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos\theta}{2+\cos\theta} d\theta.$$

For z on the unit circle, $z = e^{i\theta}$, $\overline{z} = \frac{1}{z}$ and therefore, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Thus $dz = ie^{i\theta}d\theta$ and so $d\theta = \frac{dz}{iz}$. Note this is proceeding formally to get a complex integral which reduces to the one of interest. It follows that a complex integral which reduces to the one desired is

$$\frac{1}{2i}\int_\gamma \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{z} = \frac{1}{2i}\int_\gamma \frac{z^2+1}{z\left(4z+z^2+1\right)} dz$$

where γ is the unit circle. Now the integrand has poles of order 1 at those points where $z\left(4z+z^2+1\right) = 0$. These points are

$$0, -2+\sqrt{3}, -2-\sqrt{3}.$$

Only the first two are inside the unit circle. It is also clear the function has simple poles at these points. Therefore,

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z\left(\frac{z^2+1}{z\left(4z+z^2+1\right)}\right) = 1.$$

$$\operatorname{Res}\left(f, -2+\sqrt{3}\right) =$$

$$\lim_{z\to -2+\sqrt{3}} \left(z - \left(-2+\sqrt{3}\right)\right) \frac{z^2+1}{z\left(4z+z^2+1\right)} = -\frac{2}{3}\sqrt{3}.$$

It follows

$$\begin{aligned}
\int_0^\pi \frac{\cos\theta}{2+\cos\theta} d\theta &= \frac{1}{2i}\int_\gamma \frac{z^2+1}{z\left(4z+z^2+1\right)} dz \\
&= \frac{1}{2i} 2\pi i \left(1 - \frac{2}{3}\sqrt{3}\right) \\
&= \pi\left(1 - \frac{2}{3}\sqrt{3}\right).
\end{aligned}$$

Other rational functions of the trig functions will work out by this method also.

Sometimes you have to be clever about which version of an analytic function that reduces to a real function you should use. The following is such an example.

Example 20.21 *The integral here is*

$$\int_0^\infty \frac{\ln x}{1+x^4} dx.$$

The same curve used in the integral involving $\frac{\sin x}{x}$ earlier will create problems with the log since the usual version of the log is not defined on the negative real axis. This does not need to be of concern however. Simply use another branch of the logarithm. Leave out the ray from 0 along the negative y axis and use Theorem 19.5 to define $L(z)$ on this set. Thus $L(z) = \ln |z| + i \arg_1(z)$ where $\arg_1(z)$ will be the angle, θ, between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ such that $z = |z| e^{i\theta}$. Now the only singularities contained in this curve are

$$\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}, -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$$

and the integrand, f has simple poles at these points. Thus using the same procedure as in the other examples,

$$\operatorname{Res}\left(f, \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi$$

and

$$\operatorname{Res}\left(f, \frac{-1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi.$$

Consider the integral along the small semicircle of radius r. This reduces to

$$\int_\pi^0 \frac{\ln|r| + it}{1 + (re^{it})^4} \left(rie^{it}\right) dt$$

which clearly converges to zero as $r \to 0$ because $r \ln r \to 0$. Therefore, taking the limit as $r \to 0$,

$$\int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz + \lim_{r\to 0+} \int_{-R}^{-r} \frac{\ln(-t) + i\pi}{1+t^4} dt +$$

$$\lim_{r\to 0+} \int_r^R \frac{\ln t}{1+t^4} dt = 2\pi i \left(\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi + \frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi\right).$$

Observing that $\int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz \to 0$ as $R \to \infty$,

$$e(R) + 2 \lim_{r \to 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \int_{-\infty}^0 \frac{1}{1+t^4} dt = \left(-\frac{1}{8} + \frac{1}{4} i\right) \pi^2 \sqrt{2}$$

where $e(R) \to 0$ as $R \to \infty$. From an earlier example this becomes

$$e(R) + 2 \lim_{r \to 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \left(\frac{\sqrt{2}}{4}\pi\right) = \left(-\frac{1}{8} + \frac{1}{4} i\right) \pi^2 \sqrt{2}.$$

Now letting $r \to 0+$ and $R \to \infty$,

$$\begin{aligned} 2 \int_0^\infty \frac{\ln t}{1+t^4} dt &= \left(-\frac{1}{8} + \frac{1}{4} i\right) \pi^2 \sqrt{2} - i\pi \left(\frac{\sqrt{2}}{4}\pi\right) \\ &= -\frac{1}{8} \sqrt{2} \pi^2, \end{aligned}$$

and so

$$\int_0^\infty \frac{\ln t}{1+t^4} dt = -\frac{1}{16} \sqrt{2} \pi^2,$$

which is probably not the first thing you would thing of. You might try to imagine how this could be obtained using elementary techniques.

The next example illustrates the use of what is referred to as a branch cut. It includes many examples.

Example 20.22 *Mellin transformations are of the form*

$$\int_0^\infty f(x) x^\alpha \frac{dx}{x}.$$

Sometimes it is possible to evaluate such a transform in terms of the constant, α.

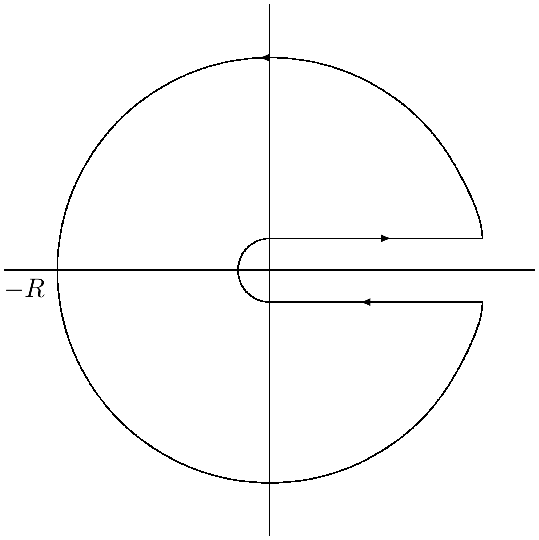
Assume f is an analytic function except at isolated singularities, none of which are on $(0, \infty)$. Also assume that f has the growth conditions,

$$|f(z)| \leq \frac{C}{|z|^b}, b > \alpha$$

for all large $|z|$ and assume that

$$|f(z)| \leq \frac{C'}{|z|^{b_1}}, b_1 < \alpha$$

for all $|z|$ sufficiently small. It turns out there exists an explicit formula for this Mellin transformation under these conditions. Consider the following contour.



In this contour the small semicircle in the center has radius ε which will converge to 0. Denote by γ_R the large circular path which starts at the upper edge of the slot and continues to the lower edge. Denote by γ_ε the small semicircular contour and denote by $\gamma_{\varepsilon R+}$ the straight part of the contour from 0 to R which provides the top edge of the slot. Finally denote by $\gamma_{\varepsilon R-}$ the straight part of the contour from R to 0 which provides the bottom edge of the slot. The interesting aspect of this problem is the definition of $f(z) z^{\alpha-1}$. Let

$$z^{\alpha-1} \equiv e^{(\ln|z|+i\arg(z))(\alpha-1)} = e^{(\alpha-1)\log(z)}$$

where $\arg(z)$ is the angle of z in $(0, 2\pi)$. Thus you use a branch of the logarithm which is defined on $\mathbb{C}\setminus(0,\infty)$. Then it is routine to verify from the assumed estimates that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z) z^{\alpha-1} dz = 0$$

and

$$\lim_{\varepsilon\to 0+} \int_{\gamma_\varepsilon} f(z) z^{\alpha-1} dz = 0.$$

Also, it is routine to verify

$$\lim_{\varepsilon\to 0+} \int_{\gamma_{\varepsilon R+}} f(z) z^{\alpha-1} dz = \int_0^R f(x) x^{\alpha-1} dx$$

and

$$\lim_{\varepsilon\to 0+} \int_{\gamma_{\varepsilon R-}} f(z) z^{\alpha-1} dz = -e^{i2\pi(\alpha-1)} \int_0^R f(x) x^{\alpha-1} dx.$$

Therefore, letting Σ_R denote the sum of the residues of $f(z) z^{\alpha-1}$ which are contained in the disk of radius R except for the possible residue at 0,

$$e(R) + \left(1 - e^{i2\pi(\alpha-1)}\right) \int_0^R f(x) x^{\alpha-1} dx = 2\pi i \Sigma_R$$

where $e(R) \to 0$ as $R \to \infty$. Now letting $R \to \infty$,

$$\lim_{R\to\infty} \int_0^R f(x) x^{\alpha-1} dx = \frac{2\pi i}{1 - e^{i2\pi(\alpha-1)}} \Sigma = \frac{\pi e^{-\pi i \alpha}}{\sin(\pi\alpha)} \Sigma$$

where Σ denotes the sum of all the residues of $f(z) z^{\alpha-1}$ except for the residue at 0.

The next example is similar to the one on the Mellin transform. In fact it is a Mellin transform but is worked out independently of the above to emphasize a slightly more informal technique related to the contour.

Example 20.23 $\int_0^\infty \frac{x^{p-1}}{1+x} dx,\ p \in (0,1)$.

Since the exponent of x in the numerator is larger than -1. The integral does converge. However, the techniques of real analysis don't tell us what it converges to. The contour to be used is as follows: From $(\varepsilon, 0)$ to $(r, 0)$ along the x axis and then from $(r, 0)$ to $(r, 0)$ counter clockwise along the circle of radius r, then from $(r, 0)$ to $(\varepsilon, 0)$ along the x axis and from $(\varepsilon, 0)$ to $(\varepsilon, 0)$, clockwise along the circle of radius ε. You should draw a picture of this contour. The interesting thing about this is that z^{p-1} cannot be defined all the way around 0. Therefore, use a branch of z^{p-1} corresponding to the branch of the logarithm obtained by deleting the positive x axis. Thus

$$z^{p-1} = e^{(\ln|z| + iA(z))(p-1)}$$

where $z = |z| e^{iA(z)}$ and $A(z) \in (0, 2\pi)$. Along the integral which goes in the positive direction on the x axis, let $A(z) = 0$ while on the one which goes in the negative direction, take $A(z) = 2\pi$. This is the appropriate choice obtained by replacing the line from $(\varepsilon, 0)$ to $(r, 0)$ with two lines having a small gap joined by a circle of radius ε and then taking a limit as the gap closes. You should verify that the two integrals taken along the circles of radius ε and r converge to 0 as $\varepsilon \to 0$ and as $r \to \infty$. Therefore, taking the limit,

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx + \int_\infty^0 \frac{x^{p-1}}{1+x} \left(e^{2\pi i(p-1)}\right) dx = 2\pi i \operatorname{Res}(f, -1).$$

Calculating the residue of the integrand at -1, and simplifying the above expression,

$$\left(1 - e^{2\pi i(p-1)}\right) \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)i\pi}.$$

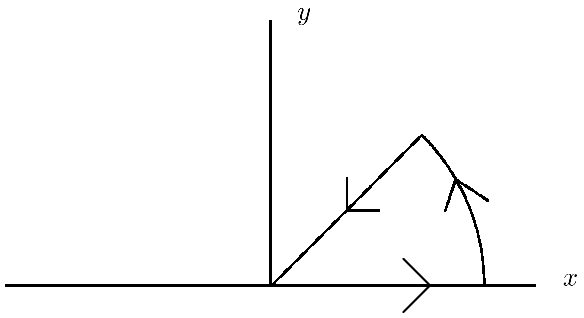
Upon simplification

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}.$$

Example 20.24 *The Fresnel integrals are*

$$\int_0^\infty \cos\left(x^2\right) dx, \int_0^\infty \sin\left(x^2\right) dx.$$

To evaluate these integrals consider $f(z) = e^{iz^2}$ on the curve which goes from the origin to the point r on the x axis and from this point to the point $r\left(\frac{1+i}{\sqrt{2}}\right)$ along a circle of radius r, and from there back to the origin as illustrated in the following picture.



Thus the curve to integrate over is shaped like a slice of pie. Denote by γ_r the curved part. Since f is analytic,

$$\begin{aligned}
0 &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{i\left(t\left(\frac{1+i}{\sqrt{2}}\right)\right)^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\
&= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{-t^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\
&= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \frac{\sqrt{\pi}}{2} \left(\frac{1+i}{\sqrt{2}}\right) + e(r)
\end{aligned}$$

where $e(r) \to 0$ as $r \to \infty$. Here we used the fact that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. Now consider the first of these integrals.

$$\begin{aligned}
\left|\int_{\gamma_r} e^{iz^2} dz\right| &= \left|\int_0^{\frac{\pi}{4}} e^{i\left(re^{it}\right)^2} rie^{it} dt\right| \\
&\leq r \int_0^{\frac{\pi}{4}} e^{-r^2 \sin 2t} dt \\
&= \frac{r}{2} \int_0^1 \frac{e^{-r^2 u}}{\sqrt{1-u^2}} du
\end{aligned}$$

$$\leq \frac{r}{2} \int_0^{r^{-(3/2)}} \frac{1}{\sqrt{1-u^2}} du + \frac{r}{2} \left(\int_0^1 \frac{1}{\sqrt{1-u^2}}\right) e^{-\left(r^{1/2}\right)}$$

which converges to zero as $r \to \infty$. Therefore, taking the limit as $r \to \infty$,

$$\frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right) = \int_0^\infty e^{ix^2} dx$$

and so

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2 dx.$$

The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for $\cot \pi z$ known as the Mittag- Leffler expansion of $\cot \pi z$.

Example 20.25 *Let γ_N be the contour which goes from $-N - \frac{1}{2} - Ni$ horizontally to $N + \frac{1}{2} - Ni$ and from there, vertically to $N + \frac{1}{2} + Ni$ and then horizontally to $-N - \frac{1}{2} + Ni$ and finally vertically to $-N - \frac{1}{2} - Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the following integral.*

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z \left(\alpha^2 - z^2\right)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag Leffler,

$$\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \frac{2}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}. \tag{20.12}$$

You should verify that $\cot \pi z$ is bounded on this contour and that therefore, $I_N \to 0$ as $N \to \infty$. Now you compute the residues of the integrand at $\pm\alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, you can evaluate I_N by using these. It is left as an exercise to calculate these residues and find that the residue at $\pm\alpha$ is

$$\frac{-\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore,

$$0 = \lim_{N\to\infty} I_N = \lim_{N\to\infty} 2\pi i \left[\sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi \alpha}{\alpha}\right]$$

which establishes the following formula of Mittag Leffler.

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

Writing this in a slightly nicer form, yields 20.12.

20.3 The Spectral Radius Of A Bounded Linear Transformation

As a very important application of the theory of Laurent series, I will give a short description of the spectral radius. This is a fundamental result which must be understood in order to prove convergence of various important numerical methods such as the Gauss Seidel or Jacobi methods.

Definition 20.26 *Let X be a complex Banach space and let $A \in \mathcal{L}(X,X)$. Then*

$$r(A) \equiv \left\{ \lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X,X) \right\}$$

This is called the resolvent set. The spectrum of A, denoted by $\sigma(A)$ is defined as all the complex numbers which are not in the resolvent set. Thus

$$\sigma(A) \equiv \mathbb{C} \setminus r(A)$$

Lemma 20.27 *$\lambda \in r(A)$ if and only if $\lambda I - A$ is one to one and onto X. Also if $|\lambda| > ||A||$, then $\lambda \in \sigma(A)$. If the Neumann series,*

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k$$

converges, then

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k = (\lambda I - A)^{-1}.$$

Proof: Note that to be in $r(A), \lambda I - A$ must be one to one and map X onto X since otherwise, $(\lambda I - A)^{-1} \notin \mathcal{L}(X,X)$.

By the open mapping theorem, if these two algebraic conditions hold, then $(\lambda I - A)^{-1}$ is continuous and so this proves the first part of the lemma. Now suppose $|\lambda| > ||A||$. Consider the Neumann series

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k.$$

By the root test, Theorem 18.3 on Page 386 this series converges to an element of $\mathcal{L}(X,X)$ denoted here by B. Now suppose the series converges. Letting $B_n \equiv \frac{1}{\lambda} \sum_{k=0}^{n} \left(\frac{A}{\lambda} \right)^k$,

$$\begin{aligned}
(\lambda I - A) B_n &= B_n (\lambda I - A) = \sum_{k=0}^{n} \left(\frac{A}{\lambda} \right)^k - \sum_{k=0}^{n} \left(\frac{A}{\lambda} \right)^{k+1} \\
&= I - \left(\frac{A}{\lambda} \right)^{n+1} \to I
\end{aligned}$$

as $n \to \infty$ because the convergence of the series requires the n^{th} term to converge to 0. Therefore,

$$(\lambda I - A) B = B (\lambda I - A) = I$$

which shows $\lambda I - A$ is both one to one and onto and the Neumann series converges to $(\lambda I - A)^{-1}$. This proves the lemma.

This lemma also shows that $\sigma(A)$ is bounded. In fact, $\sigma(A)$ is closed.

Lemma 20.28 *$r(A)$ is open. In fact, if $\lambda \in r(A)$ and $|\mu - \lambda| < \left|\left|(\lambda I - A)^{-1}\right|\right|^{-1}$, then $\mu \in r(A)$.*

Proof: First note

$$\begin{aligned}(\mu I - A) &= \left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right)(\lambda I - A) &(20.13)\\ &= (\lambda I - A)\left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right) &(20.14)\end{aligned}$$

Also from the assumption about $|\lambda - \mu|$,

$$\left|\left|(\lambda - \mu)(\lambda I - A)^{-1}\right|\right| \leq |\lambda - \mu| \left|\left|(\lambda I - A)^{-1}\right|\right| < 1$$

and so by the root test,

$$\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda I - A)^{-1}\right)^k$$

converges to an element of $\mathcal{L}(X, X)$. As in Lemma 20.27,

$$\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda I - A)^{-1}\right)^k = \left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right)^{-1}.$$

Therefore, from 20.13,

$$(\mu I - A)^{-1} = (\lambda I - A)^{-1}\left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right)^{-1}.$$

This proves the lemma.

Corollary 20.29 *$\sigma(A)$ is a compact set.*

Proof: Lemma 20.27 shows $\sigma(A)$ is bounded and Lemma 20.28 shows it is closed.

Definition 20.30 *The spectral radius, denoted by $\rho(A)$ is defined by*

$$\rho(A) \equiv \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Since $\sigma(A)$ is compact, this maximum exists. Note from Lemma 20.27, $\rho(A) \leq ||A||$.

There is a simple formula for the spectral radius.

Lemma 20.31 *If* $|\lambda| > \rho(A)$, *then the Neumann series,*

$$\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$$

converges.

Proof: This follows directly from Theorem 20.18 on Page 461 and the observation above that $\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k = (\lambda I - A)^{-1}$ for all $|\lambda| > ||A||$. Thus the analytic function, $\lambda \to (\lambda I - A)^{-1}$ has a Laurent expansion on $|\lambda| > \rho(A)$ by Theorem 20.18 and it must coincide with $\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$ on $|\lambda| > ||A||$ so the Laurent expansion of $\lambda \to (\lambda I - A)^{-1}$ must equal $\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$ on $|\lambda| > \rho(A)$. This proves the lemma.

The theorem on the spectral radius follows. It is due to Gelfand.

Theorem 20.32 $\rho(A) = \lim_{n\to\infty} ||A^n||^{1/n}$.

Proof: If

$$|\lambda| < \limsup_{n\to\infty} ||A^n||^{1/n}$$

then by the root test, the Neumann series does not converge and so by Lemma 20.31 $|\lambda| \leq \rho(A)$. Thus

$$\rho(A) \geq \limsup_{n\to\infty} ||A^n||^{1/n}.$$

Now let p be a positive integer. Then $\lambda \in \sigma(A)$ implies $\lambda^p \in \sigma(A^p)$ because

$$\begin{aligned}
\lambda^p I - A^p &= (\lambda I - A)\left(\lambda^{p-1} + \lambda^{p-2}A + \cdots + A^{p-1}\right) \\
&= \left(\lambda^{p-1} + \lambda^{p-2}A + \cdots + A^{p-1}\right)(\lambda I - A)
\end{aligned}$$

It follows from Lemma 20.27 applied to A^pthat for $\lambda \in \sigma(A), |\lambda^p| \leq ||A^p||$ and so $|\lambda| \leq ||A^p||^{1/p}$. Therefore, $\rho(A) \leq ||A^p||^{1/p}$ and since p is arbitrary,

$$\liminf_{p\to\infty} ||A^p||^{1/p} \geq \rho(A) \geq \limsup_{n\to\infty} ||A^n||^{1/n}.$$

This proves the theorem.

20.4 Exercises

1. Example 20.19 found the integral of a rational function of a certain sort. The technique used in this example typically works for rational functions of the form $\frac{f(x)}{g(x)}$ where $\deg(g(x)) \geq \deg f(x) + 2$ provided the rational function has no poles on the real axis. State and prove a theorem based on these observations.

2. Fill in the missing details of Example 20.25 about $I_N \to 0$. Note how important it was that the contour was chosen just right for this to happen. Also verify the claims about the residues.

3. Suppose f has a pole of order m at $z = a$. Define $g(z)$ by
$$g(z) = (z-a)^m f(z).$$
Show
$$\text{Res}(f,a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$
Hint: Use the Laurent series.

4. Give a proof of Theorem 20.6. **Hint:** Let p be a pole. Show that near p, a pole of order m,
$$\frac{f'(z)}{f(z)} = \frac{-m + \sum_{k=1}^{\infty} b_k (z-p)^k}{(z-p) + \sum_{k=2}^{\infty} c_k (z-p)^k}$$
Show that $\text{Res}(f,p) = -m$. Carry out a similar procedure for the zeros.

5. Use Rouche's theorem to prove the fundamental theorem of algebra which says that if $p(z) = z^n + a_{n-1}z^{n-1} \cdots + a_1 z + a_0$, then p has n zeros in $\mathbb{C}$. **Hint:** Let $q(z) = -z^n$ and let γ be a large circle, $\gamma(t) = re^{it}$ for r sufficiently large.

6. Consider the two polynomials $z^5 + 3z^2 - 1$ and $z^5 + 3z^2$. Show that on $|z| = 1$, the conditions for Rouche's theorem hold. Now use Rouche's theorem to verify that $z^5 + 3z^2 - 1$ must have two zeros in $|z| < 1$.

7. Consider the polynomial, $z^{11} + 7z^5 + 3z^2 - 17$. Use Rouche's theorem to find a bound on the zeros of this polynomial. In other words, find r such that if z is a zero of the polynomial, $|z| < r$. Try to make r fairly small if possible.

8. Verify that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. **Hint:** Use polar coordinates.

9. Use the contour described in Example 20.19 to compute the exact values of the following improper integrals.

   (a) $\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2} dx$

   (b) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$

   (c) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a, b > 0$

10. Evaluate the following improper integrals.

    (a) $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} dx$

(b) $\int_0^\infty \frac{x \sin x}{(x^2+a^2)^2} dx$

11. Find the Cauchy principle value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} dx$$

defined as

$$\lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{1-\varepsilon} \frac{\sin x}{(x^2+1)(x-1)} dx + \int_{1+\varepsilon}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} dx \right).$$

12. Find a formula for the integral $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$ where n is a nonnegative integer.

13. Find $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$.

14. If $m < n$ for m and n integers, show

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx = \frac{\pi}{2n} \frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

15. Find $\int_{-\infty}^{\infty} \frac{1}{(1+x^4)^2} dx$.

16. Find $\int_0^\infty \frac{\ln(x)}{1+x^2} dx = 0$.

17. Suppose f has an isolated singularity at α. Show the singularity is essential if and only if the principal part of the Laurent series of f has infinitely many terms. That is, show $f(z) = \sum_{k=0}^{\infty} a_k (z-\alpha)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z-\alpha)^k}$ where infinitely many of the b_k are nonzero.

18. Suppose Ω is a bounded open set and f_n is analytic on Ω and continuous on $\overline{\Omega}$. Suppose also that $f_n \to f$ uniformly on $\overline{\Omega}$ and that $f \neq 0$ on $\partial\Omega$. Show that for all n large enough, f_n and f have the same number of zeros on Ω provided the zeros are counted according to multiplicity.

Complex Mappings

21.1 Conformal Maps

If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U, an open set of $\mathbb{C}$, and if $f : U \to \mathbb{C}$ is analytic, consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, (x, y). Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \to \mathbb{C}$ is analytic. What can be said about the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$? It turns out this angle is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. To see this, note $(x, y) \cdot (a, b) = \frac{1}{2}(z\overline{w} + \overline{z}w)$ where $z = x + iy$ and $w = a + ib$. Therefore, letting θ be the cosine between the two vectors, $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$, it follows from calculus that

$$
\begin{aligned}
& \cos\theta \\
= & \frac{(f \circ \gamma)'(t_0) \cdot (f \circ \eta)'(s_0)}{\left|(f \circ \eta)'(s_0)\right| \left|(f \circ \gamma)'(t_0)\right|} \\
= & \frac{1}{2} \frac{f'(\gamma(t_0))\gamma'(t_0)\overline{f'(\eta(s_0))\eta'(s_0)} + \overline{f'(\gamma(t_0))\gamma'(t_0)} f'(\eta(s_0))\eta'(s_0)}{|f'(\gamma(t_0))| \, |f'(\eta(s_0))|} \\
= & \frac{1}{2} \frac{f'(z)\overline{f'(z)}\gamma'(t_0)\overline{\eta'(s_0)} + \overline{f'(z)} f'(z)\overline{\gamma'(t_0)}\eta'(s_0)}{|f'(z)| \, |f'(z)|} \\
= & \frac{1}{2} \frac{\gamma'(t_0)\overline{\eta'(s_0)} + \eta'(s_0)\overline{\gamma'(t_0)}}{1}
\end{aligned}
$$

which equals the angle between the vectors, $\gamma'(t_0)$ and $\eta'(t_0)$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. .

Actually, they also preserve orientations. If $z = x + iy$ and $w = a + ib$ are two complex numbers, then $(x, y, 0)$ and $(a, b, 0)$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $(x, y, 0) \times (a, b, 0)$, yields a vector normal to the two given vectors such that the triple, $(x, y, 0)$, $(a, b, 0)$, and $(x, y, 0) \times (a, b, 0)$ satisfies the right hand rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product will produce a vector which is a multiple of $\mathbf{k}$, the unit vector in the direction of the z axis. In fact, you can verify by computing both sides that, letting $z = x + iy$ and $w = a + ib$,

$$(x, y, 0) \times (a, b, 0) = \operatorname{Re}(zi\overline{w})\,\mathbf{k}.$$

Therefore, in the above situation,

$$\begin{aligned} & (f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) \\ = \; & \operatorname{Re}\left(f'(\gamma(t_0))\,\gamma'(t_0)\, i\overline{f'(\eta(s_0))\eta'(s_0)}\right)\mathbf{k} \\ = \; & |f'(z)|^2 \operatorname{Re}\left(\gamma'(t_0)\, i\overline{\eta'(s_0)}\right)\mathbf{k} \end{aligned}$$

which shows that the orientation of $\gamma'(t_0)$, $\eta'(s_0)$ is the same as the orientation of $(f \circ \gamma)'(t_0), (f \circ \eta)'(s_0)$. Mappings which preserve both orientation and angles are called conformal mappings and this has shown that analytic functions are conformal mappings if the derivative does not vanish.

21.2 Fractional Linear Transformations

21.2.1 Circles And Lines

These mappings map lines and circles to either lines or circles.

Definition 21.1 *A fractional linear transformation is a function of the form*

$$f(z) = \frac{az + b}{cz + d} \tag{21.1}$$

where $ad - bc \neq 0$.

Note that if $c = 0$, this reduces to a linear transformation $(a/d)\,z + (b/d)$. Special cases of these are defined as follows.

$$\text{dilations: } z \to \delta z,\ \delta \neq 0,\ \text{ inversions: } z \to \frac{1}{z},$$

$$\text{translations: } z \to z + \rho.$$

The next lemma is the key to understanding fractional linear transformations.

Lemma 21.2 *The fractional linear transformation, 21.1 can be written as a finite composition of dilations, inversions, and translations.*

Proof: Let

$$S_1(z) = z + \frac{d}{c}, S_2(z) = \frac{1}{z}, S_3(z) = \frac{(bc - ad)}{c^2} z$$

and

$$S_4(z) = z + \frac{a}{c}$$

in the case where $c \neq 0$. Then $f(z)$ given in 21.1 is of the form

$$f(z) = S_4 \circ S_3 \circ S_2 \circ S_1.$$

Here is why.

$$S_2(S_1(z)) = S_2\left(z + \frac{d}{c}\right) \equiv \frac{1}{z + \frac{d}{c}} = \frac{c}{zc + d}.$$

Now consider

$$S_3\left(\frac{c}{zc+d}\right) \equiv \frac{(bc - ad)}{c^2}\left(\frac{c}{zc+d}\right) = \frac{bc - ad}{c(zc+d)}.$$

Finally, consider

$$S_4\left(\frac{bc - ad}{c(zc+d)}\right) \equiv \frac{bc - ad}{c(zc+d)} + \frac{a}{c} = \frac{b + az}{zc + d}.$$

In case that $c = 0$, $f(z) = \frac{a}{d}z + \frac{b}{d}$ which is a translation composed with a dilation. Because of the assumption that $ad - bc \neq 0$, it follows that since $c = 0$, both a and $d \neq 0$. This proves the lemma.

This lemma implies the following corollary.

Corollary 21.3 *Fractional linear transformations map circles and lines to circles or lines.*

Proof: It is obvious that dilations and translations map circles to circles and lines to lines. What of inversions? If inversions have this property, the above lemma implies a general fractional linear transformation has this property as well.

Note that all circles and lines may be put in the form

$$\alpha\left(x^2 + y^2\right) - 2ax - 2by = r^2 - \left(a^2 + b^2\right)$$

where $\alpha = 1$ gives a circle centered at (a, b) with radius r and $\alpha = 0$ gives a line. In terms of complex variables you may therefore consider all possible circles and lines in the form

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0, \tag{21.2}$$

To see this let $\beta = \beta_1 + i\beta_2$ where $\beta_1 \equiv -a$ and $\beta_2 \equiv b$. Note that even if α is not 0 or 1 the expression still corresponds to either a circle or a line because you can divide by α if $\alpha \neq 0$. Now I verify that replacing z with $\frac{1}{z}$ results in an expression of the form in 21.2. Thus, let $w = \frac{1}{z}$ where z satisfies 21.2. Then

$$\left(\alpha + \beta\overline{w} + \overline{\beta}w + \gamma w\overline{w}\right) = \frac{1}{z\overline{z}}\left(\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma\right) = 0$$

and so w also satisfies a relation like 21.2. One simply switches α with γ and β with $\overline{\beta}$. Note the situation is slightly different than with dilations and translations. In the case of an inversion, a circle becomes either a line or a circle and similarly, a line becomes either a circle or a line. This proves the corollary.

The next example is quite important.

Example 21.4 *Consider the fractional linear transformation,* $w = \frac{z-i}{z+i}$.

First consider what this mapping does to the points of the form $z = x + i0$. Substituting into the expression for w,

$$w = \frac{x-i}{x+i} = \frac{x^2 - 1 - 2xi}{x^2+1},$$

a point on the unit circle. Thus this transformation maps the real axis to the unit circle.

The upper half plane is composed of points of the form $x + iy$ where $y > 0$. Substituting in to the transformation,

$$w = \frac{x + i\,(y-1)}{x + i\,(y+1)},$$

which is seen to be a point on the interior of the unit disk because $|y-1| < |y+1|$ which implies $|x + i\,(y+1)| > |x + i\,(y-1)|$. Therefore, this transformation maps the upper half plane to the interior of the unit disk.

One might wonder whether the mapping is one to one and onto. The mapping is clearly one to one because it has an inverse, $z = -i\frac{w+1}{w-1}$ for all w in the interior of the unit disk. Also, a short computation verifies that z so defined is in the upper half plane. Therefore, this transformation maps $\{z \in \mathbb{C} \text{ such that } \operatorname{Im} z > 0\}$ one to one and onto the unit disk $\{z \in \mathbb{C} \text{ such that } |z| < 1\}$.

A fancy way to do part of this is to use Theorem 19.11. $\limsup_{z\to a} \left|\frac{z-i}{z+i}\right| \leq 1$ whenever a is the real axis or ∞. Therefore, $\left|\frac{z-i}{z+i}\right| \leq 1$. This is a little shorter.

21.2.2 Three Points To Three Points

There is a simple procedure for determining fractional linear transformations which map a given set of three points to another set of three points. The problem is as follows: There are three distinct points in the extended complex plane, z_1, z_2, and z_3 and it is desired to find a fractional linear transformation such that $z_i \to w_i$ for $i = 1, 2, 3$ where here w_1, w_2, and w_3 are three distinct points in the extended complex plane. Then the procedure says that to find the desired fractional linear transformation solve the following equation for w.

$$\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$

The result will be a fractional linear transformation with the desired properties. If any of the points equals ∞, then the quotient containing this point should be adjusted.

Why should this procedure work? Here is a heuristic argument to indicate why you would expect this to happen rather than a rigorous proof. The reader may want to tighten the argument to give a proof. First suppose $z = z_1$. Then the right side equals zero and so the left side also must equal zero. However, this requires $w = w_1$. Next suppose $z = z_2$. Then the right side equals 1. To get a 1 on the left, you need $w = w_2$. Finally suppose $z = z_3$. Then the right side involves division by 0. To get the same bad behavior, on the left, you need $w = w_3$.

Example 21.5 *Let* $\operatorname{Im}\xi > 0$ *and consider the fractional linear transformation which takes* ξ *to 0,* $\overline{\xi}$ *to* ∞ *and* 0 *to* $\xi/\overline{\xi}$, .

The equation for w is

$$\frac{w-0}{w-\left(\xi/\overline{\xi}\right)} = \frac{z-\xi}{z-0} \cdot \frac{\overline{\xi}-0}{\overline{\xi}-\xi}$$

After some computations,

$$w = \frac{z-\xi}{z-\overline{\xi}}.$$

Note that this has the property that $\frac{x-\xi}{x-\overline{\xi}}$ is always a point on the unit circle because it is a complex number divided by its conjugate. Therefore, this fractional linear transformation maps the real line to the unit circle. It also takes the point, ξ to 0 and so it must map the upper half plane to the unit disk. You can verify the mapping is onto as well.

Example 21.6 *Let* $z_1 = 0, z_2 = 1,$ *and* $z_3 = 2$ *and let* $w_1 = 0, w_2 = i,$ *and* $w_3 = 2i$.

Then the equation to solve is

$$\frac{w}{w-2i} \cdot \frac{-i}{i} = \frac{z}{z-2} \cdot \frac{-1}{1}$$

Solving this yields $w = iz$ which clearly works.

21.3 Riemann Mapping Theorem

From the open mapping theorem analytic functions map regions to other regions or else to single points. The Riemann mapping theorem states that for every simply connected region, Ω which is not equal to all of $\mathbb{C}$ there exists an analytic function, f such that $f(\Omega) = B(0,1)$ and in addition to this, f is one to one. The proof involves several ideas which have been developed up to now. The proof is based on the following important theorem, a case of Montel's theorem. Before, beginning, note that the Riemann mapping theorem is a classic example of a major existence theorem. In mathematics there are two sorts of questions, those related to whether something exists and those involving methods for finding it. The real questions are often related to questions of existence. There is a long and involved history for proofs of this theorem. The first proofs were based on the Dirichlet principle and turned out to be incorrect, thanks to Weierstrass who pointed out the errors. For more on the history of this theorem, see Hille [23].

The following theorem is really wonderful. It is about the existence of a subsequence having certain salubrious properties. It is this wonderful result which will give the existence of the mapping desired. The other parts of the argument are technical details to set things up and use this theorem.

21.3.1 Montel's Theorem

Theorem 21.7 *Let Ω be an open set in $\mathbb{C}$ and let $\mathcal{F}$ denote a set of analytic functions mapping Ω to $B(0,M) \subseteq \mathbb{C}$. Then there exists a sequence of functions from $\mathcal{F}$, $\{f_n\}_{n=1}^{\infty}$ and an analytic function, f such that $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of Ω.*

Proof: First note there exists a sequence of compact sets, K_n such that $K_n \subseteq \operatorname{int} K_{n+1} \subseteq \Omega$ for all n where here $\operatorname{int} K$ denotes the interior of the set K, the union of all open sets contained in K and $\cup_{n=1}^{\infty} K_n = \Omega$. In fact, you can verify that $\overline{B(0,n)} \cap \left\{z \in \Omega : \operatorname{dist}\left(z, \Omega^C\right) \leq \frac{1}{n}\right\}$ works for K_n. Then there exist positive numbers, δ_n such that if $z \in K_n$, then $\overline{B(z,\delta_n)} \subseteq \operatorname{int} K_{n+1}$. Now denote by $\mathcal{F}_n$ the set of restrictions of functions of $\mathcal{F}$ to K_n. Then let $z \in K_n$ and let $\gamma(t) \equiv z + \delta_n e^{it}, t \in [0, 2\pi]$. It follows that for $z_1 \in B(z, \delta_n)$, and $f \in \mathcal{F}$,

$$\begin{aligned} |f(z) - f(z_1)| &= \left| \frac{1}{2\pi i} \int_\gamma f(w) \left(\frac{1}{w-z} - \frac{1}{w-z_1} \right) dw \right| \\ &\leq \frac{1}{2\pi} \left| \int_\gamma f(w) \frac{z - z_1}{(w-z)(w-z_1)} dw \right| \end{aligned}$$

Letting $|z_1 - z| < \frac{\delta_n}{2}$,

$$\begin{aligned} |f(z) - f(z_1)| &\leq \frac{M}{2\pi} 2\pi \delta_n \frac{|z - z_1|}{\delta_n^2/2} \\ &\leq 2M \frac{|z - z_1|}{\delta_n}. \end{aligned}$$

It follows that $\mathcal{F}_n$ is equicontinuous and uniformly bounded so by the Arzela Ascoli theorem there exists a sequence, $\{f_{nk}\}_{k=1}^{\infty} \subseteq \mathcal{F}$ which converges uniformly on K_n. Let $\{f_{1k}\}_{k=1}^{\infty}$ converge uniformly on K_1. Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\{f_{2k}\}_{k=1}^{\infty}$ which also converges uniformly on K_2. Continue in this way to obtain $\{f_{nk}\}_{k=1}^{\infty}$ which converges uniformly on $K_1, \cdots, K_n$. Now the sequence $\{f_{nn}\}_{n=m}^{\infty}$ is a subsequence of $\{f_{mk}\}_{k=1}^{\infty}$ and so it converges uniformly on K_m for all m. Denoting f_{nn} by f_n for short, this is the sequence of functions promised by the theorem. It is clear $\{f_n\}_{n=1}^{\infty}$ converges uniformly on every compact subset of Ω because every such set is contained in K_m for all m large enough. Let $f(z)$ be the point to which $f_n(z)$ converges. Then f is a continuous function defined on Ω. Is f is analytic? Yes it is by Lemma 18.18. Alternatively, you could let $T \subseteq \Omega$ be a triangle. Then

$$\int_{\partial T} f(z)\, dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)\, dz = 0.$$

Therefore, by Morera's theorem, f is analytic.

As for the uniform convergence of the derivatives of f, recall Theorem 18.52 about the existence of a cycle. Let K be a compact subset of $\operatorname{int}(K_n)$ and let $\{\gamma_k\}_{k=1}^{m}$ be closed oriented curves contained in

$$\operatorname{int}(K_n) \setminus K$$

such that $\sum_{k=1}^{m} n(\gamma_k, z) = 1$ for every $z \in K$. Also let η denote the distance between $\cup_j \gamma_j^*$ and K. Then for $z \in K$,

$$\begin{aligned} \left| f^{(k)}(z) - f_n^{(k)}(z) \right| &= \left| \frac{k!}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(w) - f_n(w)}{(w-z)^{k+1}} dw \right| \\ &\leq \frac{k!}{2\pi} \|f_k - f\|_{K_n} \sum_{j=1}^{m} (\text{length of } \gamma_k) \frac{1}{\eta^{k+1}}. \end{aligned}$$

where here $\|f_k - f\|_{K_n} \equiv \sup\{|f_k(z) - f(z)| : z \in K_n\}$. Thus you get uniform convergence of the derivatives.

Since the family, $\mathcal{F}$ satisfies the conclusion of Theorem 21.7 it is known as a normal family of functions. More generally,

Definition 21.8 *Let $\mathcal{F}$ denote a collection of functions which are analytic on Ω, a region. Then $\mathcal{F}$ is normal if every sequence contained in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of Ω.*

The following result is about a certain class of fractional linear transformations. Recall Lemma 19.18 which is listed here for convenience.

Lemma 21.9 *For $\alpha \in B(0,1)$, let*

$$\phi_\alpha(z) \equiv \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Then ϕ_α maps $B(0,1)$ one to one and onto $B(0,1)$, $\phi_\alpha^{-1} = \phi_{-\alpha}$, and

$$\phi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

The next lemma, known as Schwarz's lemma is interesting for its own sake but will also be an important part of the proof of the Riemann mapping theorem. It was stated and proved earlier but for convenience it is given again here.

