Computational Methods and Function Theory
Volume 4 (2004), No. 1, 151–158

Landau's Theorem for Planar Harmonic Mappings

Michael Dorff and Maria Nowak

To Walter Hengartner

Abstract. Landau gave a lower estimate for the radius of a schlicht disk centered at the origin and contained in the image of the unit disk under a bounded holomorphic function f normalized by $f(0) = f'(0) - 1 = 1$. Chen, Gauthier, and Hengartner established analogous versions for bounded harmonic functions. We improve upon their estimates.

Keywords. Landau Theorem, planar harmonic mappings.

2000 MSC. 30C99, 30C62.

1. Introduction

The classical Landau Theorem for bounded holomorphic functions states that if f is a holomorphic function in the unit disk $\mathbb{D}$ with $f(0) = 0$, $f'(0) = 1$, and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent in the disk $|z| < \rho_0$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}},$$

and $f(|z| < \rho_0)$ contains a disk $|w| < R_0$ with

$$R_0 = M\rho_0^2.$$

This result is sharp (see [3, 7]). Furthermore, for holomorphic functions in $\mathbb{D}$ with the only restriction that $f'(0) = 1$, there is the Bloch Theorem which asserts the existence of a positive constant b such that $f(\mathbb{D})$ contains a schlicht disk, that is, a disk of radius b which is the univalent image of some region in $\mathbb{D}$. The Bloch constant is defined as the supremum of all such b (see [6] and [5]).

Harmonic mappings can be regarded as generalizations of holomorphic functions. A function $f(z) = u(z) + iv(z)$ defined on a domain G in the complex plane is a harmonic mapping if and only if f is twice continuously differentiable and

Received March 19, 2003.
This research was partially supported by a COBASE grant from the U.S. National Academies of Science.

ISSN 1617-9447/$ 2.50

$\Delta f = 4f_{z\bar{z}} = 0$. If G is simply connected, then f can be written as $f = h + \overline{g}$, where h and g are holomorphic on G.

Chen, Gauthier and Hengartner [4] showed by giving an example that the Bloch Theorem does not hold for normalized harmonic functions. However, they did obtain a Bloch Theorem for open planar harmonic mappings and two versions of the Landau Theorem for harmonic mappings. In this paper we give slightly better results for these Landau Theorems. The proofs contained in Section 2 rely on some results obtained in [4] while the proofs in Section 3 are based on coefficient estimates given in Lemma 3.

2. Versions of Landau Theorems for harmonic functions

Let f be a harmonic function in the unit disk $\mathbb{D}$. Then $f = h+\overline{g}$, where g and h are holomorphic on $\mathbb{D}$. For such f, define

$$\Lambda_f = \max_{0\le\theta\le 2\pi} |f_z + e^{-2i\theta} f_{\bar{z}}| = |f_z| + |f_{\bar{z}}| = |h'| + |g'|$$

and

$$\lambda_f = \min_{0\le\theta\le 2\pi} |f_z + e^{-2i\theta} f_{\bar{z}}| = \big||f_z| - |f_{\bar{z}}|\big| = \big||h'| - |g'|\big|.$$

The Jacobian of f is given by

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2.$$

Note that $|J_f| = \Lambda_f \lambda_f$. It is known [8] that a harmonic mapping is locally univalent in $\mathbb{D}$ if and only if its Jacobian does not vanish anywhere in $\mathbb{D}$.

The following Schwarz Lemma for harmonic mappings is proved in [4].

Theorem A (Schwarz Lemma)**.** *Let f be a harmonic mapping of the unit disk $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$\Lambda_f(0) \le \frac{4}{\pi}, \tag{1}$$

$$\Lambda_f(z) \le \frac{8}{\pi(1-|z|^2)}, \qquad \textit{for } z \in \mathbb{D}, \tag{2}$$

$$|f(z)| \le \frac{4}{\pi}\arctan|z| \le \frac{4}{\pi}|z|, \qquad \textit{for } z \in \mathbb{D}. \tag{3}$$

For $r > 0$ let $\mathbb{D}_r$ denote the disk with center at the origin and radius r. We will use the following result obtained in [4]:

