

# LINEAR INVARIANCE AND INTEGRAL OPERATORS OF UNIVALENT FUNCTIONS

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ABSTRACT. Different methods have been used in studying the univalence of the integral

$$(1) \quad \mathcal{J}_{\alpha,\beta}(f)(z) = \int_0^z \left(f'(t)\right)^\alpha \left(\frac{f(t)}{t}\right)^\beta dt, \quad \alpha, \beta \in \mathbb{R},$$

where  $f$  belongs to one of the known families of holomorphic and univalent functions  $f(z) = z + a_2z^2 + \dots$  in the unit disk  $\mathbb{D} = \{z : |z| < 1\}$  (see [5]).

In this paper, we study a larger set than (1), namely the set of the minimal invariant family which contains (1), where  $f$  belongs to the linear invariant family, and thereby we obtain information about the univalence of (1). In particular, we determine the order of this minimal invariant family in the cases of univalent and convex univalent functions in  $\mathbb{D}$ . As a result, we find the radius of close-to-convexity and the lower bound for the radius of univalence for the minimal invariant family in the case of convex univalent functions. This allows us to determine the exact region for  $(\alpha, \beta)$  where the corresponding minimal invariant family is univalent and close-to-convex. These results are sharp and generalize those which were obtained in [10].

## 1. Introduction

Let  $S$  denote the class of holomorphic and univalent functions  $f$  in the unit disk  $\mathbb{D} = \{z : |z| < 1\}$  which have the form

$$(2) \quad f(z) = z + a_2z^2 + \dots, \quad z \in \mathbb{D},$$

and let  $S^c \subset S$  be the subclass consisting of convex functions. These two classes are examples of linear invariant family as described by Pommerenke [12]. If  $\mathcal{M}$  is a linear invariant family of locally univalent functions of the form (2), then the order of such family is defined by

$$(3) \quad \text{ord } \mathcal{M} = \sup_{f \in \mathcal{M}} |a_2|.$$

We have  $\text{ord } S = 2$  and  $\text{ord } S^c = 1$ .

For  $\alpha, \beta \in \mathbb{R}$  and  $f \in S(\text{or } S^c)$ , consider the integral operator

$$(4) \quad F_{\alpha,\beta}(z) = \mathcal{J}_{\alpha,\beta}(f)(z) = \int_0^z \left(f'(t)\right)^\alpha \left(\frac{f(t)}{t}\right)^\beta dt, \quad z \in \mathbb{D}.$$

Properties such as univalence, convexity, and close-to-convexity, of the function  $F_{\alpha,\beta}$  have been intensively studied (see [5] for references). The following three operators are of particular interest and their univalence (in the case of  $f \in S$ ) have not yet been settled:

- (a) the Biernacki integral,  $F_{0,1}$ , was claimed by Biernacki to be univalent, but this claim was disproved by a counterexample by Krzyz and Lewandowski [7]. The exact radius of univalence for  $F_{0,1}$  is not known. Lewandowski [8] has given the best known estimate for  $r_u(F_{0,1}) > 0.91$ .
- (b) the Royster-Pfaltzgraff integral,  $F_{\alpha,0}$ , is known to be univalent for  $|\alpha| \leq 1/4$  ( $\alpha \in \mathbb{C}$ ) [11] and to be nonunivalent if  $|\alpha| > 1/3, \alpha \neq 1$  [14]. The lower bound for the radius of univalence,  $r_u(F_{\alpha,0})$ , is given in [13].
- (c) the Danikas-Ruscheweyh integral,  $F_{1,-1}$ , has been conjectured recently to be univalent [3].

Also, the operator,  $\mathcal{J}_{1,-2}(f), f \in S^c$ , is important and appears in the paper of Hall [6].