Lemma 21.10 *Suppose $F : B(0,1) \to B(0,1)$, F is analytic, and $F(0) = 0$. Then for all $z \in B(0,1)$,*

$$|F(z)| \leq |z|, \tag{21.3}$$

and

$$|F'(0)| \leq 1. \tag{21.4}$$

If equality holds in 21.4 then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$F(z) = \lambda z. \tag{21.5}$$

Proof: First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,

$$\left|\frac{F(z)}{z}\right| \leq \max_{t\in[0,2\pi]} \frac{\left|F\left(re^{it}\right)\right|}{r} \leq \frac{1}{r}.$$

Then letting $r \to 1$,

$$\left|\frac{F(z)}{z}\right| \leq 1$$

this shows 21.3 and it also verifies 21.4 on taking the limit as $z \to 0$. If equality holds in 21.4, then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, λ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$. This proves the lemma.

Definition 21.11 *A region, Ω has the square root property if whenever $f, \frac{1}{f} : \Omega \to \mathbb{C}$ are both analytic[1], it follows there exists $\phi : \Omega \to \mathbb{C}$ such that ϕ is analytic and $f(z) = \phi^2(z)$.*

The next theorem will turn out to be equivalent to the Riemann mapping theorem.

21.3.2 Regions With Square Root Property

Theorem 21.12 *Let $\Omega \neq \mathbb{C}$ for Ω a region and suppose Ω has the square root property. Then for $z_0 \in \Omega$ there exists $h : \Omega \to B(0,1)$ such that h is one to one, onto, analytic, and $h(z_0) = 0$.*

Proof: Define $\mathcal{F}$ to be the set of functions, f such that $f : \Omega \to B(0,1)$ is one to one and analytic. The first task is to show $\mathcal{F}$ is nonempty. Then, using Montel's theorem it will be shown there is a function in $\mathcal{F}$, h, such that $|h'(z_0)| \geq \left|\psi'(z_0)\right|$

[1] This implies f has no zero on Ω.

for all $\psi \in \mathcal{F}$. When this has been done it will be shown that h is actually onto. This will prove the theorem.

Claim 1: $\mathcal{F}$ is nonempty.

Proof of Claim 1: Since $\Omega \neq \mathbb{C}$ it follows there exists $\xi \notin \Omega$. Then it follows $z-\xi$ and $\frac{1}{z-\xi}$ are both analytic on Ω. Since Ω has the square root property, there exists an analytic function, $\phi : \Omega \to \mathbb{C}$ such that $\phi^2(z) = z-\xi$ for all $z \in \Omega$, $\phi(z) = \sqrt{z-\xi}$. Since $z-\xi$ is not constant, neither is ϕ and it follows from the open mapping theorem that $\phi(\Omega)$ is a region. Note also that ϕ is one to one because if $\phi(z_1) = \phi(z_2)$, then you can square both sides and conclude $z_1 - \xi = z_2 - \xi$ implying $z_1 = z_2$.

Now pick $a \in \phi(\Omega)$. Thus $\sqrt{z_a - \xi} = a$. I claim there exists a positive lower bound to $\left|\sqrt{z-\xi} + a\right|$ for $z \in \Omega$. If not, there exists a sequence, $\{z_n\} \subseteq \Omega$ such that

$$\sqrt{z_n - \xi} + a = \sqrt{z_n - \xi} + \sqrt{z_a - \xi} \equiv \varepsilon_n \to 0.$$

Then

$$\sqrt{z_n - \xi} = \left(\varepsilon_n - \sqrt{z_a - \xi}\right) \tag{21.6}$$

and squaring both sides,

$$z_n - \xi = \varepsilon_n^2 + z_a - \xi - 2\varepsilon_n \sqrt{z_a - \xi}.$$

Consequently, $(z_n - z_a) = \varepsilon_n^2 - 2\varepsilon_n\sqrt{z_a - \xi}$ which converges to 0. Taking the limit in 21.6, it follows $2\sqrt{z_a - \xi} = 0$ and so $\xi = z_a$, a contradiction to $\xi \notin \Omega$. Choose $r > 0$ such that for all $z \in \Omega$, $\left|\sqrt{z-\xi} + a\right| > r > 0$. Then consider

$$\psi(z) \equiv \frac{r}{\sqrt{z-\xi} + a}. \tag{21.7}$$

This is one to one, analytic, and maps Ω into $B(0,1)$ ($\left|\sqrt{z-\xi} + a\right| > r$). Thus $\mathcal{F}$ is not empty and this proves the claim.

Claim 2: Let $z_0 \in \Omega$. There exists a finite positive real number, η, defined by

$$\eta \equiv \sup\left\{\left|\psi'(z_0)\right| : \psi \in \mathcal{F}\right\} \tag{21.8}$$

and an analytic function, $h \in \mathcal{F}$ such that $|h'(z_0)| = \eta$. Furthermore, $h(z_0) = 0$.

Proof of Claim 2: First you show $\eta < \infty$. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and r is small enough that $B(z_0, r) \subseteq \Omega$. Then for $\psi \in \mathcal{F}$, the Cauchy integral formula for the derivative implies

$$\psi'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(w)}{(w-z_0)^2}dw$$

and so $\left|\psi'(z_0)\right| \leq (1/2\pi)\, 2\pi r\left(1/r^2\right) = 1/r$. Therefore, $\eta < \infty$ as desired. For ψ defined above in 21.7

$$\psi'(z_0) = \frac{-r\phi'(z_0)}{(\phi(z_0) + a)^2} = \frac{-r(1/2)\left(\sqrt{z_0 - \xi}\right)^{-1}}{(\phi(z_0)+a)^2} \neq 0.$$

Therefore, $\eta > 0$. It remains to verify the existence of the function, h.

By Theorem 21.7, there exists a sequence, $\{\psi_n\}$, of functions in $\mathcal{F}$ and an analytic function, h, such that

$$\left|\psi_n' (z_0)\right| \to \eta \tag{21.9}$$

and

$$\psi_n \to h, \psi_n' \to h', \tag{21.10}$$

uniformly on all compact subsets of Ω. It follows

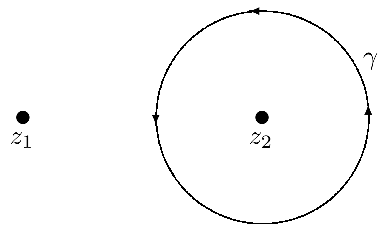
$$|h' (z_0)| = \lim_{n\to\infty} \left|\psi_n' (z_0)\right| = \eta \tag{21.11}$$

and for all $z \in \Omega$,

$$|h (z)| = \lim_{n\to\infty} |\psi_n (z)| \leq 1. \tag{21.12}$$

By 21.11, h is not a constant. Therefore, in fact, $|h (z)| < 1$ for all $z \in \Omega$ in 21.12 by the open mapping theorem.

Next it must be shown that h is one to one in order to conclude $h \in \mathcal{F}$. Pick $z_1 \in \Omega$ and suppose z_2 is another point of Ω. Since the zeros of $h - h (z_1)$ have no limit point, there exists a circular contour bounding a circle which contains z_2 but not z_1 such that γ^* contains no zeros of $h - h (z_1)$.



Using the theorem on counting zeros, Theorem 19.20, and the fact that ψ_n is one to one,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{\psi_n' (w)}{\psi_n (w) - \psi_n (z_1)} dw \\ &= \frac{1}{2\pi i} \int_\gamma \frac{h' (w)}{h (w) - h (z_1)} dw, \end{aligned}$$

which shows that $h - h (z_1)$ has no zeros in $B (z_2, r)$. In particular z_2 is not a zero of $h - h (z_1)$. This shows that h is one to one since $z_2 \neq z_1$ was arbitrary. Therefore, $h \in \mathcal{F}$. It only remains to verify that $h (z_0) = 0$.

If $h (z_0) \neq 0$,consider $\phi_{h(z_0)} \circ h$ where ϕ_α is the fractional linear transformation defined in Lemma 21.9. By this lemma it follows $\phi_{h(z_0)} \circ h \in \mathcal{F}$. Now using the chain rule,

$$\begin{aligned}\left|\left(\phi_{h(z_0)}\circ h\right)'(z_0)\right| &= \left|\phi_{h(z_0)}'(h(z_0))\right|\left|h'(z_0)\right| \\ &= \left|\frac{1}{1-\left|h(z_0)\right|^2}\right|\left|h'(z_0)\right| \\ &= \left|\frac{1}{1-\left|h(z_0)\right|^2}\right|\eta > \eta\end{aligned}$$

Contradicting the definition of η. This proves Claim 2.

Claim 3: The function, h just obtained maps Ω onto $B(0,1)$.

Proof of Claim 3: To show h is onto, use the fractional linear transformation of Lemma 21.9. Suppose h is not onto. Then there exists $\alpha \in B(0,1)\setminus h(\Omega)$. Then $0 \neq \phi_\alpha \circ h(z)$ for all $z \in \Omega$ because

$$\phi_\alpha \circ h(z) = \frac{h(z)-\alpha}{1-\overline{\alpha}h(z)}$$

and it is assumed $\alpha \notin h(\Omega)$. Therefore, since Ω has the square root property, you can consider an analytic function $z \to \sqrt{\phi_\alpha \circ h(z)}$. This function is one to one because both ϕ_α and h are. Also, the values of this function are in $B(0,1)$ by Lemma 21.9 so it is in $\mathcal{F}$.

Now let

$$\psi \equiv \phi_{\sqrt{\phi_\alpha \circ h(z_0)}} \circ \sqrt{\phi_\alpha \circ h}. \tag{21.13}$$

Thus

$$\psi(z_0) = \phi_{\sqrt{\phi_\alpha \circ h(z_0)}} \circ \sqrt{\phi_\alpha \circ h(z_0)} = 0$$

and ψ is a one to one mapping of Ω into $B(0,1)$ so ψ is also in $\mathcal{F}$. Therefore,

$$\left|\psi'(z_0)\right| \leq \eta,\ \left|\left(\sqrt{\phi_\alpha \circ h}\right)'(z_0)\right| \leq \eta. \tag{21.14}$$

Define $s(w) \equiv w^2$. Then using Lemma 21.9, in particular, the description of $\phi_\alpha^{-1} = \phi_{-\alpha}$, you can solve 21.13 for h to obtain

$$\begin{aligned}h(z) &= \phi_{-\alpha}\circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}} \circ \psi \\ &= \left(\overbrace{\phi_{-\alpha}\circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}}^{\equiv F} \circ \psi\right)(z) \\ &= (F \circ \psi)(z) \end{aligned} \tag{21.15}$$

Now

$$F(0) = \phi_{-\alpha}\circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(0) = \phi_\alpha^{-1}(\phi_\alpha \circ h(z_0)) = h(z_0) = 0$$

and F maps $B(0,1)$ into $B(0,1)$. Also, F is not one to one because it maps $B(0,1)$ to $B(0,1)$ and has s in its definition. Thus there exists $z_1 \in B(0,1)$ such that $\phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(z_1) = -\frac{1}{2}$ and another point $z_2 \in B(0,1)$ such that $\phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(z_2) = \frac{1}{2}$. However, thanks to $s, F(z_1) = F(z_2)$.

Since $F(0) = h(z_0) = 0$, you can apply the Schwarz lemma to F. Since F is not one to one, it can't be true that $F(z) = \lambda z$ for $|\lambda| = 1$ and so by the Schwarz lemma it must be the case that $|F'(0)| < 1$. But this implies from 21.15 and 21.14 that

$$\begin{aligned} \eta &= |h'(z_0)| = |F'(\psi(z_0))| \left|\psi'(z_0)\right| \\ &= |F'(0)| \left|\psi'(z_0)\right| < \left|\psi'(z_0)\right| \leq \eta, \end{aligned}$$

a contradiction. This proves the theorem.

The following lemma yields the usual form of the Riemann mapping theorem.

Lemma 21.13 *Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$. Then Ω has the square root property.*

Proof: Let f and $\frac{1}{f}$ both be analytic on Ω. Then $\frac{f'}{f}$ is analytic on Ω so by Corollary 18.50, there exists $\widetilde{F}$, analytic on Ω such that $\widetilde{F}' = \frac{f'}{f}$ on Ω. Then $\left(fe^{-\widetilde{F}}\right)' = 0$ and so $f(z) = Ce^{\widetilde{F}} = e^{a+ib}e^{\widetilde{F}}$. Now let $F = \widetilde{F} + a + ib$. Then F is still a primitive of f'/f and $f(z) = e^{F(z)}$. Now let $\phi(z) \equiv e^{\frac{1}{2}F(z)}$. Then ϕ is the desired square root and so Ω has the square root property.

Corollary 21.14 *(Riemann mapping theorem) Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$ and let $z_0 \in \Omega$. Then there exists a function, $f : \Omega \to B(0,1)$ such that f is one to one, analytic, and onto with $f(z_0) = 0$. Furthermore, f^{-1} is also analytic.*

Proof: From Theorem 21.12 and Lemma 21.13 there exists a function, $f : \Omega \to B(0,1)$ which is one to one, onto, and analytic such that $f(z_0) = 0$. The assertion that f^{-1} is analytic follows from the open mapping theorem.

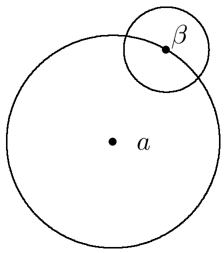
21.4 Analytic Continuation

21.4.1 Regular And Singular Points

Given a function which is analytic on some set, can you extend it to an analytic function defined on a larger set? Sometimes you can do this. It was done in the proof of the Cauchy integral formula. There are also reflection theorems like those discussed in the exercises starting with Problem 10 on Page 422. Here I will give a systematic way of extending an analytic function to a larger set. I will emphasize simply connected regions. The subject of analytic continuation is much larger than the introduction given here. A good source for much more on this is found in Alfors

[2]. The approach given here is suggested by Rudin [36] and avoids many of the standard technicalities.

Definition 21.15 *Let f be analytic on $B(a,r)$ and let $\beta \in \partial B(a,r)$. Then β is called a regular point of f if there exists some $\delta > 0$ and a function, g analytic on $B(\beta,\delta)$ such that $g = f$ on $B(\beta,\delta) \cap B(a,r)$. Those points of $\partial B(a,r)$ which are not regular are called singular.*



Theorem 21.16 *Suppose f is analytic on $B(a,r)$ and the power series*

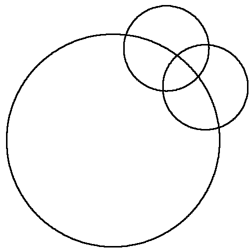
$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

has radius of convergence r. Then there exists a singular point on $\partial B(a,r)$.

Proof: If not, then for every $z \in \partial B(a,r)$ there exists $\delta_z > 0$ and g_z analytic on $B(z,\delta_z)$ such that $g_z = f$ on $B(z,\delta_z) \cap B(a,r)$. Since $\partial B(a,r)$ is compact, there exist $z_1, \cdots, z_n$, points in $\partial B(a,r)$ such that $\{B(z_k,\delta_{z_k})\}_{k=1}^n$ covers $\partial B(a,r)$. Now define

$$g(z) \equiv \begin{cases} f(z) \text{ if } z \in B(a,r) \\ g_{z_k}(z) \text{ if } z \in B(z_k,\delta_{z_k}) \end{cases}$$

Is this well defined? If $z \in B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j})$, is $g_{z_i}(z) = g_{z_j}(z)$? Consider the following picture representing this situation.



You see that if $z \in B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j})$ then $I \equiv B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j}) \cap B(a,r)$ is a nonempty open set. Both g_{z_i} and g_{z_j} equal f on I. Therefore, they must be equal on $B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j})$ because I has a limit point. Therefore, g is well defined and analytic on an open set containing $\overline{B(a,r)}$. Since g agrees with f on $B(a,r)$, the power series for g is the same as the power series for f and converges on a ball which is larger than $B(a,r)$ contrary to the assumption that the radius of convergence of the above power series equals r. This proves the theorem.

21.4.2 Continuation Along A Curve

Next I will describe what is meant by continuation along a curve. The following definition is standard and is found in Rudin [36].

Definition 21.17 *A function element is an ordered pair, (f,D) where D is an open ball and f is analytic on D. (f_0,D_0) and (f_1,D_1) are direct continuations of each other if $D_1 \cap D_0 \neq \emptyset$ and $f_0 = f_1$ on $D_1 \cap D_0$. In this case I will write $(f_0,D_0) \sim (f_1,D_1)$. A chain is a finite sequence, of disks, $\{D_0,\cdots,D_n\}$ such that $D_{i-1} \cap D_i \neq \emptyset$. If (f_0,D_0) is a given function element and there exist function elements, (f_i,D_i) such that $\{D_0,\cdots,D_n\}$ is a chain and $(f_{j-1},D_{j-1}) \sim (f_j,D_j)$ then (f_n,D_n) is called the analytic continuation of (f_0,D_0) along the chain $\{D_0,\cdots,D_n\}$. Now suppose γ is an oriented curve with parameter interval $[a,b]$ and there exists a chain, $\{D_0,\cdots,D_n\}$ such that $\gamma^* \subseteq \cup_{k=1}^{n} D_k, \gamma(a)$ is the center of $D_0, \gamma(b)$ is the center of D_n, and there is an increasing list of numbers in $[a,b]$, $a = s_0 < s_1 \cdots < s_n = b$ such that $\gamma([s_i,s_{i+1}]) \subseteq D_i$ and (f_n,D_n) is an analytic continuation of (f_0,D_0) along the chain. Then (f_n,D_n) is called an analytic continuation of (f_0,D_0) along the curve γ. (γ will always be a continuous curve. Nothing more is needed.)*

In the above situation it does not follow that if $D_n \cap D_0 \neq \emptyset$, that $f_n = f_0$! However, there are some cases where this will happen. This is the monodromy theorem which follows. This is as far as I will go on the subject of analytic continuation. For more on this subject including a development of the concept of Riemann surfaces, see Alfors [2].

Lemma 21.18 *Suppose $(f,B(0,r))$ for $r<1$ is a function element and $(f,B(0,r))$ can be analytically continued along every curve in $B(0,1)$ that starts at 0. Then there exists an analytic function, g defined on $B(0,1)$ such that $g = f$ on $B(0,r)$.*

Proof: Let

$$\begin{aligned} R &= \sup\{r_1 \geq r \text{ such that there exists } g_{r_1} \\ &\quad \text{analytic on } B(0,r_1) \text{ which agrees with } f \text{ on } B(0,r).\} \end{aligned}$$

Define $g_R(z) \equiv g_{r_1}(z)$ where $|z| < r_1$. This is well defined because if you use r_1 and r_2, both g_{r_1} and g_{r_2} agree with f on $B(0,r)$, a set with a limit point and so the two functions agree at every point in both $B(0,r_1)$ and $B(0,r_2)$. Thus g_R is analytic on $B(0,R)$. If $R<1$, then by the assumption there are no singular points on $B(0,R)$ and so Theorem 21.16 implies the radius of convergence of the power series for g_R is larger than R contradicting the choice of R. Therefore, $R=1$ and this proves the lemma. Let $g = g_R$.

The following theorem is the main result in this subject, the monodromy theorem.

Theorem 21.19 *Let Ω be a simply connected proper subset of $\mathbb{C}$ and suppose $(f, B(a,r))$ is a function element with $B(a,r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through a. Then there exists G analytic on Ω such that G agrees with f on $B(a,r)$.*

Proof: By the Riemann mapping theorem, there exists $h : \Omega \to B(0,1)$ which is analytic, one to one and onto such that $f(a) = 0$. Since h is an open map, there exists $\delta > 0$ such that

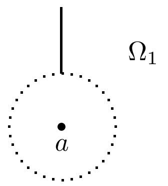
$$B(0,\delta) \subseteq h(B(a,r)).$$

It follows $f \circ h^{-1}$ can be analytically continued along every curve through 0. By Lemma 21.18 there exists g analytic on $B(0,1)$ which agrees with $f \circ h^{-1}$ on $B(0,\delta)$. Define $G(z) \equiv g(h(z))$. For $z = h^{-1}(w)$, it follows $G\left(h^{-1}(w)\right) = g(w)$. If $w \in B(0,\delta)$, then $G\left(h^{-1}(w)\right) = f \circ h^{-1}(w)$ and so $G = f$ on $h^{-1}(B(0,\delta))$, an open set contained in $B(a,r)$. Therefore, $G = f$ on $B(a,r)$ because $h^{-1}(B(0,\delta))$ has a limit point. This proves the theorem.

Actually, you sometimes want to consider the case where $\Omega = \mathbb{C}$. This requires a small modification to obtain from the above theorem.

Corollary 21.20 *Suppose $(f, B(a,r))$ is a function element with $B(a,r) \subseteq \mathbb{C}$. Suppose also that this function element can be analytically continued along every curve through a. Then there exists G analytic on $\mathbb{C}$ such that G agrees with f on $B(a,r)$.*

Proof: Let $\Omega_1 \equiv \{z \in \mathbb{C} : a + it : t > a\}$ and $\Omega_2 \equiv \{z \in \mathbb{C} : a - it : t > a\}$. Here is a picture of Ω_1.



A picture of Ω_2 is similar except the line extends down from the boundary of $B(a,r)$.

Thus $B(a,r) \subseteq \Omega_i$ and Ω_i is simply connected and proper. By Theorem 21.19 there exist analytic functions, G_i analytic on Ω_i such that $G_i = f$ on $B(a,r)$. Thus $G_1 = G_2$ on $B(a,r)$, a set with a limit point. Therefore, $G_1 = G_2$ on $\Omega_1 \cap \Omega_2$. Now let $G(z) = G_i(z)$ where $z \in \Omega_i$. This is well defined and analytic on $\mathbb{C}$. This proves the corollary.

21.5 The Picard Theorems

The Picard theorem says that if f is an entire function and there are two complex numbers not contained in $f(\mathbb{C})$, then f is constant. This is certainly one of the most amazing things which could be imagined. However, this is only the little

Picard theorem. The big Picard theorem is even more incredible. This one asserts that to be non constant the entire function must take every value of $\mathbb{C}$ but two infinitely many times! I will begin with the little Picard theorem. The method of proof I will use is the one found in Saks and Zygmund [38], Conway [11] and Hille [23]. This is not the way Picard did it in 1879. That approach is very different and is presented at the end of the material on elliptic functions. This approach is much more recent dating it appears from around 1924.

Lemma 21.21 *Let f be analytic on a region containing $\overline{B(0,r)}$ and suppose*

$$|f'(0)| = b > 0, f(0) = 0,$$

and $|f(z)| \leq M$ for all $z \in \overline{B(0,r)}$. Then $f(B(0,r)) \supseteq B\left(0, \frac{r^2 b^2}{6M}\right)$.

Proof: By assumption,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \ |z| \leq r. \tag{21.16}$$

Then by the Cauchy integral formula for the derivative,

$$a_k = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f(w)}{w^{k+1}} dw$$

where the integral is in the counter clockwise direction. Therefore,

$$|a_k| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|f\left(re^{i\theta}\right)\right|}{r^k} d\theta \leq \frac{M}{r^k}.$$

In particular, $br \leq M$. Therefore, from 21.16

$$\begin{aligned} |f(z)| &\geq b|z| - \sum_{k=2}^{\infty} \frac{M}{r^k} |z|^k = b|z| - \frac{M\left(\frac{|z|}{r}\right)^2}{1 - \frac{|z|}{r}} \\ &= b|z| - \frac{M|z|^2}{r^2 - r|z|} \end{aligned}$$

Suppose $|z| = \frac{r^2 b}{4M} < r$. Then this is no larger than

$$\frac{1}{4} b^2 r^2 \frac{3M - br}{M(4M - br)} \geq \frac{1}{4} b^2 r^2 \frac{3M - M}{M(4M - M)} = \frac{r^2 b^2}{6M}.$$

Let $|w| < \frac{r^2 b}{4M}$. Then for $|z| = \frac{r^2 b}{4M}$ and the above,

$$|w| = |(f(z) - w) - f(z)| < \frac{r^2 b}{4M} \leq |f(z)|$$

and so by Rouche's theorem, $z \to f(z) - w$ and $z \to f(z)$ have the same number of zeros in $B\left(0, \frac{r^2 b}{4M}\right)$. But f has at least one zero in this ball and so this shows there exists at least one $z \in B\left(0, \frac{r^2 b}{4M}\right)$ such that $f(z) - w = 0$. This proves the lemma.

21.5.1 Two Competing Lemmas

Lemma 21.21 is a really nice lemma but there is something even better, Bloch's lemma. This lemma does not depend on the bound of f. Like the above two lemmas it is interesting for its own sake and in addition is the key to a fairly short proof of Picard's theorem. It features the number $\frac{1}{24}$. The best constant is not currently known.

Lemma 21.22 *Let f be analytic on an open set containing $\overline{B(0,R)}$ and suppose $|f'(0)| > 0$. Then there exists $a \in B(0,R)$ such that*

$$f(B(0,R)) \supseteq B\left(f(a), \frac{|f'(0)| R}{24}\right).$$

Proof: Let $K(\rho) \equiv \max\{|f'(z)| : |z| = \rho\}$. For simplicity, let $C_\rho \equiv \{z : |z| = \rho\}$.

Claim: K is continuous from the left.

Proof of claim: Let $z_\rho \in C_\rho$ such that $|f'(z_\rho)| = K(\rho)$. Then by the maximum modulus theorem, if $\lambda \in (0,1)$,

$$|f'(\lambda z_\rho)| \leq K(\lambda\rho) \leq K(\rho) = |f'(z_\rho)|.$$

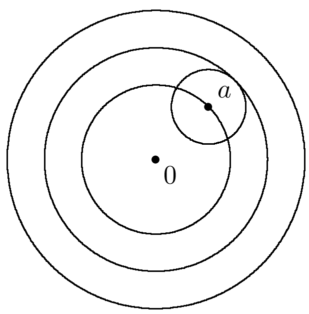
Letting $\lambda \to 1$ yields the claim.

Let ρ_0 be the largest such that $(R - \rho_0) K(\rho_0) = R|f'(0)|$. (Note $(R-0)K(0) = R|f'(0)|$.) Thus $\rho_0 < R$ because $(R-R)K(R) = 0$. Let $|a| = \rho_0$ such that $|f'(a)| = K(\rho_0)$. Thus

$$|f'(a)|(R - \rho_0) = |f'(0)| R \tag{21.17}$$

Now let $r = \frac{R-\rho_0}{2}$. From 21.17,

$$|f'(a)| r = \frac{1}{2}|f'(0)| R,\ B(a,r) \subseteq B(0, \rho_0 + r) \subseteq B(0,R). \tag{21.18}$$



Therefore, if $z \in B(a,r)$, it follows from the maximum modulus theorem and the definition of ρ_0 that

$$\begin{aligned} |f'(z)| &\leq K(\rho_0 + r) < \frac{R|f'(0)|}{R-\rho_0 - r} = \frac{2R|f'(0)|}{R-\rho_0} \\ &= \frac{2R|f'(0)|}{2r} = \frac{R|f'(0)|}{r} \end{aligned} \tag{21.19}$$

Let $g(z) = f(a+z) - f(a)$ where $z \in B(0,r)$. Then $|g'(0)| = |f'(a)| > 0$ and for $z \in \overline{B(0,r)}$,

$$|g(z)| \leq \left| \int_{\gamma(a,z)} g'(w)\, dw \right| \leq |z-a| \frac{R|f'(0)|}{r} = R|f'(0)|.$$

By Lemma 21.21 and 21.18,

$$\begin{aligned} g(B(0,r)) &\supseteq B\left(0, \frac{r^2 |f'(a)|^2}{6R|f'(0)|}\right) \\ &= B\left(0, \frac{r^2 \left(\frac{1}{2r}|f'(0)|R\right)^2}{6R|f'(0)|}\right) = B\left(0, \frac{|f'(0)|R}{24}\right) \end{aligned}$$

Now $g(B(0,r)) = f(B(a,r)) - f(a)$ and so this implies

$$f(B(0,R)) \supseteq f(B(a,r)) \supseteq B\left(f(a), \frac{|f'(0)|R}{24}\right).$$

This proves the lemma.

Here is a slightly more general version which allows the center of the open set to be arbitrary.

Lemma 21.23 *Let f be analytic on an open set containing $\overline{B(z_0,R)}$ and suppose $|f'(z_0)| > 0$. Then there exists $a \in B(z_0,R)$ such that*

$$f(B(z_0,R)) \supseteq B\left(f(a), \frac{|f'(z_0)|R}{24}\right).$$

Proof: You look at $g(z) \equiv f(z_0+z) - f(z_0)$ for $z \in B(0,R)$. Then $g'(0) = f'(z_0)$ and so by Lemma 21.22 there exists $a_1 \in B(0,R)$ such that

$$g(B(0,R)) \supseteq B\left(g(a_1), \frac{|f'(z_0)|R}{24}\right).$$

Now $g(B(0,R)) = f(B(z_0,R)) - f(z_0)$ and $g(a_1) = f(a) - f(z_0)$ for some $a \in B(z_0,R)$ and so

$$\begin{aligned} f(B(z_0,R)) - f(z_0) &\supseteq B\left(g(a_1), \frac{|f'(z_0)|R}{24}\right) \\ &= B\left(f(a) - f(z_0), \frac{|f'(z_0)|R}{24}\right) \end{aligned}$$

which implies
$$f\left(B\left(z_0,R\right)\right)\supseteq B\left(f\left(a\right),\frac{\left|f'\left(z_0\right)\right|R}{24}\right)$$
as claimed. This proves the lemma.

No attempt was made to find the best number to multiply by $R\left|f'\left(z_0\right)\right|$. A discussion of this is given in Conway [11]. See also [23]. Much larger numbers than 1/24 are available and there is a conjecture due to Alfors about the best value. The conjecture is that 1/24 can be replaced with
$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\left(1+\sqrt{3}\right)^{1/2}\Gamma\left(\frac{1}{4}\right)}\approx .471\,86$$
You can see there is quite a gap between the constant for which this lemma is proved above and what is thought to be the best constant.

Bloch's lemma above gives the existence of a ball of a certain size inside the image of a ball. By contrast the next lemma leads to conditions under which the values of a function do not contain a ball of certain radius. It concerns analytic functions which do not achieve the values 0 and 1.

Lemma 21.24 *Let $\mathcal{F}$ denote the set of functions, f defined on Ω, a simply connected region which do not achieve the values 0 and 1. Then for each such function, it is possible to define a function analytic on Ω, $H(z)$ by the formula*
$$H\left(z\right)\equiv\log\left[\sqrt{\frac{\log\left(f\left(z\right)\right)}{2\pi i}}-\sqrt{\frac{\log\left(f\left(z\right)\right)}{2\pi i}-1}\right].$$
There exists a constant C independent of $f\in\mathcal{F}$ such that $H\left(\Omega\right)$ does not contain any ball of radius C.

Proof: Let $f\in\mathcal{F}$. Then since f does not take the value 0, there exists g_1 a primitive of f'/f. Thus
$$\frac{d}{dz}\left(e^{-g_1}f\right)=0$$
so there exists a,b such that $f\left(z\right)e^{-g_1(z)}=e^{a+bi}$. Letting $g\left(z\right)=g_1\left(z\right)+a+ib$, it follows $e^{g(z)}=f\left(z\right)$. Let $\log\left(f\left(z\right)\right)=g\left(z\right)$. Then for $n\in\mathbb{Z}$, the integers,
$$\frac{\log\left(f\left(z\right)\right)}{2\pi i},\frac{\log\left(f\left(z\right)\right)}{2\pi i}-1\neq n$$
because if equality held, then $f\left(z\right)=1$ which does not happen. It follows $\frac{\log(f(z))}{2\pi i}$ and $\frac{\log(f(z))}{2\pi i}-1$ are never equal to zero. Therefore, using the same reasoning, you can define a logarithm of these two quantities and therefore, a square root. Hence there exists a function analytic on Ω,
$$\sqrt{\frac{\log\left(f\left(z\right)\right)}{2\pi i}}-\sqrt{\frac{\log\left(f\left(z\right)\right)}{2\pi i}-1}. \tag{21.20}$$

For n a positive integer, this function cannot equal $\sqrt{n} \pm \sqrt{n-1}$ because if it did, then

$$\left(\sqrt{\frac{\log (f(z))}{2\pi i}} - \sqrt{\frac{\log (f(z))}{2\pi i} - 1}\right) = \sqrt{n} \pm \sqrt{n-1} \tag{21.21}$$

and you could take reciprocals of both sides to obtain

$$\left(\sqrt{\frac{\log (f(z))}{2\pi i}} + \sqrt{\frac{\log (f(z))}{2\pi i} - 1}\right) = \sqrt{n} \mp \sqrt{n-1}. \tag{21.22}$$

Then adding 21.21 and 21.22

$$2\sqrt{\frac{\log (f(z))}{2\pi i}} = 2\sqrt{n}$$

which contradicts the above observation that $\frac{\log(f(z))}{2\pi i}$ is not equal to an integer.

Also, the function of 21.20 is never equal to zero. Therefore, you can define the logarithm of this function also. It follows

$$H(z) \equiv \log\left(\sqrt{\frac{\log (f(z))}{2\pi i}} - \sqrt{\frac{\log (f(z))}{2\pi i} - 1}\right) \neq \ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$$

where m is an arbitrary integer and n is a positive integer. Now

$$\lim_{n\to\infty} \ln\left(\sqrt{n} + \sqrt{n-1}\right) = \infty$$

and $\lim_{n\to\infty} \ln\left(\sqrt{n} - \sqrt{n-1}\right) = -\infty$ and so $\mathbb{C}$ is covered by rectangles having vertices at points $\ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ as described above. Each of these rectangles has height equal to 2π and a short computation shows their widths are bounded. Therefore, there exists C independent of $f \in \mathcal{F}$ such that C is larger than the diameter of all these rectangles. Hence $H(\Omega)$ cannot contain any ball of radius larger than C.

21.5.2 The Little Picard Theorem

Now here is the little Picard theorem. It is easy to prove from the above.

Theorem 21.25 *If h is an entire function which omits two values then h is a constant.*

Proof: Suppose the two values omitted are a and b and that h is not constant. Let $f(z) = (h(z) - a)/(b - a)$. Then f omits the two values 0 and 1. Let H be defined in Lemma 21.24. Then $H(z)$ is clearly not of the form $az+b$ because then it would have values equal to the vertices $\ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ or else be constant neither of which happen if h is not constant. Therefore, by Liouville's theorem, H' must be unbounded. Pick ξ such that $|H'(\xi)| > 24C$ where C is such that $H(\mathbb{C})$

contains no balls of radius larger than C. But by Lemma 21.23 $H(B(\xi,1))$ must contain a ball of radius $\frac{|H'(\xi)|}{24} > \frac{24C}{24} = C$, a contradiction. This proves Picard's theorem.

The following is another formulation of this theorem.

Corollary 21.26 *If f is a meromophic function defined on $\mathbb{C}$ which omits three distinct values, a, b, c, then f is a constant.*

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c}\frac{b-c}{b-a}$. Then $\phi(c) = \infty, \phi(a) = 0$, and $\phi(b) = 1$. Now consider the function, $h = \phi \circ f$. Then h misses the three points $\infty, 0$, and 1. Since h is meromorphic and does not have ∞ in its values, it must actually be analytic. Thus h is an entire function which misses the two values 0 and 1. Therefore, h is constant by Theorem 21.25.

21.5.3 Schottky's Theorem

Lemma 21.27 *Let f be analytic on an open set containing $\overline{B(0,R)}$ and suppose that f does not take on either of the two values 0 or 1. Also suppose $|f(0)| \leq \beta$. Then letting $\theta \in (0,1)$, it follows*

$$|f(z)| \leq M(\beta,\theta)$$

for all $z \in B(0,\theta R)$, where $M(\beta,\theta)$ is a function of only the two variables β,θ. (In particular, there is no dependence on R.)

Proof: Consider the function, $H(z)$ used in Lemma 21.24 given by

$$H(z) \equiv \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right). \tag{21.23}$$

You notice there are two explicit uses of logarithms. Consider first the logarithm inside the radicals. Choose this logarithm such that

$$\log(f(0)) = \ln|f(0)| + i\arg(f(0)),\ \arg(f(0)) \in (-\pi,\pi]. \tag{21.24}$$

You can do this because

$$e^{\log(f(0))} = f(0) = e^{\ln|f(0)|}e^{i\alpha} = e^{\ln|f(0)|+i\alpha}$$

and by replacing α with $\alpha + 2m\pi$ for a suitable integer, m it follows the above equation still holds. Therefore, you can assume 21.24. Similar reasoning applies to the logarithm on the outside of the parenthesis. It can be assumed $H(0)$ equals

$$\ln\left|\sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1}\right| + i\arg\left(\sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1}\right) \tag{21.25}$$

where the imaginary part is no larger than π in absolute value.

Now if $\xi \in B(0,R)$ is a point where $H'(\xi) \neq 0$, then by Lemma 21.22

$$H(B(\xi, R-|\xi|)) \supseteq B\left(H(a), \frac{|H'(\xi)|(R-|\xi|)}{24}\right)$$

where a is some point in $B(\xi, R-|\xi|)$. But by Lemma 21.24 $H(B(\xi, R-|\xi|))$ contains no balls of radius C where C depended only on the maximum diameters of those rectangles having vertices $\ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ for n a positive integer and m an integer. Therefore,

$$\frac{|H'(\xi)|(R-|\xi|)}{24} < C$$

and consequently

$$|H'(\xi)| < \frac{24C}{R-|\xi|}.$$

Even if $H'(\xi) = 0$, this inequality still holds. Therefore, if $z \in B(0,R)$ and $\gamma(0,z)$ is the straight segment from 0 to z,

$$\begin{aligned} |H(z) - H(0)| &= \left|\int_{\gamma(0,z)} H'(w)\,dw\right| = \left|\int_0^1 H'(tz)\,zdt\right| \\ &\leq \int_0^1 |H'(tz)\,z|\,dt \leq \int_0^1 \frac{24C}{R-t|z|}\,|z|\,dt \\ &= 24C\ln\left(\frac{R}{R-|z|}\right). \end{aligned}$$

Therefore, for $z \in \partial B(0,\theta R)$,

$$|H(z)| \leq |H(0)| + 24C\ln\left(\frac{1}{1-\theta}\right). \tag{21.26}$$

By the maximum modulus theorem, the above inequality holds for all $|z| < \theta R$ also.

Next I will use 21.23 to get an inequality for $|f(z)|$ in terms of $|H(z)|$. From 21.23,

$$H(z) = \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)$$

and so

$$\begin{aligned} 2H(z) &= \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)^2 \\ -2H(z) &= \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)^{-2} \\ &= \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\sqrt{\frac{\log(f(z))}{2\pi i}}+\sqrt{\frac{\log(f(z))}{2\pi i}-1}\right)^2 \\
&+\left(\sqrt{\frac{\log(f(z))}{2\pi i}}-\sqrt{\frac{\log(f(z))}{2\pi i}-1}\right)^2 \\
=\;& \exp(2H(z))+\exp(-2H(z))
\end{aligned}$$

and

$$\left(\frac{\log(f(z))}{\pi i}-1\right)=\frac{1}{2}\left(\exp(2H(z))+\exp(-2H(z))\right).$$

Thus

$$\log(f(z))=\pi i+\frac{\pi i}{2}\left(\exp(2H(z))+\exp(-2H(z))\right)$$

which shows

$$\begin{aligned}
|f(z)| &= \left|\exp\left[\frac{\pi i}{2}\left(\exp(2H(z))+\exp(-2H(z))\right)\right]\right| \\
&\leq \exp\left|\frac{\pi i}{2}\left(\exp(2H(z))+\exp(-2H(z))\right)\right| \\
&\leq \exp\left|\frac{\pi}{2}\left(|\exp(2H(z))|+|\exp(-2H(z))|\right)\right| \\
&\leq \exp\left|\frac{\pi}{2}\left(\exp(2|H(z)|)+\exp(|-2H(z)|)\right)\right| \\
&= \exp(\pi\exp 2|H(z)|).
\end{aligned}$$

Now from 21.26 this is dominated by

$$\begin{aligned}
&\exp\left(\pi\exp 2\left(|H(0)|+24C\ln\left(\frac{1}{1-\theta}\right)\right)\right) \\
=\;&\exp\left(\pi\exp(2|H(0)|)\exp\left(48C\ln\left(\frac{1}{1-\theta}\right)\right)\right) \qquad (21.27)
\end{aligned}$$

Consider $\exp(2|H(0)|)$. I want to obtain an inequality for this which involves β. This is where I will use the convention about the logarithms discussed above. From 21.25,

$$2|H(0)|=2\left|\log\left(\sqrt{\frac{\log(f(0))}{2\pi i}}-\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right)\right|$$

$$\begin{aligned}
&\leq 2\left(\left(\ln\left|\sqrt{\frac{\log(f(0))}{2\pi i}}-\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right|\right)^2+\pi^2\right)^{1/2} \\
&\leq 2\left(\left|\ln\left(\left|\sqrt{\frac{\log(f(0))}{2\pi i}}\right|+\left|\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right|\right)\right|^2+\pi^2\right)^{1/2} \\
&\leq 2\left|\ln\left(\left|\sqrt{\frac{\log(f(0))}{2\pi i}}\right|+\left|\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right|\right)\right|+2\pi \\
&\leq \ln\left(2\left(\left|\frac{\log(f(0))}{2\pi i}\right|+\left|\frac{\log(f(0))}{2\pi i}-1\right|\right)\right)+2\pi \\
&= \ln\left(\left(\left|\frac{\log(f(0))}{\pi i}\right|+\left|\frac{\log(f(0))}{\pi i}-2\right|\right)\right)+2\pi \qquad (21.28)
\end{aligned}$$

Consider $\left|\frac{\log(f(0))}{\pi i}\right|$

$$\frac{\log(f(0))}{\pi i}=-\frac{\ln|f(0)|}{\pi}i+\frac{\arg(f(0))}{\pi}$$

and so

$$\begin{aligned}
\left|\frac{\log(f(0))}{\pi i}\right| &= \left(\left|\frac{\ln|f(0)|}{\pi}\right|^2+\left(\frac{\arg(f(0))}{\pi}\right)^2\right)^{1/2} \\
&\leq \left(\left|\frac{\ln\beta}{\pi}\right|^2+\left(\frac{\pi}{\pi}\right)^2\right)^{1/2} \\
&= \left(\left|\frac{\ln\beta}{\pi}\right|^2+1\right)^{1/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left|\frac{\log(f(0))}{\pi i}-2\right| &\leq \left(\left|\frac{\ln\beta}{\pi}\right|^2+(2+1)^2\right)^{1/2} \\
&= \left(\left|\frac{\ln\beta}{\pi}\right|^2+9\right)^{1/2}
\end{aligned}$$

It follows from 21.28 that

$$2|H(0)|\leq\ln\left(2\left(\left|\frac{\ln\beta}{\pi}\right|^2+9\right)^{1/2}\right)+2\pi.$$

Hence from 21.27

$$|f(z)|\leq$$

$$\exp\left(\pi\exp\left(\ln\left(2\left(\left|\frac{\ln\beta}{\pi}\right|^2+9\right)^{1/2}\right)+2\pi\right)\exp\left(48C\ln\left(\frac{1}{1-\theta}\right)\right)\right)$$

and so, letting $M(\beta,\theta)$ be given by the above expression on the right, the lemma is proved.

