CHAPTER 4

Harmonic Univalent Mappings

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4.1. Introduction

Complex-valued analytic functions have many very nice properties that are not necessarily true for real-valued functions. For example, if you can differentiate the complex-valued function one time, then you can differentiate it infinitely many times. In addition, complex-valued analytic functions can always be represented as a Taylor series, and they are conformal (that is, they preserve angles). Why does an analytic function have these properties? If $f = u + iv$ is an analytic function, then its real part, $u(x, y)$ and its imaginary part, $v(x, y)$, satisfy Laplace's equation and thus are both harmonic. Also, u and v satisfy the Cauchy-Riemann equations and are therefore harmonic conjugates of each other. In this chapter we discuss some ideas and problems related to a collection of univalent (i.e., $1-1$) complex-valued functions, $f = u + iv$, where u and v satisfy Laplace's equation but not necessarily the Cauchy-Riemann equations. This collection of functions are known as *harmonic univalent functions* or mappings, and contain the collection of analytic univalent functions as a subset. Analytic univalent functions have been studied since the early 1900's, and there are thousands of research papers written on the subject. The study of harmonic univalent mappings is a fairly recent area of research. So, it is natural to consider the properties of analytic univalent functions as a starting point for our study of harmonic univalent mappings. A general theme will be "What properties of analytic univalent functions are still true for this larger class of harmonic univalent functions?"

Section 4.2 presents some background about the collection of univalent analytic functions. Section 4.3 introduces the fundamentals of harmonic univalent functions. The study of harmonic univalent functions from the perspective of univalent complex-valued analytic functions is a new area of research. Finding examples of such functions is not easy, but a very useful method of doing so is discussed in Section 4.4. These three sections should be read first. After that, the remaining sections can be read in any order and are independent of each other. There are three applets used in this chapter:

- *ComplexTool* is used to plot the image of domains in $\mathbb{C}$ under complex-valued functions.
- *ShearTool* is used to plot the image of domains in C under a complex-valued harmonic function that is formed by shearing an analytic function and its dilatation; the user enters the corresponding analytic function and dilatation without having to solve explicitly for the harmonic function.
- *LinComboTool* is used to plot and explore the convex combination of complex-valued harmonic polygonal maps.

They can be accessed online at
http://www.jimrolf.com/explorationsInComplexVariables/chapter4.html. Each section contains examples, exercises, and explorations that involve using the applets. You should do all of the exercises and explorations many of which present functions and concepts that will be used later in the chapter (there are additional exercises at the end of the chapter). In the study of harmonic univalent functions, there are many open problems. Some of these are specifically mentioned. In addition, there are short projects and long projects that are suitable as research problems for undergraduates to explore.

The goal of this chapter is not to give a comprehensive or step-by-step approach to this topic, but rather to get the reader engaged with the general notions, questions, and techniques of the area – but even more so, to encourage the reader to actively pose as well as pursue their own questions. To better understand the nature and purpose of this text, the reader should be sure to read the Introduction before proceeding. The study of harmonic univalent functions has many interesting problems that can be investigated by undergraduates through the use of computers and the applets. I anticipate that some students will explore the ideas in this chapter and that this will lead them to prove some new results in the field.

4.2. The Family S of Analytic, Normalized, Univalent Functions

We will be discussing mapping problems in complex analysis. This deals with the properties of a collection of functions that map one domain onto certain other image domains. Before we get into these properties, we need some background material.

Definition 4.1.

(1) Let $G \subset \mathbb{C}$ be a simply-connected domain.
(2) Let $\mathbb{D} = \{z : |z| < 1\}$, the unit disk.
(3) A function f is *univalent* in G if f is one-to-one in G. That is, if $f(z_1) = f(z_2)$, then $z_1 = z_2$, $\forall z_1, z_2 \in G$.

Univalent analytic functions are nice, because they guarantee the existence of an inverse function that is analytic.

Example 4.2. Suppose we want to prove that $f(z) = (1 + z)^2$ is univalent in $\mathbb{D}$. A standard argument for that is to let $z_1, z_2 \in \mathbb{D}$ and suppose $f(z_1) = f(z_2)$. This means that

$$
\begin{aligned}
f(z_1) = f(z_2) &\Rightarrow (1+z_1)^2 = (1+z_2)^2 \\
&\Rightarrow 1 + 2z_1 + z_1^2 = 1 + 2z_2 + z_2^2 \\
&\Rightarrow z_1^2 - z_2^2 + 2(z_1 - z_2) = 0 \\
&\Rightarrow (z_1 - z_2)(z_1 + z_2 + 2) = 0.
\end{aligned}
$$

Since $|z_1|, |z_2| < 1$, we know that $z_1 + z_2 + 2 \neq 0$. Hence, we must have $z_1 - z_2 = 0$, or $z_1 = z_2$.

We can graph the image of $\mathbb{D}$ under the map $f(z) = (1+z)^2$ by using the accompanying applet *ComplexTool*. To do so, open *ComplexTool* (see Figure 4.1).