In this paper we first determine the order of the minimal invariant families  $\mathcal{M}_{\alpha,\beta}(S)$  and  $\mathcal{M}_{\alpha,\beta}(S^c)$  containing the sets

$$(5) \quad \mathcal{J}_{\alpha,\beta}(S) = \{F_{\alpha,\beta} : f \in S\}, \quad \mathcal{J}_{\alpha,\beta}(S^c) = \{F_{\alpha,\beta} : f \in S^c\},$$

respectively, where  $F_{\alpha,\beta}$  is given in (4). From this, we obtain the sharp value of the radius of close-to-convexity and the bound for the radius of univalence for the family  $\mathcal{M}_{\alpha,\beta}(S^c)$ , which allows us to determine the exact region for  $(\alpha, \beta)$  in which  $\mathcal{J}_{\alpha,\beta}(f), f \in S^c$  is univalent (and even close-to-convex) in  $\mathbb{D}$ . The last result generalizes the earlier results about univalence of  $\mathcal{J}_{\alpha,0}$  and  $\mathcal{J}_{0,\beta}, f \in S^c$  [10].

## 2. Background Results

We will use following lemmas.

**Lemma A.** (Prokhorov, Szynal [13])

Let  $\mathcal{M}$  be a linear invariant family of functions  $f$  of the form (2) such that  $f(z)/z \neq 0, z \in \mathbb{D}$  and let  $\mathcal{J}_{\alpha,\beta}(\mathcal{M})$  denote the set of functions  $F_{\alpha,\beta}$  given by

$$(6) \quad G_{\alpha,\beta}(z) = \hat{\mathcal{J}}_{\alpha,\beta}(f)(z) = \left(\frac{\xi}{f(\xi)}\right)^\beta \int_0^z \frac{(f'(t))^\alpha}{(1-\bar{\xi}t)^{2-2\alpha-\beta}} \left(\frac{f(t)-f(\xi)}{t-\xi}\right)^\beta dt.$$

Then the family  $\mathcal{M}_{\alpha,\beta}$  of functions  $G_{\alpha,\beta}$  is the minimal invariant family containing the class  $\mathcal{J}_{\alpha,\beta}(\mathcal{M})$ , where  $\xi$  is an arbitrary point from  $\mathbb{D} \setminus \{0\}$ .

**Lemma B.** (e.g., Goluzin [4])

If  $f \in S$  and  $w \notin f(\mathbb{D})$ , then

$$\left|a_2 + \frac{1}{w}\right| \leq 2.$$

In particular,  $|w| \geq 1/4$  with equality only for Koebe functions.

**Lemma C.** (Barnard, Schober [1])

If  $f \in S^c$  and  $w \notin f(\mathbb{D})$ , then

$$\left|a_2 + \frac{1}{w}\right| \leq \tau = \frac{2}{x_0} \sin x_0 - \cos x_0 \approx 1.3270,$$

where  $x_0 \approx 2.0816$  is the unique root of the equation  $x \cot x = 1 - \frac{1}{2}x^2$ . The result is sharp.

**Lemma D.** (Sheil-Small [15], Suffridge [16])

Let  $f \in S^c$  and  $\xi \in \mathbb{D}$  be fixed. Then the function

$$g(z) = \frac{\xi z}{f(\xi)} \cdot \frac{f(z) - f(\xi)}{z - \xi}$$

belongs to the class of starlike functions of order  $1/2$ .

From Lemmas D and A, we obtain

**Lemma 1.** Let  $f \in S^c$  and  $\xi \in \mathbb{D}$ . Then the family  $\mathcal{M}_{\alpha,\beta}(S^c)$  is given by the formula

$$(7) \quad G_{\alpha,\beta}(z) = \hat{\mathcal{J}}_{\alpha,\beta}(f)(z) = \int_0^z \frac{(f'(t))^\alpha}{(1 - \bar{\xi}t)^{2-2\alpha-\beta}} \left( \frac{g(t)}{t} \right)^\beta dt,$$

where  $g$  is a starlike function of order  $1/2$ .

**Remark.** Note how the family of the integrals given in (4) and the minimal invariant family, which contains (4), given in (7) differ from each other (every convex function is starlike of order  $1/2$  in  $\mathbb{D}$ ).