The following theorem will be referred to as Schottky's theorem. It looks just like the above lemma except it is only assumed that f is analytic on $B(0,R)$ rather than on an open set containing $\overline{B(0,R)}$. Also, the case of an arbitrary center is included along with arbitrary points which are not attained as values of the function.

Theorem 21.28 *Let f be analytic on $B(z_0,R)$ and suppose that f does not take on either of the two distinct values a or b. Also suppose $|f(z_0)|\leq\beta$. Then letting $\theta\in(0,1)$, it follows*

$$|f(z)|\leq M(a,b,\beta,\theta)$$

for all $z\in B(z_0,\theta R)$, where $M(a,b,\beta,\theta)$ is a function of only the variables β,θ,a,b. (In particular, there is no dependence on R.)

Proof: First you can reduce to the case where the two values are 0 and 1 by considering

$$h(z)\equiv\frac{f(z)-a}{b-a}.$$

If there exists an estimate of the desired sort for h, then there exists such an estimate for f. Of course here the function, M would depend on a and b. Therefore, there is no loss of generality in assuming the points which are missed are 0 and 1.

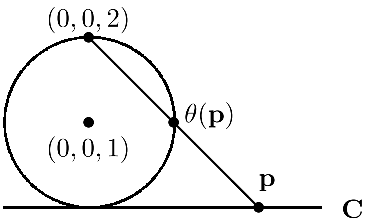
Apply Lemma 21.27 to $B(0,R_1)$ for the function, $g(z)\equiv f(z_0+z)$ and $R_1<R$. Then if $\beta\geq|f(z_0)|=|g(0)|$, it follows $|g(z)|=|f(z_0+z)|\leq M(\beta,\theta)$ for every $z\in B(0,\theta R_1)$. Now let $\theta\in(0,1)$ and choose $R_1<R$ large enough that $\theta R=\theta_1R_1$ where $\theta_1\in(0,1)$. Then if $|z-z_0|<\theta R$, it follows

$$|f(z)|\leq M(\beta,\theta_1).$$

Now let $R_1\to R$ so $\theta_1\to\theta$.

21.5.4 A Brief Review

First recall the definition of the metric on $\widehat{\mathbb{C}}$. For convenience it is listed here again. Consider the unit sphere, S^2 given by $(z-1)^2+y^2+x^2=1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, p in the complex plane to the point $(0,0,2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty)\equiv(0,0,2)$.



Then θ^{-1} is sometimes called sterographic projection. The mapping θ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, z_n converges to ∞ if and only if θz_n converges to $(0,0,2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta(z_n) \to \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 21.29 *Let $z, w \in \widehat{\mathbb{C}}$. Then let $d(x,y) \equiv |\theta(z) - \theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^3$.*

Theorem 21.30 $\left(\widehat{\mathbb{C}}, d\right)$ *is a compact, hence complete metric space.*

Proof: Suppose $\{z_n\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\{\theta(z_n)\}$ is a sequence in S^2 which is compact. Therefore, there exists a subsequence, $\{\theta z_{n_k}\}$ and a point, $z \in S^2$ such that $\theta z_{n_k} \to \theta z$ in S^2 which implies immediately that $d(z_{n_k}, z) \to 0$. A compact metric space must be complete.

Also recall the interesting fact that meromorphic functions are continuous with values in $\widehat{\mathbb{C}}$ which is reviewed here for convenience. It came from the theory of classification of isolated singularities.

Theorem 21.31 *Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \widehat{\mathbb{C}}$ be meromorphic. Then f is continuous with respect to the metric, d on $\widehat{\mathbb{C}}$.*

Proof: Let $z_n \to z$ where $z \in \Omega$. Then if z is a pole, it follows from Theorem 18.38 that

$$d(f(z_n), \infty) \equiv d(f(z_n), f(z)) \to 0.$$

If z is not a pole, then $f(z_n) \to f(z)$ in $\mathbb{C}$ which implies $|\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \to 0$. Recall that θ is continuous on $\mathbb{C}$.

The fundamental result behind all the theory about to be presented is the Ascoli Arzela theorem also listed here for convenience.

Definition 21.32 *Let (X,d) be a complete metric space. Then it is said to be locally compact if $\overline{B(x,r)}$ is compact for each $r > 0$.*

Thus if you have a locally compact metric space, then if $\{a_n\}$ is a bounded sequence, it must have a convergent subsequence.

Let K be a compact subset of $\mathbb{R}^n$ and consider the continuous functions which have values in a locally compact metric space, (X,d) where d denotes the metric on X. Denote this space as $C(K,X)$.

Definition 21.33 *For $f, g \in C(K, X)$, where K is a compact subset of $\mathbb{R}^n$ and X is a locally compact complete metric space define*

$$\rho_K(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})) : \mathbf{x} \in K\}.$$

The Ascoli Arzela theorem, Theorem 5.22 is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 21.34 *Let $A \subseteq C(K, X)$ for K a compact subset of $\mathbb{R}^n$. Then A is said to be uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x} - \mathbf{y}| < \delta$ and $f \in A$,*

$$d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon.$$

The set, A is said to be uniformly bounded if for some $M < \infty$, and $a \in X$,

$$f(\mathbf{x}) \in B(a, M)$$

for all $f \in A$ and $\mathbf{x} \in K$.

The Ascoli Arzela theorem follows.

Theorem 21.35 *Suppose K is a nonempty compact subset of $\mathbb{R}^n$ and $A \subseteq C(K, X)$, is uniformly bounded and uniformly equicontinuous where X is a locally compact complete metric space. Then if $\{f_k\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, f_{k_l} such that*

$$\lim_{l \to \infty} \rho_K(f_{k_l}, f) = 0.$$

In the cases of interest here, $X = \widehat{\mathbb{C}}$ with the metric defined above.

21.5.5 Montel's Theorem

The following lemma is another version of Montel's theorem. It is this which will make possible a proof of the big Picard theorem.

Lemma 21.36 *Let Ω be a region and let $\mathcal{F}$ be a set of functions analytic on Ω none of which achieve the two distinct values, a and b. If $\{f_n\} \subseteq \mathcal{F}$ then one of the following hold: Either there exists a function, f analytic on Ω and a subsequence, $\{f_{n_k}\}$ such that for any compact subset, K of Ω,*

$$\lim_{k \to \infty} ||f_{n_k} - f||_{K,\infty} = 0. \tag{21.29}$$

or there exists a subsequence $\{f_{n_k}\}$ such that for all compact subsets K,

$$\lim_{k \to \infty} \rho_K(f_{n_k}, \infty) = 0. \tag{21.30}$$

Proof: Let $B(z_0, 2R) \subseteq \Omega$. There are two cases to consider. The first case is that there exists a subsequence, n_k such that $\{f_{n_k}(z_0)\}$ is bounded. The second case is that $\lim_{n\to\infty} |f_{n_k}(z_0)| = \infty$.

Consider the first case. By Theorem 21.28 $\{f_{n_k}(z)\}$ is uniformly bounded on $\overline{B(z_0, R)}$ because by this theorem, and letting $\theta = 1/2$ applied to $B(z_0, 2R)$, it follows $|f_{n_k}(z)| \leq M\left(a, b, \frac{1}{2}, \beta\right)$ where β is an upper bound to the numbers, $|f_{n_k}(z_0)|$. The Cauchy integral formula implies the existence of a uniform bound on the $\left\{f'_{n_k}\right\}$ which implies the functions are equicontinuous and uniformly bounded. Therefore, by the Ascoli Arzela theorem there exists a further subsequence which converges uniformly on $\overline{B(z_0, R)}$ to a function, f analytic on $B(z_0, R)$. Thus denoting this subsequence by $\{f_{n_k}\}$ to save on notation,

$$\lim_{k\to\infty} ||f_{n_k} - f||_{\overline{B(z_0,R)},\infty} = 0. \tag{21.31}$$

Consider the second case. In this case, it follows $\{1/f_n(z_0)\}$ is bounded on $\overline{B(z_0, R)}$ and so by the same argument just given $\{1/f_n(z)\}$ is uniformly bounded on $\overline{B(z_0, R)}$. Therefore, a subsequence converges uniformly on $\overline{B(z_0, R)}$. But $\{1/f_n(z)\}$ converges to 0 and so this requires that $\{1/f_n(z)\}$ must converge uniformly to 0. Therefore,

$$\lim_{k\to\infty} \rho_{\overline{B(z_0,R)}}(f_{n_k}, \infty) = 0. \tag{21.32}$$

Now let $\{D_k\}$ denote a countable set of closed balls, $D_k = \overline{B(z_k, R_k)}$ such that $B(z_k, 2R_k) \subseteq \Omega$ and $\cup_{k=1}^{\infty} \operatorname{int}(D_k) = \Omega$. Using a Cantor diagonal process, there exists a subsequence, $\{f_{n_k}\}$ of $\{f_n\}$ such that for each D_j, one of the above two alternatives holds. That is, either

$$\lim_{k\to\infty} ||f_{n_k} - g_j||_{D_j,\infty} = 0 \tag{21.33}$$

or,

$$\lim_{k\to\infty} \rho_{D_j}(f_{n_k}, \infty). \tag{21.34}$$

Let $A = \{\cup \operatorname{int}(D_j) : 21.33 \text{ holds}\}$, $B = \{\cup \operatorname{int}(D_j) : 21.34 \text{ holds}\}$. Note that the balls whose union is A cannot intersect any of the balls whose union is B. Therefore, one of A or B must be empty since otherwise, Ω would not be connected.

If K is any compact subset of Ω, it follows K must be a subset of some finite collection of the D_j. Therefore, one of the alternatives in the lemma must hold. That the limit function, f must be analytic follows easily in the same way as the proof in Theorem 21.7 on Page 484. You could also use Morera's theorem. This proves the lemma.

21.5.6 The Great Big Picard Theorem

The next theorem is the main result which the above lemmas lead to. It is the Big Picard theorem, also called the Great Picard theorem.Recall $B'(a, r)$ is the deleted ball consisting of all the points of the ball except the center.

Theorem 21.37 *Suppose f has an isolated essential singularity at 0. Then for every $R > 0$, and $\beta \in \mathbb{C}$, $f^{-1}(\beta) \cap B'(0, R)$ is an infinite set except for one possible exceptional β.*

Proof: Suppose this is not true. Then there exists $R_1 > 0$ and two points, α and β such that $f^{-1}(\beta) \cap B'(0, R_1)$ and $f^{-1}(\alpha) \cap B'(0, R_1)$ are both finite sets. Then shrinking R_1 and calling the result R, there exists $B(0, R)$ such that

$$f^{-1}(\beta) \cap B'(0, R) = \emptyset, \ f^{-1}(\alpha) \cap B'(0, R) = \emptyset.$$

Now let A_0 denote the annulus $\left\{z \in \mathbb{C} : \frac{R}{2^2} < |z| < \frac{3R}{2^2}\right\}$ and let A_n denote the annulus $\left\{z \in \mathbb{C} : \frac{R}{2^{2+n}} < |z| < \frac{3R}{2^{2+n}}\right\}$. The reason for the 3 is to insure that $A_n \cap A_{n+1} \neq \emptyset$. This follows from the observation that $3R/2^{2+1+n} > R/2^{2+n}$. Now define a set of functions on A_0 as follows:

$$f_n(z) \equiv f\left(\frac{z}{2^n}\right).$$

By the choice of R, this set of functions missed the two points α and β. Therefore, by Lemma 21.36 there exists a subsequence such that one of the two options presented there holds.

First suppose $\lim_{k\to\infty} ||f_{n_k} - f||_{K,\infty} = 0$ for all K a compact subset of A_0 and f is analytic on A_0. In particular, this happens for γ_0 the circular contour having radius $R/2$. Thus f_{n_k} must be bounded on this contour. But this says the same thing as $f(z/2^{n_k})$ is bounded for $|z| = R/2$, this holding for each $k = 1, 2, \cdots$. Thus there exists a constant, M such that on each of a shrinking sequence of concentric circles whose radii converge to 0, $|f(z)| \leq M$. By the maximum modulus theorem, $|f(z)| \leq M$ at every point between successive circles in this sequence. Therefore, $|f(z)| \leq M$ in $B'(0, R)$ contradicting the Weierstrass Casorati theorem.

The other option which might hold from Lemma 21.36 is that $\lim_{k\to\infty} \rho_K(f_{n_k}, \infty) = 0$ for all K compact subset of A_0. Since f has an essential singularity at 0 the zeros of f in $B(0, R)$ are isolated. Therefore, for all k large enough, f_{n_k} has no zeros for $|z| < 3R/2^2$. This is because the values of f_{n_k} are the values of f on A_{n_k}, a small anulus which avoids all the zeros of f whenever k is large enough. Only consider k this large. Then use the above argument on the analytic functions $1/f_{n_k}$. By the assumption that $\lim_{k\to\infty} \rho_K(f_{n_k}, \infty) = 0$, it follows $\lim_{k\to\infty} ||1/f_{n_k} - 0||_{K,\infty} = 0$ and so as above, there exists a shrinking sequence of concentric circles whose radii converge to 0 and a constant, M such that for z on any of these circles, $|1/f(z)| \leq M$. This implies that on some deleted ball, $B'(0, r)$ where $r \leq R$, $|f(z)| \geq 1/M$ which again violates the Weierstrass Casorati theorem. This proves the theorem.

As a simple corollary, here is what this remarkable theorem says about entire functions.

Corollary 21.38 *Suppose f is entire and nonconstant and not a polynomial. Then f assumes every complex value infinitely many times with the possible exception of one.*

Proof: Since f is entire, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Define for $z \neq 0$,

$$g(z) \equiv f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n.$$

Thus 0 is an isolated essential singular point of g. By the big Picard theorem, Theorem 21.37 it follows g takes every complex number but possibly one an infinite number of times. This proves the corollary.

Note the difference between this and the little Picard theorem which says that an entire function which is not constant must achieve every value but two.

21.6 Exercises

1. Prove that in Theorem 21.7 it suffices to assume $\mathcal{F}$ is uniformly bounded on each compact subset of Ω.

2. Find conditions on a, b, c, d such that the fractional linear transformation, $\frac{az+b}{cz+d}$ maps the upper half plane onto the upper half plane.

3. Let D be a simply connected region which is a proper subset of $\mathbb{C}$. Does there exist an entire function, f which maps $\mathbb{C}$ onto D? Why or why not?

4. Verify the conclusion of Theorem 21.7 involving the higher order derivatives.

5. What if $\Omega = \mathbb{C}$? Does there exist an analytic function, f mapping Ω one to one and onto $B(0,1)$? Explain why or why not. Was $\Omega \neq \mathbb{C}$ used in the proof of the Riemann mapping theorem?

6. Verify that $|\phi_\alpha(z)| = 1$ if $|z| = 1$. Apply the maximum modulus theorem to conclude that $|\phi_\alpha(z)| \leq 1$ for all $|z| < 1$.

7. Suppose that $|f(z)| \leq 1$ for $|z| = 1$ and $f(\alpha) = 0$ for $|\alpha| < 1$. Show that $|f(z)| \leq |\phi_\alpha(z)|$ for all $z \in B(0,1)$. **Hint:** Consider $\frac{f(z)(1-\overline{\alpha}z)}{z-\alpha}$ which has a removable singularity at α. Show the modulus of this function is bounded by 1 on $|z| = 1$. Then apply the maximum modulus theorem.

8. Let U and V be open subsets of $\mathbb{C}$ and suppose $u : U \to \mathbb{R}$ is harmonic while h is an analytic map which takes V one to one onto U. Show that $u \circ h$ is harmonic on V.

9. Show that for a harmonic function, u defined on $B(0,R)$, there exists an analytic function, $h = u + iv$ where

$$v(x,y) \equiv \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt.$$

10. Suppose Ω is a simply connected region and u is a real valued function defined on Ω such that u is harmonic. Show there exists an analytic function, f such that $u = \operatorname{Re} f$. Show this is not true if Ω is not a simply connected region. **Hint:** You might use the Riemann mapping theorem and Problems 8 and 9. For the second part it might be good to try something like $u(x, y) = \ln\left(x^2 + y^2\right)$ on the annulus $1 < |z| < 2$.

11. Show that $w = \frac{1+z}{1-z}$ maps $\{z \in \mathbb{C} : \operatorname{Im} z > 0 \text{ and } |z| < 1\}$ to the first quadrant, $\{z = x + iy : x, y > 0\}$.

12. Let $f(z) = \frac{az+b}{cz+d}$ and let $g(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$. Show that $f \circ g(z)$ equals the quotient of two expressions, the numerator being the top entry in the vector
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$
and the denominator being the bottom entry. Show that if you define
$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \equiv \frac{az+b}{cz+d},$$
then $\phi(AB) = \phi(A) \circ \phi(B)$. Find an easy way to find the inverse of $f(z) = \frac{az+b}{cz+d}$ and give a condition on the a, b, c, d which insures this function has an inverse.

13. The modular group[2] is the set of fractional linear transformations, $\frac{az+b}{cz+d}$ such that a, b, c, d are integers and $ad - bc = 1$. Using Problem 12 or brute force show this modular group is really a group with the group operation being composition. Also show the inverse of $\frac{az+b}{cz+d}$ is $\frac{dz-b}{-cz+a}$.

14. Let Ω be a region and suppose f is analytic on Ω and that the functions f_n are also analytic on Ω and converge to f uniformly on compact subsets of Ω. Suppose f is one to one. Can it be concluded that for an arbitrary compact set, $K \subseteq \Omega$ that f_n is one to one for all n large enough?

15. The Vitali theorem says that if Ω is a region and $\{f_n\}$ is a uniformly bounded sequence of functions which converges pointwise on a set, $S \subseteq \Omega$ which has a limit point in Ω, then in fact, $\{f_n\}$ must converge uniformly on compact subsets of Ω to an analytic function. Prove this theorem. **Hint:** If the sequence fails to converge, show you can get two different subsequences converging uniformly on compact sets to different functions. Then argue these two functions coincide on S.

16. Does there exist a function analytic on $B(0,1)$ which maps $B(0,1)$ onto $B'(0,1)$, the open unit ball in which 0 has been deleted?

[2]This is the terminology used in Rudin's book Real and Complex Analysis.

Approximation By Rational Functions

22.1 Runge's Theorem

Consider the function, $\frac{1}{z} = f(z)$ for z defined on $\Omega \equiv B(0,1) \setminus \{0\} = B'(0,1)$. Clearly f is analytic on Ω. Suppose you could approximate f uniformly by polynomials on $\overline{\text{ann}\left(0, \frac{1}{2}, \frac{3}{4}\right)}$, a compact subset of Ω. Then, there would exist a suitable polynomial $p(z)$, such that $\left|\frac{1}{2\pi i}\int_\gamma f(z) - p(z)\,dz\right| < \frac{1}{10}$ where here γ is a circle of radius $\frac{2}{3}$. However, this is impossible because $\frac{1}{2\pi i}\int_\gamma f(z)\,dz = 1$ while $\frac{1}{2\pi i}\int_\gamma p(z)\,dz = 0$. This shows you can't expect to be able to uniformly approximate analytic functions on compact sets using polynomials. This is just horrible! In real variables, you can approximate any **continuous function** on a compact set with a polynomial. However, that is just the way it is. It turns out that the ability to approximate an analytic function on Ω with polynomials is dependent on Ω being simply connected.

All these theorems work for f having values in a complex Banach space. However, I will present them in the context of functions which have values in $\mathbb{C}$. The changes necessary to obtain the extra generality are very minor.

Definition 22.1 *Approximation will be taken with respect to the following norm.*

$$||f - g||_{K,\infty} \equiv \sup\{||f(z) - g(z)|| : z \in K\}$$

22.1.1 Approximation With Rational Functions

It turns out you can approximate analytic functions by rational functions, quotients of polynomials. The resulting theorem is one of the most profound theorems in complex analysis. The basic idea is simple. The Riemann sums for the Cauchy integral formula are rational functions. The idea used to implement this observation is that if you have a compact subset, K of an open set, Ω there exists a cycle composed of closed oriented curves $\{\gamma_j\}_{j=1}^n$ which are contained in $\Omega \setminus K$ such that for every $z \in K, \sum_{k=1}^{n} n(\gamma_k, z) = 1$. One more ingredient is needed and this is a theorem which lets you keep the approximation but move the poles.

To begin with, consider the part about the cycle of closed oriented curves. Recall Theorem 18.52 which is stated for convenience.

Theorem 22.2 *Let K be a compact subset of an open set, Ω. Then there exist continuous, closed, bounded variation oriented curves $\left\{\gamma_j\right\}_{j=1}^{m}$ for which $\gamma_j^* \cap K = \emptyset$ for each j, $\gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^{m} n(p, \gamma_k) = 1.$$

and

$$\sum_{k=1}^{m} n(z, \gamma_k) = 0$$

for all $z \notin \Omega$.

This theorem implies the following.

Theorem 22.3 *Let $K \subseteq \Omega$ where K is compact and Ω is open. Then there exist oriented closed curves, γ_k such that $\gamma_k^* \cap K = \emptyset$ but $\gamma_k^* \subseteq \Omega$, such that for all $z \in K$,*

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{p} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{22.1}$$

Proof: This follows from Theorem 18.52 and the Cauchy integral formula. As shown in the proof, you can assume the γ_k are linear mappings but this is not important.

Next I will show how the Cauchy integral formula leads to approximation by rational functions, quotients of polynomials.

Lemma 22.4 *Let K be a compact subset of an open set, Ω and let f be analytic on Ω. Then there exists a rational function, Q whose poles are not in K such that*

$$||Q - f||_{K,\infty} < \varepsilon.$$

Proof: By Theorem 22.3 there are oriented curves, γ_k described there such that for all $z \in K$,

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{p} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{22.2}$$

Defining $g(w, z) \equiv \frac{f(w)}{w-z}$ for $(w, z) \in \cup_{k=1}^{p} \gamma_k^* \times K$, it follows since the distance between K and $\cup_k \gamma_k^*$ is positive that g is uniformly continuous and so there exists a $\delta > 0$ such that if $||\mathcal{P}|| < \delta$, then for all $z \in K$,

$$\left| f(z) - \frac{1}{2\pi i} \sum_{k=1}^{p} \sum_{j=1}^{n} \frac{f(\gamma_k(\tau_j))(\gamma_k(t_i) - \gamma_k(t_{i-1}))}{\gamma_k(\tau_j) - z} \right| < \frac{\varepsilon}{2}.$$

The complicated expression is obtained by replacing each integral in 22.2 with a Riemann sum. Simplifying the appearance of this, it follows there exists a rational function of the form

$$R(z)=\sum_{k=1}^{M}\frac{A_k}{w_k-z}$$

where the w_k are elements of components of $\mathbb{C}\setminus K$ and A_k are complex numbers or in the case where f has values in X, these would be elements of X such that

$$||R-f||_{K,\infty}<\frac{\varepsilon}{2}.$$

This proves the lemma.

22.1.2 Moving The Poles And Keeping The Approximation

Lemma 22.4 is a nice lemma but needs refining. In this lemma, the Riemann sum handed you the poles. It is much better if you can pick the poles. The following theorem from advanced calculus, called Merten's theorem, will be used

22.1.3 Merten's Theorem.

Theorem 22.5 *Suppose $\sum_{i=r}^{\infty} a_i$ and $\sum_{j=r}^{\infty} b_j$ both converge absolutely*[1]*. Then*

$$\left(\sum_{i=r}^{\infty}a_i\right)\left(\sum_{j=r}^{\infty}b_j\right)=\sum_{n=r}^{\infty}c_n$$

where

$$c_n=\sum_{k=r}^{n}a_kb_{n-k+r}.$$

Proof: Let $p_{nk}=1$ if $r\le k\le n$ and $p_{nk}=0$ if $k>n$. Then

$$c_n=\sum_{k=r}^{\infty}p_{nk}a_kb_{n-k+r}.$$

[1] Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten's theorem and may be read in the 1974 book by Apostol listed in the bibliography.

Also,

$$\begin{aligned}
\sum_{k=r}^{\infty}\sum_{n=r}^{\infty}p_{nk}\left|a_k\right|\left|b_{n-k+r}\right| &= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{n=r}^{\infty}p_{nk}\left|b_{n-k+r}\right| \\
&= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{n=k}^{\infty}\left|b_{n-k+r}\right| \\
&= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{n=k}^{\infty}\left|b_{n-(k-r)}\right| \\
&= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{m=r}^{\infty}\left|b_m\right| < \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=r}^{\infty}c_n &= \sum_{n=r}^{\infty}\sum_{k=r}^{n}a_k b_{n-k+r} = \sum_{n=r}^{\infty}\sum_{k=r}^{\infty}p_{nk}a_k b_{n-k+r} \\
&= \sum_{k=r}^{\infty}a_k\sum_{n=r}^{\infty}p_{nk}b_{n-k+r} = \sum_{k=r}^{\infty}a_k\sum_{n=k}^{\infty}b_{n-k+r} \\
&= \sum_{k=r}^{\infty}a_k\sum_{m=r}^{\infty}b_m
\end{aligned}$$

and this proves the theorem.

It follows that $\sum_{n=r}^{\infty}c_n$ converges absolutely. Also, you can see by induction that you can multiply any number of absolutely convergent series together and obtain a series which is absolutely convergent. Next, here are some similar results related to Merten's theorem.

Lemma 22.6 *Let $\sum_{n=0}^{\infty}a_n(z)$ and $\sum_{n=0}^{\infty}b_n(z)$ be two convergent series for $z\in K$ which satisfy the conditions of the Weierstrass M test. Thus there exist positive constants, A_n and B_n such that $|a_n(z)|\leq A_n, |b_n(z)|\leq B_n$ for all $z\in K$ and $\sum_{n=0}^{\infty}A_n<\infty, \sum_{n=0}^{\infty}B_n<\infty$. Then defining the Cauchy product,*

$$c_n(z)\equiv\sum_{k-0}^{n}a_{n-k}(z)\,b_k(z),$$

it follows $\sum_{n=0}^{\infty}c_n(z)$ also converges absolutely and uniformly on K because $c_n(z)$ satisfies the conditions of the Weierstrass M test. Therefore,

$$\sum_{n=0}^{\infty}c_n(z)=\left(\sum_{k=0}^{\infty}a_k(z)\right)\left(\sum_{n=0}^{\infty}b_n(z)\right). \tag{22.3}$$

Proof:

$$|c_n(z)|\leq\sum_{k=0}^{n}|a_{n-k}(z)|\,|b_k(z)|\leq\sum_{k=0}^{n}A_{n-k}B_k.$$

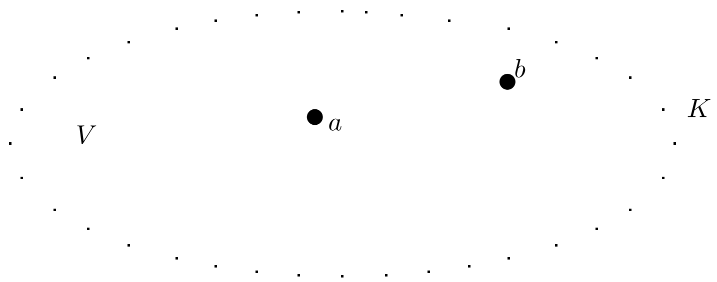
Also,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{n-k}B_k &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}A_{n-k}B_k\\ &= \sum_{k=0}^{\infty}B_k\sum_{n=0}^{\infty}A_n<\infty.\end{aligned}$$

The claim of 22.3 follows from Merten's theorem. This proves the lemma.

Corollary 22.7 *Let P be a polynomial and let $\sum_{n=0}^{\infty}a_n(z)$ converge uniformly and absolutely on K such that the a_n satisfy the conditions of the Weierstrass M test. Then there exists a series for $P\left(\sum_{n=0}^{\infty}a_n(z)\right),\sum_{n=0}^{\infty}c_n(z)$, which also converges absolutely and uniformly for $z\in K$ because $c_n(z)$ also satisfies the conditions of the Weierstrass M test.*

The following picture is descriptive of the following lemma. This lemma says that if you have a rational function with one pole off a compact set, then you can approximate on the compact set with another rational function which has a different pole.



Lemma 22.8 *Let R be a rational function which has a pole only at $a\in V$, a component of $\mathbb{C}\setminus K$ where K is a compact set. Suppose $b\in V$. Then for $\varepsilon>0$ given, there exists a rational function, Q, having a pole only at b such that*

$$||R-Q||_{K,\infty}<\varepsilon. \tag{22.4}$$

If it happens that V is unbounded, then there exists a polynomial, P such that

$$||R-P||_{K,\infty}<\varepsilon. \tag{22.5}$$

Proof: Say that $b\in V$ satisfies $\mathcal{P}$ if for all $\varepsilon>0$ there exists a rational function, Q_b, having a pole only at b such that

$$||R-Q_b||_{K,\infty}<\varepsilon$$

Now define a set,

$$S\equiv\{b\in V:b\text{ satisfies }\mathcal{P}\ \}.$$

Observe that $S \neq \emptyset$ because $a \in S$.

I claim S is open. Suppose $b_1 \in S$. Then there exists a $\delta > 0$ such that

$$\left|\frac{b_1 - b}{z - b}\right| < \frac{1}{2} \tag{22.6}$$

for all $z \in K$ whenever $b \in B(b_1, \delta)$. In fact, it suffices to take $|b - b_1| < \operatorname{dist}(b_1, K)/4$ because then

$$\begin{aligned} \left|\frac{b_1 - b}{z - b}\right| &< \left|\frac{\operatorname{dist}(b_1, K)/4}{z - b}\right| \leq \frac{\operatorname{dist}(b_1, K)/4}{|z - b_1| - |b_1 - b|} \\ &\leq \frac{\operatorname{dist}(b_1, K)/4}{\operatorname{dist}(b_1, K) - \operatorname{dist}(b_1, K)/4} \leq \frac{1}{3} < \frac{1}{2}. \end{aligned}$$

Since b_1 satisfies $\mathcal{P}$, there exists a rational function Q_{b_1} with the desired properties. It is shown next that you can approximate Q_{b_1} with Q_b thus yielding an approximation to R by the use of the triangle inequality,

$$||R - Q_{b_1}||_{K,\infty} + ||Q_{b_1} - Q_b||_{K,\infty} \geq ||R - Q_b||_{K,\infty}.$$

Since Q_{b_1} has poles only at b_1, it follows it is a sum of functions of the form $\frac{\alpha_n}{(z-b_1)^n}$. Therefore, it suffices to consider the terms of Q_{b_1} or that Q_{b_1} is of the special form

$$Q_{b_1}(z) = \frac{1}{(z - b_1)^n}.$$

However,

$$\frac{1}{(z - b_1)^n} = \frac{1}{(z - b)^n \left(1 - \frac{b_1 - b}{z - b}\right)^n}$$

Now from the choice of b_1, the series

$$\sum_{k=0}^{\infty} \left(\frac{b_1 - b}{z - b}\right)^k = \frac{1}{\left(1 - \frac{b_1 - b}{z - b}\right)}$$

converges absolutely independent of the choice of $z \in K$ because

$$\left|\left(\frac{b_1 - b}{z - b}\right)^k\right| < \frac{1}{2^k}.$$

By Corollary 22.7 the same is true of the series for $\frac{1}{\left(1 - \frac{b_1 - b}{z - b}\right)^n}$. Thus a suitable partial sum can be made uniformly on K as close as desired to $\frac{1}{(z-b_1)^n}$. This shows that b satisfies $\mathcal{P}$ whenever b is close enough to b_1 verifying that S is open.

Next it is shown S is closed in V. Let $b_n \in S$ and suppose $b_n \to b \in V$. Then since $b_n \in S$, there exists a rational function, Q_{b_n} such that

$$||Q_{b_n} - R||_{K,\infty} < \frac{\varepsilon}{2}.$$

Then for all n large enough,

$$\frac{1}{2}\operatorname{dist}(b,K) \geq |b_n - b|$$

and so for all n large enough,

$$\left|\frac{b-b_n}{z-b_n}\right| < \frac{1}{2},$$

for all $z \in K$. Pick such a b_n. As before, it suffices to assume Q_{b_n}, is of the form $\frac{1}{(z-b_n)^n}$. Then

$$Q_{b_n}(z) = \frac{1}{(z-b_n)^n} = \frac{1}{(z-b)^n\left(1-\frac{b_n-b}{z-b}\right)^n}$$

and because of the estimate, there exists M such that for all $z \in K$

$$\left|\frac{1}{\left(1-\frac{b_n-b}{z-b}\right)^n} - \sum_{k=0}^{M} a_k \left(\frac{b_n-b}{z-b}\right)^k\right| < \frac{\varepsilon\left(\operatorname{dist}(b,K)\right)^n}{2}. \tag{22.7}$$

Therefore, for all $z \in K$

$$\begin{aligned}
\left|Q_{b_n}(z) - \frac{1}{(z-b)^n}\sum_{k=0}^{M} a_k\left(\frac{b_n-b}{z-b}\right)^k\right| &= \\
\left|\frac{1}{(z-b)^n\left(1-\frac{b_n-b}{z-b}\right)^n} - \frac{1}{(z-b)^n}\sum_{k=0}^{M} a_k\left(\frac{b_n-b}{z-b}\right)^k\right| &\leq \\
\frac{\varepsilon\left(\operatorname{dist}(b,K)\right)^n}{2}\frac{1}{\operatorname{dist}(b,K)^n} &= \frac{\varepsilon}{2}
\end{aligned}$$

and so, letting $Q_b(z) = \frac{1}{(z-b)^n}\sum_{k=0}^{M} a_k \left(\frac{b_n-b}{z-b}\right)^k$,

$$\begin{aligned}
||R - Q_b||_{K,\infty} &\leq ||R - Q_{b_n}||_{K,\infty} + ||Q_{b_n} - Q_b||_{K,\infty} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

showing that $b \in S$. Since S is both open and closed in V it follows that, since $S \neq \emptyset$, $S = V$. Otherwise V would fail to be connected.

It remains to consider the case where V is unbounded. Pick $b \in V$ large enough that

$$\left|\frac{z}{b}\right| < \frac{1}{2} \tag{22.8}$$

for all $z \in K$. From what was just shown, there exists a rational function, Q_b having a pole only at b such that $||Q_b - R||_{K,\infty} < \frac{\varepsilon}{2}$. It suffices to assume that Q_b is of the form

$$\begin{aligned} Q_b(z) &= \frac{p(z)}{(z-b)^n} = p(z)(-1)^n \frac{1}{b^n} \frac{1}{\left(1-\frac{z}{b}\right)^n} \\ &= p(z)(-1)^n \frac{1}{b^n} \left(\sum_{k=0}^{\infty} \left(\frac{z}{b}\right)^k \right)^n \end{aligned}$$

Then by an application of Corollary 22.7 there exists a partial sum of the power series for Q_b which is uniformly close to Q_b on K. Therefore, you can approximate Q_b and therefore also R uniformly on K by a polynomial consisting of a partial sum of the above infinite sum. This proves the theorem.

If f is a polynomial, then f has a pole at ∞. This will be discussed more later.

22.1.4 Runge's Theorem

Now what follows is the first form of Runge's theorem.

Theorem 22.9 *Let K be a compact subset of an open set, Ω and let $\{b_j\}$ be a set which consists of one point from each component of $\widehat{\mathbb{C}} \setminus K$. Let f be analytic on Ω. Then for each $\varepsilon > 0$, there exists a rational function, Q whose poles are all contained in the set, $\{b_j\}$ such that*

$$||Q - f||_{K,\infty} < \varepsilon. \tag{22.9}$$

If $\widehat{\mathbb{C}} \setminus K$ has only one component, then Q may be taken to be a polynomial.

Proof: By Lemma 22.4 there exists a rational function of the form

$$R(z) = \sum_{k=1}^{M} \frac{A_k}{w_k - z}$$

where the w_k are elements of components of $\mathbb{C} \setminus K$ and A_k are complex numbers such that

$$||R - f||_{K,\infty} < \frac{\varepsilon}{2}.$$

Consider the rational function, $R_k(z) \equiv \frac{A_k}{w_k - z}$ where $w_k \in V_j$, one of the components of $\mathbb{C} \setminus K$, the given point of V_j being b_j. By Lemma 22.8, there exists a function, Q_k which is either a rational function having its only pole at b_j or a polynomial, depending on whether V_j is bounded such that

$$||R_k - Q_k||_{K,\infty} < \frac{\varepsilon}{2M}.$$

Letting $Q(z) \equiv \sum_{k=1}^{M} Q_k(z)$,

$$||R - Q||_{K,\infty} < \frac{\varepsilon}{2}.$$

It follows
$$||f-Q||_{K,\infty} \leq ||f-R||_{K,\infty} + ||R-Q||_{K,\infty} < \varepsilon.$$
In the case of only one component of $\mathbb{C} \setminus K$, this component is the unbounded component and so you can take Q to be a polynomial. This proves the theorem.

The next version of Runge's theorem concerns the case where the given points are contained in $\widehat{\mathbb{C}} \setminus \Omega$ for Ω an open set rather than a compact set. Note that here there could be uncountably many components of $\widehat{\mathbb{C}} \setminus \Omega$ because the components are no longer open sets. An easy example of this phenomenon in one dimension is where $\Omega = [0,1] \setminus P$ for P the Cantor set. Then you can show that $\mathbb{R} \setminus \Omega$ has uncountably many components. Nevertheless, Runge's theorem will follow from Theorem 22.9 with the aid of the following interesting lemma.

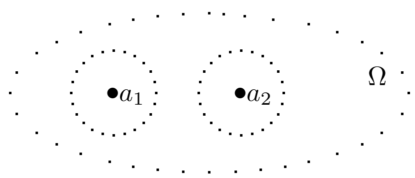
Lemma 22.10 *Let Ω be an open set in $\mathbb{C}$. Then there exists a sequence of compact sets, $\{K_n\}$ such that*
$$\Omega = \cup_{k=1}^{\infty} K_n, \ \cdots, K_n \subseteq \operatorname{int} K_{n+1} \cdots, \tag{22.10}$$
and for any $K \subseteq \Omega$,
$$K \subseteq K_n, \tag{22.11}$$
for all n sufficiently large, and every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$.

Proof: Let
$$V_n \equiv \{z : |z| > n\} \cup \bigcup_{z \notin \Omega} B\left(z, \frac{1}{n}\right).$$
Thus $\{z : |z| > n\}$ contains the point, ∞. Now let
$$K_n \equiv \widehat{\mathbb{C}} \setminus V_n = \mathbb{C} \setminus V_n \subseteq \Omega.$$
You should verify that 22.10 and 22.11 hold. It remains to show that every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$. Let D be a component of $\widehat{\mathbb{C}} \setminus K_n \equiv V_n$.

If $\infty \notin D$, then D contains no point of $\{z : |z| > n\}$ because this set is connected and D is a component. (If it did contain a point of this set, it would have to contain the whole set.) Therefore, $D \subseteq \bigcup_{z \notin \Omega} B\left(z, \frac{1}{n}\right)$ and so D contains some point of $B\left(z, \frac{1}{n}\right)$ for some $z \notin \Omega$. Therefore, since this ball is connected, it follows D must contain the whole ball and consequently D contains some point of Ω^C. (The point z at the center of the ball will do.) Since D contains $z \notin \Omega$, it must contain the component, H_z, determined by this point. The reason for this is that
$$H_z \subseteq \widehat{\mathbb{C}} \setminus \Omega \subseteq \widehat{\mathbb{C}} \setminus K_n$$
and H_z is connected. Therefore, H_z can only have points in one component of $\widehat{\mathbb{C}} \setminus K_n$. Since it has a point in D, it must therefore, be totally contained in D. This verifies the desired condition in the case where $\infty \notin D$.

Now suppose that $\infty \in D$. $\infty \notin \Omega$ because Ω is given to be a set in $\mathbb{C}$. Letting H_∞ denote the component of $\widehat{\mathbb{C}} \setminus \Omega$ determined by ∞, it follows both D and H_∞ contain ∞. Therefore, the connected set, H_∞ cannot have any points in another component of $\widehat{\mathbb{C}} \setminus K_n$ and it is a set which is contained in $\widehat{\mathbb{C}} \setminus K_n$ so it must be contained in D. This proves the lemma.

