REMARK ON THE HIGHER ORDER SCHWARZIAN DERIVATIVES FOR CONVEX UNIVALENT FUNCTIONS

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Abstract. We observe that in contrast to the class S, the extremal functions for the bound of higher order Schwarzian derivatives for the class S^c of convex univalent functions are different. We prove the sharp bound for three first consecutive derivatives.

Let S denote the class of holomorphic and univalent functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, z \in \mathbb{D},$$

and $S^c \subset S$ the class consisting of convex functions.

Let

$$S(f)(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, z \in \mathbb{D}$$

denote the Schwarzian derivative for f, and let the higher order Schwarzian derivative be defined inductively (see [5]) as:

$$\begin{aligned} \sigma_{n+1}(f) &= (\sigma_n(f))' - (n-1)\sigma_n(f) \cdot \frac{f''}{f'}, \ n \geq 4 \\ \sigma_3(f) &= S(f). \end{aligned} \tag{1}$$

In [5] it was proved that the upper bound for $|\sigma_n(f)|, f \in S$ is attained for the Koebe function for each $n = 3, 4, \ldots$.

In this note we show that situation is different when we deal with the class of convex univalent functions. Because of linear invariance of the class S^c one can restrict the considerations to $\sigma_n(f)(0) := S_n$. We have the following

Theorem 1. *If $f \in S^c$, then the following sharp estimates hold:*

$$\begin{aligned} |S_3| &= |6(a_3 - a_2^2)| \leq 2 \\ |S_4| &= 24|a_4 - 3a_3a_2 + 2a_2^3| \leq 4 \\ |S_5| &= 24|5a_5 - 20a_4a_2 - 9a_3^2 + 48a_3a_2^2 - 24a_2^4| \leq 12 \end{aligned}$$

The extremal functions (up to rotations) have the form

$$f_n(z) = \int_0^z (1 - t^{n-1})^{-\frac{2}{n-1}} dt, \ n = 3, 4, 5, \tag{2}$$

respectively.

Proof. From (1) one can easily find

$$
\begin{aligned}
\sigma_4(f) &= \frac{f''''}{f'} - 6\,\frac{f'''f''}{f'^2} + 6\left(\frac{f''}{f'}\right)^3 \\
\sigma_5(f) &= \frac{f'''''}{f'} - \frac{10f''''f''}{f'^2} - 6\left(\frac{f'''}{f'}\right)^2 + 48\,\frac{f'''f''^2}{f'^3} - 36\left(\frac{f''}{f'}\right)^4 .
\end{aligned} \tag{3}
$$

Note that in [5] there are two misprints in the last formula.

Therefore we have from (3):

$$
\begin{aligned}
S_3 &= 6(a_3 - a_2^2) \\
S_4 &= 24(a_4 - 3a_2a_3 + 2a_2^3) \\
S_5 &= 24(5a_5 - 20a_4a_2 - 9a_3^2 + 48a_3a_2^2 - 24a_2^4).
\end{aligned} \tag{4}
$$

We are going to use the connection of the class S^c and functions with positive real part in $\mathbb{D}$, as well as the functions satisfying the Schwarz lemma condition.

Namely we have

$$
f \in S^c \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} = p(z) = \frac{1+\omega(z)}{1-\omega(z)},\ z \in \mathbb{D}, \tag{5}
$$

where $p(z) = 1 + p_1z + p_2z^2 + \cdots$, $\mathrm{Re}\{p(z)\} > 0, z \in \mathbb{D}$(i.e., $p \in P$, the class of functions with positive real part) and $\omega(z) = c_1z + c_2z^2 + \cdots, |\omega(z)| < 1, z \in \mathbb{D}$ (i.e., $\omega \in \Omega$, the class of Schwarz functions).

From (5) we find

$$S_3 = 2c_2$$

and because $|c_2| \le 1 - |c_1|^2$,

$$|S_3| \le 2,$$

which is as well the well-known result of Hummel [1]. The extermal function is

$$f_3(z) = \int_0^z \frac{dt}{1-t^2} = \frac{1}{2}\,\log\frac{1+z}{1-z}.$$

The functional S_4 has a special form of the functional $|a_4 + sa_2a_3 + ua_2^3|, u, s \in \mathbb{R}$ which was estimated sharply for each $s, u \in \mathbb{R}$ in [4] and therefore the result follows by taking $s = -3, u = 2$ in Theorem 1 in [4].

The extremal function is determined by taking $\omega(z) = z^3$ in (5) which gives (2). Finally in order to get the bound for $|S_5|$ which is complicated we transform it to the class Ω of Schwarz functions $\omega(z)$.

By equating the coefficients in (5) one can find the relations:

$$
\begin{aligned}
a_2 &= c_1 \\
a_3 &= \frac{1}{3}\,(c_2 + 3c_1^2) \\
a_4 &= \frac{1}{6}\,(c_3 + 5c_1c_2 + 6c_1^3) \\
a_5 &= \frac{1}{10}\left(c_4 + \frac{14}{3}\,c_3c_1 + \frac{43}{3}\,c_2c_1^2 + 2c_2^2 + 10c_1^4\right),
\end{aligned}
$$

which transform S_5 as given by (4) to a nicer form

$$
S_5 = 12(c_4 - 2c_3c_1 + c_2c_1^2). \tag{6}
$$

Now we can try to estimate (6) by the use of the Caratheodory inequalities applied to the class Ω as it was done in [4]. However, this leads to very complicated calculations. But one can observe that within the class Ω the functional $|c_4-2c_3c_1+c_2c_1^2|$ and $|c_4+2c_3c_1+c_2c_1^2|$ have the same upper bound, because if $\omega(z)\in\Omega$, then $\omega_1(z)=-\omega(-z)\in\Omega$.

On the other hand, comparing the coefficients p_k and c_k in (5) one gets

$$\begin{aligned} p_1 &= 2c_1,\\ p_2 &= 2(c_2+c_1^2),\\ p_3 &= 2(c_3+2c_1c_2+c_1^3)\\ p_4 &= 2(c_4+2c_1c_2+c_2^2+3c_1^2+c_1^4) \end{aligned}$$

from which we obtain that

$$2(c_4+2c_3c_1+c_1^3)=p_4-\frac{1}{2}\,p_2^2.$$

Leuthwiler and Schober [3] gave the precise bound for $\left|p_4-\frac{1}{2}\,p_2^2\right|\le 2$, which implies that $|c_4+2c_3c_1+c_1^3|=|c_4-2c_3c_1+c_1^3|\le 1$. This completes the proof. The extremal function is obtained by taking $\omega(z)=z^4$ in (5).

Note that writing S_5 with the coefficients of p_k leads to another "bad" expression. □

Remark. We conjecture that for every $n=6,\dots$ the maximal value of $|S_n|$ is attained by the function given by (2).

Remark. The general approach to the bound of S_4 or S_5 would lead within the class P to consideration of functions of the form $p(z)=\sum_{k=1}^{n}\lambda_k\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}$, $n\le 4$ or 5, which is very difficult to handle because it involves long and tedious calculations.

Remark. One can observe that the bound for $|\sigma_n(f)|$ given in [5] follows directly from the formula (1) in [5] and the result of R. Klouth and K.-J. Wirths [2].

References

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