**Lemma 2.** The order of the family  $\mathcal{M}$  is given by the formula

$$(8) \quad \text{ord } \mathcal{M}_{\alpha,\beta} = \sup_{\substack{f \in \mathcal{M} \\ \xi \in \mathbb{D}}} \left| \alpha a_2 + (1 - \alpha)\bar{\xi} + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right|.$$

*Proof.* By (3) we have to calculate  $\frac{1}{2}F''_{\alpha,\beta}(0)$  which gives us the result. □

**Lemma 3.** If  $f(z) = k(z) = z/(1 - z)^2$ , then

$$(9) \quad \text{ord } (\hat{\mathcal{J}}_{\alpha,\beta}(k)) = \begin{cases} 1 + 2|\beta| & \text{if } \alpha = 0, \beta \in \mathbb{R}, \\ |1 - \alpha| - 2|\alpha| \left(1 + \frac{\beta}{\alpha}\right) & \text{if } -\infty < \frac{\beta}{\alpha} \leq -2, \\ |1 - \alpha| + 2|\alpha| & \text{if } -2 \leq \frac{\beta}{\alpha} \leq 0, \\ |1 - \alpha| + 2|\alpha| \left(1 + \frac{\beta}{\alpha}\right) & \text{if } \frac{\beta}{\alpha} \geq 0. \end{cases}$$

*Proof.* Formula (8) gives

$$\begin{aligned} \text{ord } (\hat{\mathcal{J}}_{\alpha,\beta}(k)) &= \sup_{\xi \in \mathbb{D}} \left| 2\alpha + (1 - \alpha)\bar{\xi} + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{(1 - \xi)^2}{\xi} \right) \right| \\ &= \sup_{0 \leq \theta \leq 2\pi} |2\alpha + (1 - \alpha)e^{-i\theta} - \beta \cos \theta| \\ &\leq |1 - \alpha| + \sup_{0 \leq \theta \leq 2\pi} |2\alpha + \beta(1 - \cos \theta)| \end{aligned}$$

which implies the result. □

For completeness, let us write

**Lemma 4.** If  $f(z) = l(z) = z/(1 - z)$ , then

$$(10) \quad \text{ord } (\hat{\mathcal{J}}_{\alpha,\beta}(l)) = \begin{cases} 2\alpha + \beta - 1 & \text{if } \beta \geq -2\alpha + 2, \\ 1 & \text{if } -2\alpha \leq \beta \leq -2\alpha + 2, \\ 1 - 2\alpha - \beta & \text{if } \beta \leq -2\alpha. \end{cases}$$

### 3. New Results

Now we find  $\text{ord } \mathcal{M}_{\alpha,\beta}(S)$  and  $\text{ord } \mathcal{M}_{\alpha,\beta}(S^c)$ .

**Theorem 1.** *We have*

$$\text{ord } \mathcal{M}_{\alpha,\beta}(S) = \text{ord}(\hat{\mathcal{J}}_{\alpha,\beta}(k)).$$

*Proof.* Let

$$\nu_{\alpha,\beta}(f) = \sup_{\xi \in \mathbb{D}} \left| \alpha a_2 + (1 - \alpha)\bar{\xi} + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right|.$$

Then

$$\text{ord } \mathcal{M}_{\alpha,\beta}(S) = \sup_{f \in S} \nu_{\alpha,\beta}(f).$$

Next, notice that

$$\nu_{\alpha,\beta}(f) \leq |1 - \alpha| + \sup_{\xi \in \mathbb{D}} \left| \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right|.$$

If  $\beta = 0$ , then  $\nu_{\alpha,\beta}(f) \leq |1 - \alpha| + 2|\alpha| = \text{ord } \mathcal{M}_{\alpha,\beta}(S)$ . If  $\beta \neq 0$ , we observe that the least upper bound for

$$\left| \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right|$$

is attained if  $|\xi| \rightarrow 1^-$ . To see this, suppose to the contrary that there exists  $\xi_0 \in \mathbb{D}$  such that

$$\begin{aligned} \nu_{\alpha,\beta}(f) &= \left| \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi_0} - \bar{\xi}_0 - \frac{1}{f(\xi_0)} \right) \right| \\ &= \text{Re} \left[ e^{i\gamma} \left( \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi_0} - \bar{\xi}_0 - \frac{1}{f(\xi_0)} \right) \right) \right], \quad \gamma \in \mathbb{R}. \end{aligned}$$