The following picture is a very simple example of the sort of thing considered by Runge's theorem. The picture is of a region which has a couple of holes.



However, there could be many more holes than two. In fact, there could be infinitely many. Nor does it follow that the components of the complement of Ω need to have any interior points. Therefore, the picture is certainly not representative.

Theorem 22.11 *(Runge) Let Ω be an open set, and let A be a set which has one point in each component of $\widehat{\mathbb{C}} \setminus \Omega$ and let f be analytic on Ω. Then there exists a sequence of rational functions, $\{R_n\}$ having poles only in A such that R_n converges uniformly to f on compact subsets of Ω.*

Proof: Let K_n be the compact sets of Lemma 22.10 where each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$. It follows each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a point of A. Therefore, by Theorem 22.9 there exists R_n a rational function with poles only in A such that

$$||R_n - f||_{K_n,\infty} < \frac{1}{n}.$$

It follows, since a given compact set, K is a subset of K_n for all n large enough, that $R_n \to f$ uniformly on K. This proves the theorem.

Corollary 22.12 *Let Ω be simply connected and f analytic on Ω. Then there exists a sequence of polynomials, $\{p_n\}$ such that $p_n \to f$ uniformly on compact sets of Ω.*

Proof: By definition of what is meant by simply connected, $\widehat{\mathbb{C}} \setminus \Omega$ is connected and so there are no bounded components of $\widehat{\mathbb{C}} \setminus \Omega$. Therefore, in the proof of Theorem 22.11 when you use Theorem 22.9, you can always have R_n be a polynomial by Lemma 22.8.

22.2 The Mittag-Leffler Theorem

22.2.1 A Proof From Runge's Theorem

This theorem is fairly easy to prove once you have Theorem 22.9. Given a set of complex numbers, does there exist a meromorphic function having its poles equal to this set of numbers? The Mittag-Leffler theorem provides a very satisfactory answer to this question. Actually, it says somewhat more. You can specify, not just the location of the pole but also the kind of singularity the meromorphic function is to have at that pole.

Theorem 22.13 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in an open subset of $\mathbb{C}$, Ω. Suppose also that $P \subseteq \Omega \subseteq \mathbb{C}$. For each z_k, denote by $S_k(z)$ a function of the form*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{a_j^k}{(z - z_k)^j}.$$

Then there exists a meromorphic function, Q defined on Ω such that the poles of Q are the points, $\{z_k\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of Q at z_k equals $S_k(z)$. In other words, for z near $z_k, Q(z) = g_k(z) + S_k(z)$ for some function, g_k analytic near z_k.

Proof: Let $\{K_n\}$ denote the sequence of compact sets described in Lemma 22.10. Thus $\cup_{n=1}^{\infty} K_n = \Omega$, $K_n \subseteq \text{int}(K_{n+1}) \subseteq K_{n+1} \cdots$, and the components of $\widehat{\mathbb{C}} \setminus K_n$ contain the components of $\widehat{\mathbb{C}} \setminus \Omega$. Renumbering if necessary, you can assume each $K_n \neq \emptyset$. Also let $K_0 = \emptyset$. Let $P_m \equiv P \cap (K_m \setminus K_{m-1})$ and consider the rational function, R_m defined by

$$R_m(z) \equiv \sum_{z_k \in K_m \setminus K_{m-1}} S_k(z).$$

Since each K_m is compact, it follows P_m is finite and so the above really is a rational function. Now for $m > 1$,this rational function is analytic on some open set containing K_{m-1}. There exists a set of points, A one point in each component of $\widehat{\mathbb{C}} \setminus \Omega$. Consider $\widehat{\mathbb{C}} \setminus K_{m-1}$. Each of its components contains a component of $\widehat{\mathbb{C}} \setminus \Omega$ and so for each of these components of $\widehat{\mathbb{C}} \setminus K_{m-1}$, there exists a point of A which is contained in it. Denote the resulting set of points by A'. By Theorem 22.9 there exists a rational function, Q_m whose poles are all contained in the set, $A' \subseteq \Omega^C$ such that

$$||R_m - Q_m||_{K_{m-1},\infty} < \frac{1}{2^m}.$$

The meromorphic function is

$$Q(z) \equiv R_1(z) + \sum_{k=2}^{\infty} (R_k(z) - Q_k(z)).$$

It remains to verify this function works. First consider K_1. Then on K_1, the above sum converges uniformly. Furthermore, the terms of the sum are analytic in some open set containing K_1. Therefore, the infinite sum is analytic on this open set and so for $z \in K_1$ The function, f is the sum of a rational function, R_1, having poles at

P_1 with the specified singular terms and an analytic function. Therefore, Q works on K_1. Now consider K_m for $m > 1$. Then

$$Q(z) = R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z)) + \sum_{k=m+2}^{\infty} (R_k(z) - Q_k(z)).$$

As before, the infinite sum converges uniformly on K_{m+1} and hence on some open set, O containing K_m. Therefore, this infinite sum equals a function which is analytic on O. Also,

$$R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z))$$

is a rational function having poles at $\cup_{k=1}^{m} P_k$ with the specified singularities because the poles of each Q_k are not in Ω. It follows this function is meromorphic because it is analytic except for the points in P. It also has the property of retaining the specified singular behavior.

22.2.2 A Direct Proof Without Runge's Theorem

There is a direct proof of this important theorem which is not dependent on Runge's theorem in the case where $\Omega = \mathbb{C}$. I think it is arguably easier to understand and the Mittag-Leffler theorem is very important so I will give this proof here.

Theorem 22.14 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in $\mathbb{C}$ which satisfies $\lim_{n\to\infty} |z_n| = \infty$. For each z_k, denote by $S_k(z)$ a polynomial in $\frac{1}{z-z_k}$ which is of the form*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{a_j^k}{(z-z_k)^j}.$$

Then there exists a meromorphic function, Q defined on $\mathbb{C}$ such that the poles of Q are the points, $\{z_k\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of Q at z_k equals $S_k(z)$. In other words, for z near z_k,

$$Q(z) = g_k(z) + S_k(z)$$

for some function, g_k analytic in some open set containing z_k.

Proof: First consider the case where none of the $z_k = 0$. Letting

$$K_k \equiv \{z : |z| \leq |z_k|/2\},$$

there exists a power series for $\frac{1}{z-z_k}$ which converges uniformly and absolutely on this set. Here is why:

$$\frac{1}{z-z_k} = \left(\frac{-1}{1-\frac{z}{z_k}}\right)\frac{1}{z_k} = \frac{-1}{z_k}\sum_{l=0}^{\infty}\left(\frac{z}{z_k}\right)^l$$

and the Weierstrass M test can be applied because

$$\left|\frac{z}{z_k}\right| < \frac{1}{2}$$

on this set. Therefore, by Corollary 22.7, $S_k(z)$, being a polynomial in $\frac{1}{z-z_k}$, has a power series which converges uniformly to $S_k(z)$ on K_k. Therefore, there exists a polynomial, $P_k(z)$ such that

$$||P_k - S_k||_{\overline{B(0,|z_k|/2)},\infty} < \frac{1}{2^k}.$$

Let

$$Q(z) \equiv \sum_{k=1}^{\infty} (S_k(z) - P_k(z)). \tag{22.12}$$

Consider $z \in K_m$ and let N be large enough that if $k > N$, then $|z_k| > 2|z|$

$$Q(z) = \sum_{k=1}^{N} (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).$$

On K_m, the second sum converges uniformly to a function analytic on $\operatorname{int}(K_m)$ (interior of K_m) while the first is a rational function having poles at $z_1, \cdots, z_N$. Since any compact set is contained in K_m for large enough m, this shows $Q(z)$ is meromorphic as claimed and has poles with the given singularities.

Now consider the case where the poles are at $\{z_k\}_{k=0}^{\infty}$ with $z_0 = 0$. Everything is similar in this case. Let

$$Q(z) \equiv S_0(z) + \sum_{k=1}^{\infty} (S_k(z) - P_k(z)).$$

The series converges uniformly on every compact set because of the assumption that $\lim_{n\to\infty} |z_n| = \infty$ which implies that any compact set is contained in K_k for k large enough. Choose N such that $z \in \operatorname{int}(K_N)$ and $z_n \notin K_N$ for all $n \geq N+1$. Then

$$Q(z) = S_0(z) + \sum_{k=1}^{N} (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).$$

The last sum is analytic on $\operatorname{int}(K_N)$ because each function in the sum is analytic due to the fact that none of its poles are in K_N. Also, $S_0(z) + \sum_{k=1}^{N} (S_k(z) - P_k(z))$ is a finite sum of rational functions so it is a rational function and P_k is a polynomial so z_m is a pole of this function with the correct singularity whenever $z_m \in \operatorname{int}(K_N)$.

22.2.3 Functions Meromorphic On $\widehat{\mathbb{C}}$

Sometimes it is useful to think of isolated singular points at ∞.

Definition 22.15 *Suppose f is analytic on $\{z \in \mathbb{C} : |z| > r\}$. Then f is said to have a removable singularity at ∞ if the function, $g(z) \equiv f\left(\frac{1}{z}\right)$ has a removable singularity at 0. f is said to have a pole at ∞ if the function, $g(z) = f\left(\frac{1}{z}\right)$ has a pole at 0. Then f is said to be meromorphic on $\widehat{\mathbb{C}}$ if all its singularities are isolated and either poles or removable.*

So what is f like for these cases? First suppose f has a removable singularity at ∞. Then $zg(z)$ converges to 0 as $z \to 0$. It follows $g(z)$ must be analytic near 0 and so can be given as a power series. Thus $f(z)$ is of the form $f(z) = g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$. Next suppose f has a pole at ∞. This means $g(z)$ has a pole at 0 so $g(z)$ is of the form $g(z) = \sum_{k=1}^{m} \frac{b_k}{z^k} + h(z)$ where $h(z)$ is analytic near 0. Thus in the case of a pole at ∞, $f(z)$ is of the form $f(z) = g\left(\frac{1}{z}\right) = \sum_{k=1}^{m} b_k z^k + \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$.

It turns out that the functions which are meromorphic on $\widehat{\mathbb{C}}$ are all rational functions. To see this suppose f is meromorphic on $\widehat{\mathbb{C}}$ and note that there exists $r > 0$ such that $f(z)$ is analytic for $|z| > r$. This is required if ∞ is to be isolated. Therefore, there are only finitely many poles of f for $|z| \leq r$, $\{a_1, \cdots, a_m\}$, because by assumption, these poles are isolated and this is a compact set. Let the singular part of f at a_k be denoted by $S_k(z)$. Then $f(z) - \sum_{k=1}^{m} S_k(z)$ is analytic on all of $\mathbb{C}$. Therefore, it is bounded on $|z| \leq r$. In one case, f has a removable singularity at ∞. In this case, f is bounded as $z \to \infty$ and $\sum_k S_k$ also converges to 0 as $z \to \infty$. Therefore, by Liouville's theorem, $f(z) - \sum_{k=1}^{m} S_k(z)$ equals a constant and so $f - \sum_k S_k$ is a constant. Thus f is a rational function. In the other case that f has a pole at ∞, $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n - \sum_{k=1}^{m} S_k(z)$. Now $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k$ is analytic on $\mathbb{C}$ and so is bounded on $|z| \leq r$. But now $\sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n - \sum_{k=1}^{m} S_k(z)$ converges to 0 as $z \to \infty$ and so by Liouville's theorem, $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k$ must equal a constant and again, $f(z)$ equals a rational function.

22.2.4 A Great And Glorious Theorem About Simply Connected Regions

Here is given a laundry list of properties which are equivalent to an open set being simply connected. Recall Definition 18.48 on Page 417 which said that an open set, Ω is simply connected means $\widehat{\mathbb{C}} \setminus \Omega$ is connected. Recall also that this is not the same thing at all as saying $\mathbb{C} \setminus \Omega$ is connected. Consider the outside of a disk for example. I will continue to use this definition for simply connected because it is the most convenient one for complex analysis. However, there are many other equivalent conditions. First here is an interesting lemma which is interesting for its own sake. Recall $n(p, \gamma)$ means the winding number of γ about p. Now recall Theorem 18.52 implies the following lemma in which B^C is playing the role of Ω in Theorem 18.52.

Lemma 22.16 *Let K be a compact subset of B^C, the complement of a closed set. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each j, $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^{m} n(\Gamma_k, p) = 1.$$

while for all $z \in B$

$$\sum_{k=1}^{m} n(\Gamma_k, z) = 0.$$

Definition 22.17 *Let γ be a closed curve in an open set, $\Omega, \gamma : [a, b] \to \Omega$. Then γ is said to be homotopic to a point, p in Ω if there exists a continuous function, $H : [0,1] \times [a,b] \to \Omega$ such that $H(0,t) = p, H(\alpha, a) = H(\alpha, b)$, and $H(1,t) = \gamma(t)$. This function, H is called a homotopy.*

Lemma 22.18 *Suppose γ is a closed continuous bounded variation curve in an open set, Ω which is homotopic to a point. Then if $a \notin \Omega$, it follows $n(a, \gamma) = 0$.*

Proof: Let H be the homotopy described above. The problem with this is that it is not known that $H(\alpha, \cdot)$ is of bounded variation. There is no reason it should be. Therefore, it might not make sense to take the integral which defines the winding number. There are various ways around this. Extend H as follows. $H(\alpha, t) = H(\alpha, a)$ for $t < a, H(\alpha, t) = H(\alpha, b)$ for $t > b$. Let $\varepsilon > 0$.

$$H_\varepsilon(\alpha, t) \equiv \frac{1}{2\varepsilon} \int_{-2\varepsilon + t + \frac{2\varepsilon}{(b-a)}(t-a)}^{t + \frac{2\varepsilon}{(b-a)}(t-a)} H(\alpha, s)\, ds, \ H_\varepsilon(0, t) = p.$$

Thus $H_\varepsilon(\alpha, \cdot)$ is a closed curve which has bounded variation and when $\alpha = 1$, this converges to γ uniformly on $[a, b]$. Therefore, for ε small enough, $n(a, H_\varepsilon(1, \cdot)) = n(a, \gamma)$ because they are both integers and as $\varepsilon \to 0, n(a, H_\varepsilon(1, \cdot)) \to n(a, \gamma)$. Also, $H_\varepsilon(\alpha, t) \to H(\alpha, t)$ uniformly on $[0,1] \times [a,b]$ because of uniform continuity of H. Therefore, for small enough ε, you can also assume $H_\varepsilon(\alpha, t) \in \Omega$ for all α, t. Now $\alpha \to n(a, H_\varepsilon(\alpha, \cdot))$ is continuous. Hence it must be constant because the winding number is integer valued. But

$$\lim_{\alpha \to 0} \frac{1}{2\pi i} \int_{H_\varepsilon(\alpha, \cdot)} \frac{1}{z - a} dz = 0$$

because the length of $H_\varepsilon(\alpha, \cdot)$ converges to 0 and the integrand is bounded because $a \notin \Omega$. Therefore, the constant can only equal 0. This proves the lemma.

Now it is time for the great and glorious theorem on simply connected regions. The following equivalence of properties is taken from Rudin [36]. There is a slightly different list in Conway [11] and a shorter list in Ash [6].

Theorem 22.19 *The following are equivalent for an open set, Ω.*

1. *Ω is homeomorphic to the unit disk, $B(0,1)$.*

2. *Every closed curve contained in Ω is homotopic to a point in Ω.*

3. *If $z \notin \Omega$, and if γ is a closed bounded variation continuous curve in Ω, then $n(\gamma, z) = 0$.*

4. *Ω is simply connected, ($\widehat{\mathbb{C}} \setminus \Omega$ is connected and Ω is connected.)*

5. *Every function analytic on Ω can be uniformly approximated by polynomials on compact subsets.*

6. *For every f analytic on Ω and every closed continuous bounded variation curve, γ,*
$$\int_{\gamma} f(z)\, dz = 0.$$

7. *Every function analytic on Ω has a primitive on Ω.*

8. *If $f, 1/f$ are both analytic on Ω, then there exists an analytic, g on Ω such that $f = \exp(g)$.*

9. *If $f, 1/f$ are both analytic on Ω, then there exists ϕ analytic on Ω such that $f = \phi^2$.*

Proof: 1⇒2**.** Assume 1 and let γ be a closed curve in Ω. Let h be the homeomorphism, $h : B(0,1) \to \Omega$. Let $H(\alpha, t) = h\left(\alpha\left(h^{-1}\gamma(t)\right)\right)$. This works.

2⇒3 This is Lemma 22.18.

3⇒4. Suppose 3 but 4 fails to hold. Then if $\widehat{\mathbb{C}} \setminus \Omega$ is not connected, there exist disjoint nonempty sets, A and B such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. It follows each of these sets must be closed because neither can have a limit point in Ω nor in the other. Also, one and only one of them contains ∞. Let this set be B. Thus A is a closed set which must also be bounded. Otherwise, there would exist a sequence of points in A, $\{a_n\}$ such that $\lim_{n\to\infty} a_n = \infty$ which would contradict the requirement that no limit points of A can be in B. Therefore, A is a compact set contained in the open set, $B^C \equiv \{z \in \mathbb{C} : z \notin B\}$. Pick $p \in A$. By Lemma 22.16 there exist continuous bounded variation closed curves $\{\Gamma_k\}_{k=1}^m$ which are contained in B^C, do not intersect A and such that
$$1 = \sum_{k=1}^{m} n(p, \Gamma_k)$$
However, if these curves do not intersect A and they also do not intersect B then they must be all contained in Ω. Since $p \notin \Omega$, it follows by 3 that for each k, $n(p, \Gamma_k) = 0$, a contradiction.

4⇒5 This is Corollary 22.12 on Page 520.

5⇒6 Every polynomial has a primitive and so the integral over any closed bounded variation curve of a polynomial equals 0. Let f be analytic on Ω. Then let $\{f_n\}$ be a sequence of polynomials converging uniformly to f on γ^*. Then

$$0 = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

6⇒7 Pick $z_0 \in \Omega$. Letting $\gamma(z_0, z)$ be a bounded variation continuous curve joining z_0 to z in Ω, you define a primitive for f as follows.

$$F(z) = \int_{\gamma(z_0,z)} f(w)\,dw.$$

This is well defined by 6 and is easily seen to be a primitive. You just write the difference quotient and take a limit using 6.

$$\begin{aligned}
\lim_{w\to 0} \frac{F(z+w) - F(z)}{w} &= \lim_{w\to 0} \frac{1}{w} \left(\int_{\gamma(z_0,z+w)} f(u)\,du - \int_{\gamma(z_0,z)} f(u)\,du \right) \\
&= \lim_{w\to 0} \frac{1}{w} \int_{\gamma(z,z+w)} f(u)\,du \\
&= \lim_{w\to 0} \frac{1}{w} \int_0^1 f(z+tw)\,w dt = f(z).
\end{aligned}$$

7⇒8 Suppose then that $f, 1/f$ are both analytic. Then f'/f is analytic and so it has a primitive by 7. Let this primitive be g_1. Then

$$\begin{aligned}
\left(e^{-g_1} f\right)' &= e^{-g_1}(-g_1') f + e^{-g_1} f' \\
&= -e^{-g_1} \left(\frac{f'}{f}\right) f + e^{-g_1} f' = 0.
\end{aligned}$$

Therefore, since Ω is connected, it follows $e^{-g_1} f$ must equal a constant. (Why?) Let the constant be e^{a+ibi}. Then $f(z) = e^{g_1(z)} e^{a+ib}$. Therefore, you let $g(z) = g_1(z) + a + ib$.

8⇒9 Suppose then that $f, 1/f$ are both analytic on Ω. Then by 8 $f(z) = e^{g(z)}$. Let $\phi(z) \equiv e^{g(z)/2}$.

9⇒1 There are two cases. First suppose $\Omega = \mathbb{C}$. This satisfies condition 9 because if $f, 1/f$ are both analytic, then the same argument involved in 8⇒9 gives the existence of a square root. A homeomorphism is $h(z) \equiv \frac{z}{\sqrt{1+|z|^2}}$. It obviously maps onto $B(0,1)$ and is continuous. To see it is 1 - 1 consider the case of z_1 and z_2 having different arguments. Then $h(z_1) \neq h(z_2)$. If $z_2 = tz_1$ for a positive $t \neq 1$, then it is also clear $h(z_1) \neq h(z_2)$. To show h^{-1} is continuous, note that if you have an open set in $\mathbb{C}$ and a point in this open set, you can get a small open set containing this point by allowing the modulus and the argument to lie in some open interval. Reasoning this way, you can verify h maps open sets to open sets. In the case where $\Omega \neq \mathbb{C}$, there exists a one to one analytic map which maps Ω onto $B(0,1)$ by the Riemann mapping theorem. This proves the theorem.

22.3 Exercises

1. Let $a \in \mathbb{C}$. Show there exists a sequence of polynomials, $\{p_n\}$ such that $p_n(a) = 1$ but $p_n(z) \to 0$ for all $z \neq a$.

2. Let l be a line in $\mathbb{C}$. Show there exists a sequence of polynomials $\{p_n\}$ such that $p_n(z) \to 1$ on one side of this line and $p_n(z) \to -1$ on the other side of the line. **Hint:** The complement of this line is simply connected.

3. Suppose Ω is a simply connected region, f is analytic on Ω, $f \neq 0$ on Ω, and $n \in \mathbb{N}$. Show that there exists an analytic function, g such that $g(z)^n = f(z)$ for all $z \in \Omega$. That is, you can take the n^{th} root of $f(z)$. If Ω is a region which contains 0, is it possible to find $g(z)$ such that g is analytic on Ω and $g(z)^2 = z$?

4. Suppose Ω is a region (connected open set) and f is an analytic function defined on Ω such that $f(z) \neq 0$ for any $z \in \Omega$. Suppose also that for every positive integer, n there exists an analytic function, g_n defined on Ω such that $g_n^n(z) = f(z)$. Show that then it is possible to define an analytic function, L on $f(\Omega)$ such that $e^{L(f(z))} = f(z)$ for all $z \in \Omega$.

5. You know that $\phi(z) \equiv \frac{z-i}{z+i}$ maps the upper half plane onto the unit ball. Its inverse, $\psi(z) = i\frac{1+z}{1-z}$ maps the unit ball onto the upper half plane. Also for z in the upper half plane, you can define a square root as follows. If $z = |z| e^{i\theta}$ where $\theta \in (0, \pi)$, let $z^{1/2} \equiv |z|^{1/2} e^{i\theta/2}$ so the square root maps the upper half plane to the first quadrant. Now consider
$$z \to \exp\left(-i \log\left[i\left(\frac{1+z}{1-z}\right)\right]^{1/2}\right). \tag{22.13}$$
Show this is an analytic function which maps the unit ball onto an annulus. Is it possible to find a one to one analytic map which does this?

Infinite Products

The Mittag-Leffler theorem gives existence of a meromorphic function which has specified singular part at various poles. It would be interesting to do something similar to zeros of an analytic function. That is, given the order of the zero at various points, does there exist an analytic function which has these points as zeros with the specified orders? You know that if you have the zeros of the polynomial, you can factor it. Can you do something similar with analytic functions which are just limits of polynomials? These questions involve the concept of an infinite product.

Definition 23.1 $\prod_{n=1}^{\infty}(1+u_n) \equiv \lim_{n\to\infty}\prod_{k=1}^{n}(1+u_k)$ *whenever this limit exists. If $u_n = u_n(z)$ for $z \in H$, we say the infinite product converges uniformly on H if the partial products, $\prod_{k=1}^{n}(1+u_k(z))$ converge uniformly on H.*

The main theorem is the following.

Theorem 23.2 *Let $H \subseteq \mathbb{C}$ and suppose that $\sum_{n=1}^{\infty}|u_n(z)|$ converges uniformly on H where $u_n(z)$ bounded on H. Then*

$$P(z) \equiv \prod_{n=1}^{\infty}(1+u_n(z))$$

converges uniformly on H. If $(n_1, n_2, \cdots)$ is any permutation of $(1, 2, \cdots)$, then for all $z \in H$,

$$P(z) = \prod_{k=1}^{\infty}(1+u_{n_k}(z))$$

and P has a zero at z_0 if and only if $u_n(z_0) = -1$ for some n.

Proof: First a simple estimate:

$$\begin{aligned}
&\prod_{k=m}^{n}\left(1+\left|u_k\left(z\right)\right|\right) \\
= \;& \exp\left(\ln\left(\prod_{k=m}^{n}\left(1+\left|u_k\left(z\right)\right|\right)\right)\right) = \exp\left(\sum_{k=m}^{n}\ln\left(1+\left|u_k\left(z\right)\right|\right)\right) \\
\leq \;& \exp\left(\sum_{k=m}^{\infty}\left|u_k\left(z\right)\right|\right) < e
\end{aligned}$$

for all $z \in H$ provided m is large enough. Since $\sum_{k=1}^{\infty}\left|u_k\left(z\right)\right|$ converges uniformly on H, $\left|u_k\left(z\right)\right| < \frac{1}{2}$ for all $z \in H$ provided k is large enough. Thus you can take $\log\left(1+u_k\left(z\right)\right)$. Pick N_0 such that for $n > m \geq N_0$,

$$\left|u_m\left(z\right)\right| < \frac{1}{2},\ \prod_{k=m}^{n}\left(1+\left|u_k\left(z\right)\right|\right) < e. \tag{23.1}$$

Now having picked N_0, the assumption the u_n are bounded on H implies there exists a constant, C, independent of $z \in H$ such that for all $z \in H$,

$$\prod_{k=1}^{N_0}\left(1+\left|u_k\left(z\right)\right|\right) < C. \tag{23.2}$$

Let $N_0 < M < N$. Then

$$\begin{aligned}
&\left|\prod_{k=1}^{N}\left(1+u_k\left(z\right)\right) - \prod_{k=1}^{M}\left(1+u_k\left(z\right)\right)\right| \\
\leq \;& \prod_{k=1}^{N_0}\left(1+\left|u_k\left(z\right)\right|\right)\left|\prod_{k=N_0+1}^{N}\left(1+u_k\left(z\right)\right) - \prod_{k=N_0+1}^{M}\left(1+u_k\left(z\right)\right)\right| \\
\leq \;& C\left|\prod_{k=N_0+1}^{N}\left(1+u_k\left(z\right)\right) - \prod_{k=N_0+1}^{M}\left(1+u_k\left(z\right)\right)\right| \\
\leq \;& C\left(\prod_{k=N_0+1}^{M}\left(1+\left|u_k\left(z\right)\right|\right)\right)\left|\prod_{k=M+1}^{N}\left(1+u_k\left(z\right)\right) - 1\right| \\
\leq \;& Ce\left|\prod_{k=M+1}^{N}\left(1+\left|u_k\left(z\right)\right|\right) - 1\right|.
\end{aligned}$$

Since $1 \leq \prod_{k=M+1}^{N}\left(1+\left|u_k(z)\right|\right) \leq e$, it follows the term on the far right is dominated by

$$
\begin{aligned}
& Ce^2\left|\ln\left(\prod_{k=M+1}^{N}\left(1+\left|u_k(z)\right|\right)\right)-\ln 1\right| \\
\leq \; & Ce^2\sum_{k=M+1}^{N}\ln\left(1+\left|u_k(z)\right|\right) \\
\leq \; & Ce^2\sum_{k=M+1}^{N}\left|u_k(z)\right|<\varepsilon
\end{aligned}
$$

uniformly in $z \in H$ provided M is large enough. This follows from the simple observation that if $1<x<e$, then $x-1 \leq e(\ln x-\ln 1)$. Therefore, $\left\{\prod_{k=1}^{m}\left(1+u_k(z)\right)\right\}_{m=1}^{\infty}$ is uniformly Cauchy on H and therefore, converges uniformly on H. Let $P(z)$ denote the function it converges to.

What about the permutations? Let $\{n_1, n_2, \cdots\}$ be a permutation of the indices. Let $\varepsilon>0$ be given and let N_0 be such that if $n>N_0$,

$$\left|\prod_{k=1}^{n}\left(1+u_k(z)\right)-P(z)\right|<\varepsilon$$

for all $z \in H$. Let $\{1,2,\cdots,n\} \subseteq \left\{n_1, n_2, \cdots, n_{p(n)}\right\}$ where $p(n)$ is an increasing sequence. Then from 23.1 and 23.2,

$$
\begin{aligned}
& \left|P(z)-\prod_{k=1}^{p(n)}\left(1+u_{n_k}(z)\right)\right| \\
\leq \; & \left|P(z)-\prod_{k=1}^{n}\left(1+u_k(z)\right)\right|+\left|\prod_{k=1}^{n}\left(1+u_k(z)\right)-\prod_{k=1}^{p(n)}\left(1+u_{n_k}(z)\right)\right| \\
\leq \; & \varepsilon+\left|\prod_{k=1}^{n}\left(1+u_k(z)\right)-\prod_{k=1}^{p(n)}\left(1+u_{n_k}(z)\right)\right| \\
\leq \; & \varepsilon+\left|\prod_{k=1}^{n}\left(1+\left|u_k(z)\right|\right)\right|\left|1-\prod_{n_k>n}\left(1+u_{n_k}(z)\right)\right| \\
\leq \; & \varepsilon+\left|\prod_{k=1}^{N_0}\left(1+\left|u_k(z)\right|\right)\right|\left|\prod_{k=N_0+1}^{n}\left(1+\left|u_k(z)\right|\right)\right|\left|1-\prod_{n_k>n}\left(1+u_{n_k}(z)\right)\right| \\
\leq \; & \varepsilon+Ce\left|\prod_{n_k>n}\left(1+\left|u_{n_k}(z)\right|\right)-1\right| \leq \varepsilon+Ce\left|\prod_{k=n+1}^{M(p(n))}\left(1+\left|u_{n_k}(z)\right|\right)-1\right|
\end{aligned}
$$

where $M(p(n))$ is the largest index in the permuted list, $\{n_1, n_2, \cdots, n_{p(n)}\}$. then from 23.1, this last term is dominated by

$$\begin{aligned}
& \varepsilon + Ce^2 \left| \ln \left(\prod_{k=n+1}^{M(p(n))} (1 + |u_{n_k}(z)|) \right) - \ln 1 \right| \\
\leq \quad & \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} \ln (1 + |u_{n_k}|) \leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} |u_{n_k}| < 2\varepsilon
\end{aligned}$$

for all n large enough uniformly in $z \in H$. Therefore, $\left| P(z) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| < 2\varepsilon$ whenever n is large enough. This proves the part about the permutation.

It remains to verify the assertion about the points, z_0, where $P(z_0) = 0$. Obviously, if $u_n(z_0) = -1$, then $P(z_0) = 0$. Suppose then that $P(z_0) = 0$ and $M > N_0$. Then

$$\left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| =$$

$$\begin{aligned}
& \left| \prod_{k=1}^{M} (1 + u_k(z_0)) - \prod_{k=1}^{\infty} (1 + u_k(z_0)) \right| \\
\leq \quad & \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \left| 1 - \prod_{k=M+1}^{\infty} (1 + u_k(z_0)) \right| \\
\leq \quad & \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \left| \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - 1 \right| \\
\leq \quad & e \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \left| \ln \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - \ln 1 \right| \\
\leq \quad & e \left(\sum_{k=M+1}^{\infty} \ln (1 + |u_k(z)|) \right) \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \\
\leq \quad & e \sum_{k=M+1}^{\infty} |u_k(z)| \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \\
\leq \quad & \frac{1}{2} \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|
\end{aligned}$$

whenever M is large enough. Therefore, for such M,

$$\prod_{k=1}^{M} (1 + u_k(z_0)) = 0$$

and so $u_k(z_0) = -1$ for some $k \leq M$. This proves the theorem.

23.1 Analytic Function With Prescribed Zeros

Suppose you are given complex numbers, $\{z_n\}$ and you want to find an analytic function, f such that these numbers are the zeros of f. How can you do it? The problem is easy if there are only finitely many of these zeros, $\{z_1, z_2, \cdots, z_m\}$. You just write $(z - z_1)(z - z_2) \cdots (z - z_m)$. Now if none of the $z_k = 0$ you could also write it at $\prod_{k=1}^{m}\left(1 - \frac{z}{z_k}\right)$ and this might have a better chance of success in the case of infinitely many prescribed zeros. However, you would need to verify something like $\sum_{n=1}^{\infty}\left|\frac{z}{z_n}\right| < \infty$ which might not be so. The way around this is to adjust the product, making it $\prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right)e^{g_k(z)}$ where $g_k(z)$ is some analytic function. Recall also that for $|x| < 1$, $\ln\left((1-x)^{-1}\right) = \sum_{n=1}^{\infty}\frac{x^n}{n}$. If you had x/x_n small and real, then $1 = (1 - x/x_n)\exp\left(\ln\left((1 - x/x_n)^{-1}\right)\right)$ and $\prod_{k=1}^{\infty} 1$ of course converges but loses all the information about zeros. However, this is why it is not too unreasonable to consider factors of the form

$$\left(1 - \frac{z}{z_k}\right)e^{\sum_{k=1}^{p_k}\left(\frac{z}{z_k}\right)^k \frac{1}{k}}$$

where p_k is suitably chosen.

First here are some estimates.

Lemma 23.3 *For $z \in \mathbb{C}$,*

$$|e^z - 1| \leq |z|\, e^{|z|}, \tag{23.3}$$

and if $|z| \leq 1/2$,

$$\left|\sum_{k=m}^{\infty}\frac{z^k}{k}\right| \leq \frac{1}{m}\frac{|z|^m}{1 - |z|} \leq \frac{2}{m}|z|^m \leq \frac{1}{m}\frac{1}{2^{m-1}}. \tag{23.4}$$

Proof: Consider 23.3.

$$|e^z - 1| = \left|\sum_{k=1}^{\infty}\frac{z^k}{k!}\right| \leq \sum_{k=1}^{\infty}\frac{|z|^k}{k!} = e^{|z|} - 1 \leq |z|\, e^{|z|}$$

the last inequality holding by the mean value theorem. Now consider 23.4.

$$\begin{aligned}\left|\sum_{k=m}^{\infty}\frac{z^k}{k}\right| &\leq \sum_{k=m}^{\infty}\frac{|z|^k}{k} \leq \frac{1}{m}\sum_{k=m}^{\infty}|z|^k \\ &= \frac{1}{m}\frac{|z|^m}{1 - |z|} \leq \frac{2}{m}|z|^m \leq \frac{1}{m}\frac{1}{2^{m-1}}.\end{aligned}$$

This proves the lemma.

The functions, E_p in the next definition are called the elementary factors.

Definition 23.4 *Let* $E_0(z) \equiv 1 - z$ *and for* $p \geq 1$,

$$E_p(z) \equiv (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)$$

In terms of this new symbol, here is another estimate. A sharper inequality is available in Rudin [36] but it is more difficult to obtain.

Corollary 23.5 *For* E_p *defined above and* $|z| \leq 1/2$,

$$|E_p(z) - 1| \leq 3|z|^{p+1}.$$

Proof: From elementary calculus, $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for all $|x| < 1$. Therefore, for $|z| < 1$,

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n},\ \log\left((1-z)^{-1}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n},$$

because the function $\log(1-z)$ and the analytic function, $-\sum_{n=1}^{\infty} \frac{z^n}{n}$ both are equal to $\ln(1-x)$ on the real line segment $(-1,1)$, a set which has a limit point. Therefore, using Lemma 23.3,

$$\begin{aligned}
& |E_p(z) - 1| \\
= & \left|(1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) - 1\right| \\
= & \left|(1-z)\exp\left(\log\left((1-z)^{-1}\right) - \sum_{n=p+1}^{\infty} \frac{z^n}{n}\right) - 1\right| \\
= & \left|\exp\left(-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right) - 1\right| \\
\leq & \left|-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right| e^{\left|-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right|} \\
\leq & \frac{1}{p+1} \cdot 2 \cdot e^{1/(p+1)} |z|^{p+1}. \leq 3|z|^{p+1}
\end{aligned}$$

This proves the corollary.

With this estimate, it is easy to prove the Weierstrass product formula.

Theorem 23.6 *Let* $\{z_n\}$ *be a sequence of nonzero complex numbers which have no limit point in* $\mathbb{C}$ *and let* $\{p_n\}$ *be a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{R}{|z_n|}\right)^{p_n+1} < \infty \tag{23.5}$$

for all $R \in \mathbb{R}$. Then

$$P(z) \equiv \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

is analytic on $\mathbb{C}$ and has a zero at each point, z_n and at no others. If w occurs m times in $\{z_n\}$, then P has a zero of order m at w.

Proof: Since $\{z_n\}$ has no limit point, it follows $\lim_{n\to\infty} |z_n| = \infty$. Therefore, if $p_n = n-1$ the condition, 23.5 holds for this choice of p_n. Now by Theorem 23.2, the infinite product in this theorem will converge uniformly on $|z| \leq R$ if the same is true of the sum,

$$\sum_{n=1}^{\infty} \left| E_{p_n}\left(\frac{z}{z_n}\right) - 1 \right|. \tag{23.6}$$

But by Corollary 23.5 the n^{th} term of this sum satisfies

$$\left| E_{p_n}\left(\frac{z}{z_n}\right) - 1 \right| \leq 3 \left| \frac{z}{z_n} \right|^{p_n+1}.$$

Since $|z_n| \to \infty$, there exists N such that for $n > N, |z_n| > 2R$. Therefore, for $|z| < R$ and letting $0 < a = \min\{|z_n| : n \leq N\}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \left| E_{p_n}\left(\frac{z}{z_n}\right) - 1 \right| &\leq 3\sum_{n=1}^{N} \left| \frac{R}{a} \right|^{p_n+1} \\ +3\sum_{n=N}^{\infty} \left(\frac{R}{2R}\right)^{p_n+1} &< \infty.\end{aligned}$$

By the Weierstrass M test, the series in 23.6 converges uniformly for $|z| < R$ and so the same is true of the infinite product. It follows from Lemma 18.18 on Page 396 that $P(z)$ is analytic on $|z| < R$ because it is a uniform limit of analytic functions.

Also by Theorem 23.2 the zeros of the analytic $P(z)$ are exactly the points, $\{z_n\}$, listed according to multiplicity. That is, if z_n is a zero of order m, then if it is listed m times in the formula for $P(z)$, then it is a zero of order m for P. This proves the theorem.

The following corollary is an easy consequence and includes the case where there is a zero at 0.

Corollary 23.7 *Let $\{z_n\}$ be a sequence of nonzero complex numbers which have no limit point and let $\{p_n\}$ be a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+p_n} < \infty \tag{23.7}$$

for all $r \in \mathbb{R}$. Then

$$P(z) \equiv z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

is analytic Ω and has a zero at each point, z_n and at no others along with a zero of order m at 0. If w occurs m times in $\{z_n\}$, then P has a zero of order m at w.

The above theory can be generalized to include the case of an arbitrary open set. First, here is a lemma.

Lemma 23.8 *Let Ω be an open set. Also let $\{z_n\}$ be a sequence of points in Ω which is bounded and which has no point repeated more than finitely many times such that $\{z_n\}$ has no limit point in Ω. Then there exist $\{w_n\} \subseteq \partial\Omega$ such that $\lim_{n\to\infty} |z_n - w_n| = 0$.*

Proof: Since $\partial\Omega$ is closed, there exists $w_n \in \partial\Omega$ such that $\operatorname{dist}(z_n, \partial\Omega) = |z_n - w_n|$. Now if there is a subsequence, $\{z_{n_k}\}$ such that $|z_{n_k} - w_{n_k}| \geq \varepsilon$ for all k, then $\{z_{n_k}\}$ must possess a limit point because it is a bounded infinite set of points. However, this limit point can only be in Ω because $\{z_{n_k}\}$ is bounded away from $\partial\Omega$. This is a contradiction. Therefore, $\lim_{n\to\infty} |z_n - w_n| = 0$. This proves the lemma.

Corollary 23.9 *Let $\{z_n\}$ be a sequence of complex numbers contained in Ω, an open subset of $\mathbb{C}$ which has no limit point in Ω. Suppose each z_n is repeated no more than finitely many times. Then there exists a function f which is analytic on Ω whose zeros are exactly $\{z_n\}$. If $w \in \{z_n\}$ and w is listed m times, then w is a zero of order m of f.*

Proof: There is nothing to prove if $\{z_n\}$ is finite. You just let $f(z) = \prod_{j=1}^{m} (z - z_j)$ where $\{z_n\} = \{z_1, \cdots, z_m\}$.

Pick $w \in \Omega \setminus \{z_n\}_{n=1}^{\infty}$ and let $h(z) \equiv \frac{1}{z-w}$. Since w is not a limit point of $\{z_n\}$, there exists $r > 0$ such that $B(w, r)$ contains no points of $\{z_n\}$. Let $\Omega_1 \equiv \Omega \setminus \{w\}$. Now h is not constant and so $h(\Omega_1)$ is an open set by the open mapping theorem. In fact, h maps each component of Ω to a region. $|z_n - w| > r$ for all z_n and so $|h(z_n)| < r^{-1}$. Thus the sequence, $\{h(z_n)\}$ is a bounded sequence in the open set $h(\Omega_1)$. It has no limit point in $h(\Omega_1)$ because this is true of $\{z_n\}$ and Ω_1. By Lemma 23.8 there exist $w_n \in \partial(h(\Omega_1))$ such that $\lim_{n\to\infty} |w_n - h(z_n)| = 0$. Consider for $z \in \Omega_1$

$$f(z) \equiv \prod_{n=1}^{\infty} E_n\left(\frac{h(z_n) - w_n}{h(z) - w_n}\right). \tag{23.8}$$

Letting K be a compact subset of Ω_1, $h(K)$ is a compact subset of $h(\Omega_1)$ and so if $z \in K$, then $|h(z) - w_n|$ is bounded below by a positive constant. Therefore, there exists N large enough that for all $z \in K$ and $n \geq N$,

$$\left|\frac{h(z_n) - w_n}{h(z) - w_n}\right| < \frac{1}{2}$$

and so by Corollary 23.5, for all $z \in K$ and $n \geq N$,

$$\left|E_n\left(\frac{h(z_n) - w_n}{h(z) - w_n}\right) - 1\right| \leq 3\left(\frac{1}{2}\right)^n. \tag{23.9}$$

Therefore,

$$\sum_{n=1}^{\infty} \left| E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right) - 1 \right|$$

converges uniformly for $z \in K$. This implies $\prod_{n=1}^{\infty} E_n \left(\frac{h(z_n)-w_n}{h(z)-w_n} \right)$ also converges uniformly for $z \in K$ by Theorem 23.2. Since K is arbitrary, this shows f defined in 23.8 is analytic on Ω_1.

