Harmonic univalent mappings onto asymmetric vertical strips

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Abstract

Let Ω_α be the asymmetrical vertical strips defined by $\Omega_\alpha = \{w : \frac{\alpha-\pi}{2\sin\alpha} < \text{Re } w < \frac{\alpha}{2\sin\alpha}\}$, where $\pi/2 \leq \alpha < \pi$, and let D be the unit disk. We characterize the class $S_H(D, \Omega_\alpha)$ of univalent harmonic orientation-preserving functions f which map D onto Ω_α and are normalized by $f(0) = 0$, $f_{\overline{z}}(0) = 0$, and $f_z(0) > 0$. Then we use this characterization to demonstrate a few other results..

1 Introduction

Let S_H be the class of complex-valued harmonic functions f which are univalent and orientation-preserving mappings of the unit disk $D = \{z : |z| < 1\}$ and are normalized by $f(0) = 0$ and $f_z(0) = 1$. Clunie and Sheil-Small [1] showed that functions in such a class have the form

$$f = h + \overline{g},$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_b z^k$$

are analytic in D. They also showed that the orientation-preserving condition implies that $|b_1| < 1$ and so $(f - \overline{b_1 f})/(1 - |b_1|^2) \in S_H$. Hence it is customary to just consider the subclass

$$S_H^O = \{f \in S_H \text{ with } f_{\overline{z}}(0) = 0\}.$$

The uniqueness result of the Riemann Mapping Theorem does not extend to these classes of harmonic functions, and several authors have studied the subclass of functions that map D onto specific domains. In particular, Hengartner and Schober [3] considered the strip domain $\Omega = \{w : |\text{ Im } w| < \pi/4\}$. We will apply their results to derive a family of functions that includes all mappings in S_H^O from D onto vertical strip domains that are asymmetric with respect to the imaginary axis. Using this, we will characterize all mappings in S_H^O whose image is either a right-half plane or the entire plane minus a slit lying on the negative real axis.

2 Asymmetric vertical strip mappings

In [3], Hengartner and Schober investigated the family $S_H(D,\Omega)$ of normalized harmonic univalent mappings from the unit disk D onto the horizontal strip $\Omega = \{w : |\text{ Im } w| < \pi/4\}$. By the use of a rotation and a composition on their family of functions, we derive analogous results about the family of normalized univalent mappings from D onto the vertical asymmetric strips.

In particular, let $f \in S_H(D,\Omega_\alpha)$, the family of normalized univalent mappings from D onto the the vertical asymmetric strips $\Omega_\alpha = \{w : \frac{\alpha-\pi}{2\sin\alpha} < \text{ Re } (w) < \frac{\alpha}{2\sin\alpha}\}$, where $\frac{\pi}{2} \leq \alpha < \pi$. Recall that $f = h + \overline{g}$, where h, g are in the space of analytic functions, $H(D)$, on D, and that $|a(z)| = |g'(z)/h'(z)| < 1$. Now, $f = \text{ Re } (h+g) + i \text{ Im } (h-g)$. So

$$
\begin{aligned}
h(z) - g(z) &= \int \frac{h'(z) - g'(z)}{h'(z) + g'(z)} [h'(z) + g'(z)] dz \\
&= \int \frac{1 - a(z)}{1 + a(z)} \cdot \varphi'(z) dz,
\end{aligned}
$$

where $\varphi(z) = h(z) + g(z)$.