But the right hand side of this equation is a harmonic function which cannot attain its maximum inside  $\mathbb{D}$ . This contradiction proves our observation and we can write:

$$\begin{aligned} \sup_{\xi \in \mathbb{D}} \left| \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right| &= \lim_{|\xi| \rightarrow 1} \sup \left| \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right| \\ &= \lim_{|\xi| \rightarrow 1^-} \sup \left| \alpha a_2 - \frac{\beta}{2} \cdot \frac{1}{f(\xi)} \right| \\ &= \begin{cases} 2|\beta| & \text{if } \alpha = 0, \\ |\alpha| \sup_{c \in \partial f(\mathbb{D})} \left| a_2 - \frac{\beta}{2\alpha} \cdot \frac{1}{f(c)} \right| & \text{if } \alpha \neq 0. \end{cases} \end{aligned}$$

Using Lemma B and the fact that

$$\sup_{c \in \partial f(\mathbb{D})} \left| a_2 - \frac{\beta}{2\alpha} \cdot \frac{1}{c} \right| = \sup_{w \notin f(\mathbb{D})} \left| a_2 - \frac{\beta}{2\alpha} \cdot \frac{1}{w} \right|$$

we have to consider the following cases:

(a) if  $-2 \leq \beta/\alpha \leq 0$ , then by Lemma B and the inequality  $|a_2| \leq 2$  we have

$$\left| a_2 - \frac{\beta}{2\alpha w} \right| = \left| \left( 1 - \frac{\beta}{2\alpha} + \frac{\beta}{2\alpha} \right) a_2 - \frac{\beta}{2\alpha w} \right| \leq \left( -\frac{\beta}{2\alpha} \right) \left| a_2 + \frac{1}{w} \right| + \left( 1 + \frac{\beta}{2\alpha} \right) |a_2| \leq 2;$$

(b) if  $\beta/\alpha \leq -2$ , then Lemma B and  $|w| \geq 1/4$  yields

$$\left| a_2 - \frac{\beta}{2\alpha} \frac{1}{w} \right| \leq \left| a_2 + \frac{1}{w} \right| + \frac{1}{|w|} \left( -1 - \frac{\beta}{2\alpha} \right) \leq -2 - 2\frac{\beta}{\alpha};$$

(c) if  $\beta/\alpha \geq 0$ , then as above

$$\left| a_2 - \frac{\beta}{2\alpha} \frac{1}{w} \right| \leq \left| a_2 \right| + \frac{\beta}{2\alpha} \frac{1}{w} \leq 2 + 2\frac{\beta}{\alpha}.$$

This completes the proof. All bounds are sharp and obtained by Koebe functions. The case in which  $\alpha = 1$  was obtained in [9].  $\square$

For the class of convex univalent functions the situation is different.

**Theorem 2.**

$$(11) \quad \text{ord } \mathcal{M}_{\alpha, \beta}(S^c) = \begin{cases} 1 + |\beta| & \text{if } \alpha = 0, \beta \in \mathbb{R}, \\ |\alpha| \left( 1 + \frac{\beta}{\alpha} \right) + |1 - \alpha| & \text{if } \frac{\beta}{\alpha} \geq 0, \\ |\alpha| \left[ 1 + \frac{\beta}{2\alpha} (1 - \tau) \right] + |1 - \alpha| & \text{if } -2 \leq \frac{\beta}{\alpha} \leq 0, \\ |\alpha| \left[ \tau - 2 - \frac{\beta}{\alpha} \right] + |1 - \alpha| & \text{if } -\infty \leq \frac{\beta}{\alpha} \leq -2. \end{cases}$$

For the first two cases the bounds are sharp.