Also if z_n is listed m times so it is a zero of multiplicity m and w_n is the point from $\partial(h(\Omega_1))$ closest to $h(z_n)$, then there are m factors in 23.8 which are of the form

$$\begin{aligned} E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right) &= \left(1 - \frac{h(z_n) - w_n}{h(z) - w_n} \right) e^{g_n(z)} \\ &= \left(\frac{h(z) - h(z_n)}{h(z) - w_n} \right) e^{g_n(z)} \\ &= \frac{z_n - z}{(z - w)(z_n - w)} \left(\frac{1}{h(z) - w_n} \right) e^{g_n(z)} \\ &= (z - z_n) G_n(z) \end{aligned} \tag{23.10}$$

where G_n is an analytic function which is not zero at and near z_n. Therefore, f has a zero of order m at z_n. This proves the theorem except for the point, w which has been left out of Ω_1. It is necessary to show f is analytic at this point also and right now, f is not even defined at w.

The $\{w_n\}$ are bounded because $\{h(z_n)\}$ is bounded and $\lim_{n \to \infty} |w_n - h(z_n)| = 0$ which implies $|w_n - h(z_n)| \leq C$ for some constant, C. Therefore, there exists $\delta > 0$ such that if $z \in B'(w, \delta)$, then for all n,

$$\left| \frac{h(z_n) - w}{\left(\frac{1}{z-w} \right) - w_n} \right| = \left| \frac{h(z_n) - w_n}{h(z) - w_n} \right| < \frac{1}{2}.$$

Thus 23.9 holds for all $z \in B'(w, \delta)$ and n so by Theorem 23.2, the infinite product in 23.8 converges uniformly on $B'(w, \delta)$. This implies f is bounded in $B'(w, \delta)$ and so w is a removable singularity and f can be extended to w such that the result is analytic. It only remains to verify $f(w) \neq 0$. After all, this would not do because it would be another zero other than those in the given list. By 23.10, a partial product is of the form

$$\prod_{n=1}^{N} \left(\frac{h(z) - h(z_n)}{h(z) - w_n} \right) e^{g_n(z)} \tag{23.11}$$

where

$$g_n(z) \equiv \left(\frac{h(z_n) - w_n}{h(z) - w_n} + \frac{1}{2} \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right)^2 + \cdots + \frac{1}{n} \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right)^n \right)$$

Each of the quotients in the definition of $g_n(z)$ converges to 0 as $z \to w$ and so the partial product of 23.11 converges to 1 as $z \to w$ because $\left(\frac{h(z)-h(z_n)}{h(z)-w_n}\right) \to 1$ as $z \to w$.

If $f(w) = 0$, then if z is close enough to w, it follows $|f(z)| < \frac{1}{2}$. Also, by the uniform convergence on $B'(w, \delta)$, it follows that for some N, the partial product up to N must also be less than $1/2$ in absolute value for all z close enough to w and as noted above, this does not occur because such partial products converge to 1 as $z \to w$. Hence $f(w) \neq 0$. This proves the corollary.

Recall the definition of a meromorphic function on Page 410. It was a function which is analytic everywhere except at a countable set of isolated points at which the function has a pole. It is clear that the quotient of two analytic functions yields a meromorphic function but is this the only way it can happen?

Theorem 23.10 *Suppose Q is a meromorphic function on an open set, Ω. Then there exist analytic functions on $\Omega, f(z)$ and $g(z)$ such that $Q(z) = f(z)/g(z)$ for all z not in the set of poles of Q.*

Proof: Let Q have a pole of order $m(z)$ at z. Then by Corollary 23.9 there exists an analytic function, g which has a zero of order $m(z)$ at every $z \in \Omega$. It follows gQ has a removable singularity at the poles of Q. Therefore, there is an analytic function, f such that $f(z) = g(z)Q(z)$. This proves the theorem.

Corollary 23.11 *Suppose Ω is a region and Q is a meromorphic function defined on Ω such that the set, $\{z \in \Omega : Q(z) = c\}$ has a limit point in Ω. Then $Q(z) = c$ for all $z \in \Omega$.*

Proof: From Theorem 23.10 there are analytic functions, f, g such that $Q = \frac{f}{g}$. Therefore, the zero set of the function, $f(z) - cg(z)$ has a limit point in Ω and so $f(z) - cg(z) = 0$ for all $z \in \Omega$. This proves the corollary.

23.2 Factoring A Given Analytic Function

The next theorem is the Weierstrass factorization theorem which can be used to factor a given analytic function f. If f has a zero of order m when $z = 0$, then you could factor out a z^m and from there consider the factorization of what remains when you have factored out the z^m. Therefore, the following is the main thing of interest.

Theorem 23.12 *Let f be analytic on $\mathbb{C}$, $f(0) \neq 0$, and let the zeros of f, be $\{z_k\}$, listed according to order. (Thus if z is a zero of order m, it will be listed m times in the list, $\{z_k\}$.) Choosing nonnegative integers, p_n such that for all $r > 0$,*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{p_n+1} < \infty,$$

There exists an entire function, g such that

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right). \tag{23.12}$$

Note that $e^{g(z)} \neq 0$ for any z and this is the interesting thing about this function.

Proof: $\{z_n\}$ cannot have a limit point because if there were a limit point of this sequence, it would follow from Theorem 18.23 that $f(z) = 0$ for all z, contradicting the hypothesis that $f(0) \neq 0$. Hence $\lim_{n\to\infty} |z_n| = \infty$ and so

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+n-1} = \sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{n} < \infty$$

by the root test. Therefore, by Theorem 23.6

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

a function analytic on $\mathbb{C}$ by picking $p_n = n-1$ or perhaps some other choice. ($p_n = n-1$ works but there might be another choice that would work.) Then f/P has only removable singularities in $\mathbb{C}$ and no zeros thanks to Theorem 23.6. Thus, letting $h(z) = f(z)/P(z)$, Corollary 18.50 implies that h'/h has a primitive, $\widetilde{g}$. Then

$$\left(he^{-\widetilde{g}}\right)' = 0$$

and so

$$h(z) = e^{a+ib}e^{\widetilde{g}(z)}$$

for some constants, a, b. Therefore, letting $g(z) = \widetilde{g}(z) + a + ib$, $h(z) = e^{g(z)}$ and thus 23.12 holds. This proves the theorem.

Corollary 23.13 *Let f be analytic on $\mathbb{C}$, f has a zero of order m at 0, and let the other zeros of f be $\{z_k\}$, listed according to order. (Thus if z is a zero of order l, it will be listed l times in the list, $\{z_k\}$.) Also let*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+p_n} < \infty \tag{23.13}$$

for any choice of $r > 0$. Then there exists an entire function, g such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right). \tag{23.14}$$

Proof: Since f has a zero of order m at 0, it follows from Theorem 18.23 that $\{z_k\}$ cannot have a limit point in $\mathbb{C}$ and so you can apply Theorem 23.12 to the function, $f(z)/z^m$ which has a removable singularity at 0. This proves the corollary.

23.2.1 Factoring Some Special Analytic Functions

Factoring a polynomial is in general a hard task. It is true it is easy to prove the factors exist but finding them is another matter. Corollary 23.13 gives the existence of factors of a certain form but it does not tell how to find them. This should not be surprising. You can't expect things to get easier when you go from polynomials to analytic functions. Nevertheless, it is possible to factor some popular analytic functions. These factorizations are based on the following Mitag-Leffler expansions. By an auspicious choice of the contour and the method of residues it is possible to obtain a very interesting formula for $\cot \pi z$.

Example 23.14 *Let γ_N be the contour which goes from $-N-\frac{1}{2}-Ni$ horizontally to $N+\frac{1}{2}-Ni$ and from there, vertically to $N+\frac{1}{2}+Ni$ and then horizontally to $-N-\frac{1}{2}+Ni$ and finally vertically to $-N-\frac{1}{2}-Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the integral*

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z \left(\alpha^2 - z^2\right)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag-Leffler,

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi \alpha. \tag{23.15}$$

First you show that $\cot \pi z$ is bounded on this contour. This is easy using the formula for $\cot(z) = \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$. Therefore, $I_N \to 0$ as $N \to \infty$ because the integrand is of order $1/N^2$ while the diameter of γ_N is of order N. Next you compute the residues of the integrand at $\pm\alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, using the residue theorem, you can evaluate I_N by using these. You can calculate these residues and find that the residue at $\pm\alpha$ is

$$\frac{-\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore

$$0 = \lim_{N\to\infty} I_N = \lim_{N\to\infty} 2\pi i \left[\sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi \alpha}{\alpha} \right]$$

which establishes the following formula of Mittag Leffler.

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

Writing this in a slightly nicer form, you obtain 23.15.

This is a very interesting formula. This will be used to factor $\sin(\pi z)$. The zeros of this function are at the integers. Therefore, considering 23.13 you can pick $p_n = 1$ in the Weierstrass factorization formula. Therefore, by Corollary 23.13 there exists an analytic function $g(z)$ such that

$$\sin(\pi z) = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \tag{23.16}$$

where the z_n are the nonzero integers. Remember you can permute the factors in these products. Therefore, this can be written more conveniently as

$$\sin(\pi z) = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

and it is necessary to find $g(z)$. Differentiating both sides of 23.16

$$\begin{aligned} \pi \cos(\pi z) &= e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) + z g'(z) e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) \\ &\quad + z e^{g(z)} \sum_{n=1}^{\infty} - \left(\frac{2z}{n^2}\right) \prod_{k \neq n} \left(1 - \left(\frac{z}{k}\right)^2\right) \end{aligned}$$

Now divide both sides by $\sin(\pi z)$ to obtain

$$\begin{aligned} \pi \cot(\pi z) &= \frac{1}{z} + g'(z) - \sum_{n=1}^{\infty} \frac{2z/n^2}{(1 - z^2/n^2)} \\ &= \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \end{aligned}$$

By 23.15, this yields $g'(z) = 0$ for z not an integer and so $g(z) = c$, a constant. So far this yields

$$\sin(\pi z) = z e^{c} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

and it only remains to find c. Divide both sides by πz and take a limit as $z \to 0$. Using the power series of $\sin(\pi z)$, this yields

$$1 = \frac{e^c}{\pi}$$

and so $c = \ln \pi$. Therefore,

$$\sin(\pi z) = z\pi \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right). \tag{23.17}$$

Example 23.15 *Find an interesting formula for* $\tan(\pi z)$.

This is easy to obtain from the formula for $\cot(\pi z)$.

$$\cot\left(\pi\left(z+\frac{1}{2}\right)\right) = -\tan \pi z$$

for z real and therefore, this formula holds for z complex also. Therefore, for $z+\frac{1}{2}$ not an integer

$$\pi \cot\left(\pi\left(z+\frac{1}{2}\right)\right) = \frac{2}{2z+1} + \sum_{n=1}^{\infty} \frac{2z+1}{\left(\frac{2z+1}{2}\right)^2 - n^2}$$

23.3 The Existence Of An Analytic Function With Given Values

The Weierstrass product formula, Theorem 23.6, along with the Mittag-Leffler theorem, Theorem 22.13 can be used to obtain an analytic function which has given values on a countable set of points, having no limit point. This is clearly an amazing result and indicates how potent these theorems are. In fact, you can show that it isn't just the values of the function which may be specified at the points in this countable set of points but the derivatives up to any finite order.

Theorem 23.16 *Let* $P \equiv \{z_k\}_{k=1}^{\infty}$ *be a set of points in* $\mathbb{C}$,*which has no limit point. For each* z_k, *consider*

$$\sum_{j=0}^{m_k} a_j^k (z-z_k)^j. \tag{23.18}$$

Then there exists an analytic function defined on $\mathbb{C}$ *such that the Taylor series of* f *at* z_k *has the first* m_k *terms given by 23.18.*[1]

Proof: By the Weierstrass product theorem, Theorem 23.6, there exists an analytic function, f defined on all of Ω such that f has a zero of order m_k+1 at z_k. Consider this z_k Thus for z near z_k,

$$f(z) = \sum_{j=m_k+1}^{\infty} c_j (z-z_k)^j$$

where $c_{m_k+1} \neq 0$. You choose $b_1, b_2, \cdots, b_{m_k+1}$ such that

$$f(z)\left(\sum_{l=1}^{m_k+1} \frac{b_l}{(z-z_k)^k}\right) = \sum_{j=0}^{m_k} a_j^k (z-z_k)^j + \sum_{k=m_k+1}^{\infty} c_j^k (z-z_k)^j.$$

[1]This says you can specify the first m_k derivatives of the function at the point z_k.

Thus you need

$$\sum_{l=1}^{m_k+1} \sum_{j=m_k+1}^{\infty} c_j b_l (z - z_k)^{j-l} = \sum_{r=0}^{m_k} a_r^k (z - z_k)^r + \text{Higher order terms.}$$

It follows you need to solve the following system of equations for $b_1, \cdots, b_{m_k+1}$.

$$\begin{gathered} c_{m_k+1} b_{m_k+1} = a_0^k \\ c_{m_k+2} b_{m_k+1} + c_{m_k+1} b_{m_k} = a_1^k \\ c_{m_k+3} b_{m_k+1} + c_{m_k+2} b_{m_k} + c_{m_k+1} b_{m_k-1} = a_2^k \\ \vdots \\ c_{m_k+m_k+1} b_{m_k+1} + c_{m_k+m_k} b_{m_k} + \cdots + c_{m_k+1} b_1 = a_{m_k}^k \end{gathered}$$

Since $c_{m_k+1} \neq 0$, it follows there exists a unique solution to the above system. You first solve for b_{m_k+1} in the top. Then, having found it, you go to the next and use $c_{m_k+1} \neq 0$ again to find b_{m_k} and continue in this manner. Let $S_k(z)$ be determined in this manner for each z_k. By the Mittag-Leffler theorem, there exists a Meromorphic function, g such that g has exactly the singularities, $S_k(z)$. Therefore, $f(z) g(z)$ has removable singularities at each z_k and for z near z_k, the first m_k terms of fg are as prescribed. This proves the theorem.

Corollary 23.17 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in Ω, an open set such that P has no limit points in Ω. For each z_k, consider*

$$\sum_{j=0}^{m_k} a_j^k (z - z_k)^j . \tag{23.19}$$

Then there exists an analytic function defined on Ω such that the Taylor series of f at z_k has the first m_k terms given by 23.19.

Proof: The proof is identical to the above except you use the versions of the Mittag-Leffler theorem and Weierstrass product which pertain to open sets.

Definition 23.18 *Denote by $H(\Omega)$ the analytic functions defined on Ω, an open subset of $\mathbb{C}$. Then $H(\Omega)$ is a commutative ring[2] with the usual operations of addition and multiplication. A set, $I \subseteq H(\Omega)$ is called a finitely generated ideal of the ring if I is of the form*

$$\left\{ \sum_{k=1}^{n} g_k f_k : f_k \in H(\Omega) \text{ for } k = 1, 2, \cdots, n \right\}$$

where $g_1, \cdots, g_n$ are given functions in $H(\Omega)$. This ideal is also denoted as $[g_1, \cdots, g_n]$ and is called the ideal generated by the functions, $\{g_1, \cdots, g_n\}$. Since there are finitely many of these functions it is called a finitely generated ideal. A principal ideal is one which is generated by a single function. An example of such a thing is $[1] = H(\Omega)$.

[2]It is not a field because you can't divide two analytic functions and get another one.

Then there is the following interesting theorem.

Theorem 23.19 *Every finitely generated ideal in $H(\Omega)$ for Ω a connected open set (region) is a principal ideal.*

Proof: Let $I = [g_1, \cdots, g_n]$ be a finitely generated ideal as described above. Then if any of the functions has no zeros, this ideal would consist of $H(\Omega)$ because then $g_i^{-1} \in H(\Omega)$ and so $1 \in I$. It follows all the functions have zeros. If any of the functions has a zero of infinite order, then the function equals zero on Ω because Ω is connected and can be deleted from the list. Similarly, if the zeros of any of these functions have a limit point in Ω, then the function equals zero and can be deleted from the list. Thus, without loss of generality, all zeros are of finite order and there are no limit points of the zeros in Ω. Let $m(g_i, z)$ denote the order of the zero of g_i at z. If g_i has no zero at z, then $m(g_i, z) = 0$.

I claim that if no point of Ω is a zero of all the g_i, then the conclusion of the theorem is true and in fact $[g_1, \cdots, g_n] = [1] = H(\Omega)$. The claim is obvious if $n = 1$ because this assumption that no point is a zero of all the functions implies $g \neq 0$ and so g^{-1} is analytic. Hence $1 \in [g_1]$. Suppose it is true for $n-1$ and consider $[g_1, \cdots, g_n]$ where no point of Ω is a zero of all the g_i. Even though this may be true of $\{g_1, \cdots, g_n\}$, it may not be true of $\{g_1, \cdots, g_{n-1}\}$. By Corollary 23.9 there exists ϕ, a function analytic on Ω such that $m(\phi, z) = \min\{m(g_i, z), i = 1, 2, \cdots, n-1\}$. Thus the functions $\{g_1/\phi, \cdots, g_{n-1}/\phi\}$ .are all analytic. Could they all equal zero at some point, z? If so, pick i where $m(\phi, z) = m(g_i, z)$. Thus g_i/ϕ is not equal to zero at z after all and so these functions are analytic there is no point of Ω which is a zero of all of them. By induction, $[g_1/\phi, \cdots, g_{n-1}/\phi] = H(\Omega)$. (Also there are no new zeros obtained in this way.)

Now this means there exist functions $f_i \in H(\Omega)$ such that

$$\sum_{i=1}^{n} f_i \left(\frac{g_i}{\phi}\right) = 1$$

and so $\phi = \sum_{i=1}^{n} f_i g_i$. Therefore, $[\phi] \subseteq [g_1, \cdots, g_{n-1}]$. On the other hand, if $\sum_{k=1}^{n-1} h_k g_k \in [g_1, \cdots, g_{n-1}]$ you could define $h \equiv \sum_{k=1}^{n-1} h_k (g_k/\phi)$, an analytic function with the property that $h\phi = \sum_{k=1}^{n-1} h_k g_k$ which shows $[\phi] = [g_1, \cdots, g_{n-1}]$. Therefore,

$$[g_1, \cdots, g_n] = [\phi, g_n]$$

Now ϕ has no zeros in common with g_n because the zeros of ϕ are contained in the set of zeros for $g_1, \cdots, g_{n-1}$. Now consider a zero, α of ϕ. It is not a zero of g_n and so near α, these functions have the form

$$\phi(z) = \sum_{k=m}^{\infty} a_k (z-\alpha)^k,\ g_n(z) = \sum_{k=0}^{\infty} b_k (z-\alpha)^k,\ b_0 \neq 0.$$

I want to determine coefficients for an analytic function, h such that

$$m(1 - hg_n, \alpha) \geq m(\phi, \alpha). \tag{23.20}$$

Let

$$h(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$$

and the c_k must be determined. Using Merten's theorem, the power series for $1-hg_n$ is of the form

$$1 - b_0 c_0 - \sum_{j=1}^{\infty} \left(\sum_{r=0}^{j} b_{j-r} c_r \right) (z-\alpha)^j .$$

First determine c_0 such that $1 - c_0 b_0 = 0$. This is no problem because $b_0 \neq 0$. Next you need to get the coefficients of $(z-\alpha)$ to equal zero. This requires

$$b_1 c_0 + b_0 c_1 = 0.$$

Again, there is no problem because $b_0 \neq 0$. In fact, $c_1 = (-b_1 c_0 / b_0)$. Next consider the second order terms if $m \geq 2$.

$$b_2 c_0 + b_1 c_1 + b_0 c_2 = 0$$

Again there is no problem in solving, this time for c_2 because $b_0 \neq 0$. Continuing this way, you see that in every step, the c_k which needs to be solved for is multiplied by $b_0 \neq 0$. Therefore, by Corollary 23.9 there exists an analytic function, h satisfying 23.20. Therefore, $(1 - hg_n)/\phi$ has a removable singularity at every zero of ϕ and so may be considered an analytic function. Therefore,

$$1 = \frac{1 - hg_n}{\phi} \phi + hg_n \in [\phi, g_n] = [g_1 \cdots g_n]$$

which shows $[g_1 \cdots g_n] = H(\Omega) = [1]$. It follows the claim is established.

Now suppose $\{g_1 \cdots g_n\}$ are just elements of $H(\Omega)$. As explained above, it can be assumed they all have zeros of finite order and the zeros have no limit point in Ω since if these occur, you can delete the function from the list. By Corollary 23.9 there exists $\phi \in H(\Omega)$ such that $m(\phi, z) \leq \min\{m(g_i, z) : i = 1, \cdots, n\}$. Then g_k/ϕ has a removable singularity at each zero of g_k and so can be regarded as an analytic function. Also, as before, there is no point which is a zero of each g_k/ϕ and so by the first part of this argument, $[g_1/\phi \cdots g_n/\phi] = H(\Omega)$. As in the first part of the argument, this implies $[g_1 \cdots g_n] = [\phi]$ which proves the theorem. $[g_1 \cdots g_n]$ is a principal ideal as claimed.

The following corollary follows from the above theorem. You don't need to assume Ω is connected.

Corollary 23.20 *Every finitely generated ideal in $H(\Omega)$ for Ω an open set is a principal ideal.*

Proof: Let $[g_1, \cdots, g_n]$ be a finitely generated ideal in $H(\Omega)$. Let $\{U_k\}$ be the components of Ω. Then applying the above to each component, there exists $h_k \in H(U_k)$ such that restricting each g_i to U_k, $[g_1, \cdots, g_n] = [h_k]$. Then let $h(z) = h_k(z)$ for $z \in U_k$. This is an analytic function which works.

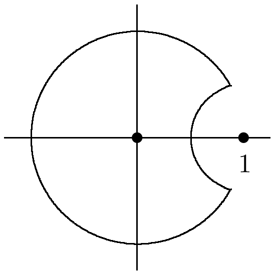
23.4 Jensen's Formula

This interesting formula relates the zeros of an analytic function to an integral. The proof given here follows Alfors, [2]. First, here is a technical lemma.

Lemma 23.21
$$\int_{-\pi}^{\pi} \ln\left|1-e^{i\theta}\right| d\theta = 0.$$

Proof: First note that the only problem with the integrand occurs when $\theta = 0$. However, this is an integrable singularity so the integral will end up making sense. Letting $z = e^{i\theta}$, you could get the above integral as a limit as $\varepsilon \to 0$ of the following contour integral where γ_ε is the contour shown in the following picture with the radius of the big circle equal to 1 and the radius of the little circle equal to ε..

$$\int_{\gamma_\varepsilon} \frac{\ln|1-z|}{iz} dz.$$



On the indicated contour, $1-z$ lies in the half plane $\operatorname{Re} z > 0$ and so $\log(1-z) = \ln|1-z| + i\arg(1-z)$. The above integral equals

$$\int_{\gamma_\varepsilon} \frac{\log(1-z)}{iz} dz - \int_{\gamma_\varepsilon} \frac{\arg(1-z)}{z} dz$$

The first of these integrals equals zero because the integrand has a removable singularity at 0. The second equals

$$i\int_{-\pi}^{-\eta_\varepsilon} \arg\left(1-e^{i\theta}\right) d\theta + i\int_{\eta_\varepsilon}^{\pi} \arg\left(1-e^{i\theta}\right) d\theta$$
$$+\varepsilon i\int_{-\frac{\pi}{2}-\lambda_\varepsilon}^{-\pi} \theta d\theta + \varepsilon i\int_{\pi}^{\frac{\pi}{2}-\lambda_\varepsilon} \theta d\theta$$

where $\eta_\varepsilon, \lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. The last two terms converge to 0 as $\varepsilon \to 0$ while the first two add to zero. To see this, change the variable in the first integral and then recall that when you multiply complex numbers you add the arguments. Thus you end up integrating arg (real valued function) which equals zero.

In this material on Jensen's equation, ε will denote a small positive number. Its value is not important as long as it is positive. Therefore, it may change from place to place. Now suppose f is analytic on $B(0, r+\varepsilon)$, and f has no zeros on $\overline{B(0,r)}$. Then you can define a branch of the logarithm which makes sense for complex numbers near $f(z)$. Thus $z \to \log(f(z))$ is analytic on $B(0, r+\varepsilon)$. Therefore, its real part, $u(x,y) \equiv \ln|f(x+iy)|$ must be harmonic. Consider the following lemma.

Lemma 23.22 *Let u be harmonic on $B(0, r+\varepsilon)$. Then*

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta.$$

Proof: For a harmonic function, u defined on $B(0, r+\varepsilon)$, there exists an analytic function, $h = u + iv$ where

$$v(x,y) \equiv \int_0^y u_x(x,t)\, dt - \int_0^x u_y(t,0)\, dt.$$

By the Cauchy integral theorem,

$$h(0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{h(z)}{z} dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} h\left(re^{i\theta}\right) d\theta.$$

Therefore, considering the real part of h,

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta.$$

This proves the lemma.

Now this shows the following corollary.

Corollary 23.23 *Suppose f is analytic on $B(0, r+\varepsilon)$ and has no zeros on $\overline{B(0,r)}$. Then*

$$\ln|f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| \tag{23.21}$$

What if f has some zeros on $|z| = r$ but none on $B(0,r)$? It turns out 23.21 is still valid. Suppose the zeros are at $\left\{re^{i\theta_k}\right\}_{k=1}^m$, listed according to multiplicity. Then let

$$g(z) = \frac{f(z)}{\prod_{k=1}^m \left(z - re^{i\theta_k}\right)}.$$

It follows g is analytic on $B(0, r+\varepsilon)$ but has no zeros in $\overline{B(0,r)}$. Then 23.21 holds for g in place of f. Thus

$$\begin{aligned}
&\ln|f(0)| - \sum_{k=1}^{m} \ln|r| \\
= \;& \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m} \ln\left|re^{i\theta} - re^{i\theta_k}\right| d\theta \\
= \;& \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m} \ln\left|e^{i\theta} - e^{i\theta_k}\right| d\theta - \sum_{k=1}^{m}\ln|r| \\
= \;& \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m} \ln\left|e^{i\theta} - 1\right| d\theta - \sum_{k=1}^{m}\ln|r|
\end{aligned}$$

Therefore, 23.21 will continue to hold exactly when $\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\ln\left|e^{i\theta}-1\right| d\theta = 0$. But this is the content of Lemma 23.21. This proves the following lemma.

Lemma 23.24 *Suppose f is analytic on $B(0, r+\varepsilon)$ and has no zeros on $B(0,r)$. Then*

$$\ln|f(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left|f\left(re^{i\theta}\right)\right| \tag{23.22}$$

With this preparation, it is now not too hard to prove Jensen's formula. Suppose there are n zeros of f in $B(0,r)$, $\{a_k\}_{k=1}^{n}$, listed according to multiplicity, none equal to zero. Let

$$F(z) \equiv f(z)\prod_{i=1}^{n}\frac{r^2 - \overline{a_i}z}{r(z-a_i)}.$$

Then F is analytic on $B(0, r+\varepsilon)$ and has no zeros in $B(0,r)$. The reason for this is that $f(z)/\prod_{i=1}^{n} r(z-a_i)$ has no zeros there and $r^2 - \overline{a_i}z$ cannot equal zero if $|z| < r$ because if this expression equals zero, then

$$|z| = \frac{r^2}{|a_i|} > r.$$

The other interesting thing about $F(z)$ is that when $z = re^{i\theta}$,

$$\begin{aligned}
F\left(re^{i\theta}\right) &= f\left(re^{i\theta}\right)\prod_{i=1}^{n}\frac{r^2 - \overline{a_i}re^{i\theta}}{r\left(re^{i\theta}-a_i\right)} \\
&= f\left(re^{i\theta}\right)\prod_{i=1}^{n}\frac{r - \overline{a_i}e^{i\theta}}{\left(re^{i\theta}-a_i\right)} = f\left(re^{i\theta}\right)e^{i\theta}\prod_{i=1}^{n}\frac{re^{-i\theta} - \overline{a_i}}{re^{i\theta}-a_i}
\end{aligned}$$

so $\left|F\left(re^{i\theta}\right)\right| = \left|f\left(re^{i\theta}\right)\right|$.

Theorem 23.25 *Let f be analytic on $B(0, r + \varepsilon)$ and suppose $f(0) \neq 0$. If the zeros of f in $B(0, r)$ are $\{a_k\}_{k=1}^{n}$, listed according to multiplicity, then*

$$\ln |f(0)| = -\sum_{i=1}^{n} \ln \left(\frac{r}{|a_i|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(re^{i\theta} \right) \right| d\theta.$$

Proof: From the above discussion and Lemma 23.24,

$$\ln |F(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \left| f\left(re^{i\theta} \right) \right| d\theta$$

But $F(0) = f(0) \prod_{i=1}^{n} \frac{r}{a_i}$ and so $\ln |F(0)| = \ln |f(0)| + \sum_{i=1}^{n} \ln \left| \frac{r}{a_i} \right|$. Therefore,

$$\ln |f(0)| = -\sum_{i=1}^{n} \ln \left| \frac{r}{a_i} \right| + \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(re^{i\theta} \right) \right| d\theta$$

as claimed.

Written in terms of exponentials this is

$$|f(0)| \prod_{k=1}^{n} \left| \frac{r}{a_k} \right| = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(re^{i\theta} \right) \right| d\theta \right).$$

23.5 Blaschke Products

The Blaschke[3] product is a way to produce a function which is bounded and analytic on $B(0,1)$ which also has given zeros in $B(0,1)$. The interesting thing here is that there may be infinitely many of these zeros. Thus, unlike the above case of Jensen's inequality, the function is not analytic on $\overline{B(0,1)}$. Recall for purposes of comparison, Liouville's theorem which says bounded entire functions are constant. The Blaschke product gives examples of bounded functions on $B(0,1)$ which are definitely not constant.

Theorem 23.26 *Let $\{\alpha_n\}$ be a sequence of nonzero points in $B(0,1)$ with the property that*

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$

Then for $k \geq 0$, an integer

$$B(z) \equiv z^k \prod_{k=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}$$

is a bounded function which is analytic on $B(0,1)$ which has zeros only at 0 if $k > 0$ and at the α_n.

[3] Wilhelm Blaschke, 1915

Proof: From Theorem 23.2 the above product will converge uniformly on $B(0,r)$ for $r<1$ to an analytic function if

$$\sum_{k=1}^{\infty}\left|\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}-1\right|$$

converges uniformly on $B(0,r)$. But for $|z|<r$,

$$\begin{aligned}
&\left|\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}-1\right| \\
= &\left|\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}-\frac{\alpha_n(1-\overline{\alpha_n}z)}{\alpha_n(1-\overline{\alpha_n}z)}\right| \\
= &\left|\frac{|\alpha_n|\alpha_n-|\alpha_n|z-\alpha_n+|\alpha_n|^2 z}{(1-\overline{\alpha_n}z)\alpha_n}\right| \\
= &\left|\frac{|\alpha_n|\alpha_n-\alpha_n-|\alpha_n|z+|\alpha_n|^2 z}{(1-\overline{\alpha_n}z)\alpha_n}\right| \\
= &\ ||\alpha_n|-1|\left|\frac{\alpha_n+z|\alpha_n|}{(1-\overline{\alpha_n}z)\alpha_n}\right| \\
= &\ ||\alpha_n|-1|\left|\frac{1+z(|\alpha_n|/\alpha_n)}{(1-\overline{\alpha_n}z)}\right| \\
\leq &\ ||\alpha_n|-1|\left|\frac{1+|z|}{1-|z|}\right| \leq ||\alpha_n|-1|\left|\frac{1+r}{1-r}\right|
\end{aligned}$$

and so the assumption on the sum gives uniform convergence of the product on $B(0,r)$ to an analytic function. Since $r<1$ is arbitrary, this shows $B(z)$ is analytic on $B(0,1)$ and has the specified zeros because the only place the factors equal zero are at the α_n or 0.

Now consider the factors in the product. The claim is that they are all no larger in absolute value than 1. This is very easy to see from the maximum modulus theorem. Let $|\alpha|<1$ and $\phi(z)=\frac{\alpha-z}{1-\overline{\alpha}z}$. Then ϕ is analytic near $B(0,1)$ because its only pole is $1/\overline{\alpha}$. Consider $z=e^{i\theta}$. Then

$$\left|\phi\left(e^{i\theta}\right)\right|=\left|\frac{\alpha-e^{i\theta}}{1-\overline{\alpha}e^{i\theta}}\right|=\left|\frac{1-\alpha e^{-i\theta}}{1-\overline{\alpha}e^{i\theta}}\right|=1.$$

Thus the modulus of $\phi(z)$ equals 1 on $\partial B(0,1)$. Therefore, by the maximum modulus theorem, $|\phi(z)|<1$ if $|z|<1$. This proves the claim that the terms in the product are no larger than 1 and shows the function determined by the Blaschke product is bounded. This proves the theorem.

Note in the conditions for this theorem the one for the sum, $\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$. The Blaschke product gives an analytic function, whose absolute value is bounded by 1 and which has the α_n as zeros. What if you had a bounded function, analytic on $B(0,1)$ which had zeros at $\{\alpha_k\}$? Could you conclude the condition on the sum?

The answer is yes. In fact, you can get by with less than the assumption that f is bounded but this will not be presented here. See Rudin [36]. This theorem is an exciting use of Jensen's equation.

Theorem 23.27 *Suppose f is an analytic function on $B(0,1), f(0) \neq 0$, and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the zeros of f are $\{\alpha_k\}_{k=1}^{\infty}$, listed according to multiplicity. Then $\sum_{k=1}^{\infty}(1-|\alpha_k|) < \infty$.*

Proof: If there are only finitely many zeros, there is nothing to prove so assume there are infinitely many. Also let the zeros be listed such that $|\alpha_n| \leq |\alpha_{n+1}| \cdots$ Let $n(r)$ denote the number of zeros in $B(0,r)$. By Jensen's formula,

$$\ln|f(0)| + \sum_{i=1}^{n(r)} \ln r - \ln|\alpha_i| = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta \leq \ln(M).$$

Therefore, by the mean value theorem,

$$\sum_{i=1}^{n(r)} \frac{1}{r}(r-|\alpha_i|) \leq \sum_{i=1}^{n(r)} \ln r - \ln|\alpha_i| \leq \ln(M) - \ln|f(0)|$$

As $r \to 1-, n(r) \to \infty$, and so an application of Fatous lemma yields

$$\sum_{i=1}^{\infty}(1-|\alpha_i|) \leq \lim\inf_{r\to 1-} \sum_{i=1}^{n(r)} \frac{1}{r}(r-|\alpha_i|) \leq \ln(M) - \ln|f(0)|.$$

This proves the theorem.

You don't need the assumption that $f(0) \neq 0$.

Corollary 23.28 *Suppose f is an analytic function on $B(0,1)$ and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the nonzero zeros[4] of f are $\{\alpha_k\}_{k=1}^{\infty}$, listed according to multiplicity. Then $\sum_{k=1}^{\infty}(1-|\alpha_k|) < \infty$.*

Proof: Suppose f has a zero of order m at 0. Then consider the analytic function, $g(z) \equiv f(z)/z^m$ which has the same zeros except for 0. The argument goes the same way except here you use g instead of f and only consider $r > r_0 > 0$.

[4]This is a fun thing to say: nonzero zeros.

Thus from Jensen's equation,

$$\begin{aligned}
& \ln|g(0)| + \sum_{i=1}^{n(r)} \ln r - \ln|\alpha_i| \\
= & \frac{1}{2\pi}\int_0^{2\pi} \ln\left|g\left(re^{i\theta}\right)\right| d\theta \\
= & \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} m\ln(r) \\
\leq & M + \frac{1}{2\pi}\int_0^{2\pi} m\ln\left(r^{-1}\right) \\
\leq & M + m\ln\left(\frac{1}{r_0}\right).
\end{aligned}$$

Now the rest of the argument is the same.

An interesting restatement yields the following amazing result.

Corollary 23.29 *Suppose f is analytic and bounded on $B(0,1)$ having zeros $\{\alpha_n\}$. Then if $\sum_{k=1}^{\infty}(1-|\alpha_n|)=\infty$, it follows f is identically equal to zero.*

23.5.1 The Müntz-Szasz Theorem Again

Corollary 23.29 makes possible an easy proof of a remarkable theorem named above which yields a wonderful generalization of the Weierstrass approximation theorem. In what follows $b>0$. The Weierstrass approximation theorem states that linear combinations of $1, t, t^2, t^3, \cdots$ (polynomials) are dense in $C([0,b])$. Let $\lambda_1<\lambda_2<\lambda_3<\cdots$ be an increasing list of positive real numbers. This theorem tells when linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ are dense in $C([0,b])$. The proof which follows is like the one given in Rudin [36]. There is a much longer one in Cheney [12] which discusses more aspects of the subject. This other approach is much more elementary and does not depend in any way on the theory of functions of a complex variable. There are those of us who automatically prefer real variable techniques. Nevertheless, this proof by Rudin is a very nice and insightful application of the preceding material. Cheney refers to the theorem as the second Müntz theorem. I guess Szasz must also have been involved.

Theorem 23.30 *Let $\lambda_1<\lambda_2<\lambda_3<\cdots$ be an increasing list of positive real numbers and let $a>0$. If*

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty, \tag{23.23}$$

then linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ are dense in $C([0,b])$.

Proof: Let X denote the closure of linear combinations of $\left\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\right\}$ in $C([0,b])$. If $X\neq C([0,b])$, then letting $f\in C([0,b])\setminus X$, define $\Lambda\in C([0,b])'$ as follows. First let $\Lambda_0 : X + \mathbb{C}f$ be given by $\Lambda_0 (g + \alpha f) = \alpha \, ||f||_\infty$. Then

$$\begin{aligned} \sup_{||g+\alpha f|| \leq 1} |\Lambda_0 (g + \alpha f)| &= \sup_{||g+\alpha f|| \leq 1} |\alpha| \, ||f||_\infty \\ &= \sup_{||g/\alpha + f|| \leq \frac{1}{|\alpha|}} |\alpha| \, ||f||_\infty \\ &= \sup_{||g+f|| \leq \frac{1}{|\alpha|}} |\alpha| \, ||f||_\infty \end{aligned}$$

Now $\operatorname{dist}(f, X) > 0$ because X is closed. Therefore, there exists a lower bound, $\eta > 0$ to $||g + f||$ for $g \in X$. Therefore, the above is no larger than

$$\sup_{|\alpha| \leq \frac{1}{\eta}} |\alpha| \, ||f||_\infty = \left(\frac{1}{\eta} \right) ||f||_\infty$$

which shows that $||\Lambda_0|| \leq \left(\frac{1}{\eta} \right) ||f||_\infty$. By the Hahn Banach theorem Λ_0 can be extended to $\Lambda \in C([0, b])'$ which has the property that $\Lambda(X) = 0$ but $\Lambda(f) = ||f|| \neq 0$. By the Weierstrass approximation theorem, there exists a polynomial, p such that $\Lambda(p) \neq 0$. Therefore, if it can be shown that whenever $\Lambda(X) = 0$, it is the case that $\Lambda(p) = 0$ for all polynomials, it must be the case that X is dense in $C([0, b])$.

By the Riesz representation theorem the elements of $C([0, b])'$ are complex measures. Suppose then that for μ a complex measure it follows that for all t^{λ_k},

$$\int_{[0,b]} t^{\lambda_k} d\mu = 0.$$

I want to show that then

$$\int_{[0,b]} t^k d\mu = 0$$

for all positive integers. It suffices to modify μ is necessary to have $\mu(\{0\}) = 0$ since this will not change any of the above integrals. Let $\mu_1(E) = \mu(E \cap (0, b])$ and use μ_1. I will continue using the symbol, μ.

For $\operatorname{Re}(z) > 0$, define

$$F(z) \equiv \int_{[0,b]} t^z d\mu = \int_{(0,b]} t^z d\mu$$

The function $t^z = \exp(z \ln(t))$ is analytic. I claim that $F(z)$ is also analytic for $\operatorname{Re} z > 0$. Apply Morea's theorem. Let T be a triangle in $\operatorname{Re} z > 0$. Then

$$\int_{\partial T} F(z) \, dz = \int_{\partial T} \int_{(0,b]} e^{(z \ln(t))} \xi d |\mu| \, dz$$

Now $\int_{\partial T}$ can be split into three integrals over intervals of $\mathbb{R}$ and so this integral is essentially a Lebesgue integral taken with respect to Lebesgue measure. Furthermore,

$e^{(z\ln(t))}$ is a continuous function of the two variables and ξ is a function of only the one variable, t. Thus the integrand is product measurable. The iterated integral is also absolutely integrable because $\left|e^{(z\ln(t))}\right| \leq e^{x\ln t} \leq e^{x\ln b}$ where $x + iy = z$ and x is given to be positive. Thus the integrand is actually bounded. Therefore, you can apply Fubini's theorem and write

$$\begin{aligned}\int_{\partial T} F(z)\,dz &= \int_{\partial T}\int_{(0,b]} e^{(z\ln(t))}\xi d|\mu|\,dz \\ &= \int_{(0,b]} \xi \int_{\partial T} e^{(z\ln(t))} dz d|\mu| = 0.\end{aligned}$$

By Morea's theorem, F is analytic on $\operatorname{Re} z > 0$ which is given to have zeros at the λ_k.