Theorem B (Chen, Gauthier, Hengartner)**.** *Let f be a harmonic mapping of the unit disk $\mathbb{D}$ such that $f(0) = 0$, $\lambda_f(0) = 1$ and $\Lambda_f(z) \le \Lambda$ for $z \in \mathbb{D}$. Then f is univalent on a disk $\mathbb{D}_{\rho_0}$ with*

$$\rho_0 = \frac{\pi}{4(1+\Lambda)},$$

and $f(\mathbb{D}_{\rho_0})$ contains a schlicht disk $\mathbb{D}_{R_0}$ with

$$R_0 = \frac{\pi}{8(1+\Lambda)}.$$

Now we can prove the following version of a Landau Theorem for harmonic mappings.

Theorem 1. *Assume that f is a harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$, $J_f(0) = 1$ and $|f(z)| \le M$ for $z \in \mathbb{D}$. Then f is univalent in the disk $\mathbb{D}_{r_1}$ where*

$$r_1 = \frac{\pi^3}{4\sqrt{2}(\pi^2 + 64M^2)}$$

and $f(\mathbb{D}_{r_1})$ contains a schlicht disk $\mathbb{D}_{R_1}$, where

$$R_1 = \frac{\pi^4}{32\sqrt{2}M(\pi^2 + 64M^2)}.$$

Proof. If we define

$$F(z) = \frac{\sqrt{2}f\left(\frac{z}{\sqrt{2}}\right)}{\lambda_f(0)}, \qquad z \in \mathbb{D},$$

then $\lambda_F(0) = 1$. By Theorem A,

$$\frac{1}{\lambda_f(0)} = \Lambda_f(0) \le \frac{4M}{\pi}. \tag{4}$$

Hence

$$\Lambda_F(z) = \frac{\Lambda_f\left(\frac{z}{\sqrt{2}}\right)}{\lambda_f(0)} \le \frac{\frac{16M}{\pi}}{\lambda_f(0)} \le \frac{64M^2}{\pi^2}.$$

Thus, using Theorem B, we get that F is univalent in the disk

$$D := \left\{ z \in \mathbb{C} : \ |z| < \frac{\pi^3}{4(\pi^2 + 64M^2)} \right\}$$

and $F(D)$ contains the schlicht disk $|w| < \pi^3/[8(\pi^2 + 64M^2)]$. Consequently, f is univalent in $\mathbb{D}_{r_1}$ and using (4), we find that $f(\mathbb{D}_{r_1})$ contains a schlicht disk $\mathbb{D}_{R_1}$. ■

In a much the same way one can obtain a second Landau Theorem.

Theorem 2. *Assume that f is a harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$, $\lambda_f(0) = 1$ and $|f(z)| \le M$ for $z \in \mathbb{D}$. Then f is univalent in the disk $\mathbb{D}_{r_2}$ where*

$$r_2 = \frac{\pi^2}{4\sqrt{2}(\pi + 16M)}$$

and $f(\mathbb{D}_{r_2})$ contains a schlicht disk $\mathbb{D}_{R_2}$, where

$$R_2 = \frac{\pi^2}{8\sqrt{2}(\pi + 16M)}.$$

Remark 1. It is known that if f is analytic in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$, then $M \geq 1$. Moreover, the example $f(z) = z$ shows that 1 is the best constant. Note that if f is harmonic in $\mathbb{D}$, $f(0) = 0$, $J_f(0) = 1$ and $|f(z)| \leq M$, then $M \geq \pi/4$. Indeed, by (1) in Theorem A we get

$$1 = |a_1|^2 - |b_1|^2 \leq |a_1|^2 \leq \Lambda_f^2(0) \leq \left(\frac{4}{\pi}\right)^2 M^2,$$

which implies that $M \geq \pi/4$. If we require that $f(0) = 0, \Lambda_f(0) = 1$ and $|f(z)| \leq M$, then also $M \geq \pi/4$ and in this case the constant $\pi/4$ cannot be improved as shown by the function

$$f(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) - \frac{1}{2}\ln\left(\frac{1+\overline{z}}{1-\overline{z}}\right)$$

which maps $\mathbb{D}$ onto the open interval with endpoints $-i\pi/4$ and $i\pi/4$. This function also shows that the normalization $\Lambda_f(0) = 1$ does not imply the existence of a schlicht disk for bounded harmonic functions.