*Proof.* As in the proof of Theorem 1, we have

$$\begin{aligned} \nu_{\alpha, \beta}(f) &\leq |1 - \alpha| + \sup_{\xi \in \mathbb{D}} \left| \alpha a_2 + \frac{\beta}{2} \left( \frac{1}{\xi} - \bar{\xi} - \frac{1}{f(\xi)} \right) \right| \\ &= |1 - \alpha| + \begin{cases} |\beta| & \text{if } \alpha = 0, \\ |\alpha| \sup_{w \notin f(\mathbb{D})} \left| a_2 - \frac{\beta}{2\alpha} \cdot \frac{1}{w} \right| & \text{if } \alpha \neq 0. \end{cases} \end{aligned}$$

Now the upper bound for  $|a_2 - \beta/(2\alpha) \cdot 1/w|$  will depend on the inequality in Lemma C and the well known fact that  $|w| \geq 1/2$  for  $f \in S^c$ . However, because these two inequalities have different extremal functions, our bounds will be sharp only in the cases mentioned in the statement of this theorem. Therefore, we have:

(a) if  $\beta/\alpha \geq 0$ , then

$$\left| a_2 - \frac{\beta}{2\alpha} \frac{1}{w} \right| \leq \left| a_2 \right| + \left| \frac{\beta}{2\alpha} \right| \frac{1}{|w|} \leq 1 + \frac{\beta}{\alpha};$$

(b) if  $-2 \leq \beta/\alpha \leq 0$ , then

$$\left| a_2 - \frac{\beta}{2\alpha} \frac{1}{w} \right| \leq \left( -\frac{\beta}{2\alpha} \right) \left| a_2 + \frac{1}{w} \right| + \left( 1 + \frac{\beta}{2\alpha} \right) |a_2| \leq 1 + \frac{\beta}{2\alpha} (1 - \tau);$$

(c) if  $-\infty < \beta/\alpha \leq -2$ , then

$$\left| a_2 - \frac{\beta}{2\alpha} \frac{1}{w} \right| \leq \left| a_2 + \frac{1}{w} \right| - \left( 1 + \frac{\beta}{2\alpha} \right) \frac{1}{|w|} \leq \tau - 2 - \frac{\beta}{\alpha},$$

and the proof of Theorem 2 is complete.  $\square$

The following corollaries are the results of Theorems 1 and 2 and the Lemmas.

**Corollary 1.** *If  $f \in S^c$  (or  $S$ ), then*

$$(12) \quad \mathcal{M}_{\alpha,0} := \left\{ G_{\alpha,0}(z) = \int_0^z \frac{(f'(t))^\alpha}{(1-\bar{\xi}z)^{2-2\alpha}} dt ; f \in S^c \text{ (or } S) \right\},$$

and

$$\text{ord } \mathcal{M}_{\alpha,0}(S^c) = |1-\alpha| + |\alpha|; \quad \text{ord } \mathcal{M}_{\alpha,0}(S) = |1-\alpha| + 2|\alpha|.$$

**Corollary 2.** *If  $f \in S^c$ , then for all  $\alpha \in [0,1]$  and  $\bar{\xi} \in \mathbb{D}$ ,*

$$G_{\alpha,0}(z) = \int_0^z \frac{(f'(t))^\alpha}{(1-\bar{\xi}z)^{2-2\alpha}} dt \in S^c.$$

**Remark.** If  $f \in S$  and  $\alpha \in [0,1]$ , then  $\text{ord } \mathcal{M}_{\alpha,0}(S) = 1 + \alpha$  and therefore the transformation (12) decreases the order ( $\text{ord } S = 2$ ). This information implies by a theorem from Pfaltzgraff [11], the following slight extension of his  $1/4$ -theorem: *If  $G_{\alpha,0}(z) \in \mathcal{M}_{\alpha,0}(S)$  and  $\alpha \in [0,1]$ , then the integral  $\int_0^z G'_{\alpha,0}{}^\lambda(t) dt$  is univalent for  $|\lambda| \leq \frac{1}{2(1+\alpha)}$ , ( $\lambda \in \mathbb{C}$ ).*

**Remark.** From Theorems 1 and 2 we have:

$$\begin{aligned} \text{ord } \mathcal{M}_{0,1}(S) &= 3, & \text{ord } \mathcal{M}_{1,-1}(S) &= 2, \\ \text{ord } \mathcal{M}_{1,-2}(S^c) &\leq \tau, & \text{ord } \mathcal{M}_{1,2}(S^c) &= 3. \end{aligned}$$

Therefore, we have examples of operators  $\mathcal{J}_{\alpha,\beta}$  for which the minimal invariant family  $\hat{\mathcal{J}}_{\alpha,\beta}$  has larger, smaller, or the same order as  $S$  or  $S^c$ .