Now φ is the conformal map from D onto Ω_α, normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$. To see this, note that if we consider the map $F(w) = \zeta = \xi + i\eta = \varphi(f^{-1}(w))$, then f consists of the successive transformations $(u,v) \to (w,\overline{w}) \to (z,\overline{z}) \to (\varphi,\overline{\varphi}) \to (\xi,\eta)$ so that

$$
\begin{pmatrix} \frac{\partial\xi}{\partial u} & \frac{\partial\xi}{\partial v} \\ \frac{\partial\eta}{\partial u} & \frac{\partial\eta}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} h' + g' & 0 \\ 0 & \overline{h' + g'} \end{pmatrix} \cdot \begin{pmatrix} \frac{\overline{h'}}{\Delta} & -\frac{\overline{g'}}{\Delta} \\ -\frac{g'}{\Delta} & \frac{h'}{\Delta} \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}
$$

and thus $\frac{\partial\xi}{\partial u} = 1$, $\frac{\partial\xi}{\partial v} = 0$ and $\frac{\partial\eta}{\partial v} = \frac{|h'+g'|^2}{\Delta}$ where $\Delta = |h'|^2 - |g'|^2 = \text{ Re } [(\overline{h'+g'})(h'-g')]$. Therefore, φ is a univalent map from D onto a vertical strip. Because of the normalization of φ, we see that φ is the map $(1/2)\log[(1+z)/(1-z)]$ rotated by$-iz$, composed with the Möbius transformation $(z+p)/(1+\overline{p}z)$, where $0 < p < 1$, and normalized.

Hence, any map f in $S_H(D,\Omega_\alpha)$ is of the form

$$\begin{aligned} f(z) &= \text{Re } \varphi(z) + i \text{ Im} \int \frac{1-a(z)}{1+a(z)} \cdot \varphi'(z)dz. \\ &= \varphi(z) - 2i \text{ Im} \int \frac{a(z)}{1+a(z)} \cdot \varphi'(z)dz. \end{aligned} \tag{1}$$

Since a is in $H(D)$, $|a(z)| < 1$ on D, and $a(0) = 0$, we have

$$\frac{1-a(z)}{1+a(z)} = \int_{|\eta|=1} \frac{1+\eta z}{1-\eta z} d\mu(\eta),$$

where $\mathcal{P}$ is the set of probability measures on the Borel sets of $|\eta| = 1$.

Definition 2.1. For $z \in D$ and $|\eta| = 1$, define the kernel

$$\begin{aligned} K(z,\eta) &= \int_0^z \frac{1+\eta w}{1-\eta w} \frac{1}{(1+we^{i\alpha})(1+we^{-i\alpha})} dw \\ &= \begin{cases} \dfrac{\cos\alpha}{2\sin^2\alpha} \log\left(\dfrac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right) \\ \qquad + \dfrac{1}{i\sin\alpha}\left(\dfrac{ze^{i\alpha}}{1+ze^{i\alpha}}\right), & \text{if } \eta = -e^{i\alpha} \\ \\ \dfrac{\cos\alpha}{2\sin^2\alpha} \log\left(\dfrac{1+ze^{-i\alpha}}{1+ze^{i\alpha}}\right) \\ \qquad - \dfrac{1}{i\sin\alpha}\left(\dfrac{ze^{-i\alpha}}{1+ze^{-i\alpha}}\right), & \text{if } \eta = -e^{-i\alpha} \\ \\ \dfrac{1}{2i\sin\alpha}\left(\dfrac{1-\eta e^{i\alpha}}{1+\eta e^{i\alpha}}\right) \log\left(\dfrac{1-\eta z}{1+ze^{-i\alpha}}\right) \\ \qquad - \dfrac{1}{2i\sin\alpha}\left(\dfrac{1-\eta e^{-i\alpha}}{1+\eta e^{-i\alpha}}\right) \log\left(\dfrac{1-\eta z}{1+ze^{i\alpha}}\right), \\ \qquad\qquad \text{if } \eta \neq -e^{\pm i\alpha} \end{cases} \end{aligned}$$

Define the family

$$\begin{aligned} \mathcal{F}_\alpha = \{f : f(z) = \text{ Re } & \left[\frac{1}{2i\sin\alpha} \log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)\right] \\ & +i \text{ Im} \int_{|\eta|=1} K(z,\eta)\, d\mu(\eta), \quad \mu \in \mathcal{P}\} \end{aligned}$$

where $\mathcal{P}$ is the set of probability measures on the Borel sets of $|\eta| = 1$.