Remark 2. In [4], Chen, Gauthier, and Hengartner proved similar theorems with different values for r_i and R_i. In particular, they established the results for

$$R_1 = \frac{\pi^4}{512mM^3},$$

and

$$R_2 = \frac{\pi^2}{32mM},$$

where $m = (11 + 3\sqrt{13})(\sqrt{4 - \sqrt{13}})/2 \approx 6.85$. Our values for R_1 and R_2 are slightly larger for all $M \geq 1$. Our result can be improved somewhat by replacing $\sqrt{2}$ with $1/r$ in $F(z)$ and finding the value of r that maximizes the new r_i and R_i functions. This r-value depends upon M but is always less than 0.64 for $M \geq \pi/4$. However, these computations are messy and the improvements are not very significant especially in light of Theorems 4 and 5.

3. Main results

Recall that the Bloch space $\mathcal{B}$ of $\mathbb{D}$ consists of functions h analytic in $\mathbb{D}$ and such that

$$\|h\|_{\mathcal{B}} = \sup\{(1 - |z|^2)|h'(z)| : z \in \mathbb{D}\} < \infty.$$

Let $L^\infty(\mathbb{D})$ denote the set of all measurable and bounded functions on $\mathbb{D}$. It is well known that $\mathcal{B} = P(L^\infty(\mathbb{D}))$, where P is the Bergman projection given by

$$PF(z) = \int_{\mathbb{D}} \frac{F(w)}{(1 - \overline{w}z)^2}\, d\sigma(w), \qquad z \in \mathbb{D}.$$

Here $d\sigma$ denotes the normalized area measure on $\mathbb{D}$ (see [2, p. 13] or [9]).

If $f = h + \overline{g}$ is harmonic on $\mathbb{D}$ with $|f(z)| \le M$, then $h, g \in \mathcal{B}$. Indeed, assuming that $h(0) = g(0) = 0$ we get

$$h(z) = Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2}\, d\sigma(w), \qquad z \in \mathbb{D}, \tag{5}$$

and

$$g(z) = P\overline{f(z)} = \int_{\mathbb{D}} \frac{\overline{f(w)}}{(1-\overline{w}z)^2}\, d\sigma(w), \qquad z \in \mathbb{D}.$$

It is known that the coefficients of a Bloch function are bounded (see [1]). The following lemma yields coefficient estimates for the functions h and g in the case when $f = h + \bar{g}$ or Λ_f is bounded on $\mathbb{D}$.

Lemma 3. *Assume that $f = h + \bar{g}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in \mathbb{D}$.*

(a) *If $|f(z)| \le M$ for $z \in \mathbb{D}$, then*

$$|a_n|, |b_n| \le 2M, \qquad n = 1, 2, \ldots.$$

(b) *If $\Lambda_f(z) \le \Lambda$ for $z \in \mathbb{D}$, then*

$$|a_n| + |b_n| \le \frac{4\Lambda}{n+3}, \qquad n = 2, 3, \ldots.$$

Proof. Differentiating under the integral sign in (5) gives

$$h^{(n)}(z) = \int_{\mathbb{D}} \frac{(n+1)!\, \overline{w}^n f(w)}{(1-\overline{w}z)^{n+2}}\, d\sigma(w), \qquad n = 1, 2, \ldots.$$

Consequently,

$$\begin{aligned} |a_n| = \frac{|h^{(n)}(0)|}{n!} &= (n+1) \left| \int_{\mathbb{D}} f(w)\overline{w}^n \, d\sigma(w) \right| \\ &\le M(n+1) \int_{\mathbb{D}} |\overline{w}^n| \, d\sigma(w) = \frac{2M(n+1)}{n+2} < 2M. \end{aligned}$$

Clearly, the same estimates hold for the Taylor coefficients of g.

