Chapter 1

Real Functions

One of the most important concepts in all of mathematics is the notion of function. Functions are the basic tools with which we describe reality, so that mathematics may be applied to the "real world." Once real phenomena are described as functions and the properties and behaviors of the functions are known, then we understand better how the phenomena behave. We therefore hopefully gain not only the ability to predict, but to control, the world around us.

The notion of function is built upon the foundation of the real number system, so we will be studying certain properties of the real numbers that are important to the properties of functions. The idea of function is fundamental to the notions of calculus, for calculus is "about" functions. The main ideas of calculus are tools for studying functions and their behavior. Thus calculus becomes a major tool in understanding how the world "works."

In this chapter we review the definitions and basic properties of the real number system and of functions and study the most common types of functions. For the well-prepared student, this chapter is largely a review of the elementary functions. On the other hand, because we introduce and discuss here some features of real numbers and functions that are important for calculus, it is not recommended that this chapter be skipped entirely.

1.1 Real Numbers

In this section, we review the system of real numbers and some of its properties. We also mention some of the subsystems of the real number system. Much of our purpose in this section is to establish notation and terminology.

Set Notation

We describe number systems, functions, and other mathematical objects in the language of set theory. Here is a little review of the notations and terminology

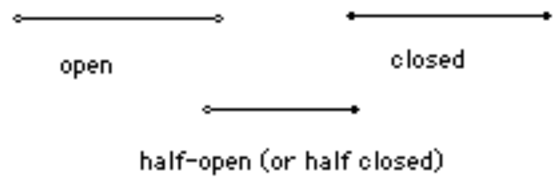

Figure 1.1: Open and closed intervals represented geometrically

associated with sets.

To say that S is a *set* means that S is a collection of objects, called *elements* of S. To say that object x is an element of S, we write $x \in S$; to say that x is not an element of S, we write $x \notin S$. If $P(x)$ is a property that is either true or false for any object x, of the form "x is ...", we denote by

$$\{x \in S | P(x)\}$$

the elements x of S such that $P(x)$ is true. If the set S is understood, we often write simply $\{x|P(x)\}$. The property $P(x)$ is called a *predicate*, and the notation $\{x|P(x)\}$ is called *set-builder* notation.

A set T is a *subset* of set S, written $T \subseteq S$, in case $x \in T$ implies $x \in S$; we say $S = T$ if both $T \subseteq S$ and $S \subseteq T$.

If A and B are sets, their *union* is the set

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

and their *intersection* is the set

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

Real Numbers and Intervals

The set we will work with most often is the set $\mathbb{R}$ of *real numbers*. For the time being, it is convenient to think of the real numbers as corresponding to the points on a line. The distance between points on the line is measured by the difference of the numbers corresponding to the points. Thus if point P corresponds to the real number a and point Q corresponds to the real number b, then the distance between P and Q is $PQ = |a - b| = \sqrt{(a-b)^2}$.

Certain subsets of $\mathbb{R}$ called *intervals* will also be used frequently in our work. If $a, b \in \mathbb{R}$ and $a < b$, we define the *open interval*

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

We represent the open interval graphically by drawing a line segment with open dots at the ends, indicating that the endpoints are missing, or excluded from the set (see Figure 1.1).

Figure 1.2: Infinite intervals (a, ∞) and $[a, \infty)$ represented geometrically

We define the *closed interval*

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}.$$

We represent the closed interval graphically by drawing a line segment with solid dots at the ends, indicating that the endpoint is present, or included in the set. If just one endpoint is included, we call the interval *half-open* or *half-closed*; just one of the two possibilities is shown in Figure 1.1.

We also work with *infinite intervals*; if $a \in \mathbb{R}$ we set

$$(a, \infty) = \{x \in \mathbb{R} | x > a\},$$

$$[a, \infty) = \{x \in \mathbb{R} | x \geq a\},$$

$$(-\infty, a) = \{x \in \mathbb{R} | x < a\},$$

$$(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}.$$

These infinite intervals correspond to rays, or half-lines, on the real line (see, for example, Figure 1.2). In this notation, we can also designate $\mathbb{R}$ itself as $(-\infty, \infty)$.

The System of Real Numbers

The *system* of real numbers is the set $\mathbb{R}$ of real numbers together with the usual operations of addition, subtraction, multiplication, and division and the usual order relation "less than." The basic properties of the real number system are as follows:

1. Addition and multiplication are associative and commutative and multiplication distributes over addition. That is, if $a, b, c \in \mathbb{R}$, then

    (a) $(a + b) + c = a + (b + c)$,

    (b) $(ab)c = a(bc)$,

    (c) $a + b = b + a$,

    (d) $ab = ba$,

    (e) $a(b + c) = ab + ac$.

2. $\mathbb{R}$ has an additive identity called *zero* and denoted 0 such that if $a \in \mathbb{R}$ then $a + 0 = a$.

3. $\mathbb{R}$ has a multiplicative identity called *one* and denoted 1 such that if $a \in \mathbb{R}$ then $a \cdot 1 = a$.

4. Each $a \in \mathbb{R}$ has an additive inverse called the *negative* of a and denoted $-a$ such that $a + (-a) = 0$.

5. Each nonzero $a \in \mathbb{R}$ has a multiplicative inverse called the *reciprocal* of a and denoted $1/a$ such that $a \cdot (1/a) = 0$.

6. $\mathbb{R}$ has a subset $\mathbb{P}$ called the *positive cone* of $\mathbb{R}$ such that if $a, b \in \mathbb{P}$ then $a+b, ab \in \mathbb{P}$ and for any $a \in \mathbb{R}$, exactly one of the statements $a \in \mathbb{P}$, $-a \in \mathbb{P}$, $a = 0$ is true. If $a \in \mathbb{P}$, we say a is *positive*. (Further properties of order are found in the exercises.)