Now let $\phi(z) = \frac{1+z}{1-z}$. Then ϕ maps $B(0,1)$ one to one onto $\operatorname{Re} z > 0$. To see this let $0 < r < 1$.

$$\phi\left(re^{i\theta}\right) = \frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1-r^2+i2r\sin\theta}{1+r^2-2r\cos\theta}$$

and so $\operatorname{Re}\phi\left(re^{i\theta}\right) > 0$. Now the inverse of ϕ is $\phi^{-1}(z) = \frac{z-1}{z+1}$. For $\operatorname{Re} z > 0$,

$$\left|\phi^{-1}(z)\right|^2 = \frac{z-1}{z+1}\cdot\frac{\overline{z}-1}{\overline{z}+1} = \frac{|z|^2 - 2\operatorname{Re} z + 1}{|z|^2 + 2\operatorname{Re} z + 1} < 1.$$

Consider $F \circ \phi$, an analytic function defined on $B(0,1)$. This function is given to have zeros at z_n where $\phi(z_n) = \frac{1+z_n}{1-z_n} = \lambda_n$. This reduces to $z_n = \frac{-1+\lambda_n}{1+\lambda_n}$. Now

$$1 - |z_n| \geq \frac{c}{1+\lambda_n}$$

for a positive constant, c. It is given that $\sum \frac{1}{\lambda_n} = \infty$. so it follows $\sum(1 - |z_n|) = \infty$ also. Therefore, by Corollary 23.29, $F \circ \phi = 0$. It follows $F = 0$ also. In particular, $F(k)$ for k a positive integer equals zero. This has shown that if $\Lambda \in C([0,b])'$ and Λ sends 1 and all the t^{λ_n} to 0, then Λ sends 1 and all t^k for k a positive integer to zero. As explained above, X is dense in $C((0,b])$.

The converse of this theorem is also true and is proved in Rudin [36].

23.6 Exercises

1. Suppose f is an entire function with $f(0) = 1$. Let

$$M(r) = \max\{|f(z)| : |z| = r\}.$$

Use Jensen's equation to establish the following inequality.

$$M(2r) \geq 2^{n(r)}$$

where $n(r)$ is the number of zeros of f in $\overline{B(0,r)}$.

2. The version of the Blaschke product presented above is that found in most complex variable texts. However, there is another one in [30]. Instead of $\frac{\alpha_n - z}{1-\overline{\alpha_n} z}\frac{|\alpha_n|}{\alpha_n}$ you use
$$\frac{\alpha_n - z}{\frac{1}{\overline{\alpha_n}} - z}$$
Prove a version of Theorem 23.26 using this modification.

3. The Weierstrass approximation theorem holds for polynomials of n variables on any compact subset of $\mathbb{R}^n$. Give a multidimensional version of the Müntz-Szasz theorem which will generalize the Weierstrass approximation theorem for n dimensions. You might just pick a compact subset of $\mathbb{R}^n$ in which all components are positive. You have to do something like this because otherwise, t^λ might not be defined.

4. Show $\cos(\pi z) = \prod_{k=1}^\infty \left(1 - \frac{4z^2}{(2k-1)^2}\right)$.

5. Recall $\sin(\pi z) = z\pi \prod_{n=1}^\infty \left(1 - \left(\frac{z}{n}\right)^2\right)$. Use this to derive Wallis product, $\frac{\pi}{2} = \prod_{k=1}^\infty \frac{4k^2}{(2k-1)(2k+1)}$.

6. The order of an entire function, f is defined as
$$\inf\left\{a \geq 0 : |f(z)| \leq e^{|z|^a} \text{ for all large enough } |z|\right\}$$
If no such a exists, the function is said to be of infinite order. Show the order of an entire function is also equal to $\limsup_{r\to\infty} \frac{\ln(\ln(M(r)))}{\ln(r)}$ where $M(r) \equiv \max\{|f(z)| : |z| = r\}$.

7. Suppose Ω is a simply connected region and let f be meromorphic on Ω. Suppose also that the set, $S \equiv \{z \in \Omega : f(z) = c\}$ has a limit point in Ω. Can you conclude $f(z) = c$ for all $z \in \Omega$?

8. This and the next collection of problems are dealing with the gamma function. Show that
$$\left|\left(1 + \frac{z}{n}\right)e^{\frac{-z}{n}} - 1\right| \leq \frac{C(z)}{n^2}$$
and therefore,
$$\sum_{n=1}^\infty \left|\left(1 + \frac{z}{n}\right)e^{\frac{-z}{n}} - 1\right| < \infty$$
with the convergence uniform on compact sets.

9. $\uparrow$ Show $\prod_{n=1}^\infty \left(1 + \frac{z}{n}\right)e^{\frac{-z}{n}}$ converges to an analytic function on $\mathbb{C}$ which has zeros only at the negative integers and that therefore,
$$\prod_{n=1}^\infty \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

is a meromorphic function (Analytic except for poles) having simple poles at the negative integers.

10. $\uparrow$Show there exists γ such that if
$$\Gamma(z) \equiv \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}},$$
then $\Gamma(1) = 1$. Thus Γ is a meromorphic function having simple poles at the negative integers. **Hint:** $\prod_{n=1}^{\infty} (1+n) e^{-1/n} = c = e^{\gamma}$.

11. $\uparrow$Now show that
$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right]$$

12. $\uparrow$Justify the following argument leading to Gauss's formula
$$\Gamma(z) = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \left(\frac{k}{k+z}\right) e^{\frac{z}{k}}\right) \frac{e^{-\gamma z}}{z}$$
$$\begin{aligned} &= \lim_{n\to\infty} \left(\frac{n!}{(1+z)(2+z)\cdots(n+z)} e^{z\left(\sum_{k=1}^{n} \frac{1}{k}\right)}\right) \frac{e^{-\gamma z}}{z} \\ &= \lim_{n\to\infty} \frac{n!}{(1+z)(2+z)\cdots(n+z)} e^{z\left(\sum_{k=1}^{n} \frac{1}{k}\right)} e^{-z\left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right]} \\ &= \lim_{n\to\infty} \frac{n! n^z}{(1+z)(2+z)\cdots(n+z)}. \end{aligned}$$

13. $\uparrow$ Verify from the Gauss formula above that $\Gamma(z+1) = \Gamma(z) z$ and that for n a nonnegative integer, $\Gamma(n+1) = n!$.

14. $\uparrow$ The usual definition of the gamma function for positive x is
$$\Gamma_1(x) \equiv \int_0^{\infty} e^{-t} t^{x-1} dt.$$
Show $\left(1-\frac{t}{n}\right)^n \leq e^{-t}$ for $t \in [0,n]$. Then show
$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1} dt = \frac{n! n^x}{x(x+1)\cdots(x+n)}.$$
Use the first part to conclude that
$$\Gamma_1(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)} = \Gamma(x).$$
Hint: To show $\left(1-\frac{t}{n}\right)^n \leq e^{-t}$ for $t \in [0,n]$, verify this is equivalent to showing $(1-u)^n \leq e^{-nu}$ for $u \in [0,1]$.

15. ↑Show $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$ whenever $\operatorname{Re} z > 0$. **Hint:** You have already shown that this is true for positive real numbers. Verify this formula for $\operatorname{Re} z > 0$ yields an analytic function.

16. ↑Show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then find $\Gamma\left(\frac{5}{2}\right)$.

17. Show that $\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} ds = \sqrt{2\pi}$. **Hint:** Denote this integral by I and observe that $I^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy$. Then change variables to polar coordinates, $x = r\cos(\theta)$, $y = r\sin\theta$.

18. ↑ Now that you know what the gamma function is, consider in the formula for $\Gamma(\alpha+1)$ the following change of variables. $t = \alpha + \alpha^{1/2} s$. Then in terms of the new variable, s, the formula for $\Gamma(\alpha+1)$ is

$$e^{-\alpha}\alpha^{\alpha+\frac{1}{2}} \int_{-\sqrt{\alpha}}^{\infty} e^{-\sqrt{\alpha}s}\left(1+\frac{s}{\sqrt{\alpha}}\right)^{\alpha} ds$$

$$= e^{-\alpha}\alpha^{\alpha+\frac{1}{2}} \int_{-\sqrt{\alpha}}^{\infty} e^{\alpha\left[\ln\left(1+\frac{s}{\sqrt{\alpha}}\right)-\frac{s}{\sqrt{\alpha}}\right]} ds$$

Show the integrand converges to $e^{-\frac{s^2}{2}}$. Show that then

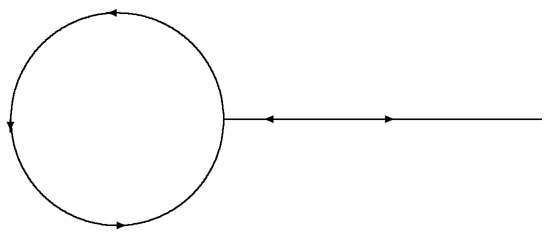
$$\lim_{\alpha\to\infty} \frac{\Gamma(\alpha+1)}{e^{-\alpha}\alpha^{\alpha+(1/2)}} = \int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} ds = \sqrt{2\pi}.$$

Hint: You will need to obtain a dominating function for the integral so that you can use the dominated convergence theorem. You might try considering $s \in (-\sqrt{\alpha}, \sqrt{\alpha})$ first and consider something like $e^{1-(s^2/4)}$ on this interval. Then look for another function for $s > \sqrt{\alpha}$. This formula is known as Stirling's formula.

19. This and the next several problems develop the zeta function and give a relation between the zeta and the gamma function. Define for $0 < r < 2\pi$

$$\begin{aligned} I_r(z) \equiv & \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^{re^{i\theta}}-1} ire^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt \qquad (23.24) \\ & + \int_\infty^r \frac{e^{(z-1)\ln t}}{e^t-1} dt \end{aligned}$$

Show that I_r is an entire function. The reason $0 < r < 2\pi$ is that this prevents $e^{re^{i\theta}} - 1$ from equaling zero. The above is just a precise description of the contour integral, $\int_\gamma \frac{w^{z-1}}{e^w - 1} dw$ where γ is the contour shown below.

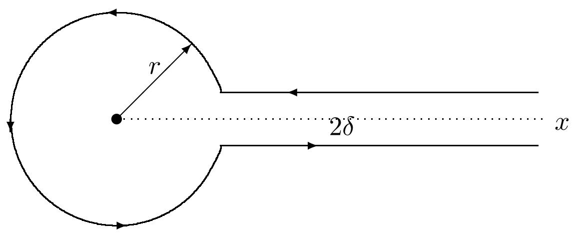


in which on the integrals along the real line, the argument is different in going from r to ∞ than it is in going from ∞ to r. Now I have not defined such contour integrals over contours which have infinite length and so have chosen to simply write out explicitly what is involved. You have to work with these integrals given above anyway but the contour integral just mentioned is the motivation for them. **Hint:** You may want to use convergence theorems from real analysis if it makes this more convenient but you might not have to.

20. $\uparrow$ In the context of Problem 19 define for small $\delta > 0$

$$I_{r\delta}(z) \equiv \int_{\gamma_{r,\delta}} \frac{w^{z-1}}{e^w - 1} dw$$

where $\gamma_{r\delta}$ is shown below.

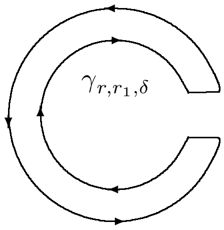


Show that $\lim_{\delta\to 0} I_{r\delta}(z) = I_r(z)$. **Hint:** Use the dominated convergence theorem if it makes this go easier. This is not a hard problem if you use these theorems but you can probably do it without them with more work.

21. $\uparrow$ In the context of Problem 20 show that for $r_1 < r$, $I_{r\delta}(z) - I_{r_1\delta}(z)$ is a contour integral,

$$\int_{\gamma_{r,r_1,\delta}} \frac{w^{z-1}}{e^w - 1} dw$$

where the oriented contour is shown below.



In this contour integral, w^{z-1} denotes $e^{(z-1)\log(w)}$ where $\log(w) = \ln|w| + i\arg(w)$ for $\arg(w) \in (0, 2\pi)$. Explain why this integral equals zero. From Problem 20 it follows that $I_r = I_{r_1}$. Therefore, you can define an entire function, $I(z) \equiv I_r(z)$ for all r positive but sufficiently small. **Hint:** Remember the Cauchy integral formula for analytic functions defined on simply connected regions. You could argue there is a simply connected region containing $\gamma_{r,r_1,\delta}$.

22. $\uparrow$ In case $\operatorname{Re} z > 1$, you can get an interesting formula for $I(z)$ by taking the limit as $r \to 0$. Recall that

$$\begin{aligned} I_r(z) \equiv & \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^{re^{i\theta}} - 1} ire^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt \\ & + \int_\infty^r \frac{e^{(z-1)\ln t}}{e^t - 1} dt \end{aligned} \tag{23.25}$$

and now it is desired to take a limit in the case where $\operatorname{Re} z > 1$. Show the first integral above converges to 0 as $r \to 0$. Next argue the sum of the two last integrals converges to

$$\left(e^{(z-1)2\pi i} - 1\right) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt.$$

Thus

$$I(z) = \left(e^{z2\pi i} - 1\right) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt \tag{23.26}$$

when $\operatorname{Re} z > 1$.

23. $\uparrow$ So what does all this have to do with the zeta function and the gamma function? The zeta function is defined for $\operatorname{Re} z > 1$ by

$$\sum_{n=1}^\infty \frac{1}{n^z} \equiv \zeta(z).$$

By Problem 15, whenever $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Change the variable and conclude

$$\Gamma(z)\frac{1}{n^z}=\int_0^\infty e^{-ns}s^{z-1}ds.$$

Therefore, for $\operatorname{Re} z>1$,

$$\zeta(z)\Gamma(z)=\sum_{n=1}^\infty\int_0^\infty e^{-ns}s^{z-1}ds.$$

Now show that you can interchange the order of the sum and the integral. This is possibly most easily done by using Fubini's theorem. Show that $\sum_{n=1}^\infty\int_0^\infty\left|e^{-ns}s^{z-1}\right|ds<\infty$ and then use Fubini's theorem. I think you could do it other ways though. It is possible to do it without any reference to Lebesgue integration. Thus

$$\begin{aligned}\zeta(z)\Gamma(z) &= \int_0^\infty s^{z-1}\sum_{n=1}^\infty e^{-ns}ds\\ &= \int_0^\infty \frac{s^{z-1}e^{-s}}{1-e^{-s}}ds=\int_0^\infty\frac{s^{z-1}}{e^s-1}ds\end{aligned}$$

By 23.26,

$$\begin{aligned}I(z) &= \left(e^{z2\pi i}-1\right)\int_0^\infty\frac{e^{(z-1)\ln(t)}}{e^t-1}dt\\ &= \left(e^{z2\pi i}-1\right)\zeta(z)\Gamma(z)\\ &= \left(e^{2\pi iz}-1\right)\zeta(z)\Gamma(z)\end{aligned}$$

whenever $\operatorname{Re} z>1$.

24. $\uparrow$ Now show there exists an entire function, $h(z)$ such that

$$\zeta(z)=\frac{1}{z-1}+h(z)$$

for $\operatorname{Re} z>1$. Conclude $\zeta(z)$ extends to a meromorphic function defined on all of $\mathbb{C}$ which has a simple pole at $z=1$, namely, the right side of the above formula. **Hint:** Use Problem 10 to observe that $\Gamma(z)$ is never equal to zero but has simple poles at every nonnegative integer. Then for $\operatorname{Re} z>1$,

$$\zeta(z)\equiv\frac{I(z)}{\left(e^{2\pi iz}-1\right)\Gamma(z)}.$$

By 23.26 ζ has no poles for $\operatorname{Re} z>1$. The right side of the above equation is defined for all z. There are no poles except possibly when z is a nonnegative integer. However, these points are not poles either because of Problem 10 which states that Γ has simple poles at these points thus cancelling the simple zeros of $\left(e^{2\pi iz}-1\right)$. The only remaining possibility for a pole for ζ is at $z=1$. Show it has a simple pole at this point. You can use the formula for $I(z)$

$$
\begin{aligned}
I(z) \quad &\equiv \quad \int_0^{2\pi}\frac{e^{(z-1)(\ln r+i\theta)}}{e^{re^{i\theta}}-1}ire^{i\theta}d\theta+\int_r^\infty \frac{e^{(z-1)(\ln t+2\pi i)}}{e^t-1}dt \quad (23.27)\\
&+\int_\infty^r \frac{e^{(z-1)\ln t}}{e^t-1}dt
\end{aligned}
$$

Thus $I(1)$ is given by

$$
I(1)\equiv\int_0^{2\pi}\frac{1}{e^{re^{i\theta}}-1}ire^{i\theta}d\theta+\int_r^\infty\frac{1}{e^t-1}dt+\int_\infty^r\frac{1}{e^t-1}dt
$$

$=\int_{\gamma_r}\frac{dw}{e^w-1}$ where γ_r is the circle of radius r. This contour integral equals $2\pi i$ by the residue theorem. Therefore,

$$
\frac{I(z)}{\left(e^{2\pi iz}-1\right)\Gamma(z)}=\frac{1}{z-1}+h(z)
$$

where $h(z)$ is an entire function. People worry a lot about where the zeros of ζ are located. In particular, the zeros for $\operatorname{Re}z\in(0,1)$ are of special interest. The Riemann hypothesis says they are all on the line $\operatorname{Re}z=1/2$. This is a good problem for you to do next.

25. There is an important relation between prime numbers and the zeta function due to Euler. Let $\{p_n\}_{n=1}^\infty$ be the prime numbers. Then for $\operatorname{Re}z>1$,

$$
\prod_{n=1}^\infty\frac{1}{1-p_n^{-z}}=\zeta(z).
$$

To see this, consider a partial product.

$$
\prod_{n=1}^N\frac{1}{1-p_n^{-z}}=\prod_{n=1}^N\sum_{j_n=1}^\infty\left(\frac{1}{p_n^z}\right)^{j_n}.
$$

Let S_N denote all positive integers which use only $p_1,\cdots,p_N$ in their prime factorization. Then the above equals $\sum_{n\in S_N}\frac{1}{n^z}$. Letting $N\to\infty$ and using the fact that $\operatorname{Re}z>1$ so that the order in which you sum is not important (See Theorem 24.1 or recall advanced calculus.) you obtain the desired equation. Show $\sum_{n=1}^\infty\frac{1}{p_n}=\infty$.

Elliptic Functions

This chapter is to give a short introduction to elliptic functions. There is much more available. There are books written on elliptic functions. What I am presenting here follows Alfors [2] although the material is found in many books on complex analysis. Hille, [23] has a much more extensive treatment than what I will attempt here. There are also many references and historical notes available in the book by Hille. Another good source for more having much the same emphasis as what is presented here is in the book by Saks and Zygmund [38]. This is a very interesting subject because it has considerable overlap with algebra.

Before beginning, recall that an absolutely convergent series can be summed in any order and you always get the same answer. The easy way to see this is to think of the series as a Lebesgue integral with respect to counting measure and apply convergence theorems as needed. The following theorem provides the necessary results.

Theorem 24.1 *Suppose* $\sum_{n=1}^{\infty} |a_n| < \infty$ *and let* $\theta, \phi : \mathbb{N} \to \mathbb{N}$ *be one to one and onto mappings. Then* $\sum_{n=1}^{\infty} a_{\phi(n)}$ *and* $\sum_{n=1}^{\infty} a_{\theta(n)}$ *both converge and the two sums are equal.*

Proof: By the monotone convergence theorem,

$$\sum_{n=1}^{\infty} |a_n| = \lim_{n\to\infty} \sum_{k=1}^{n} \left|a_{\phi(k)}\right| = \lim_{n\to\infty} \sum_{k=1}^{n} \left|a_{\theta(k)}\right|$$

but these last two equal $\sum_{k=1}^{\infty} \left|a_{\phi(k)}\right|$ and $\sum_{k=1}^{\infty} \left|a_{\theta(k)}\right|$ respectively. Therefore, $\sum_{k=1}^{\infty} a_{\theta(k)}$ and $\sum_{k=1}^{\infty} a_{\phi(k)}$ exist ($n \to a_{\theta(n)}$ is in L^1 with respect to counting measure.) It remains to show the two are equal. There exists M such that if $n > M$ then

$$\sum_{k=n+1}^{\infty} \left|a_{\theta(k)}\right| < \varepsilon, \quad \sum_{k=n+1}^{\infty} \left|a_{\phi(k)}\right| < \varepsilon$$

$$\left|\sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n} a_{\phi(k)}\right| < \varepsilon, \ \left|\sum_{k=1}^{\infty} a_{\theta(k)} - \sum_{k=1}^{n} a_{\theta(k)}\right| < \varepsilon$$

Pick such an n denoted by n_1. Then pick $n_2 > n_1 > M$ such that

$$\{\theta(1), \cdots, \theta(n_1)\} \subseteq \{\phi(1), \cdots, \phi(n_2)\}.$$

Then

$$\sum_{k=1}^{n_2} a_{\phi(k)} = \sum_{k=1}^{n_1} a_{\theta(k)} + \sum_{\phi(k) \notin \{\theta(1), \cdots, \theta(n_1)\}} a_{\phi(k)}.$$

Therefore,

$$\left| \sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)} \right| = \left| \sum_{\phi(k) \notin \{\theta(1), \cdots, \theta(n_1)\}, k \le n_2} a_{\phi(k)} \right|$$

Now all of these $\phi(k)$ in the last sum are contained in $\{\theta(n_1+1), \cdots\}$ and so the last sum above is dominated by

$$\le \sum_{k=n_1+1}^{\infty} \left| a_{\theta(k)} \right| < \varepsilon.$$

Therefore,

$$\begin{aligned}
\left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{\infty} a_{\theta(k)} \right| &\le \left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n_2} a_{\phi(k)} \right| \\
&\quad + \left| \sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)} \right| \\
+ \left| \sum_{k=1}^{n_1} a_{\theta(k)} - \sum_{k=1}^{\infty} a_{\theta(k)} \right| &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\end{aligned}$$

and since ε is arbitrary, it follows $\sum_{k=1}^{\infty} a_{\phi(k)} = \sum_{k=1}^{\infty} a_{\theta(k)}$ as claimed. This proves the theorem.

24.1 Periodic Functions

Definition 24.2 *A function defined on $\mathbb{C}$ is said to be periodic if there exists w such that $f(z+w) = f(z)$ for all $z \in \mathbb{C}$. Denote by M the set of all periods. Thus if $w_1, w_2 \in M$ and $a, b \in \mathbb{Z}$, then $aw_1 + bw_2 \in M$. For this reason M is called the module of periods.*[1] *In all which follows it is assumed f is meromorphic.*

Theorem 24.3 *Let f be a meromorphic function and let M be the module of periods. Then if M has a limit point, then f equals a constant. If this does not happen then either there exists $w_1 \in M$ such that $\mathbb{Z}w_1 = M$ or there exist $w_1, w_2 \in M$ such that $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$ and w_1/w_2 is not real. Also if $\tau = w_2/w_1$,*

$$|\tau| \ge 1, \ \frac{-1}{2} \le \operatorname{Re} \tau \le \frac{1}{2}.$$

[1] A module is like a vector space except instead of a field of scalars, you have a ring of scalars.

Proof: Suppose f is meromorphic and M has a limit point, w_0. By Theorem 23.10 on Page 538 there exist analytic functions, p, q such that $f(z) = \frac{p(z)}{q(z)}$. Now pick z_0 such that z_0 is not a pole of f. Then letting $w_n \to w_0$ where $\{w_n\} \subseteq M$, $f(z_0 + w_n) = f(z_0)$. Therefore, $p(z_0 + w_n) = f(z_0) q(z_0 + w_n)$ and so the analytic function, $p(z) - f(z_0) q(z)$ has a zero set which has a limit point. Therefore, this function is identically equal to zero because of Theorem 18.23 on Page 401. Thus f equals a constant as claimed.

This has shown that if f is not constant, then M is discreet. Therefore, there exists $w_1 \in M$ such that $|w_1| = \min\{|w| : w \in M\}$. Suppose first that every element of M is a real multiple of w_1. Thus, if $w \in M$, it follows there exists a real number, x such that $w = xw_1$. Then there exist positive integers, $k, k+1$ such that $k \leq x < k+1$. If $x > k$, then $w - kw_1 = (x-k) w_1$ is a period having smaller absolute value than $|w_1|$ which would be a contradiction. Hence, $x = k$ and so $M = \mathbb{Z}w_1$.

Now suppose there exists $w_2 \in M$ which is not a real multiple of w_1. You can let w_2 be the element of M having this property which has smallest absolute value. Now let $w \in M$. Since w_1 and w_2 point in different directions, it follows $w = xw_1 + yw_2$ for some real numbers, x, y. Let $|m - x| \leq \frac{1}{2}$ and $|n - y| \leq \frac{1}{2}$ where m, n are integers. Therefore,

$$w = mw_1 + nw_2 + (x-m) w_1 + (y-n) w_2$$

and so

$$w - mw_1 - nw_2 = (x-m) w_1 + (y-n) w_2 \tag{24.1}$$

Now since $w_2/w_1 \notin \mathbb{R}$,

$$\begin{aligned} |(x-m) w_1 + (y-n) w_2| &< |(x-m) w_1| + |(y-n) w_2| \\ &= \frac{1}{2}|w_1| + \frac{1}{2}|w_2|. \end{aligned}$$

Therefore, from 24.1,

$$\begin{aligned} |w - mw_1 - nw_2| &= |(x-m) w_1 + (y-n) w_2| \\ &< \frac{1}{2}|w_1| + \frac{1}{2}|w_2| \leq |w_2| \end{aligned}$$

and so the period, $w - mw_1 - nw_2$ cannot be a non real multiple of w_1 because w_2 is the one which has smallest absolute value and this period has smaller absolute value than w_2. Therefore, the ratio $w - mw_1 - nw_2/w_1$ must be a real number, x. Thus

$$w - mw_1 - nw_2 = xw_1$$

Since w_1 has minimal absolute value of all periods, it follows $|x| \geq 1$. Let $k \leq x < k+1$ for some integer, k. If $x > k$, then

$$w - mw_1 - nw_2 - kw_1 = (x-k) w_1$$

which would contradict the choice of w_1 as being the period having minimal absolute value because the expression on the left in the above is a period and it equals something which has absolute value less than $|w_1|$. Therefore, $x = k$ and w is an integer linear combination of w_1 and w_2. It only remains to verify the claim about τ.

From the construction, $|w_1| \leq |w_2|$ and $|w_2| \leq |w_1 - w_2|, |w_2| \leq |w_1 + w_2|$. Therefore,

$$|\tau| \geq 1,\ |\tau| \leq |1-\tau|\,,\ |\tau| \leq |1+\tau|\,.$$

The last two of these inequalities imply $-1/2 \leq \operatorname{Re}\tau \leq 1/2$.

This proves the theorem.

Definition 24.4 *For f a meromorphic function which has the last of the above alternatives holding in which $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$, the function, f is called elliptic. This is also called doubly periodic.*

Theorem 24.5 *Suppose f is an elliptic function which has no poles. Then f is constant.*

Proof: Since f has no poles it is analytic. Now consider the parallelograms determined by the vertices, $mw_1 + nw_2$ for $m, n \in \mathbb{Z}$. By periodicity of f it must be bounded because its values are identical on each of these parallelograms. Therefore, it equals a constant by Liouville's theorem.

Definition 24.6 *Define P_a to be the parallelogram determined by the points*

$$\begin{aligned}&a + mw_1 + nw_2, a + (m+1)\,w_1 + nw_2, a + mw_1 + (n+1)\,w_2,\\&a + (m+1)\,w_1 + (n+1)\,w_2\end{aligned}$$

Such P_a will be referred to as a period parallelogram. The sum of the orders of the poles in a period parallelogram which contains no poles or zeros of f on its boundary is called the order of the function. This is well defined because of the periodic property of f.

Theorem 24.7 *The sum of the residues of any elliptic function, f equals zero on every P_a if a is chosen so that there are no poles on ∂P_a.*

Proof: Choose a such that there are no poles of f on the boundary of P_a. By periodicity,

$$\int_{\partial P_a} f(z)\,dz = 0$$

because the integrals over opposite sides of the parallelogram cancel out because the values of f are the same on these sides and the orientations are opposite. It follows from the residue theorem that the sum of the residues in P_a equals 0.

Theorem 24.8 *Let P_a be a period parallelogram for a nonconstant elliptic function, f which has order equal to m. Then f assumes every value in $f(P_a)$ exactly m times.*

Proof: Let $c \in f(P_a)$ and consider $P_{a'}$ such that $f^{-1}(c) \cap P_{a'} = f^{-1}(c) \cap P_a$ and $P_{a'}$ contains the same poles and zeros of $f - c$ as P_a but $P_{a'}$ has no zeros of $f(z) - c$ or poles of f on its boundary. Thus $f'(z) / (f(z) - c)$ is also an elliptic function and so Theorem 24.7 applies. Consider

$$\frac{1}{2\pi i}\int_{\partial P_{a'}} \frac{f'(z)}{f(z)-c}dz.$$

By the argument principle, this equals $N_z - N_p$ where N_z equals the number of zeros of $f(z) - c$ and N_p equals the number of the poles of $f(z)$. From Theorem 24.7 this must equal zero because it is the sum of the residues of $f' / (f - c)$ and so $N_z = N_p$. Now N_p equals the number of poles in P_a counted according to multiplicity.

There is an even better theorem than this one.

Theorem 24.9 *Let f be a non constant elliptic function and suppose it has poles $p_1, \cdots, p_m$ and zeros, $z_1, \cdots, z_m$ in P_α, listed according to multiplicity where ∂P_α contains no poles or zeros of f. Then $\sum_{k=1}^m z_k - \sum_{k=1}^m p_k \in M$, the module of periods.*

Proof: You can assume ∂P_a contains no poles or zeros of f because if it did, then you could consider a slightly shifted period parallelogram, $P_{a'}$ which contains no new zeros and poles but which has all the old ones but no poles or zeros on its boundary. By Theorem 20.8 on Page 454

$$\frac{1}{2\pi i}\int_{\partial P_a} z\frac{f'(z)}{f(z)}dz = \sum_{k=1}^m z_k - \sum_{k=1}^m p_k. \tag{24.2}$$

Denoting by $\gamma(z, w)$ the straight oriented line segment from z to w,

$$\begin{aligned}
&\int_{\partial P_a} z\frac{f'(z)}{f(z)}dz \\
= \quad &\int_{\gamma(a,a+w_1)} z\frac{f'(z)}{f(z)}dz + \int_{\gamma(a+w_1+w_2,a+w_2)} z\frac{f'(z)}{f(z)}dz \\
&+\int_{\gamma(a+w_1,a+w_2+w_1)} z\frac{f'(z)}{f(z)}dz + \int_{\gamma(a+w_2,a)} z\frac{f'(z)}{f(z)}dz
\end{aligned}$$

$$\begin{aligned}
= \quad &\int_{\gamma(a,a+w_1)} (z - (z + w_2))\frac{f'(z)}{f(z)}dz \\
&+\int_{\gamma(a,a+w_2)} (z - (z + w_1))\frac{f'(z)}{f(z)}dz
\end{aligned}$$

Now near these line segments $\frac{f'(z)}{f(z)}$ is analytic and so there exists a primitive, $g_{w_i}(z)$ on $\gamma(a, a + w_i)$ by Corollary 18.32 on Page 407 which satisfies $e^{g_{w_i}(z)} = f(z)$. Therefore,

$$= -w_2 (g_{w_1}(a + w_1) - g_{w_1}(a)) - w_1 (g_{w_2}(a + w_2) - g_{w_2}(a)).$$

Now by periodicity of f it follows $f(a+w_1) = f(a) = f(a+w_2)$. Hence

$$g_{w_i}(a+w_1) - g_{w_i}(a) = 2m\pi i$$

for some integer, m because

$$e^{g_{w_i}(a+w_i)} - e^{g_{w_i}(a)} = f(a+w_i) - f(a) = 0.$$

Therefore, from 24.2, there exist integers, k, l such that

$$\begin{aligned}
& \frac{1}{2\pi i}\int_{\partial P_a} z\frac{f'(z)}{f(z)}dz \\
= & \frac{1}{2\pi i}\left[-w_2\left(g_{w_1}(a+w_1) - g_{w_1}(a)\right) - w_1\left(g_{w_2}(a+w_2) - g_{w_2}(a)\right)\right] \\
= & \frac{1}{2\pi i}\left[-w_2(2k\pi i) - w_1(2l\pi i)\right] \\
= & -w_2 k - w_1 l \in M.
\end{aligned}$$

From 24.2 it follows

$$\sum_{k=1}^{m} z_k - \sum_{k=1}^{m} p_k \in M.$$

This proves the theorem.

Hille says this relation is due to Liouville. There is also a simple corollary which follows from the above theorem applied to the elliptic function, $f(z) - c$.

Corollary 24.10 *Let f be a non constant elliptic function and suppose the function, $f(z) - c$ has poles $p_1, \cdots, p_m$ and zeros, $z_1, \cdots, z_m$ on P_α, listed according to multiplicity where ∂P_α contains no poles or zeros of $f(z) - c$. Then $\sum_{k=1}^{m} z_k - \sum_{k=1}^{m} p_k \in M$, the module of periods.*

24.1.1 The Unimodular Transformations

Definition 24.11 *Suppose f is a nonconstant elliptic function and the module of periods is of the form $\{aw_1 + bw_2\}$ where a, b are integers and w_1/w_2 is not real. Then by analogy with linear algebra, $\{w_1, w_2\}$ is referred to as a basis. The unimodular transformations will refer to matrices of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where all entries are integers and

$$ad - bc = \pm 1.$$

These linear transformations are also called the modular group.

The following is an interesting lemma which ties matrices with the fractional linear transformations.

Lemma 24.12 *Define*

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \equiv \frac{az+b}{cz+d}.$$

Then

$$\phi(AB) = \phi(A) \circ \phi(B), \tag{24.3}$$

$\phi(A)(z) = z$ *if and only if*

$$A = cI$$

where I *is the identity matrix and* $c \neq 0$*. Also if* $f(z) = \frac{az+b}{cz+d}$*, then* $f^{-1}(z)$ *exists if and only if* $ad - cb \neq 0$*. Furthermore it is easy to find* f^{-1}*.*

Proof: The equation in 24.3 is just a simple computation. Now suppose $\phi(A)(z) = z$. Then letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this requires

$$az + b = z(cz + d)$$

and so $az + b = cz^2 + dz$. Since this is to hold for all z it follows $c = 0 = b$ and $a = d$. The other direction is obvious.

Consider the claim about the existence of an inverse. Let $ad - cb \neq 0$ for $f(z) = \frac{az+b}{cz+d}$. Then

$$f(z) = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

It follows $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ exists and equals $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Therefore,

$$\begin{aligned} z &= \phi(I)(z) = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)\right)(z) \\ &= \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \circ \phi\left(\left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)\right)(z) \\ &= f \circ f^{-1}(z) \end{aligned}$$

which shows f^{-1} exists and it is easy to find.

Next suppose f^{-1} exists. I need to verify the condition $ad - cb \neq 0$. If f^{-1} exists, then from the process used to find it, you see that it must be a fractional linear transformation. Letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so $\phi(A) = f$, it follows there exists a matrix B such that

$$\phi(BA)(z) = \phi(B) \circ \phi(A)(z) = z.$$

However, it was shown that this implies BA is a nonzero multiple of I which requires that A^{-1} must exist. Hence the condition must hold.

Theorem 24.13 *If f is a nonconstant elliptic function with a basis $\{w_1, w_2\}$ for the module of periods, then $\{w_1', w_2'\}$ is another basis, if and only if there exists a unimodular transformation,* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ *such that*

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{24.4}$$

Proof: Since $\{w_1, w_2\}$ is a basis, there exist integers, a, b, c, d such that 24.4 holds. It remains to show the transformation determined by the matrix is unimodular. Taking conjugates,

$$\begin{pmatrix} \overline{w_1'} \\ \overline{w_2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix}$$

Now since $\{w_1', w_2'\}$ is also given to be a basis, there exits another matrix having all integer entries, $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that

$$\begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \overline{w_1'} \\ \overline{w_2'} \end{pmatrix}$$

and

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix}.$$

However, since w_1'/w_2' is not real, it is routine to verify that

$$\det \begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} \neq 0.$$

Therefore,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

and so $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} e & f \\ g & h \end{pmatrix} = 1$. But the two matrices have all integer entries and so both determinants must equal either 1 or -1.

Next suppose

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{24.5}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unimodular. I need to verify that $\{w_1', w_2'\}$ is a basis. If $w \in M$, there exist integers, m, n such that

$$w = mw_1 + nw_2 = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

From 24.5

$$\pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and so

$$w = \pm \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$$

which is an integer linear combination of $\{w_1', w_2'\}$. It only remains to verify that w_1'/w_2' is not real.

Claim: Let w_1 and w_2 be nonzero complex numbers. Then w_2/w_1 is not real if and only if

$$w_1\overline{w_2} - \overline{w_1}w_2 = \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0$$

Proof of the claim: Let $\lambda = w_2/w_1$. Then

$$w_1\overline{w_2} - \overline{w_1}w_2 = \overline{\lambda} w_1\overline{w_1} - \overline{w_1}\lambda w_1 = \left(\overline{\lambda} - \lambda\right) |w_1|^2$$

Thus the ratio is not real if and only if $\left(\overline{\lambda} - \lambda\right) \neq 0$ if and only if $w_1\overline{w_2} - \overline{w_1}w_2 \neq 0$.

Now to verify w_2'/w_1' is not real,

$$\begin{aligned} & \det \begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} \\ = \; & \det \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \right) \\ = \; & \pm \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0 \end{aligned}$$

This proves the theorem.

24.1.2 The Search For An Elliptic Function

By Theorem 24.5 and 24.7 if you want to find a nonconstant elliptic function it must fail to be analytic and also have either no terms in its Laurent expansion which are of the form $b_1 (z - a)^{-1}$ or else these terms must cancel out. It is simplest to look for a function which simply does not have them. Weierstrass looked for a function of the form

$$\wp (z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z - w)^2} - \frac{1}{w^2} \right) \tag{24.6}$$

where w consists of all numbers of the form $aw_1 + bw_2$ for a, b integers. Sometimes people write this as $\wp(z, w_1, w_2)$ to emphasize its dependence on the periods, w_1 and w_2 but I won't do so. It is understood there exist these periods, which are given. This is a reasonable thing to try. Suppose you formally differentiate the right side. Never mind whether this is justified for now. This yields

$$\wp'(z) = \frac{-2}{z^3} - \sum_{w \neq 0} \frac{-2}{(z-w)^3} = \sum_w \frac{-2}{(z-w)^3}$$

which is clearly periodic having both periods w_1 and w_2. Therefore, $\wp(z + w_1) - \wp(z)$ and $\wp(z + w_2) - \wp(z)$ are both constants, c_1 and c_2 respectively. The reason for this is that since $\wp'$ is periodic with periods w_1 and w_2, it follows $\wp'(z + w_i) - \wp'(z) = 0$ as long as z is not a period. From 24.6 you can see right away that

$$\wp(z) = \wp(-z)$$

Indeed

$$\begin{aligned}
\wp(-z) &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z-w)^2} - \frac{1}{w^2} \right) \\
&= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z+w)^2} - \frac{1}{w^2} \right) = \wp(z).
\end{aligned}$$

and so

$$\begin{aligned}
c_1 &= \wp\left(-\frac{w_1}{2} + w_1 \right) - \wp\left(-\frac{w_1}{2} \right) \\
&= \wp\left(\frac{w_1}{2} \right) - \wp\left(-\frac{w_1}{2} \right) = 0
\end{aligned}$$

which shows the constant for $\wp(z + w_1) - \wp(z)$ must equal zero. Similarly the constant for $\wp(z + w_2) - \wp(z)$ also equals zero. Thus $\wp$ is periodic having the two periods w_1, w_2.

Of course to justify this, you need to consider whether the series of 24.6 converges. Consider the terms of the series.

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = |z| \left| \frac{2w - z}{(z-w)^2 w^2} \right|$$

If $|w| > 2|z|$, this can be estimated more. For such w,

$$\begin{aligned}
& \left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| \\
= \ & |z| \left| \frac{2w - z}{(z-w)^2 w^2} \right| \leq |z| \frac{(5/2)|w|}{|w|^2 (|w| - |z|)^2} \\
\leq \ & |z| \frac{(5/2)|w|}{|w|^2 ((1/2)|w|)^2} = |z| \frac{10}{|w|^3}.
\end{aligned}$$

It follows the series in 24.6 converges uniformly and absolutely on every compact set, K provided $\sum_{w\neq 0}\frac{1}{|w|^3}$ converges. This question is considered next.