To prove (b), define, for any real α,

$$F_\alpha(z) = h(z) + e^{i\alpha} g(z), \qquad z \in \mathbb{D}.$$

By assumption,

$$|F'_\alpha(z)| \le \Lambda, \qquad z \in \mathbb{D}.$$

If a function k, holomorphic in $\mathbb{D}$, is such that $k(0) = k'(0) = 0$ and $(1-|z|^2)k'(z)$ is an integrable function with respect to $d\sigma$, then

$$k(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)k'(w)}{\overline{w}(1-z\overline{w})^2}\, d\sigma(w), \qquad z \in \mathbb{D}$$

(see [p. 57][9]). Applying this formula to the function $F_\alpha - F'_\alpha(0)$ and differentiating under the integral sign yields

$$F_\alpha^{(n)}(z) = (n+1)! \int_{\mathbb{D}} \frac{(1-|w|^2)F'_\alpha(w)\overline{w}^{\,n-1}}{(1-z\overline{w})^{\,n+1}}\, d\sigma(w), \qquad z \in \mathbb{D},\ n = 2, 3, \dots .$$

By assumption we get

$$\left|a_n + e^{i\alpha} b_n\right| \le (n+1) \int_{\mathbb{D}} (1-|w|^2)\Lambda |w|^{n-1}\, d\sigma(w) = \Lambda \frac{4}{n+3},$$

which implies (b). ■

Observe also that the fact that $h, g \in \mathcal{B}$ follows directly from (2). Since for $z \in \mathbb{D}$

$$|h'(z)|, |g'(z)| \le \Lambda_f(z) \le \frac{8}{\pi(1-|z|^2)},$$

we see that $\|h\|_{\mathcal{B}}, \|g\|_{\mathcal{B}} \le 8/\pi$.

Now we state our main results.

Theorem 4. *Assume that f is a harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$, $J_f(0) = 1$ and $|f(z)| \le M$ for $z \in \mathbb{D}$. Then f is univalent in the disk $\mathbb{D}_{r_1}$, where*

$$r_1 = 1 - \frac{4M}{\sqrt{\pi + 16M^2}},$$

and $f(\mathbb{D}_{r_1})$ contains a schlicht disk $\mathbb{D}_{R_1}$ with

$$R_1 = \frac{\pi}{4M} + 8M - 8M\sqrt{1 + \frac{\pi}{16^2 M^2}} > \frac{\pi^2}{16^2 M^3} - \frac{\pi^3}{2 \cdot 16^3 M^5}.$$

Proof. Let $f = h + \overline{g}$, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic in $\mathbb{D}$. It follows from (1) that

$$|a_1| - |b_1| = \lambda_f(0) = \frac{J_f(0)}{\Lambda_f(0)} \ge \frac{\pi}{4M}.$$

For $z_1 \ne z_2$ in $\mathbb{D}_r$ we have

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left| \int_{[z_1,z_2]} f_z(z)dz + f_{\bar z}(z)\, d\overline{z} \right| = \left| \int_{[z_1,z_2]} h'(z)\, dz + \overline{g'(z)}\, d\overline{z} \right| \\ &\ge \left| \int_{[z_1,z_2]} h'(0)\, dz + \overline{g'(0)}\, d\overline{z} \right| \\ &\quad - \left| \int_{[z_1,z_2]} (h'(z) - h'(0))\, dz + (\overline{g'(z)} - \overline{g'(0)})\, d\overline{z} \right| \end{aligned}$$

$$\begin{aligned}
&\geq \lambda_f(0)|z_1 - z_2| \\
&\quad - \left| \int_{[z_1,z_2]} (h'(z) - h'(0))\, dz \right| - \left| \int_{[z_1,z_2]} (g'(z) - g'(0))\, dz \right| \\
&\geq \frac{\pi}{4M}|z_1 - z_2| \\
&\quad -|h(z_2) - h(z_1) - h'(0)(z_1 - z_2)| \\
&\quad -|g(z_2) - g(z_1) - g'(0)(z_1 - z_2)| \\
&= \frac{\pi}{4M}|z_1 - z_2| - \left| \sum_{n\geq 2} a_n({z_2}^n - {z_1}^n) \right| - \left| \sum_{n\geq 2} b_n({z_2}^n - {z_1}^n) \right| \\
&= \frac{\pi}{4M}|z_1 - z_2| \\
&\quad -|z_1 - z_2| \sum_{n\geq 2}(|a_n| + |b_n|) \left| {z_1}^{n-1} + {z_1}^{n-2}z_2 + \cdots + z_2^{n-1} \right| \\
&\geq \frac{\pi}{4M}|z_1 - z_2| - |z_1 - z_2| \sum_{n\geq 2}(|a_n| + |b_n|)nr^{n-1}.
\end{aligned}$$

