Note that in this simple example, the Hessian matrix is constant and so all that is left is to consider the eigenvalues. Writing the characteristic equation and solving yields the eigenvalues are $2, -2, 4$. Thus the given point is a saddle point.

## 9.4 Exercises

(1) Use the second derivative test on the critical points $(1, 1)$, and $(1, -1)$ for Example 9.6. The function is $6xy^2 - 2x^3 - 3x^4$.

(2) If $H = H^T$ and $H\mathbf{x} = \lambda\mathbf{x}$ while $H\mathbf{x} = \mu\mathbf{x}$ for $\lambda \neq \mu$, show that $\mathbf{x} \cdot \mathbf{y} = 0$.

(3) Show the points $\left(\frac{1}{2}, -\frac{21}{4}\right), (0, -4)$, and $(1, -4)$ are critical points of the following function of two variables and classify them as local minima, local maxima or saddle points.
$f(x, y) = -x^4 + 2x^3 + 39x^2 + 10yx^2 - 10yx - 40x - y^2 - 8y - 16.$

(4) Show the points $\left(\frac{1}{2}, -\frac{53}{12}\right), (0, -4)$, and $(1, -4)$ are critical points of the following function of two variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y) = -3x^4 + 6x^3 + 37x^2 + 10yx^2 - 10yx - 40x - 3y^2 - 24y - 48.$

(5) Show the points $\left(\frac{1}{2}, \frac{37}{20}\right), (0, 2)$, and $(1, 2)$ are critical points of the following function of two variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y) = 5x^4 - 10x^3 + 17x^2 - 6yx^2 + 6yx - 12x + 5y^2 - 20y + 20.$

(6) Show the points $\left(\frac{1}{2}, -\frac{17}{8}\right), (0, -2)$, and $(1, -2)$ are critical points of the following function of two variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y) = 4x^4 - 8x^3 - 4yx^2 + 4yx + 8x - 4x^2 + 4y^2 + 16y + 16.$

(7) Find the critical points of the following function of three variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y, z) = \frac{1}{3}x^2 + \frac{32}{3}x + \frac{4}{3} - \frac{16}{3}yx - \frac{58}{3}y - \frac{4}{3}zx - \frac{46}{3}z + \frac{1}{3}y^2 - \frac{4}{3}zy - \frac{5}{3}z^2.$

(8) Find the critical points of the following function of three variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y, z) = -\frac{5}{3}x^2 + \frac{2}{3}x - \frac{2}{3} + \frac{8}{3}yx + \frac{2}{3}y + \frac{14}{3}zx - \frac{28}{3}z - \frac{5}{3}y^2 + \frac{14}{3}zy - \frac{8}{3}z^2.$

(9) Find the critical points of the following function of three variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y, z) = -\frac{11}{3}x^2 + \frac{40}{3}x - \frac{56}{3} + \frac{8}{3}yx + \frac{10}{3}y - \frac{4}{3}zx + \frac{22}{3}z - \frac{11}{3}y^2 - \frac{4}{3}zy - \frac{5}{3}z^2.$

(10) Find the critical points of the following function of three variables and classify them according to whether they are local minima, local maxima or saddle points.
$f(x, y, z) = -\frac{2}{3}x^2 + \frac{28}{3}x + \frac{37}{3} + \frac{14}{3}yx + \frac{10}{3}y - \frac{4}{3}zx - \frac{26}{3}z - \frac{2}{3}y^2 - \frac{4}{3}zy + \frac{7}{3}z^2.$

(11) *Show that if $f$ has a critical point and some eigenvalue of the Hessian matrix