Claim: There exists a positive number, k such that for all pairs of integers, m,n, not both equal to zero,

$$\frac{|mw_1+nw_2|}{|m|+|n|}\geq k>0.$$

Proof of claim: If not, there exists m_k and n_k such that

$$\lim_{k\to\infty}\frac{m_k}{|m_k|+|n_k|}w_1+\frac{n_k}{|m_k|+|n_k|}w_2=0$$

However, $\left(\frac{m_k}{|m_k|+|n_k|},\frac{n_k}{|m_k|+|n_k|}\right)$ is a bounded sequence in $\mathbb{R}^2$ and so, taking a subsequence, still denoted by k, you can have

$$\left(\frac{m_k}{|m_k|+|n_k|},\frac{n_k}{|m_k|+|n_k|}\right)\to(x,y)\in\mathbb{R}^2$$

and so there are real numbers, x,y such that $xw_1+yw_2=0$ contrary to the assumption that w_2/w_1 is not equal to a real number. This proves the claim.

Now from the claim,

$$\begin{aligned}&\sum_{w\neq 0}\frac{1}{|w|^3}\\ =&\sum_{(m,n)\neq(0,0)}\frac{1}{|mw_1+nw_2|^3}\leq\sum_{(m,n)\neq(0,0)}\frac{1}{k^3\left(|m|+|n|\right)^3}\\ =&\frac{1}{k^3}\sum_{j=1}^{\infty}\sum_{|m|+|n|=j}\frac{1}{\left(|m|+|n|\right)^3}=\frac{1}{k^3}\sum_{j=1}^{\infty}\frac{4j}{j^3}<\infty.\end{aligned}$$

Now consider the series in 24.6. Letting $z\in B(0,R)$,

$$\begin{aligned}\wp(z)\;\equiv\;&\frac{1}{z^2}+\sum_{w\neq 0,|w|\leq R}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)\\ &+\sum_{w\neq 0,|w|>R}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)\end{aligned}$$

and the last series converges uniformly on $B(0,R)$ to an analytic function. Thus $\wp$ is a meromorphic function and also the argument given above involving differentiation of the series termwise is valid. Thus $\wp$ is an elliptic function as claimed. This is called the Weierstrass $\wp$ function. This has proved the following theorem.

Theorem 24.14 *The function $\wp$ defined above is an example of an elliptic function. On any compact set, $\wp$ equals a rational function added to a series which is uniformly and absolutely convergent on the compact set.*

24.1.3 The Differential Equation Satisfied By $\wp$

For z not a pole,

$$\wp'(z) = \frac{-2}{z^3} - \sum_{w\neq 0} \frac{2}{(z-w)^3}$$

Also since there are no poles of order 1 you can obtain a primitive for $\wp$, $-\zeta$.[2] To do so, recall

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w\neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

where for $|z| < R$ this is the sum of a rational function with a uniformly convergent series. Therefore, you can take the integral along any path, $\gamma(0,z)$ from 0 to z which misses the poles of $\wp$. By the uniform convergence of the above integral, you can interchange the sum with the integral and obtain

$$\zeta(z) = \frac{1}{z} + \sum_{w\neq 0} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w} \tag{24.7}$$

This function is odd. Here is why.

$$\zeta(-z) = \frac{1}{-z} + \sum_{w\neq 0} \frac{1}{-z-w} - \frac{z}{w^2} + \frac{1}{w}$$

while

$$\begin{aligned} -\zeta(z) &= \frac{1}{-z} + \sum_{w\neq 0} \frac{-1}{z-w} - \frac{z}{w^2} - \frac{1}{w} \\ &= \frac{1}{-z} + \sum_{w\neq 0} \frac{-1}{z+w} - \frac{z}{w^2} + \frac{1}{w}. \end{aligned}$$

Now consider 24.7. It will be used to find the Laurent expansion about the origin for ζ which will then be differentiated to obtain the Laurent expansion for $\wp$ at the origin. Since $w \neq 0$ and the interest is for z near 0 so $|z| < |w|$,

$$\begin{aligned} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w} &= \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w}\frac{1}{1-\frac{z}{w}} \\ &= \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k \\ &= -\frac{1}{w}\sum_{k=2}^{\infty}\left(\frac{z}{w}\right)^k \end{aligned}$$

[2] I don't know why it is traditional to refer to this antiderivative as $-\zeta$ rather than ζ but I am following the convention. I think it is to minimize the number of minus signs in the next expression.

From 24.7

$$\begin{aligned}\zeta(z) &= \frac{1}{z}+\sum_{w\neq 0}\left(-\sum_{k=2}^{\infty}\frac{z^k}{w^{k+1}}\right)\\ &= \frac{1}{z}-\sum_{k=2}^{\infty}\sum_{w\neq 0}\frac{z^k}{w^{k+1}}=\frac{1}{z}-\sum_{k=2}^{\infty}\sum_{w\neq 0}\frac{z^{2k-1}}{w^{2k}}\end{aligned}$$

because the sum over odd powers must be zero because for each $w\neq 0$, there exists $-w\neq 0$ such that the two terms $\frac{z^{2k}}{w^{2k+1}}$ and $\frac{z^{2k}}{(-w)^{2k+1}}$ cancel each other. Hence

$$\zeta(z)=\frac{1}{z}-\sum_{k=2}^{\infty}G_k z^{2k-1}$$

where $G_k=\sum_{w\neq 0}\frac{1}{w^{2k}}$. Now with this,

$$\begin{aligned}-\zeta'(z) &= \wp(z)=\frac{1}{z^2}+\sum_{k=2}^{\infty}G_k(2k-1)z^{2k-2}\\ &= \frac{1}{z^2}+3G_2z^2+5G_3z^4+\cdots\end{aligned}$$

Therefore,

$$\wp'(z)=\frac{-2}{z^3}+6G_2z+20G_3z^3+\cdots$$

$$\begin{aligned}\wp'(z)^2 &= \frac{4}{z^6}-\frac{24G_2}{z^2}-80G_3+\cdots\\ 4\wp(z)^3 &= 4\left(\frac{1}{z^2}+3G_2z^2+5G_3z^4\cdots\right)^3\\ &= \frac{4}{z^6}+\frac{36}{z^2}G_2+60G_3+\cdots\end{aligned}$$

and finally

$$60G_2\wp(z)=\frac{60G_2}{z^2}+0+\cdots$$

where in the above, the positive powers of z are not listed explicitly. Therefore,

$$\wp'(z)^2-4\wp(z)^3+60G_2\wp(z)+140G_3=\sum_{n=1}^{\infty}a_nz^n$$

In deriving the equation it was assumed $|z|<|w|$ for all $w=aw_1+bw_2$ where a,b are integers not both zero. The left side of the above equation is periodic with respect to w_1 and w_2 where w_2/w_1 is not a real number. The only possible poles of the left side are at 0, w_1, w_2, and w_1+w_2, the vertices of the parallelogram determined by w_1 and w_2. This follows from the original formula for $\wp(z)$. However, the above equation shows the left side has no pole at 0. Since the left side is periodic with periods w_1 and w_2, it follows it has no pole at the other vertices of this parallelogram either. Therefore, the left side is periodic and has no poles. Consequently, it equals a constant by Theorem 24.5. But the right side of the above equation shows this constant is 0 because this side equals zero when $z = 0$. Therefore, $\wp$ satisfies the differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3 = 0.$$

It is traditional to define $60G_2 \equiv g_2$ and $140G_3 \equiv g_3$. Then in terms of these new quantities the differential equation is

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Suppose e_1, e_2 and e_3 are zeros of the polynomial $4w^3 - g_2w - g_3 = 0$. Then the above equation can be written in the form

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \tag{24.8}$$

24.1.4 A Modular Function

The next task is to find the e_i in 24.8. First recall that $\wp$ is an even function. That is $\wp(-z) = \wp(z)$. This follows from 24.6 which is listed here for convenience.

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w\neq 0}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right) \tag{24.9}$$

Thus

$$\begin{aligned}\wp(-z) &= \frac{1}{z^2} + \sum_{w\neq 0}\left(\frac{1}{(-z-w)^2} - \frac{1}{w^2}\right)\\ &= \frac{1}{z^2} + \sum_{w\neq 0}\left(\frac{1}{(-z+w)^2} - \frac{1}{w^2}\right) = \wp(z).\end{aligned}$$

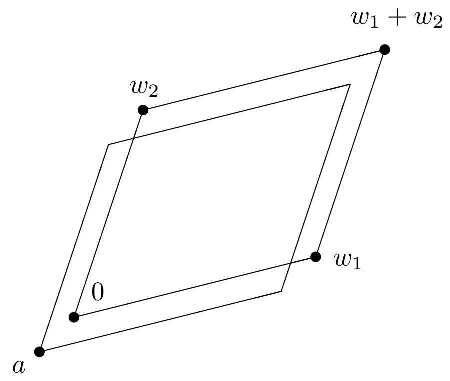
Therefore, $\wp(w_1 - z) = \wp(z - w_1) = \wp(z)$ and so $-\wp'(w_1 - z) = \wp'(z)$. Letting $z = w_1/2$, it follows $\wp'(w_1/2) = 0$. Similarly, $\wp'(w_2/2) = 0$ and $\wp'((w_1 + w_2)/2) = 0$. Therefore, from 24.8

$$0 = 4(\wp(w_1/2) - e_1)(\wp(w_1/2) - e_2)(\wp(w_1/2) - e_3).$$

It follows one of the e_i must equal $\wp(w_1/2)$. Similarly, one of the e_i must equal $\wp(w_2/2)$ and one must equal $\wp((w_1 + w_2)/2)$.

Lemma 24.15 *The numbers, $\wp(w_1/2), \wp(w_2/2)$, and $\wp((w_1 + w_2)/2)$ are distinct.*

Proof: Choose P_a, a period parallelogram which contains the pole 0, and the points $w_1/2$, $w_2/2$, and $(w_1+w_2)/2$ but no other pole of $\wp(z)$. Also ∂P_a^* does not contain any zeros of the elliptic function, $z \to \wp(z) - \wp(w_1/2)$. This can be done by shifting P_0 slightly because the poles are only at the points $aw_1 + bw_2$ for a, b integers and the zeros of $\wp(z) - \wp(w_1/2)$ are discreet.



If $\wp(w_2/2) = \wp(w_1/2)$, then $\wp(z) - \wp(w_1/2)$ has two zeros, $w_2/2$ and $w_1/2$ and since the pole at 0 is of order 2, this is the order of $\wp(z) - \wp(w_1/2)$ on P_a hence by Theorem 24.8 on Page 566 these are the only zeros of this function on P_a. It follows by Corollary 24.10 on Page 568 which says the sum of the zeros minus the sum of the poles is in M, $\frac{w_1}{2} + \frac{w_2}{2} \in M$. Thus there exist integers, a, b such that

$$\frac{w_1 + w_2}{2} = aw_1 + bw_2$$

which implies $(2a-1)w_1 + (2b-1)w_2 = 0$ contradicting w_2/w_1 not being real. Similar reasoning applies to the other pairs of points in $\{w_1/2, w_2/2, (w_1+w_2)/2\}$. For example, consider $(w_1+w_2)/2$ and choose P_a such that its boundary contains no zeros of the elliptic function, $z \to \wp(z) - \wp((w_1+w_2)/2)$ and P_a contains no poles of $\wp$ on its interior other than 0. Then if $\wp(w_2/2) = \wp((w_1+w_2)/2)$, it follows from Theorem 24.8 on Page 566 $w_2/2$ and $(w_1+w_2)/2$ are the only two zeros of $\wp(z) - \wp((w_1+w_2)/2)$ on P_a and by Corollary 24.10 on Page 568

$$\frac{w_1 + w_1 + w_2}{2} = aw_1 + bw_2 \in M$$

for some integers a, b which leads to the same contradiction as before about w_1/w_2 not being real. The other cases are similar. This proves the lemma.

Lemma 24.15 proves the e_i are distinct. Number the e_i such that

$$e_1 = \wp(w_1/2), e_2 = \wp(w_2/2)$$

and

$$e_3 = \wp((w_1+w_2)/2).$$

To summarize, it has been shown that for complex numbers, w_1 and w_2 with w_2/w_1 not real, an elliptic function, $\wp$ has been defined. Denote this function as

$\wp(z) = \wp(z, w_1, w_2)$. This in turn determined numbers, e_i as described above. Thus these numbers depend on w_1 and w_2 and as described above,

$$
\begin{aligned}
e_1(w_1, w_2) &= \wp\left(\frac{w_1}{2}, w_1, w_2\right), \ e_2(w_1, w_2) = \wp\left(\frac{w_2}{2}, w_1, w_2\right) \\
e_3(w_1, w_2) &= \wp\left(\frac{w_1 + w_2}{2}, w_1, w_2\right).
\end{aligned}
$$

Therefore, using the formula for $\wp$, 24.9,

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

you see that if the two periods w_1 and w_2 are replaced with tw_1 and tw_2 respectively, then

$$e_i(tw_1, tw_2) = t^{-2} e_i(w_1, w_2).$$

Let τ denote the complex number which equals the ratio, w_2/w_1 which was assumed in all this to not be real. Then

$$e_i(w_1, w_2) = w_1^{-2} e_i(1, \tau)$$

Now define the function, $\lambda(\tau)$

$$\lambda(\tau) \equiv \frac{e_3(1,\tau) - e_2(1,\tau)}{e_1(1,\tau) - e_2(1,\tau)} \left(= \frac{e_3(w_1, w_2) - e_2(w_1, w_2)}{e_1(w_1, w_2) - e_2(w_1, w_2)} \right). \tag{24.10}$$

This function is meromorphic for $\operatorname{Im} \tau > 0$ or for $\operatorname{Im} \tau < 0$. However, since the denominator is never equal to zero the function must actually be analytic on both the upper half plane and the lower half plane. It never is equal to 0 because $e_3 \neq e_2$ and it never equals 1 because $e_3 \neq e_1$. This is stated as an observation.

Observation 24.16 *The function, $\lambda(\tau)$ is analytic for τ in the upper half plane and never assumes the values 0 and 1.*

This is a very interesting function. Consider what happens when

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and the matrix is unimodular. By Theorem 24.13 on Page 570 $\{w_1', w_2'\}$ is just another basis for the same module of periods. Therefore, $\wp(z, w_1, w_2) = \wp(z, w_1', w_2')$ because both are defined as sums over the same values of w, just in different order which does not matter because of the absolute convergence of the sums on compact subsets of $\mathbb{C}$. Since $\wp$ is unchanged, it follows $\wp'(z)$ is also unchanged and so the numbers, e_i are also the same. However, they might be permuted in which case the function $\lambda(\tau)$ defined above would change. What would it take for $\lambda(\tau)$ to not change? In other words, for which unimodular transformations will λ be left unchanged? This happens if and only if no permuting takes place for the e_i. This occurs if $\wp\left(\frac{w_1}{2}\right) = \wp\left(\frac{w_1'}{2}\right)$ and $\wp\left(\frac{w_2}{2}\right) = \wp\left(\frac{w_2'}{2}\right)$. If

$$\frac{w_1'}{2} - \frac{w_1}{2} \in M,\ \frac{w_2'}{2} - \frac{w_2}{2} \in M$$

then $\wp\left(\frac{w_1}{2}\right) = \wp\left(\frac{w_1'}{2}\right)$ and so e_1 will be unchanged and similarly for e_2 and e_3. This occurs exactly when

$$\frac{1}{2}\left((a-1)\,w_1 + bw_2\right) \in M,\ \frac{1}{2}\left(cw_1 + (d-1)\,w_2\right) \in M.$$

This happens if a and d are odd and if b and c are even. Of course the stylish way to say this is

$$a \equiv 1 \bmod 2,\ d \equiv 1 \bmod 2,\ b \equiv 0 \bmod 2,\ c \equiv 0 \bmod 2. \tag{24.11}$$

This has shown that for unimodular transformations satisfying 24.11 λ is unchanged. Letting τ be defined as above,

$$\tau' = \frac{w_2'}{w_1'} \equiv \frac{cw_1 + dw_2}{aw_1 + bw_2} = \frac{c + d\tau}{a + b\tau}.$$

Thus for unimodular transformations, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying 24.11, or more succinctly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \tag{24.12}$$

it follows that

$$\lambda\left(\frac{c + d\tau}{a + b\tau}\right) = \lambda(\tau). \tag{24.13}$$

Furthermore, this is the only way this can happen.

Lemma 24.17 *$\lambda(\tau) = \lambda(\tau')$ if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where 24.12 holds.

Proof: It only remains to verify that if $\wp(w_1'/2) = \wp(w_1/2)$ then it is necessary that

$$\frac{w_1'}{2} - \frac{w_1}{2} \in M$$

with a similar requirement for w_2 and w_2'. If $\frac{w_1'}{2} - \frac{w_1}{2} \notin M$, then there exist integers, m, n such that

$$\frac{-w_1'}{2} + mw_1 + nw_2$$

is in the interior of P_0, the period parallelogram whose vertices are $0, w_1, w_1 + w_2$, and w_2. Therefore, it is possible to choose small a such that P_a contains the pole, 0, $\frac{w_1}{2}$, and $\frac{-w_1'}{2} + mw_1 + nw_2$ but no other poles of $\wp$ and in addition, ∂P_a^* contains no zeros of $z \to \wp(z) - \wp\left(\frac{w_1}{2}\right)$. Then the order of this elliptic function is 2. By assumption, and the fact that $\wp$ is even,

$$\wp\left(\frac{-w_1'}{2} + mw_1 + nw_2\right) = \wp\left(\frac{-w_1'}{2}\right) = \wp\left(\frac{w_1'}{2}\right) = \wp\left(\frac{w_1}{2}\right).$$

It follows both $\frac{-w_1'}{2} + mw_1 + nw_2$ and $\frac{w_1}{2}$ are zeros of $\wp(z) - \wp\left(\frac{w_1}{2}\right)$ and so by Theorem 24.8 on Page 566 these are the only two zeros of this function in P_a. Therefore, from Corollary 24.10 on Page 568

$$\frac{w_1}{2} - \frac{w_1'}{2} + mw_1 + nw_2 \in M$$

which shows $\frac{w_1}{2} - \frac{w_1'}{2} \in M$. This completes the proof of the lemma.

Note the condition in the lemma is equivalent to the condition 24.13 because you can relabel the coefficients. The message of either version is that the coefficient of τ in the numerator and denominator is odd while the constant in the numerator and denominator is even.

Next, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$ and therefore,

$$\lambda\left(\frac{2+\tau}{1}\right) = \lambda(\tau + 2) = \lambda(\tau). \tag{24.14}$$

Thus λ is periodic of period 2.

Thus λ leaves invariant a certain subgroup of the unimodular group. According to the next definition, λ is an example of something called a modular function.

Definition 24.18 *When an analytic or meromorphic function is invariant under a group of linear transformations, it is called an automorphic function. A function which is automorphic with respect to a subgroup of the modular group is called a modular function or an elliptic modular function.*

Now consider what happens for some other unimodular matrices which are not congruent to the identity mod 2. This will yield other functional equations for λ in addition to the fact that λ is periodic of period 2. As before, these functional equations come about because $\wp$ is unchanged when you change the basis for M, the module of periods. In particular, consider the unimodular matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{24.15}$$

Consider the first of these. Thus

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 + w_2 \end{pmatrix}$$

Hence $\tau' = w_2'/w_1' = (w_1 + w_2)/w_1 = 1 + \tau$. Then from the definition of λ,

$$
\begin{aligned}
\lambda(\tau') &= \lambda(1+\tau) \\
&= \frac{\wp\left(\frac{w_1'+w_2'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)}{\wp\left(\frac{w_1'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)} \\
&= \frac{\wp\left(\frac{w_1+w_2+w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= \frac{\wp\left(\frac{w_2}{2} + w_1\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= \frac{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= -\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= -\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right) + \wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= -\frac{\left(\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)}{1 + \left(\frac{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)} \\
&= \frac{\left(\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)}{\left(\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right) - 1} \\
&= \frac{\lambda(\tau)}{\lambda(\tau) - 1}. \qquad (24.16)
\end{aligned}
$$

Summarizing the important feature of the above,

$$\lambda(1+\tau) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}. \qquad (24.17)$$

Next consider the other unimodular matrix in 24.15. In this case $w_1' = w_2$ and $w_2' = w_1$. Therefore, $\tau' = w_2'/w_1' = w_1/w_2 = 1/\tau$. Then

$$\begin{aligned}
\lambda(\tau') &= \lambda(1/\tau) \\
&= \frac{\wp\left(\frac{w_1'+w_2'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)}{\wp\left(\frac{w_1'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)} \\
&= \frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_1}{2}\right)}{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1}{2}\right)} \\
&= \frac{e_3 - e_1}{e_2 - e_1} = -\frac{e_3 - e_2 + e_2 - e_1}{e_1 - e_2} \\
&= -(\lambda(\tau) - 1) = -\lambda(\tau) + 1. \qquad (24.18)
\end{aligned}$$

You could try other unimodular matrices and attempt to find other functional equations if you like but this much will suffice here.

24.1.5 A Formula For λ

Recall the formula of Mittag-Leffler for $\cot(\pi\alpha)$ given in 23.15. For convenience, here it is.

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha.$$

As explained in the derivation of this formula it can also be written as

$$\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha.$$

Differentiating both sides yields

$$\begin{aligned}
\pi^2 \csc^2(\pi\alpha) &= \sum_{n=-\infty}^{\infty} \frac{\alpha^2 + n^2}{(\alpha^2 - n^2)^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{(\alpha+n)^2 - 2\alpha n}{(\alpha+n)^2 (\alpha-n)^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{(\alpha+n)^2}{(\alpha+n)^2 (\alpha-n)^2} - \overbrace{\sum_{n=-\infty}^{\infty} \frac{2\alpha n}{(\alpha^2 - n^2)^2}}^{=0} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{(\alpha-n)^2}. \qquad (24.19)
\end{aligned}$$

Now this formula can be used to obtain a formula for $\lambda(\tau)$. As pointed out above, λ depends only on the ratio w_2/w_1 and so it suffices to take $w_1 = 1$ and

$w_2 = \tau$. Thus

$$\lambda(\tau) = \frac{\wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right)}{\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right)}. \tag{24.20}$$

From the original formula for $\wp$,

$$\begin{aligned}
& \wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\
= \quad & \frac{1}{\left(\frac{1+\tau}{2}\right)^2} - \frac{1}{\left(\frac{\tau}{2}\right)^2} + \sum_{(k,m)\neq(0,0)} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(-m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(-m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(\frac{1}{2} + \left(m + \frac{1}{2}\right)\tau - k\right)^2} - \frac{1}{\left(\left(m + \frac{1}{2}\right)\tau - k\right)^2}.
\end{aligned} \tag{24.21}$$

Similarly,

$$\begin{aligned}
& \wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\
= \quad & \frac{1}{\left(\frac{1}{2}\right)^2} - \frac{1}{\left(\frac{\tau}{2}\right)^2} + \sum_{(k,m)\neq(0,0)} \frac{1}{\left(k - \frac{1}{2} + m\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + m\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} - m\tau\right)^2} - \frac{1}{\left(k + \left(-m - \frac{1}{2}\right)\tau\right)^2} \\
= \quad & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(\frac{1}{2} + m\tau - k\right)^2} - \frac{1}{\left(\left(m + \frac{1}{2}\right)\tau - k\right)^2}.
\end{aligned} \tag{24.22}$$

Now use 24.19 to sum these over k. This yields,

$$\begin{aligned}
& \wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\
= \quad & \sum_m \frac{\pi^2}{\sin^2\left(\pi\left(\frac{1}{2} + \left(m + \frac{1}{2}\right)\tau\right)\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} \\
= \quad & \sum_m \frac{\pi^2}{\cos^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)}
\end{aligned}$$

and

$$\begin{aligned}\wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau}{2}\right) &= \sum_m \frac{\pi^2}{\sin^2\left(\pi\left(\frac{1}{2}+m\tau\right)\right)}-\frac{\pi^2}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)} \\ &= \sum_m \frac{\pi^2}{\cos^2\left(\pi m\tau\right)}-\frac{\pi^2}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}.\end{aligned}$$

The following interesting formula for λ results.

$$\lambda\left(\tau\right)=\frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}-\frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}{\sum_m \frac{1}{\cos^2\left(\pi m\tau\right)}-\frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}. \tag{24.23}$$

From this it is obvious $\lambda\left(-\tau\right)=\lambda\left(\tau\right)$. Therefore, from 24.18,

$$-\lambda\left(\tau\right)+1=\lambda\left(\frac{1}{\tau}\right)=\lambda\left(\frac{-1}{\tau}\right) \tag{24.24}$$

(It is good to recall that λ has been defined for $\tau\notin\mathbb{R}$.)

24.1.6 Mapping Properties Of λ

The two functional equations, 24.24 and 24.17 along with some other properties presented above are of fundamental importance. For convenience, they are summarized here in the following lemma.

Lemma 24.19 *The following functional equations hold for λ.*

$$\lambda\left(1+\tau\right)=\frac{\lambda\left(\tau\right)}{\lambda\left(\tau\right)-1},1=\lambda\left(\tau\right)+\lambda\left(\frac{-1}{\tau}\right) \tag{24.25}$$

$$\lambda\left(\tau+2\right)=\lambda\left(\tau\right), \tag{24.26}$$

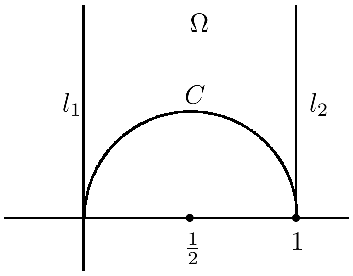
$\lambda\left(z\right)=\lambda\left(w\right)$ *if and only if there exists a unimodular matrix,*

$$\left(\begin{array}{cc} a & b \\ c & d\end{array}\right)\sim\left(\begin{array}{cc} 1 & 0 \\ 0 & 1\end{array}\right)\bmod 2$$

such that

$$w=\frac{az+b}{cz+d} \tag{24.27}$$

Consider the following picture.



In this picture, l_1 is the y axis and l_2 is the line, $x = 1$ while C is the top half of the circle centered at $\left(\frac{1}{2}, 0\right)$ which has radius $1/2$. Note the above formula implies λ has real values on l_1 which are between 0 and 1. This is because 24.23 implies

$$\begin{aligned} \lambda(ib) &= \frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)ib\right)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)ib\right)}}{\sum_m \frac{1}{\cos^2(\pi m ib)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)ib\right)}} \\ &= \frac{\sum_m \frac{1}{\cosh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} + \frac{1}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}}{\sum_m \frac{1}{\cosh^2(\pi m b)} + \frac{1}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}} \in (0,1). \qquad (24.28) \end{aligned}$$

This follows from the observation that

$$\cos(ix) = \cosh(x), \ \sin(ix) = i\sinh(x).$$

Thus it is clear from 24.28 that $\lim_{b\to 0+} \lambda(ib) = 1$.

Next I need to consider the behavior of $\lambda(\tau)$ as $\operatorname{Im}(\tau) \to \infty$. From 24.23 listed here for convenience,

$$\lambda(\tau) = \frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}{\sum_m \frac{1}{\cos^2(\pi m\tau)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}, \qquad (24.29)$$

it follows

$$\begin{aligned} \lambda(\tau) &= \frac{\frac{1}{\cos^2\left(\pi\left(-\frac{1}{2}\right)\tau\right)} - \frac{1}{\sin^2\left(\pi\left(-\frac{1}{2}\right)\tau\right)} + \frac{1}{\cos^2\left(\pi\frac{1}{2}\tau\right)} - \frac{1}{\sin^2\left(\pi\frac{1}{2}\tau\right)} + A(\tau)}{1 + B(\tau)} \\ &= \frac{\frac{2}{\cos^2\left(\pi\left(\frac{1}{2}\right)\tau\right)} - \frac{2}{\sin^2\left(\pi\left(\frac{1}{2}\right)\tau\right)} + A(\tau)}{1 + B(\tau)} \qquad (24.30) \end{aligned}$$

Where $A(\tau), B(\tau) \to 0$ as $\operatorname{Im}(\tau) \to \infty$. I took out the $m = 0$ term involving $1/\cos^2(\pi m\tau)$ in the denominator and the $m = -1$ and $m = 0$ terms in the numerator of 24.29. In fact, $e^{-i\pi(a+ib)}A(a+ib), e^{-i\pi(a+ib)}B(a+ib)$ converge to zero uniformly in a as $b \to \infty$.

Lemma 24.20 *For A, B defined in 24.30, $e^{-i\pi(a+ib)}C(a+ib) \to 0$ uniformly in a for $C = A, B$.*

Proof: From 24.23,

$$e^{-i\pi\tau}A(\tau) = \sum_{\substack{m\neq 0 \\ m\neq -1}} \frac{e^{-i\pi\tau}}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)} - \frac{e^{-i\pi\tau}}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}$$

Now let $\tau = a + ib$. Then letting $\alpha_m = \pi\left(m + \frac{1}{2}\right)$,

$$\begin{aligned} \cos(\alpha_m a + i\alpha_m b) &= \cos(\alpha_m a)\cosh(\alpha_m b) - i\sinh(\alpha_m b)\sin(\alpha_m a) \\ \sin(\alpha_m a + i\alpha_m b) &= \sin(\alpha_m a)\cosh(\alpha_m b) + i\cos(\alpha_m a)\sinh(\alpha_m b) \end{aligned}$$

Therefore,

$$\begin{aligned} \left|\cos^2(\alpha_m a + i\alpha_m b)\right| &= \cos^2(\alpha_m a)\cosh^2(\alpha_m b) + \sinh^2(\alpha_m b)\sin^2(\alpha_m a) \\ &\geq \sinh^2(\alpha_m b). \end{aligned}$$

Similarly,

$$\begin{aligned} \left|\sin^2(\alpha_m a + i\alpha_m b)\right| &= \sin^2(\alpha_m a)\cosh^2(\alpha_m b) + \cos^2(\alpha_m a)\sinh^2(\alpha_m b) \\ &\geq \sinh^2(\alpha_m b). \end{aligned}$$

It follows that for $\tau = a + ib$ and b large

$$\begin{aligned} &\left|e^{-i\pi\tau} A(\tau)\right| \\ \leq &\sum_{\substack{m\neq 0 \\ m\neq -1}} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} \\ \leq &\sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} + \sum_{m=-\infty}^{-2} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} \\ = &\ 2\sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} = 4\sum_{m=1}^{\infty} \frac{e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} \end{aligned}$$

Now a short computation shows

$$\frac{\frac{e^{\pi b}}{\sinh^2\left(\pi\left(m+1+\frac{1}{2}\right)b\right)}}{\frac{e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}} = \frac{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}{\sinh^2\left(\pi\left(m+\frac{3}{2}\right)b\right)} \leq \frac{1}{e^{3\pi b}}.$$

Therefore, for $\tau = a + ib$,

$$\begin{aligned} \left|e^{-i\pi\tau} A(\tau)\right| &\leq 4\frac{e^{\pi b}}{\sinh\left(\frac{3\pi b}{2}\right)} \sum_{m=1}^{\infty} \left(\frac{1}{e^{3\pi b}}\right)^m \\ &\leq 4\frac{e^{\pi b}}{\sinh\left(\frac{3\pi b}{2}\right)} \frac{1/e^{3\pi b}}{1-(1/e^{3\pi b})} \end{aligned}$$

which converges to zero as $b \to \infty$. Similar reasoning will establish the claim about $B(\tau)$. This proves the lemma.

Lemma 24.21 $\lim_{b\to\infty} \lambda(a+ib)\, e^{-i\pi(a+ib)} = 16$ *uniformly in* $a \in \mathbb{R}$.

Proof: From 24.30 and Lemma 24.20, this lemma will be proved if it is shown

$$\lim_{b\to\infty} \left(\frac{2}{\cos^2\left(\pi\left(\frac{1}{2}\right)(a+ib)\right)} - \frac{2}{\sin^2\left(\pi\left(\frac{1}{2}\right)(a+ib)\right)}\right) e^{-i\pi(a+ib)} = 16$$

uniformly in $a \in \mathbb{R}$. Let $\tau = a + ib$ to simplify the notation. Then the above expression equals

$$\begin{aligned}
&\left(\frac{8}{\left(e^{i\frac{\pi}{2}\tau}+e^{-i\frac{\pi}{2}\tau}\right)^2}+\frac{8}{\left(e^{i\frac{\pi}{2}\tau}-e^{-i\frac{\pi}{2}\tau}\right)^2}\right)e^{-i\pi\tau} \\
= &\left(\frac{8e^{i\pi\tau}}{\left(e^{i\pi\tau}+1\right)^2}+\frac{8e^{i\pi\tau}}{\left(e^{i\pi\tau}-1\right)^2}\right)e^{-i\pi\tau} \\
= &\frac{8}{\left(e^{i\pi\tau}+1\right)^2}+\frac{8}{\left(e^{i\pi\tau}-1\right)^2} \\
= &16\frac{1+e^{2\pi i\tau}}{\left(1-e^{2\pi i\tau}\right)^2}.
\end{aligned}$$

Now

$$\begin{aligned}
\left|\frac{1+e^{2\pi i\tau}}{\left(1-e^{2\pi i\tau}\right)^2}-1\right| &= \left|\frac{1+e^{2\pi i\tau}}{\left(1-e^{2\pi i\tau}\right)^2}-\frac{\left(1-e^{2\pi i\tau}\right)^2}{\left(1-e^{2\pi i\tau}\right)^2}\right| \\
&\leq \frac{\left|3e^{2\pi i\tau}-e^{4\pi i\tau}\right|}{\left(1-e^{-2\pi b}\right)^2} \leq \frac{3e^{-2\pi b}+e^{-4\pi b}}{\left(1-e^{-2\pi b}\right)^2}
\end{aligned}$$

and this estimate proves the lemma.

Corollary 24.22 $\lim_{b\to\infty}\lambda(a+ib) = 0$ *uniformly in* $a \in \mathbb{R}$. *Also* $\lambda(ib)$ *for* $b > 0$ *is real and is between* 0 *and 1,* λ *is real on the line,* l_2 *and on the curve,* C *and* $\lim_{b\to 0+}\lambda(1+ib) = -\infty$.

Proof: From Lemma 24.21,

$$\left|\lambda(a+ib)\,e^{-i\pi(a+ib)}-16\right| < 1$$

for all a provided b is large enough. Therefore, for such b,

$$|\lambda(a+ib)| \leq 17e^{-\pi b}.$$

24.28 proves the assertion about $\lambda(-bi)$ real.

By the first part, $\lim_{b\to\infty}|\lambda(ib)| = 0$. Now from 24.24

$$\lim_{b\to 0+}\lambda(ib) = \lim_{b\to 0+}\left(1-\lambda\left(\frac{-1}{ib}\right)\right) = \lim_{b\to 0+}\left(1-\lambda\left(\frac{i}{b}\right)\right) = 1. \tag{24.31}$$

by Corollary 24.22.

Next consider the behavior of λ on line l_2 in the above picture. From 24.17 and 24.28,

$$\lambda(1+ib) = \frac{\lambda(ib)}{\lambda(ib)-1} < 0$$

and so as $b \to 0+$ in the above, $\lambda(1+ib) \to -\infty$.

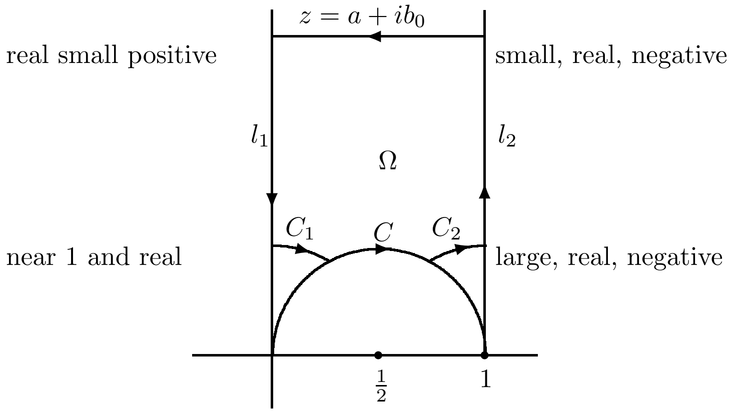
It is left as an exercise to show that the map $\tau \to 1 - \frac{1}{\tau}$ maps l_2 onto the curve, C. Therefore, by 24.25, for $\tau \in l_2$,

$$\lambda\left(1 - \frac{1}{\tau}\right) = \frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right) - 1} \tag{24.32}$$

$$= \frac{1 - \lambda(\tau)}{(1 - \lambda(\tau)) - 1} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \in \mathbb{R} \tag{24.33}$$

It follows λ is real on the boundary of Ω in the above picture. This proves the corollary.

Now, following Alfors [2], cut off Ω by considering the horizontal line segment, $z = a + ib_0$ where b_0 is very large and positive and $a \in [0,1]$. Also cut Ω off by the images of this horizontal line, under the transformations $z = \frac{1}{\tau}$ and $z = 1 - \frac{1}{\tau}$. These are arcs of circles because the two transformations are fractional linear transformations. It is left as an exercise for you to verify these arcs are situated as shown in the following picture. The important thing to notice is that for b_0 large the points of these circles are close to the origin and $(1,0)$ respectively. The following picture is a summary of what has been obtained so far on the mapping by λ.



In the picture, the descriptions are of λ acting on points of the indicated boundary of Ω. Consider the oriented contour which results from $\lambda(z)$ as z moves first up l_2 as indicated, then along the line $z = a + ib$ and then down l_1 and then along C_1 to C and along C till C_2 and then along C_2 to l_2. As indicated in the picture, this involves going from a large negative real number to a small negative real number and then over a smooth curve which stays small to a real positive number and from there to a real number near 1. $\lambda(z)$ stays fairly near 1 on C_1 provided b_0 is large so that the circle, C_1 has very small radius. Then along C, $\lambda(z)$ is real until it hits C_2. What about the behavior of λ on C_2? For $z \in C_2$, it follows from the definition of C_2 that $z = 1 - \frac{1}{\tau}$ where τ is on the line, $a + ib_0$. Therefore, by Lemma 24.21, 24.17, and 24.24

$$\begin{aligned}
\lambda(z) &= \lambda\left(1-\frac{1}{\tau}\right) = \frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right)-1} = \frac{\lambda\left(\frac{1}{\tau}\right)}{\lambda\left(\frac{1}{\tau}\right)-1} \\
&= \frac{1-\lambda(\tau)}{(1-\lambda(\tau))-1} = \frac{\lambda(\tau)-1}{\lambda(\tau)} = 1-\frac{1}{\lambda(\tau)}
\end{aligned}$$

which is approximately equal to

$$1-\frac{1}{16e^{i\pi(a+ib_0)}} = 1-\frac{e^{\pi b_0}e^{-ia\pi}}{16}.$$

These points are essentially on a large half circle in the upper half plane which has radius approximately $\frac{e^{\pi b_0}}{16}$.

Now let $w \in \mathbb{C}$ with $\operatorname{Im}(w) \neq 0$. Then for b_0 large enough, the motion over the boundary of the truncated region indicated in the above picture results in λ tracing out a large simple closed curve oriented in the counter clockwise direction which includes w on its interior if $\operatorname{Im}(w) > 0$ but which excludes w if $\operatorname{Im}(w) < 0$.

Theorem 24.23 *Let Ω be the domain described above. Then λ maps Ω one to one and onto the upper half plane of $\mathbb{C}$, $\{z \in \mathbb{C}$ such that $\operatorname{Im}(z) > 0\}$. Also, the line $\lambda(l_1) = (0,1), \lambda(l_2) = (-\infty, 0)$, and $\lambda(C) = (1,\infty)$.*

Proof: Let $\operatorname{Im}(w) > 0$ and denote by γ the oriented contour described above and illustrated in the above picture. Then the winding number of $\lambda \circ \gamma$ about w equals 1. Thus

$$\frac{1}{2\pi i}\int_{\lambda\circ\gamma}\frac{1}{z-w}dz = 1.$$

But, splitting the contour integrals into l_2,the top line, l_1, C_1, C, and C_2 and changing variables on each of these, yields

$$1 = \frac{1}{2\pi i}\int_{\gamma}\frac{\lambda'(z)}{\lambda(z)-w}dz$$

and by the theorem on counting zeros, Theorem 19.20 on Page 438, the function, $z \to \lambda(z) - w$ has exactly one zero inside the truncated Ω. However, this shows this function has exactly one zero inside Ω because b_0 was arbitrary as long as it is sufficiently large. Since w was an arbitrary element of the upper half plane, this verifies the first assertion of the theorem. The remaining claims follow from the above description of λ, in particular the estimate for λ on C_2. This proves the theorem.

Note also that the argument in the above proof shows that if $\operatorname{Im}(w) < 0$, then w is not in $\lambda(\Omega)$. However, if you consider the reflection of Ω about the y axis, then it will follow that λ maps this set one to one onto the lower half plane. The argument will make significant use of Theorem 19.22 on Page 440 which is stated here for convenience.