#### 4. Applications

Now we give some applications of the results obtained in the previous section.

**Theorem 3.** *If  $f \in S^c$  and  $G_{\alpha,\beta}$  is given by (7) (i.e.,  $G_{\alpha,\beta} \in \mathcal{M}_{\alpha,\beta}(S^c)$ ), then we have the sharp bound*

$$(13) \quad |\arg G'_{\alpha,\beta}(z)| \leq h(\alpha, \beta) \arcsin r, \quad |z| = r < 1,$$

where

$$(14) \quad h(\alpha, \beta) = 2|\alpha| + |\beta| + |2 - 2\alpha - \beta| \geq 2.$$

*Proof.* From (7) we have

$$\begin{aligned} |\arg G'_{\alpha,\beta}(z)| &\leq |\alpha| |\arg f'(z)| + |\beta| |\arg(g(z)/z)| + |2 - 2\alpha - \beta| |\arg(1 - \bar{\xi}z)| \\ &\leq 2|\alpha| \arcsin r + |\beta| \arcsin r + |2 - 2\alpha - \beta| \arcsin r \\ &= h(\alpha, \beta) \arcsin r, \end{aligned}$$

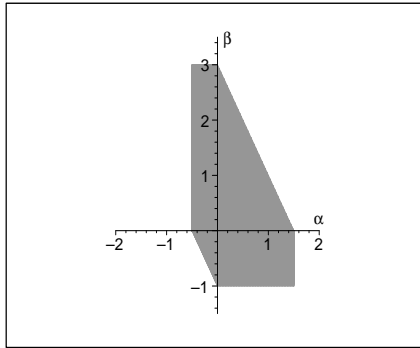
by the well-known sharp rotation theorems  $|\arg f'(z)| \leq 2 \arcsin r$ ,  $f \in S^c$  and  $|\arg(g(z)/z)| \leq \arcsin r$  for  $1/2$ -starlike functions [5].  $\square$

**Theorem 4.** *The radius of univalence,  $r_u$ , of  $\mathcal{M}_{\alpha,\beta}(S^c)$  satisfies the inequality  $r_u \geq r_{\alpha,\beta}$ , where*

$$(15) \quad r_{\alpha,\beta} = \min \left\{ 1, \tan \frac{\pi}{h(\alpha, \beta)} \right\},$$

and  $h(\alpha, \beta)$  is given by (14).

**Corollary 3.** *A function  $G_{\alpha,\beta} \in \mathcal{M}_{\alpha,\beta}(S^c)$  is univalent in  $\mathbb{D}$  if  $(\alpha, \beta) \in A$ , where  $A = \{(\alpha, \beta) : \alpha \in [0, 3/2], \beta \in [-1, 3-2\alpha]\} \cup \{(\alpha, \beta) : \alpha \in [-1/2, 0], \beta \in [-1-2\alpha, 3]\}$ .*



Region A.

*Proof.* The family  $\mathcal{M}_{\alpha,\beta}(S^c)$  is a linear invariant family and therefore by a result of Pommerenke (see [12], pp. 134-135) the radius of univalence  $r_u = r_u(\mathcal{M}_{\alpha,\beta}(S^c))$  satisfies the inequality  $r_u \geq r_{\alpha,\beta}$ , where

$$r_{\alpha,\beta} = \frac{r_0}{1 + \sqrt{1 - r_0^2}}$$

and  $r_0$  is the radius of the disk  $|z| < r_0$  in which  $G_{\alpha,\beta}(z)/z \neq 0$ . Moreover, the value of  $r_0$  is determined from the equation

$$\max_{\substack{f \in S^c \\ |z|=r}} |\arg G'_{\alpha,\beta}(z)| = 2\pi.$$

Inequality (13) gives  $r_0 = \sin \frac{2\pi}{h(\alpha,\beta)}$ . This along with (15) gives region A.

Corollary 3 and the shape of region A are equivalent to the following inequality  $2 \leq h(\alpha, \beta) \leq 4$ , which implies the univalence of  $\mathcal{M}_{\alpha,\beta}(S^c)$  in the whole disk  $\mathbb{D}$ .  $\square$

In fact, now the conclusion of Theorem 4 can be strengthened and made sharp.