From our discussion above, we obtain an isomorphism between the the family $S_H(D,\Omega)$ from Hengartner and Schober [3] and the class $S_H(D,\Omega_\alpha)$. Hence we have the following theorem.

Theorem 2.2. *The following properties hold:*

1. *If f is a univalent harmonic and orientation preserving map from the unit disk D onto $\Omega_\alpha = \{w : \frac{\alpha-\pi}{2\sin\alpha} < \; Re\ (w) < \frac{\alpha}{2\sin\alpha}\}$ such that $f(0) = 0$ and $f_z(0) > 0$, then $f_z(0) = 1$.*
2. *The set $S_H(D, \Omega_\alpha) \subset \mathcal{F}_\alpha$ with $\overline{S_H(D, \Omega_\alpha)} = \mathcal{F}_\alpha$.*
3. *If $f \in \mathcal{F}_\alpha$, then $f(D)$ is either the strip Ω_α, a halfstrip, a triangle, or a trapezium.*

3 Consequences

The results from the previous section yield a few nice consequences.

Theorem 3.1. *Every right-half plane mapping $f \in S_H^O$ can be expressed as a limit of functions in $\mathcal{F}_\alpha$. In particular, f maps ∂D into the line $Re\ w = -\frac{1}{2}$.*

Proof. This follows from the normality of the family S_H^O and an approximation theorem (theorem 3.7 in [1]). □

Corollary 3.2. *Let $f = h + \overline{g} \in S_H^O$ be a right-half plane mapping. Then*

$$f(z) = h(z) + g(z) - 2i\ Im\ g(z) = \frac{z}{1-z} - 2i\ Im \int_0^{2\pi} K(z,t)\, d\mu(t),$$

where

$$K(z,\eta) = \begin{cases} \dfrac{-\frac{1}{2}z^2}{(1-z)^2} & \text{if } \eta = 1 \\[2ex] \dfrac{z}{(1-\overline{\eta})(1-z)} + \dfrac{1}{(1-\overline{\eta})(1-\eta)} \log\left(\dfrac{1-z}{1-\eta z}\right), & \text{if } \eta \neq 1 \end{cases}$$

Proof. Let $f \in S_H(D, \Omega)$, where f is of the form in (1). The result follows from taking the limit of f as $\alpha \to \pi$. □

Corollary 3.2 provides a general description for right-half plane mappings in S_H^O, so that in such cases we know that $h(z) + g(z) = z/(1-z)$. In a similar fashion, it has been shown that all slit mappings in S_H^O whose slit lie on the negative real axis have the property that $h(z) - g(z) = 1/(1-z)^2$ ([2] or see [4]). Corollary 3.3 provides another proof of this.

Corollary 3.3. *Let $f = h + \overline{g} \in S_H^O$ be a slit mapping whose slit lies on the negative real axis. Then*

$$h(z) - g(z) = \frac{z}{(1-z)^2}.$$

Proof. Sheil-Small (REMARK 7 in [5]) showed that if $f = h + \overline{g} \in S_H^O$ is starlike, then $\hat{f} = \hat{h} - \overline{\hat{g}}$ is convex in S_H^O, where

$$\hat{h}(z) = \int_0^z \frac{h(w)}{w}\, dw \qquad \text{and} \qquad \hat{g}(z) = \int_0^z \frac{g(w)}{w}\, dw.$$

Let $f = h + \overline{g} \in S_H^O$ be a slit mapping whose slit lies on the negative real axis. Then $\hat{f}$ is convex. In particular, $\hat{f}$ is a right-half plane mapping since the process $\hat{f}(z) = \int_0^z f(w)/w\ dw$ makes the boundary of $\hat{f}$ normal to the boundary of f. Hence, by Corollary (3.2)

$$\frac{z}{1-z} = \hat{h}(z) - \hat{g}(z) = \int_0^z \frac{h(w) - g(w)}{w}\, dw.$$

Therefore,

$$\frac{z}{(1-z)^2} = h(z) - g(z).$$

□

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