Math 303 (Engineering Mathematics II)

Quiz 2

Winter 2018

TIME LIMIT 2.5 HOURS

• Answer all questions and show all your work to receive the full credit.
• Work on scratch paper will not be graded. Use the back if you need more space.
• Basic scientific calculator or testing center calculator are allowed (not graphing or symbolic ones).

For instructor use only:

<table>
<thead>
<tr>
<th>#</th>
<th>Possible</th>
<th>Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>1b</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2a</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2b</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>
1a. Consider the equation $\frac{dy}{dt} = y^2 (3-y)$.

i) Without solving the differential equation, find all equilibrium solutions and classify each one as asymptotically stable, semistable or unstable.

ii) Sketch several graphs of solutions in the ty-plane; include solutions passing through the points $(0,-1),(0,1),(0,2),(0.25),(0.4)$;

ii) determine where the graph is concave up and where it is concave down, indicating inflection points if any.

Equilibrium solns
when $\frac{dy}{dt} = 0$

or $y(0) = 0$, $y(t) = \frac{3}{y}$

$y = 3$ stable equilibrium

and $y = 0$ is semistable equil.

Behavior of solns:

a) If $y < 0$, then $y' = y^2 (3-y) > 0$

and $y(t)^2$ increases

b) If $0 < y < 3$

then $y' = y^2 (3-y)$ and $y' > 0$

c) If $y > 3$

then $y' = y^2 (3-y) < 0$

and $y(t)$ decreases

Then, any solution $y(t)$ such that $0 < y(0) < 3$

has $y'' > 0$, if $0 < y < 2$ concave up

and $y'' < 0$, if $y > 2$ concave down

Therefore, $(t^*, y(t^*)) = (2^*, 2)$ are inflection points for each $y(t)$.

Inflection point

$y'(t) = f(y)$,

then, $y''(t) = f(y) \cdot y' = f(y) f'(y)$

In our case,

$y''(t) = (2y(3-y) - y^2) y^2 (3-y)$

Thus, $y'' = 0$ if $y = 0$, $y = 3$ and

$6y - 3y^2 = 0$ or $y = 2$

$y''(t) = 3y (2-y) y^2 (3-y)$
1b. Consider the initial value problem \( y' = \sqrt{y}, \ y(0) = 0 \).

(i) By observation, the zero function \( y(t) = 0 \) is a solution of this initial value problem. Find another solution of this initial value problem. Sketch both of them.

(ii) Explain why this does not violate the existence and uniqueness theorem.

\[ y(t) = \frac{t^2}{4} \]

\[ y(t) = 0 \]

\[ \frac{dy}{\sqrt{y}} = dt, \ \text{then} \ \int y^{-1/2} \, dy = \int dt \]

\[ 2y^{1/2} = t + C, \ \text{then} \ y^{1/2} = \frac{t + C}{2} \]

or \( y(t) = \left( \frac{t + C}{2} \right)^2 \)

Using I.C.

\( 0 = y(0) = \left( \frac{c}{2} \right)^2 = \frac{c^2}{4} \)

And \( c = 0 \)

Thus, 2nd solution: \( y(t) = \frac{t^2}{4} \)

ii) \( y^1 = y^{1/2} \) conts. as a function of \( t \) and \( y \) if \( y < 0 \).

\[ \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}, \ \text{discontinuous at any pair} \ (t, 0) \ \text{in the} \ ty-\text{plane}. \ \text{In particular discontinuity at} \ (t, 0) \ \text{at} \ (t, 0), \ \text{E.R.} \ \text{and the hypothesis of the theorem does not hold.} \]
2a. Consider differential equation \( \frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \).

(i) Find all solutions of the given differential equation.

It can be solved in two ways: as a homog. eqn. introducing a variable \( v = \frac{y}{x} \) or using the technique to transform an equation into an exact equation.

(ii) Verify that initial value problem \( y(1) = 1 \) for this equation has a unique solution in some interval around \( t = 1 \).

\[ f(y) \text{ and } \frac{\partial f}{\partial y} = \frac{6y(2x^2) - 3y(3y^2 - x^2)}{4x^2y^2} = \frac{12xy - 6y^2 + 2x^2}{4x^2y} \]

Since both \( f \) and \( \frac{\partial f}{\partial y} \) are cons. around \((1, 1)\), the soln. of the diff. eqn. exists and is unique in an interval \((1-h, 1+h), \ h > 0\).

(iii) Find the solution subject to the initial condition \( y(1) = 1 \). Sketch.
2a) \[ y' = \frac{3y^2 - x^2}{2xy} \]

i) \[ 3y^2 - x^2 - 2xy \ y' = 0 \quad \text{(*)} \]

\[ M(x,y) = 3y^2 - x^2, \quad N(x,y) = -2xy \]

\[ My = 6y, \quad Nx = -2y \]

This eqn. is not exact.