**Theorem 5.** *The family  $\mathcal{M}_{\alpha,\beta}(S^c)$  consists of univalent close-to-convex functions in  $\mathbb{D}$  for  $(\alpha, \beta) \in A$ . If  $(\alpha, \beta) \notin A$  then the radius of close-to-convexity of  $\mathcal{M}_{\alpha,\beta}(S^c)$  is the unique root of the equation*

$$(16) \quad 2 \operatorname{arccot} \left[ \frac{1 - r^2}{\sqrt{h^2(\alpha, \beta)r^2 - (1 + r^2)^2}} \right] - h(\alpha, \beta) \operatorname{arccot} \left[ \frac{\frac{1}{2}h(\alpha, \beta)(1 - r^2)}{\sqrt{h^2(\alpha, \beta)r^2 - (1 + r^2)^2}} \right] = -\pi$$

The proof of Theorem 5 follows directly from inequality (13) and the following lesser known result.

**Lemma E** ([2], p. 19, Cor. 3.3). *If  $\mathcal{M}$  is a linear invariant family for which*

$$\max_{\substack{f \in \mathcal{M} \\ |z|=r < 1}} |\arg f'(z)| = 2\tau \arcsin r,$$

*then the radius of close-to-convexity of  $\mathcal{M}$  is 1 if  $1 \leq \tau \leq 2$ , and if  $\tau > 2$  the radius of close-to-convexity is the unique root of the equation*

$$2 \operatorname{arccot} w - 2\tau \operatorname{arccot}(\tau w) = -\pi,$$

where

$$w = \frac{1 - r^2}{\sqrt{4\tau^2 r^2 - (1 + r^2)^2}}.$$

**Corollary 4.** *If  $(\alpha, \beta) \in A$ , then the class  $\mathcal{M}_{\alpha, \beta}(S^c)$  consists of close-to-convex functions in  $\mathbb{D}$  and the result is sharp (by Lemma E). The extremal function is the half plane mapping  $l_\epsilon(z) = z/(1 - \epsilon z)^2$ ,  $|\epsilon| = 1$ .*

**Remark.** Because we have the inclusion  $\mathcal{J}_{\alpha, \beta}(S^c) \subset \hat{\mathcal{J}}_{\alpha, \beta}(S^c)$ , Corollary 4 remains valid for the integral given in (4) when  $f \in S^c$ . In particular, we obtain the following results.

**Corollary 5** ([10]). *If  $f \in S^c$ , then the function*

$$F_{\alpha, 0}(z) = \int_0^z (f'(t))^\alpha dt$$

*is close-to-convex in  $\mathbb{D}$  for all  $\alpha \in [-1/2, 3/2]$  and this result is sharp.*

**Corollary 6** ([10]). *If  $f \in S^c$ , then the function*

$$F_{0, \beta}(z) = \int_0^z \left( \frac{f'(t)}{t} \right)^\beta dt$$

*is close-to-convex in  $\mathbb{D}$  for all  $\beta \in [-1, 3]$  and this result is sharp.*

**Corollary 7.** *If  $f \in S^c$ , then the function*

$$F_{\alpha, 2-2\alpha}(z) = \int_0^z (f'(t))^\alpha \left( \frac{f'(t)}{t} \right)^{2-2\alpha} dt$$

*is close-to-convex in  $\mathbb{D}$  for all  $\alpha \in [-1/2, 3/2]$  and this result is sharp.*

**Remark.** One can observe that our method of employing minimal invariance family to study the univalence of the integral in (4) together with the result of Pommerenke [12] and Campbell and Ziegler [2] gives better results than the direct application of the theorem of Pfaltzgraff [11] which is so well suited to the class  $S$ .

**Remark.** Of course, in general, the radius of close-to-convexity for (4) and for its minimal invariant family (6) are different. For example, for  $F_{1, -2}$  the radius of close-to-convexity can be calculated from (4) to be  $1/\sqrt{2} \approx 0.707$ , while the radius of close-to-convexity of  $\mathcal{M}_{1, -2}(S^c)$  can be calculated from (16) with  $h = 6$  to be approximately equal to 0.553.

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