Now, using Lemma 3 (a) we obtain

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geq \frac{\pi}{4M}|z_1 - z_2| - 4M|z_1 - z_2| \sum_{n\geq 2} nr^{n-1} \\
&= \frac{\pi}{4M}|z_1 - z_2| - 4M|z_1 - z_2| \frac{2r - r^2}{(1-r)^2}.
\end{aligned}$$

Finally, observe that r_1 is a solution of the equation

$$\frac{\pi}{4M} = 4M\frac{2r - r^2}{(1-r)^2}$$

and

$$\frac{\pi}{4M} > 4M\frac{2r - r^2}{(1-r)^2},$$

if $0 < r < r_1$. This shows that $f(z_1) \neq f(z_2)$, if $z_1 \neq z_2 \in \mathbb{D}_{r_1}$.
Furthermore since $f(0) = 0$, we have

$$|f(z)| \geq |a_1 z + \overline{b}_1\overline{z}| - \left| \sum_{n=2}^{\infty} a_n z^n + \overline{b_n}\overline{z}^n \right| \geq \frac{\pi r_1}{4M} - 4M\frac{{r_1}^2}{1 - r_1} = R_1,$$

where

$$R_1 = \frac{\pi}{4M} + 8M - 8M\sqrt{1 + \frac{\pi}{16M^2}}.$$

Finally, note that the inequality stated in the theorem follows from

$$\sqrt{1+x} < 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3, \qquad x > 0.$$

■

Using a similar proof as for the previous theorem, we get the following result.

Theorem 5. *Assume that f is a harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$, $\lambda_f(0) = 1$ and $|f(z)| \leq M$ for $z \in \mathbb{D}$. Then f is univalent in the disk $\mathbb{D}_{r_2}$ where*

$$r_2 = 1 - \frac{2\sqrt{M}}{\sqrt{1+4M}}$$

and $f(\mathbb{D}_{r_2})$ contains a schlicht disk $\mathbb{D}_{R_2}$, where

$$R_2 = 8M + 1 - 8M\sqrt{1 + \frac{1}{4M}} > \frac{1}{16M} - \frac{1}{128M^2}.$$

Remark 3. For all $M \geq \pi/4$ the values for r_i and R_i in Theorems 4 and 5 are better than the results obtained in [4] and in our Theorems 1 and 2.

References

1. J. M. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.
2. S. Axler, Bergman spaces and their operators, *Surveys of some recent results in operator theory*, Vol. I, 1–50, Pitman Res. Notes Math. Ser., 171, Longman Sci. Tech., Harlow, 1988.
3. H. Chen, On the Bloch constant, *Approximation, complex analysis, and potential theory* (Montreal, QC, 2000), 129–161, NATO Sci. Ser. II Math. Phys. Chem., 37, Kluwer Acad. Publ., Dordrecht, 2001.
4. H. Chen, P. M. Gauthier, and W. Hengartner, Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.* **128** (2000) 11, 3231–3240.
5. J. B. Conway, *Functions of One Complex Variable*, 2nd ed. Graduate Texts in Mathematics, 11. Springer-Verlag, New York-Berlin, 1978.
6. W. K. Hayman, *Multivalent Functions*, 2nd ed. Cambridge Tracts in Mathematics, 110. Cambridge University Press, Cambridge, 1994.
7. E. Landau, Der Picard-Schottysche Satz und die Blochsche Konstanten, *Sitzungsber. Preuss. Akad. Wiss. Berlin Phys.-Math. Kl.* (1926), 467–474.
8. H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42** (1936), 689–692.
9. K. Zhu, *Operator Theory in Function Spaces*, 2nd ed. Graduate Texts in Mathematics, 11. Springer-Verlag, New York-Berlin, 1978.

Michael Dorff E-mail: `mdorff@math.byu.edu`
Address: *Department of Mathematics, Brigham Young University, Provo, Utah, 84602, U.S.A.*

Maria Nowak E-mail: `nowakm@golem.umcs.lublin.pl`
Address: *Department of Mathematics, Maria Curie-Skłodowska University, 20-031 Lublin, Poland.*