Let's explore if it can be converted into an exact equation.

\[ (\mu M)_y = (\mu N)_x \]

\[ My \mu + \mu' My = \mu' N + \mu Nx \]

If \( a) \mu = \mu(y) \) then \( \mu_x = 0 \)

\[ My = \left( \frac{N_x - My}{M} \right) \mu \]

\[ \mu_x = \left( \frac{My - Nx}{N} \right) \mu \]

a) If \[ \frac{N_x - My}{M} = F(y) \] we can transform the eqn. into exact.

\[ \frac{N_x - My}{M} = \frac{-2y - 6y}{3y^2 - x^2} = \frac{-8y}{3y^2 - x^2} \quad \text{Not possible} \]

b) If \[ \frac{My - Nx}{N} = G(x) \], it can be done.

In fact, \[ \frac{6y + 2y}{-2xy} = \frac{-8}{2x} = \frac{-4}{x} \]
To find \( \mu = \mu(x) \), we need to solve the linear equation

\[
\mu'(x) = -\frac{4}{x} \mu(x), \quad \text{then} \quad \frac{d\mu}{\mu} = -\frac{4}{x} \, dx
\]

\[
\int \ln |\mu| = -4 \ln |x| + C
\]

\[
\mu(x) = \tilde{C} \left( e^{\ln |x|} \right)^{-4}
\]

choose \( \tilde{C} = 1 \), then \( \mu(x) = x^{-4} \)

Multiplying by \( x^{-4} \)

\[
\frac{3y^2 - x^2}{x^4} - \frac{2xy}{x^4} \, y' = 0
\]

Verification \( M_y = 6y/x^4 \) and \( N_x = -3(-21) = 6y \)

To obtain soln:

\[
\psi_x = \frac{3y^2}{x^4} - \frac{1}{x^2}, \quad \psi_y = \frac{-2y}{x^3}
\]

\[
\int \psi(x,y) = \frac{-y^2}{x^3} + \frac{1}{x} + C(y)
\]

Then \( \psi_y = -2x^2y + C'(y) = \frac{-2y}{x^3} \)

Thus, \( C'(y) = 0 \) and \( C(y) = d \)

Integral curves are

\[
\psi(x,y) = \frac{-y^2}{x^2} + \frac{1}{x} = d
\]

iii) If \( y(1) = 1 \), then \( \frac{1}{x} - \frac{y^2}{x^3} = C \) transform into \( \frac{1}{x} - 1 = C \), then \( C = 0 \)

\[
y^2 = x^2
\]

The branch that passes through \((1,1)\) is \( y = x, \quad 0 < x < +\infty \).
\[
\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \\
\text{Alternative using Change of Variables } v = \frac{y}{x}.
\]

\[
\frac{3y^2 - x^2}{2xy} = \frac{3\left(\frac{y}{x}\right)^2 - 1}{x} \quad v = \frac{1}{x} \quad \Rightarrow \quad \frac{3v^2 - 1}{2v}
\]

\[
\begin{align*}
\text{If } \quad y(x) &= \frac{y}{x} \\
\Rightarrow \quad y(x) &= x v(x) \\
y'(x) &= v(x) + x v'(x)
\end{align*}
\]

then,

\[
\begin{align*}
v(x) + x v'(x) &= \frac{3v^2 - 1}{2v} \\
or \quad v' &= \frac{1}{x} \left[ \frac{3v^2 - 1}{2v} - v \right] = \frac{1}{x} \left[ \frac{3v^2 - 1 - 2v^2}{2v} \right] \\
or \quad v' &= \left( \frac{v^2 - 1}{2v} \right) \frac{1}{x}
\end{align*}
\]

\[
\frac{2v}{v^2 - 1} \quad \frac{dv}{dx} = \frac{1}{x} \quad \Rightarrow \quad \frac{2v}{v^2 - 1} \quad dv = \frac{1}{x} \quad dx
\]

\[
\int \left[ \ln|v^2 - 1| = \ln|x| + C \right.
\]

\[
\left. \quad e^{(v') \quad \left| v^2 - 1 \right| = c|x| \Rightarrow \quad v^2 - 1 = c|x| = cx \right]
\]

\[
\begin{align*}
or \quad v^2(x) &= cx + 1 \\
\Rightarrow \quad \frac{y^2(x)}{x^2} &= cx + 1 \\
\text{Equivalent to } \quad \frac{y^2(x)}{x^2} &= \frac{1}{x} + C
\end{align*}
\]
2b. Consider differential equation

\[(9 - t^2)y' + 2ty = 4t^2.\]

(i) Determine (without solving the problem) an interval in which the solution of the IVP consisting of the given equation and the initial condition \(y(2) = 1\) is certain to exist.

\[y' + \frac{2t}{9 - t^2} y = \frac{4t^2}{9 - t^2}, \quad y(2) = 1\]

\[p(t) = \frac{2t}{9 - t^2}, \quad q(t) = \frac{4t^2}{9 - t^2}\]

Both are not continuous at \(t = 3\)

\(\text{For I.C. } y(2) = 1\)

The longest interval where the solution exists and is unique is \((-3, 3)\).

\[\text{-4 -3 0 2 3}\]

(ii) Determine (without solving the problem) an interval in which the solution of the IVP consisting of the given equation and the initial condition \(y(-4) = 1\) is certain to exist.

In here, \(-4 \in (-\infty, 3)\)

and \(p\) and \(g\) are continuous on this interval therefore, the longest interval where the solution exists and is continuous is \((-\infty, -3)\)