Some specific subsystems of the real numbers with which we will be working are

the *natural numbers* $\mathbb{N} = \{1, 2, 3, 4, ...\}$

the *integers* $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, 4, ...\}$

the *rational numbers* $\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$

The Completeness of the Real Number System

We conclude this section with a brief discussion of one of the most important properties of the real number system, the completeness property (also called the least upper bound axiom). In order to introduce it, we need the notion of boundedness.

We say a subset S of $\mathbb{R}$ is *bounded above* in case there is a number $u \in \mathbb{R}$ such that for any $x \in S$ it is the case that $x \leq u$. The number u is called an *upper bound* of S. If S is bounded above, the least of all the upper bounds of S is called the *least upper bound* of S.

For example, the interval $S = (1, 2)$ is bounded above by any number greater than or equal to 2, and 2 is the least upper bound. The same is true of the interval $S = (1, 2]$, but in this case the set contains its least upper bound, while (1,2) does not.

We say a subset S of $\mathbb{R}$ is *bounded below* in case there is a number $l \in \mathbb{R}$ such that if $x \in S$ then $x \geq l$. The number l is called a *lower bound* of S. If S is bounded below, the greatest of all the lower bounds of S is called the *greatest lower bound* of S.

The *Completeness Property* (or *Least Upper Bound Axiom*) of the real number system states:

CP *If S is a nonempty subset of $\mathbb{R}$ that is bounded above, then S has a least upper bound.*

The completeness property will be used repeatedly in our subsequent work. We will have occasion to explore its consequences periodically as we develop the concepts of the calculus.

Exercises 1.1

1. Under what conditions is the union of two intervals another interval? Does your answer depend on the type of interval (open, closed, infinite)?

2. Under what conditions is the intersection of two intervals another interval?

3. The *complement of A in B* is the set

$$B \backslash A = \{x | x \in B \textit{ and } x \notin A\}.$$

   Under what conditions is the complement of the interval (a, b) in the interval (c, d) another interval?

4. Prove that if S is bounded above and has a least upper bound, then it has only one least upper bound.

5. Prove that if S has a greatest lower bound, it is unique.

6. Show that if S is empty, then every real number is both a lower bound and an upper bound of S.

7. Give the least upper bounds and the greatest lower bounds of the following subsets of $\mathbb{R}$. If a bound does not exist, so state.

   (a) $[0, 1]$
   (b) (0,1)
   (c) $\{1, 2, 3, \ldots\}$
   (d) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ *Ans.* 1 and 0
   (e) $(2, \infty)$

8. Prove that if S is a nonempty subset of $\mathbb{R}$ that is bounded below, then S has a greatest lower bound.

9. Use the completeness axiom to prove the following two versions of the *Nested Interval Theorem*:

   (a) If $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots$ are closed intervals such that $a_1 \leq a_2 \leq \ldots < a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1$, then there is a number c such that $c \in [a_n, b_n]$ for every n.
   (b) If $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), \ldots$ are open intervals such that $a_1 < a_2 < \ldots < a_n < \ldots < b_n < \ldots < b_2 < b_1$, then there is a number c such that $c \in (a_n, b_n)$ for every n.

10. This exercise develops the properties of the order relation $<$ in the real number system. The *positive cone* of $\mathbb{R}$ is a subset $\mathbb{P}$ of $\mathbb{R}$ such that

    (1) if $a, b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ and $ab \in \mathbb{P}$;

    (2) if $a \in \mathbb{R}$, then exactly one of the following is true:

    $$-a \in \mathbb{P}, \quad a \in \mathbb{P}, \quad a = 0.$$

    We say that if a and b are real numbers, a *is less than* b, written $a < b$, in case there is some $c \in \mathbb{P}$ such that $a + c = b$. Use properties of $\mathbb{P}$ to prove the following:

    (a) If $a \in \mathbb{P}$ then $0 < a$.

    (b) If $a, b, c \in \mathbb{R}$ and $a < b$, then $a + c < b + c$.

    (c) If $a, b \in \mathbb{R}$, $c \in \mathbb{P}$, and $a < b$, then $ac < bc$.

    (d) If $a, b \in \mathbb{R}$, exactly one of the following is true:

    $$a < b, \quad b < a, \quad a = b$$

    (e) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 \in \mathbb{P}$.

    (f) $0 \notin \mathbb{P}$ and $1 \in \mathbb{P}$.

11. Discover the flaw in the following "proof" that $1 = 2$:

    Let $a = b$. Then

    $$-ab = -b^2 \tag{1.1}$$
    $$a^2 - ab = a^2 - b^2 \tag{1.2}$$
    $$a(a - b) = (a + b)(a - b) \tag{1.3}$$
    $$a = a + b \tag{1.4}$$
    $$a = 2a \tag{1.5}$$
    $$1 = 2 \tag{1.6}$$

12. Discover the flaw in the following "proof" that $5 = 4$:

    $$-20 = -20 \tag{1.7}$$
    $$25 - 45 = 16 - 36 \tag{1.8}$$
    $$25 - 45 + \frac{81}{4} = 16 - 36 + \frac{81}{4} \tag{1.9}$$
    $$\left(5 - \frac{9}{2}\right)^2 = \left(4 - \frac{9}{2}\right)^2 \tag{1.10}$$
    $$5 - \frac{9}{2} = 4 - \frac{9}{2} \tag{1.11}$$
    $$5 = 4 \tag{1.12}$$