Theorem 24.24 *Let $f : B(a,R) \to \mathbb{C}$ be analytic and let*

$$f(z) - \alpha = (z-a)^m g(z), \ \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a,R)$. ($f(z) - \alpha$ has a zero of order m at $z = a$.) Then there exist $\varepsilon, \delta > 0$ with the property that for each z satisfying $0 < |z - \alpha| < \delta$, there exist points,

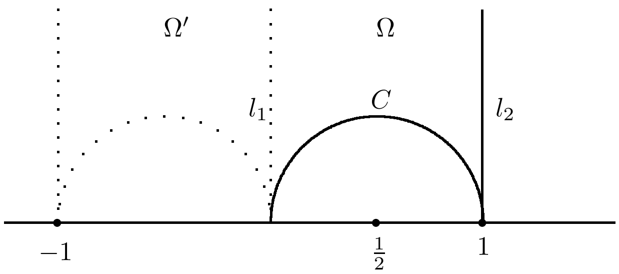
$$\{a_1, \cdots, a_m\} \subseteq B(a, \varepsilon),$$

such that

$$f^{-1}(z) \cap B(a,\varepsilon) = \{a_1, \cdots, a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot) - z$.

Corollary 24.25 *Let Ω be the region above. Consider the set of points, $Q = \overline{\Omega} \cup \Omega' \setminus \{0,1\}$ described by the following picture.*



Then $\lambda(Q) = \mathbb{C} \setminus \{0,1\}$. Also $\lambda'(z) \neq 0$ for every z in $\cup_{k=-\infty}^{\infty}(Q + 2k) \equiv H$.

Proof: By Theorem 24.23, this will be proved if it can be shown that $\lambda(\Omega') = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$. Consider λ_1 defined on Ω' by

$$\lambda_1(x+iy) \equiv \overline{\lambda(-x+iy)}.$$

Claim: λ_1 is analytic.

Proof of the claim: You just verify the Cauchy Riemann equations. Letting $\lambda(x+iy) = u(x,y) + iv(x,y)$,

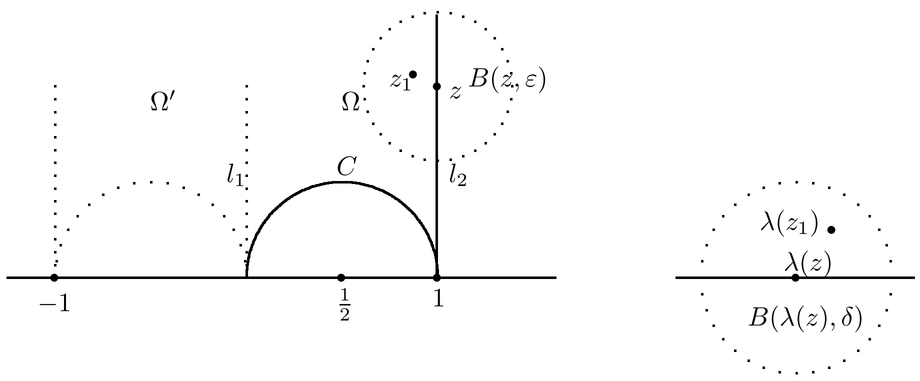
$$\begin{aligned} \lambda_1(x+iy) &= u(-x,y) - iv(-x,y) \\ &\equiv u_1(x,y) + iv(x,y). \end{aligned}$$

Then $u_{1x}(x,y) = -u_x(-x,y)$ and $v_{1y}(x,y) = -v_y(-x,y) = -u_x(-x,y)$ since λ is analytic. Thus $u_{1x} = v_{1y}$. Next, $u_{1y}(x,y) = u_y(-x,y)$ and $v_{1x}(x,y) = v_x(-x,y) = -u_y(-x,y)$ and so $u_{1y} = -v_x$.

Now recall that on l_1, λ takes real values. Therefore, $\lambda_1 = \lambda$ on l_1, a set with a limit point. It follows $\lambda = \lambda_1$ on $\Omega' \cup \Omega$. By Theorem 24.23 λ maps Ω one to one onto the upper half plane. Therefore, from the definition of $\lambda_1 = \lambda$, it follows λ maps Ω' one to one onto the lower half plane as claimed. This has shown that λ

is one to one on $\Omega \cup \Omega'$. This also verifies from Theorem 19.22 on Page 440 that $\lambda' \neq 0$ on $\Omega \cup \Omega'$.

Now consider the lines l_2 and C. If $\lambda'(z) = 0$ for $z \in l_2$, a contradiction can be obtained. Pick such a point. If $\lambda'(z) = 0$, then z is a zero of order $m \geq 2$ of the function, $\lambda - \lambda(z)$. Then by Theorem 19.22 there exist $\delta, \varepsilon > 0$ such that if $w \in B(\lambda(z), \delta)$, then $\lambda^{-1}(w) \cap B(z, \varepsilon)$ contains at least m points.



In particular, for $z_1 \in \Omega \cap B(z, \varepsilon)$ sufficiently close to $z, \lambda(z_1) \in B(\lambda(z), \delta)$ and so the function $\lambda - \lambda(z_1)$ has at least two distinct zeros. These zeros must be in $B(z, \varepsilon) \cap \Omega$ because $\lambda(z_1)$ has positive imaginary part and the points on l_2 are mapped by λ to a real number while the points of $B(z, \varepsilon) \setminus \overline{\Omega}$ are mapped by λ to the lower half plane thanks to the relation, $\lambda(z+2) = \lambda(z)$. This contradicts λ one to one on Ω. Therefore, $\lambda' \neq 0$ on l_2. Consider C. Points on C are of the form $1 - \frac{1}{\tau}$ where $\tau \in l_2$. Therefore, using 24.33,

$$\lambda\left(1 - \frac{1}{\tau}\right) = \frac{\lambda(\tau) - 1}{\lambda(\tau)}.$$

Taking the derivative of both sides,

$$\lambda'\left(1 - \frac{1}{\tau}\right)\left(\frac{1}{\tau^2}\right) = \frac{\lambda'(\tau)}{\lambda(\tau)^2} \neq 0.$$

Since λ is periodic of period 2 it follows $\lambda'(z) \neq 0$ for all $z \in \cup_{k=-\infty}^{\infty}(Q + 2k)$.

Lemma 24.26 *If* $\operatorname{Im}(\tau) > 0$ *then there exists a unimodular* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *such that*

$$\frac{c + d\tau}{a + b\tau}$$

is contained in the interior of Q. *In fact,* $\left|\frac{c+d\tau}{a+b\tau}\right| \geq 1$ *and*

$$-1/2 \leq \operatorname{Re}\left(\frac{c+d\tau}{a+b\tau}\right) \leq 1/2.$$

Proof: Letting a basis for the module of periods of $\wp$ be $\{1,\tau\}$, it follows from Theorem 24.3 on Page 564 that there exists a basis for the same module of periods, $\{w_1', w_2'\}$ with the property that for $\tau' = w_2'/w_1'$

$$|\tau'| \geq 1,\ \frac{-1}{2} \leq \operatorname{Re}\tau' \leq \frac{1}{2}.$$

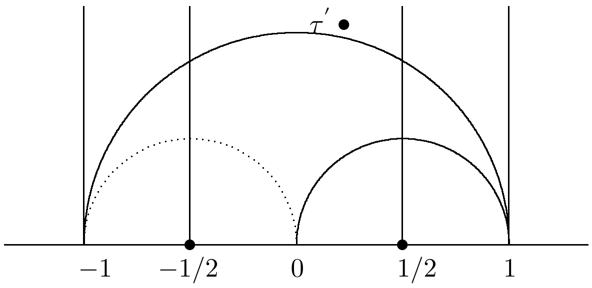
Since this is a basis for the same module of periods, there exists a unimodular matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}.$$

Hence,

$$\tau' = \frac{w_2'}{w_1'} = \frac{c+d\tau}{a+b\tau}.$$

Thus τ' is in the interior of H. In fact, it is on the interior of $\Omega' \cup \Omega \equiv Q$.



24.1.7 A Short Review And Summary

With this lemma, it is easy to extend Corollary 24.25. First, a simple observation and review is a good idea. Recall that when you change the basis for the module of periods, the Weierstrass $\wp$ function does not change and so the set of e_i used in defining λ also do not change. Letting the new basis be $\{w_1', w_2'\}$, it was shown that

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

for some unimodular transformation, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Letting $\tau = w_2/w_1$ and $\tau' = w_2'/w_1'$

$$\tau' = \frac{c+d\tau}{a+b\tau} \equiv \phi(\tau)$$

Now as discussed earlier

$$\begin{aligned}\lambda(\tau') &= \lambda(\phi(\tau)) \equiv \frac{\wp\left(\frac{w_1'+w_2'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)}{\wp\left(\frac{w_1'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)} \\ &= \frac{\wp\left(\frac{1+\tau'}{2}\right) - \wp\left(\frac{\tau'}{2}\right)}{\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau'}{2}\right)}\end{aligned}$$

These numbers in the above fraction must be the same as $\wp\left(\frac{1+\tau}{2}\right), \wp\left(\frac{\tau}{2}\right)$, and $\wp\left(\frac{1}{2}\right)$ but they might occur differently. This is because $\wp$ does not change and these numbers are the zeros of a polynomial having coefficients involving only numbers and $\wp(z)$. It could happen for example that $\wp\left(\frac{1+\tau'}{2}\right) = \wp\left(\frac{\tau}{2}\right)$ in which case this would change the value of λ. In effect, you can keep track of all possibilities by simply permuting the e_i in the formula for $\lambda(\tau)$ given by $\frac{e_3-e_2}{e_1-e_2}$. Thus consider the following permutation table.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{array}.$$

Corresponding to this list of 6 permutations, all possible formulas for $\lambda(\phi(\tau))$ can be obtained as follows. Letting $\tau' = \phi(\tau)$ where ϕ is a unimodular matrix corresponding to a change of basis,

$$\lambda(\tau') = \frac{e_3-e_2}{e_1-e_2} = \lambda(\tau) \tag{24.34}$$

$$\lambda(\tau') = \frac{e_1-e_3}{e_2-e_3} = \frac{e_3-e_2+e_2-e_1}{e_3-e_2} = 1 - \frac{1}{\lambda(\tau)} = \frac{\lambda(\tau)-1}{\lambda(\tau)} \tag{24.35}$$

$$\begin{aligned}\lambda(\tau') &= \frac{e_2-e_1}{e_3-e_1} = -\left[\frac{e_3-e_2-(e_1-e_2)}{e_1-e_2}\right]^{-1} \\ &= -\left[\lambda(\tau)-1\right]^{-1} = \frac{1}{1-\lambda(\tau)}\end{aligned} \tag{24.36}$$

$$\begin{aligned}\lambda(\tau') &= \frac{e_3-e_1}{e_2-e_1} = -\left[\frac{e_3-e_2-(e_1-e_2)}{e_1-e_2}\right] \\ &= -\left[\lambda(\tau)-1\right] = 1-\lambda(\tau)\end{aligned} \tag{24.37}$$

$$\lambda(\tau') = \frac{e_2 - e_3}{e_1 - e_3} = \frac{e_3 - e_2}{e_3 - e_2 - (e_1 - e_2)} = \frac{1}{1 - \frac{1}{\lambda(\tau)}} = \frac{\lambda(\tau)}{\lambda(\tau) - 1} \tag{24.38}$$

$$\lambda(\tau') = \frac{e_1 - e_3}{e_3 - e_2} = \frac{1}{\lambda(\tau)} \tag{24.39}$$

Corollary 24.27 *$\lambda'(\tau) \neq 0$ for all τ in the upper half plane, denoted by P_+.*

Proof: Let $\tau \in P_+$. By Lemma 24.26 there exists ϕ a unimodular transformation and τ' in the interior of Q such that $\tau' = \phi(\tau)$. Now from the definition of λ in terms of the e_i, there is at worst a permutation of the e_i and so it might be the case that $\lambda(\phi(\tau)) \neq \lambda(\tau)$ but it is the case that $\lambda(\phi(\tau)) = \xi(\lambda(\tau))$ where $\xi'(z) \neq 0$. Here ξ is one of the functions determined by 24.34 - 24.39. (Since $\lambda(\tau) \notin \{0, 1\}$, $\xi'(\lambda(z)) \neq 0$. This follows from the above possibilities for ξ listed above in 24.34 - 24.39.) All the possibilities are $\xi(z) =$

$$z, \frac{z-1}{z}, \frac{1}{1-z}, 1-z, \frac{z}{z-1}, \frac{1}{z}$$

and these are the same as the possibilities for ξ^{-1}. Therefore, $\lambda'(\phi(\tau))\phi'(\tau) = \xi'(\lambda(\tau))\lambda'(\tau)$ and so $\lambda'(\tau) \neq 0$ as claimed.

Now I will present a lemma which is of major significance. It depends on the remarkable mapping properties of the modular function and the monodromy theorem from analytic continuation. A review of the monodromy theorem will be listed here for convenience. First recall the definition of the concept of function elements and analytic continuation.

Definition 24.28 *A function element is an ordered pair, (f, D) where D is an open ball and f is analytic on D. (f_0, D_0) and (f_1, D_1) are direct continuations of each other if $D_1 \cap D_0 \neq \emptyset$ and $f_0 = f_1$ on $D_1 \cap D_0$. In this case I will write $(f_0, D_0) \sim (f_1, D_1)$. A chain is a finite sequence, of disks, $\{D_0, \cdots, D_n\}$ such that $D_{i-1} \cap D_i \neq \emptyset$. If (f_0, D_0) is a given function element and there exist function elements, (f_i, D_i) such that $\{D_0, \cdots, D_n\}$ is a chain and $(f_{j-1}, D_{j-1}) \sim (f_j, D_j)$ then (f_n, D_n) is called the analytic continuation of (f_0, D_0) along the chain $\{D_0, \cdots, D_n\}$. Now suppose γ is an oriented curve with parameter interval $[a, b]$ and there exists a chain, $\{D_0, \cdots, D_n\}$ such that $\gamma^* \subseteq \cup_{k=1}^n D_k, \gamma(a)$ is the center of $D_0, \gamma(b)$ is the center of D_n, and there is an increasing list of numbers in $[a, b], a = s_0 < s_1 \cdots < s_n = b$ such that $\gamma([s_i, s_{i+1}]) \subseteq D_i$ and (f_n, D_n) is an analytic continuation of (f_0, D_0) along the chain. Then (f_n, D_n) is called an analytic continuation of (f_0, D_0) along the curve γ. (γ will always be a continuous curve. Nothing more is needed.)*

Then the main theorem is the monodromy theorem listed next, Theorem 21.19 and its corollary on Page 493.

Theorem 24.29 *Let Ω be a simply connected subset of $\mathbb{C}$ and suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through a. Then there exists G analytic on Ω such that G agrees with f on $B(a, r)$.*

Here is the lemma.

Lemma 24.30 *Let λ be the modular function defined on P_+ the upper half plane. Let V be a simply connected region in $\mathbb{C}$ and let $f : V \to \mathbb{C}\setminus\{0,1\}$ be analytic and nonconstant. Then there exists an analytic function, $g : V \to P_+$ such that $\lambda \circ g = f$.*

Proof: Let $a \in V$ and choose r_0 small enough that $f(B(a,r_0))$ contains neither 0 nor 1. You need only let $B(a,r_0) \subseteq V$. Now there exists a unique point in Q, τ_0 such that $\lambda(\tau_0) = f(a)$. By Corollary 24.25, $\lambda'(\tau_0) \neq 0$ and so by the open mapping theorem, Theorem 19.22 on Page 440, There exists $B(\tau_0, R_0) \subseteq P_+$ such that λ is one to one on $B(\tau_0, R_0)$ and has a continuous inverse. Then picking r_0 still smaller, it can be assumed $f(B(a,r_0)) \subseteq \lambda(B(\tau_0,R_0))$. Thus there exists a local inverse for λ, λ_0^{-1} defined on $f(B(a,r_0))$ having values in $B(\tau_0,R_0) \cap \lambda^{-1}(f(B(a,r_0)))$. Then defining $g_0 \equiv \lambda_0^{-1} \circ f$, $(g_0, B(a,r_0))$ is a function element. I need to show this can be continued along every curve starting at a in such a way that each function in each function element has values in P_+.

Let $\gamma : [\alpha,\beta] \to V$ be a continuous curve starting at $a, (\gamma(\alpha) = a)$ and suppose that if $t < T$ there exists a nonnegative integer m and a function element $(g_m, B(\gamma(t), r_m))$ which is an analytic continuation of $(g_0, B(a,r_0))$ along γ where $g_m(\gamma(t)) \in P_+$ and each function in every function element for $j \leq m$ has values in P_+. Thus for some small $T > 0$ this has been achieved.

Then consider $f(\gamma(T)) \in \mathbb{C}\setminus\{0,1\}$. As in the first part of the argument, there exists a unique $\tau_T \in Q$ such that $\lambda(\tau_T) = f(\gamma(T))$ and for r small enough there is an analytic local inverse, λ_T^{-1} between $f(B(\gamma(T),r))$ and $\lambda^{-1}(f(B(\gamma(T),r))) \cap B(\tau_T, R_T) \subseteq P_+$ for some $R_T > 0$. By the assumption that the analytic continuation can be carried out for $t < T$, there exists $\{t_0, \cdots, t_m = t\}$ and function elements $(g_j, B(\gamma(t_j), r_j)), j = 0, \cdots, m$ as just described with $g_j(\gamma(t_j)) \in P_+, \lambda \circ g_j = f$ on $B(\gamma(t_j), r_j)$ such that for $t \in [t_m, T], \gamma(t) \in B(\gamma(T), r)$. Let

$$I = B(\gamma(t_m), r_m) \cap B(\gamma(T), r).$$

Then since λ_T^{-1} is a local inverse, it follows for all $z \in I$

$$\lambda(g_m(z)) = f(z) = \lambda\left(\lambda_T^{-1} \circ f(z)\right)$$

Pick $z_0 \in I$. Then by Lemma 24.19 on Page 584 there exists a unimodular mapping of the form

$$\phi(z) = \frac{az+b}{cz+d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \operatorname{mod} 2$$

such that

$$g_m(z_0) = \phi\left(\lambda_T^{-1} \circ f(z_0)\right).$$

Since both $g_m(z_0)$ and $\phi\left(\lambda_T^{-1} \circ f(z_0)\right)$ are in the upper half plane, it follows $ad - cb = 1$ and ϕ maps the upper half plane to the upper half plane. Note the pole of ϕ is real and all the sets being considered are contained in the upper half plane so ϕ is analytic where it needs to be.

Claim: For all $z \in I$,

$$g_m(z) = \phi \circ \lambda_T^{-1} \circ f(z). \tag{24.40}$$

Proof: For $z = z_0$ the equation holds. Let

$$A = \left\{z \in I : g_m(z) = \phi\left(\lambda_T^{-1} \circ f(z)\right)\right\}.$$

Thus $z_0 \in I$. If $z \in I$ and if w is close enough to z, then $w \in I$ also and so both sides of 24.40 with w in place of z are in $\lambda_m^{-1}(f(I))$. But by construction, λ is one to one on this set and since λ is invariant with respect to ϕ,

$$\lambda(g_m(w)) = \lambda\left(\lambda_T^{-1} \circ f(w)\right) = \lambda\left(\phi \circ \lambda_T^{-1} \circ f(w)\right)$$

and consequently, $w \in A$. This shows A is open. But A is also closed in I because the functions are continuous. Therefore, $A = I$ and so 24.40 is obtained.

Letting $f(z) \in f(B(\gamma(T)), r)$,

$$\lambda\left(\phi\left(\lambda_T^{-1}(f(z))\right)\right) = \lambda\left(\lambda_T^{-1}(f(z))\right) = f(z)$$

and so $\phi \circ \lambda_T^{-1}$ is a local inverse for λ on $f(B(\gamma(T)), r)$. Let the new function element be $\left(\overbrace{\phi \circ \lambda_T^{-1} \circ f}^{g_{m+1}}, B(\gamma(T), r)\right)$. This has shown the initial function element can be continued along every curve through a.

By the monodromy theorem, there exists g analytic on V such that g has values in P_+ and $g = g_0$ on $B(a, r_0)$. By the construction, it also follows $\lambda \circ g = f$. This last claim is easy to see because $\lambda \circ g = f$ on $B(a, r_0)$, a set with a limit point so the equation holds for all $z \in V$. This proves the lemma.

24.2 The Picard Theorem Again

Having done all this work on the modular function which is important for its own sake, there is an easy proof of the Picard theorem. In fact, this is the way Picard did it in 1879. I will state it slightly differently since it is no trouble to do so, [23].

Theorem 24.31 *Let f be meromorphic on $\mathbb{C}$ and suppose f misses three distinct points, a, b, c. Then f is a constant function.*

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c}\frac{b-c}{b-a}$. Then $\phi(c) = \infty, \phi(a) = 0$, and $\phi(b) = 1$. Now consider the function, $h = \phi \circ f$. Then h misses the three points $\infty, 0$, and 1. Since h is meromorphic and does not have ∞ in its values, it must actually be analytic.

Thus h is an entire function which misses the two values 0 and 1. If h is not constant, then by Lemma 24.30 there exists a function, g analytic on $\mathbb{C}$ which has values in the upper half plane, P_+ such that $\lambda \circ g = h$. However, g must be a constant because there exists ψ an analytic map on the upper half plane which maps the upper half plane to $B(0,1)$. You can use the Riemann mapping theorem or more simply, $\psi(z) = \frac{z-i}{z+i}$. Thus $\psi \circ g$ equals a constant by Liouville's theorem. Hence g is a constant and so h must also be a constant because $\lambda(g(z)) = h(z)$. This proves f is a constant also. This proves the theorem.

24.3 Exercises

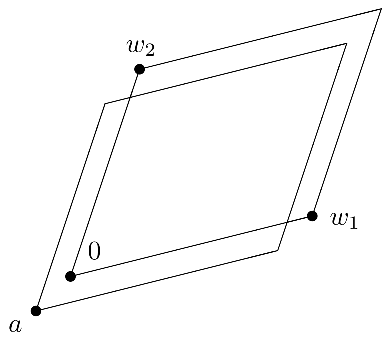
1. Show the set of modular transformations is a group. Also show those modular transformations which are congruent mod 2 to the identity as described above is a subgroup.

2. Suppose f is an elliptic function with period module M. If $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ are two bases, show that the resulting period parallelograms resulting from the two bases have the same area.

3. Given a module of periods with basis $\{w_1, w_2\}$ and letting a typical element of this module be denoted by w as described above, consider the product
$$\sigma(z) \equiv z \prod_{w \neq 0} \left(1 - \frac{z}{w}\right) e^{(z/w) + \frac{1}{2}(z/w)^2}.$$
Show this product converges uniformly on compact sets, is an entire function, and satisfies
$$\sigma'(z) / \sigma(z) = \zeta(z)$$
where $\zeta(z)$ was defined above as a primitive of $\wp(z)$ and is given by
$$\zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z - w} + \frac{z}{w^2} + \frac{1}{w}.$$

4. Show $\zeta(z + w_i) = \zeta(z) + \eta_i$ where η_i is a constant.

5. Let P_a be the parallelogram shown in the following picture.



Show that $\frac{1}{2\pi i}\int_{\partial P_a} \zeta(z)\,dz = 1$ where the contour is taken once around the parallelogram in the counter clockwise direction. Next evaluate this contour integral directly to obtain Legendre's relation,

$$\eta_1 w_2 - \eta_2 w_1 = 2\pi i.$$

6. For σ defined in Problem 3, 4 explain the following steps. For $j = 1, 2$

$$\frac{\sigma'(z+w_j)}{\sigma(z+w_j)} = \zeta(z+w_j) = \zeta(z) + \eta_j = \frac{\sigma'(z)}{\sigma(z)} + \eta_j$$

Therefore, there exists a constant, C_j such that

$$\sigma(z+w_j) = C_j \sigma(z) e^{\eta_j z}.$$

Next show σ is an odd function, $(\sigma(-z) = -\sigma(z))$ and then let $z = -w_j/2$ to find $C_j = -e^{\frac{\eta_j w_j}{2}}$ and so

$$\sigma(z+w_j) = -\sigma(z) e^{\eta_j\left(z+\frac{w_j}{2}\right)}. \tag{24.41}$$

7. Show any **even** elliptic function, f with periods w_1 and w_2 for which 0 is neither a pole nor a zero can be expressed in the form

$$f(0)\prod_{k=1}^{n}\frac{\wp(z)-\wp(a_k)}{\wp(z)-\wp(b_k)}$$

where C is some constant. Here $\wp$ is the Weierstrass function which comes from the two periods, w_1 and w_2. **Hint:** You might consider the above function in terms of the poles and zeros on a period parallelogram and recall that an entire function which is elliptic is a constant.

8. Suppose f is any elliptic function with $\{w_1, w_2\}$ a basis for the module of periods. Using Theorem 24.9 and 24.41 show that there exists constants $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ such that for some constant C,

$$f(z) = C\prod_{k=1}^{n}\frac{\sigma(z-a_k)}{\sigma(z-b_k)}.$$

Hint: You might try something like this: By Theorem 24.9, it follows that if $\{\alpha_k\}$ are the zeros and $\{b_k\}$ the poles in an appropriate period parallelogram, $\sum \alpha_k - \sum b_k$ equals a period. Replace α_k with a_k such that $\sum a_k - \sum b_k = 0$. Then use 24.41 to show that the given formula for f is bi periodic. Anyway, you try to arrange things such that the given formula has the same poles as f. Remember an entire elliptic function equals a constant.

9. Show that the map $\tau \to 1 - \frac{1}{\tau}$ maps l_2 onto the curve, C in the above picture on the mapping properties of λ.

10. Modify the proof of Theorem 24.23 to show that $\lambda(\Omega) \cap \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\} = \emptyset$.

The Hausdorff Maximal Theorem

The purpose of this appendix is to prove the equivalence between the axiom of choice, the Hausdorff maximal theorem, and the well-ordering principle. The Hausdorff maximal theorem and the well-ordering principle are very useful but a little hard to believe; so, it may be surprising that they are equivalent to the axiom of choice. First it is shown that the axiom of choice implies the Hausdorff maximal theorem, a remarkable theorem about partially ordered sets.

A nonempty set is partially ordered if there exists a partial order, $\prec$, satisfying

$$x \prec x$$

and

$$\text{if } x \prec y \text{ and } y \prec z \text{ then } x \prec z.$$

An example of a partially ordered set is the set of all subsets of a given set and $\prec \equiv \subseteq$. Note that two elements in a partially ordered sets may not be related. In other words, just because $x,\ y$ are in the partially ordered set, it does not follow that either $x \prec y$ or $y \prec x$. A subset of a partially ordered set, $\mathcal{C}$, is called a chain if $x,\ y \in \mathcal{C}$ implies that either $x \prec y$ or $y \prec x$. If either $x \prec y$ or $y \prec x$ then x and y are described as being comparable. A chain is also called a totally ordered set. $\mathcal{C}$ is a maximal chain if whenever $\widetilde{\mathcal{C}}$ is a chain containing $\mathcal{C}$, it follows the two chains are equal. In other words $\mathcal{C}$ is a maximal chain if there is no strictly larger chain.

Lemma A.1 *Let $\mathcal{F}$ be a nonempty partially ordered set with partial order $\prec$. Then assuming the axiom of choice, there exists a maximal chain in $\mathcal{F}$.*

Proof: Let $\mathcal{X}$ be the set of all chains from $\mathcal{F}$. For $\mathcal{C} \in \mathcal{X}$, let

$$S_{\mathcal{C}} = \{x \in \mathcal{F} \text{ such that } \mathcal{C} \cup \{x\} \text{ is a chain strictly larger than } \mathcal{C}\}.$$

If $S_{\mathcal{C}} = \emptyset$ for any $\mathcal{C}$, then $\mathcal{C}$ is maximal. Thus, assume $S_{\mathcal{C}} \neq \emptyset$ for all $\mathcal{C} \in \mathcal{X}$. Let $f(\mathcal{C}) \in S_{\mathcal{C}}$. (This is where the axiom of choice is being used.) Let

$$g(\mathcal{C}) = \mathcal{C} \cup \{f(\mathcal{C})\}.$$

Thus $g(\mathcal{C}) \supsetneq \mathcal{C}$ and $g(\mathcal{C}) \setminus \mathcal{C} = \{f(\mathcal{C})\} = \{\text{a single element of } \mathcal{F}\}$. A subset $\mathcal{T}$ of $\mathcal{X}$ is called a tower if

$$\emptyset \in \mathcal{T},$$

$$\mathcal{C} \in \mathcal{T} \text{ implies } g(\mathcal{C}) \in \mathcal{T},$$

and if $\mathcal{S} \subseteq \mathcal{T}$ is totally ordered with respect to set inclusion, then

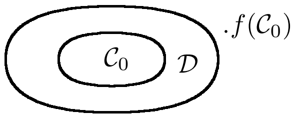
$$\cup\mathcal{S} \in \mathcal{T}.$$

Here $\mathcal{S}$ is a chain with respect to set inclusion whose elements are chains.

Note that $\mathcal{X}$ is a tower. Let $\mathcal{T}_0$ be the intersection of all towers. Thus, $\mathcal{T}_0$ is a tower, the smallest tower. Are any two sets in $\mathcal{T}_0$ comparable in the sense of set inclusion so that $\mathcal{T}_0$ is actually a chain? Let $\mathcal{C}_0$ be a set of $\mathcal{T}_0$ which is comparable to every set of $\mathcal{T}_0$. Such sets exist, $\emptyset$ being an example. Let

$$\mathcal{B} \equiv \{\mathcal{D} \in \mathcal{T}_0 : \mathcal{D} \supsetneq \mathcal{C}_0 \text{ and } f(\mathcal{C}_0) \notin \mathcal{D}\}.$$

The picture represents sets of $\mathcal{B}$. As illustrated in the picture, $\mathcal{D}$ is a set of $\mathcal{B}$ when $\mathcal{D}$ is larger than $\mathcal{C}_0$ but fails to be comparable to $g(\mathcal{C}_0)$. Thus there would be more than one chain ascending from $\mathcal{C}_0$ if $\mathcal{B} \neq \emptyset$, rather like a tree growing upward in more than one direction from a fork in the trunk. It will be shown this can't take place for any such $\mathcal{C}_0$ by showing $\mathcal{B} = \emptyset$.



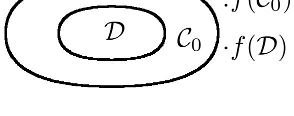
This will be accomplished by showing $\widetilde{\mathcal{T}_0} \equiv \mathcal{T}_0 \setminus \mathcal{B}$ is a tower. Since $\mathcal{T}_0$ is the smallest tower, this will require that $\widetilde{\mathcal{T}_0} = \mathcal{T}_0$ and so $\mathcal{B} = \emptyset$.

Claim: $\widetilde{\mathcal{T}_0}$ is a tower and so $\mathcal{B} = \emptyset$.

Proof of the claim: It is clear that $\emptyset \in \widetilde{\mathcal{T}_0}$ because for $\emptyset$ to be contained in $\mathcal{B}$ it would be required to properly contain $\mathcal{C}_0$ which is not possible. Suppose $\mathcal{D} \in \widetilde{\mathcal{T}_0}$. The plan is to verify $g(\mathcal{D}) \in \widetilde{\mathcal{T}_0}$.

Case 1: $f(\mathcal{D}) \in \mathcal{C}_0$. If $\mathcal{D} \subseteq \mathcal{C}_0$, then since both $\mathcal{D}$ and $\{f(\mathcal{D})\}$ are contained in $\mathcal{C}_0$, it follows $g(\mathcal{D}) \subseteq \mathcal{C}_0$ and so $g(\mathcal{D}) \notin \mathcal{B}$. On the other hand, if $\mathcal{D} \supsetneq \mathcal{C}_0$, then since $\mathcal{D} \in \widetilde{\mathcal{T}_0}$, $f(\mathcal{C}_0) \in \mathcal{D}$ and so $g(\mathcal{D})$ also contains $f(\mathcal{C}_0)$ implying $g(\mathcal{D}) \notin \mathcal{B}$. These are the only two cases to consider because $\mathcal{C}_0$ is comparable to every set of $\mathcal{T}_0$.

Case 2: $f(\mathcal{D}) \notin \mathcal{C}_0$. If $\mathcal{D} \subsetneq \mathcal{C}_0$ it can't be the case that $f(\mathcal{D}) \notin \mathcal{C}_0$ because if this were so, $g(\mathcal{D})$ would not compare to $\mathcal{C}_0$.



Hence if $f(\mathcal{D}) \notin \mathcal{C}_0$, then $\mathcal{D} \supseteq \mathcal{C}_0$. If $\mathcal{D} = \mathcal{C}_0$, then $f(\mathcal{D}) = f(\mathcal{C}_0) \in g(\mathcal{D})$ so

$g(\mathcal{D}) \notin \mathcal{B}$. Therefore, assume $\mathcal{D} \supsetneq \mathcal{C}_0$. Then, since $\mathcal{D}$ is in $\widetilde{\mathcal{T}_0}$, $f(\mathcal{C}_0) \in \mathcal{D}$ and so $f(\mathcal{C}_0) \in g(\mathcal{D})$. Therefore, $g(\mathcal{D}) \in \widetilde{\mathcal{T}_0}$.

Now suppose $\mathcal{S}$ is a totally ordered subset of $\widetilde{\mathcal{T}_0}$ with respect to set inclusion. Then if every element of $\mathcal{S}$ is contained in $\mathcal{C}_0$, so is $\cup\mathcal{S}$ and so $\cup\mathcal{S} \in \widetilde{\mathcal{T}_0}$. If, on the other hand, some chain from $\mathcal{S}$, $\mathcal{C}$, contains $\mathcal{C}_0$ properly, then since $\mathcal{C} \notin \mathcal{B}$, $f(\mathcal{C}_0) \in \mathcal{C} \subseteq \cup\mathcal{S}$ showing that $\cup\mathcal{S} \notin \mathcal{B}$ also. This has proved $\widetilde{\mathcal{T}_0}$ is a tower and since $\mathcal{T}_0$ is the smallest tower, it follows $\widetilde{\mathcal{T}_0} = \mathcal{T}_0$. This has shown roughly that no splitting into more than one ascending chain can occur at any $\mathcal{C}_0$ which is comparable to every set of $\mathcal{T}_0$. Next it is shown that every element of $\mathcal{T}_0$ has the property that it is comparable to all other elements of $\mathcal{T}_0$. This is done by showing that these elements which possess this property form a tower.

Define $\mathcal{T}_1$ to be the set of all elements of $\mathcal{T}_0$ which are comparable to every element of $\mathcal{T}_0$. (Recall the elements of $\mathcal{T}_0$ are chains from the original partial order.)

Claim: $\mathcal{T}_1$ is a tower.

Proof of the claim: It is clear that $\emptyset \in \mathcal{T}_1$ because $\emptyset$ is a subset of every set. Suppose $\mathcal{C}_0 \in \mathcal{T}_1$. It is necessary to verify that $g(\mathcal{C}_0) \in \mathcal{T}_1$. Let $\mathcal{D} \in \mathcal{T}_0$ (Thus $\mathcal{D} \subseteq \mathcal{C}_0$ or else $\mathcal{D} \supsetneq \mathcal{C}_0$.)and consider $g(\mathcal{C}_0) \equiv \mathcal{C}_0 \cup \{f(\mathcal{C}_0)\}$. If $\mathcal{D} \subseteq \mathcal{C}_0$, then $\mathcal{D} \subseteq g(\mathcal{C}_0)$ so $g(\mathcal{C}_0)$ is comparable to $\mathcal{D}$. If $\mathcal{D} \supsetneq \mathcal{C}_0$, then $\mathcal{D} \supseteq g(\mathcal{C}_0)$ by what was just shown $(\mathcal{B} = \emptyset)$. Hence $g(\mathcal{C}_0)$ is comparable to $\mathcal{D}$. Since $\mathcal{D}$ was arbitrary, it follows $g(\mathcal{C}_0)$ is comparable to every set of $\mathcal{T}_0$. Now suppose $\mathcal{S}$ is a chain of elements of $\mathcal{T}_1$ and let $\mathcal{D}$ be an element of $\mathcal{T}_0$. If every element in the chain, $\mathcal{S}$ is contained in $\mathcal{D}$, then $\cup\mathcal{S}$ is also contained in $\mathcal{D}$. On the other hand, if some set, $\mathcal{C}$, from $\mathcal{S}$ contains $\mathcal{D}$ properly, then $\cup\mathcal{S}$ also contains $\mathcal{D}$. Thus $\cup\mathcal{S} \in \mathcal{T}_1$ since it is comparable to every $\mathcal{D} \in \mathcal{T}_0$.

This shows $\mathcal{T}_1$ is a tower and proves therefore, that $\mathcal{T}_0 = \mathcal{T}_1$. Thus every set of $\mathcal{T}_0$ compares with every other set of $\mathcal{T}_0$ showing $\mathcal{T}_0$ is a chain in addition to being a tower.

Now $\cup\mathcal{T}_0, g(\cup\mathcal{T}_0) \in \mathcal{T}_0$. Hence, because $g(\cup\mathcal{T}_0)$ is an element of $\mathcal{T}_0$, and $\mathcal{T}_0$ is a chain of these, it follows $g(\cup\mathcal{T}_0) \subseteq \cup\mathcal{T}_0$. Thus

$$\cup\mathcal{T}_0 \supseteq g(\cup\mathcal{T}_0) \supsetneq \cup\mathcal{T}_0,$$

a contradiction. Hence there must exist a maximal chain after all. This proves the lemma.

If X is a nonempty set,$\leq$ is an order on X if

$$x \leq x,$$

and if $x,\ y \in X$, then

$$\text{either } x \leq y \text{ or } y \leq x$$

and

$$\text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\leq$ is a well order and say that $(X, \leq)$ is a well-ordered set if every nonempty subset of X has a smallest element. More precisely, if $S \neq \emptyset$ and $S \subseteq X$ then there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. A familiar example of a well-ordered set is the natural numbers.

Lemma A.2 *The Hausdorff maximal principle implies every nonempty set can be well-ordered.*

Proof: Let X be a nonempty set and let $a \in X$. Then $\{a\}$ is a well-ordered subset of X. Let

$$\mathcal{F} = \{S \subseteq X : \text{ there exists a well order for } S\}.$$

Thus $\mathcal{F} \neq \emptyset$. For $S_1,\ S_2 \in \mathcal{F}$, define $S_1 \prec S_2$ if $S_1 \subseteq S_2$ and there exists a well order for S_2, $\leq_2$ such that

$$(S_2, \leq_2) \text{ is well-ordered}$$

and if

$$y \in S_2 \setminus S_1 \text{ then } x \leq_2 y \text{ for all } x \in S_1,$$

and if $\leq_1$is the well order of S_1 then the two orders are consistent on S_1. Then observe that $\prec$ is a partial order on $\mathcal{F}$. By the Hausdorff maximal principle, let $\mathcal{C}$ be a maximal chain in $\mathcal{F}$ and let

$$X_\infty \equiv \cup \mathcal{C}.$$

Define an order, $\leq$, on X_∞ as follows. If $x,\ y$ are elements of X_∞, pick $S \in \mathcal{C}$ such that $x,\ y$ are both in S. Then if $\leq_S$ is the order on S, let $x \leq y$ if and only if $x \leq_S y$. This definition is well defined because of the definition of the order, $\prec$. Now let U be any nonempty subset of X_∞. Then $S \cap U \neq \emptyset$ for some $S \in \mathcal{C}$. Because of the definition of $\leq$, if $y \in S_2 \setminus S_1,\ S_i \in \mathcal{C}$, then $x \leq y$ for all $x \in S_1$. Thus, if $y \in X_\infty \setminus S$ then $x \leq y$ for all $x \in S$ and so the smallest element of $S \cap U$ exists and is the smallest element in U. Therefore X_∞ is well-ordered. Now suppose there exists $z \in X \setminus X_\infty$. Define the following order, $\leq_1$, on $X_\infty \cup \{z\}$.

$$x \leq_1 y \text{ if and only if } x \leq y \text{ whenever } x, y \in X_\infty$$

$$x \leq_1 z \text{ whenever } x \in X_\infty.$$

Then let

$$\widetilde{\mathcal{C}} = \{S \in \mathcal{C} \text{ or } X_\infty \cup \{z\}\}.$$

Then $\widetilde{\mathcal{C}}$ is a strictly larger chain than $\mathcal{C}$ contradicting maximality of $\mathcal{C}$. Thus $X \setminus X_\infty = \emptyset$ and this shows X is well-ordered by $\leq$. This proves the lemma.

With these two lemmas the main result follows.

Theorem A.3 *The following are equivalent.*

The axiom of choice

The Hausdorff maximal principle

The well-ordering principle.

Proof: It only remains to prove that the well-ordering principle implies the axiom of choice. Let I be a nonempty set and let X_i be a nonempty set for each $i \in I$. Let $X = \cup\{X_i : i \in I\}$ and well order X. Let $f(i)$ be the smallest element of X_i. Then

$$f \in \prod_{i \in I} X_i.$$

A.1 Exercises

1. Zorn's lemma states that in a nonempty partially ordered set, if every chain has an upper bound, there exists a maximal element, x in the partially ordered set. x is maximal, means that if $x \prec y$, it follows $y = x$. Show Zorn's lemma is equivalent to the Hausdorff maximal theorem.

2. Let X be a vector space. $Y \subseteq X$ is a Hamel basis if every element of X can be written in a unique way as a finite linear combination of elements in Y. Show that every vector space has a Hamel basis and that if Y, Y_1 are two Hamel bases of X, then there exists a one to one and onto map from Y to Y_1.

3. $\uparrow$ Using the Baire category theorem of the chapter on Banach spaces show that any Hamel basis of a Banach space is either finite or uncountable.

4. $\uparrow$ Consider the vector space of all polynomials defined on $[0, 1]$. Does there exist a norm, $||\cdot||$ defined on these polynomials such that with this norm, the vector space of polynomials becomes a Banach space (complete normed vector space)?

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