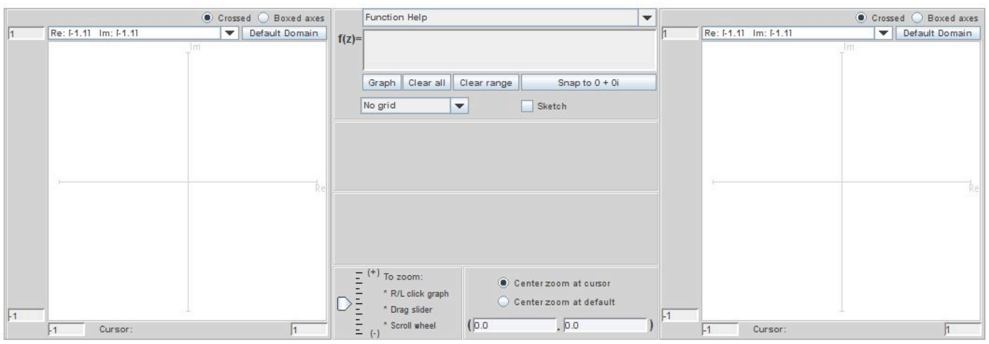


FIGURE 4.1. The applet *ComplexTool*

In the middle section near the top there is a box that has **f(z)=** before it. In this box, enter $(1+z) \wedge 2$. Below this, there is a window that states `No grid`. Click on the down arrow ▼ and choose the option `Circular grid`; an image of a circular grid should appear on the left. Next, click on the button `Graph` which is in the middle section below the function you entered earlier. The image of the circular grid should appear on the right. To reduce the size of the image, click on the down arrow ▼ above the image and chose a different size, such as `Re: [-3,3] Im: [-3,3]`. Also, you can move the axes so that the image is centered by positioning the cursor over the image, clicking on the mouse button, and dragging the image to the left (see Figure 4.2).

EXPLORATION 4.3. From Example 4.2, we know that $f(z) = (1+z)^2$ is univalent, while it can be shown that $f(z) = (1+z)^4$ is not univalent (see Exploration 4.4). Use *ComplexTool* to graph the image of $\mathbb{D}$ under the analytic function $f(z) = (1+z)^2$ and then under the analytic function $f(z) = (1+z)^4$. What aspects of these two images suggest that a function is univalent or is not univalent? Explore this idea by plotting the following further functions in *ComplexTool* and conjecture which of them

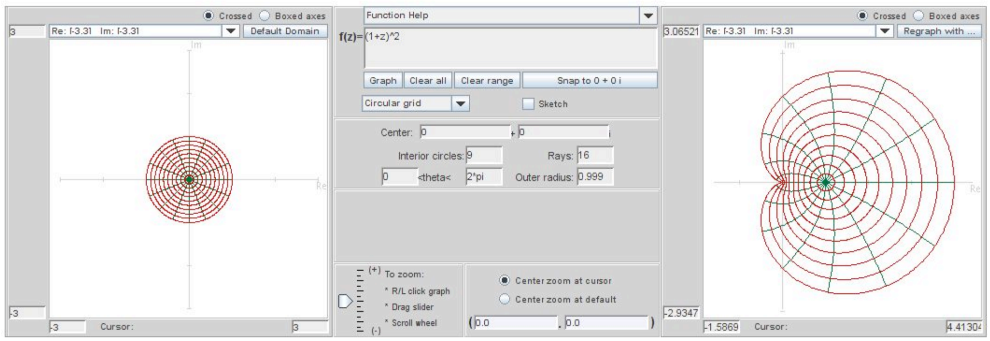


FIGURE 4.2. The image of the unit disk under the map $f(z) = (1+z)^2$.

is univalent:

(a) $g_1(z) = z - z^2$;

(b) $g_2(z) = z - \frac{1}{2}z^2$;

(c) $g_3(z) = 2z - z^2$;

(d) $g_4(z) = z + \frac{3}{4}z^2$;

(e) $g_5(z) = \frac{z}{1-z}$;

(f) $g_6(z) = \frac{z^2}{1-z}$;

(g) $g_7(z) = \frac{z}{(1-z)^2}$.

Try it out!

EXPLORATION 4.4. Prove that $f(z) = (1+z)^4$ is not univalent in $\mathbb{D}$.

One way to do this is to find two distinct points $z_1, z_2 \in \mathbb{D}$ such that $f(z_1) = f(z_2)$. You can use *ComplexTool* to help you find z_1 and z_2. Plot the image of $\mathbb{D}$ under the map $f(z) = (1+z)^4$. Check the **Sketch** box in the top middle section. The **Sketch** command allows you to draw a shape in the original domain on the left and see the image under the function of that shape on the right. For example, draw a line from along the imaginary axis from 0 to i and then draw a line from along the imaginary axis from 0 to $-i$; you should see that the two image curves meet at $f(i) = f(-i)$. Compute $f(i)$ and $f(-i)$ to prove that this is true. However, this does not prove that f is not univalent in $\mathbb{D}$ since $i, -i \notin \mathbb{D}$.

Use the **Sketch** feature of *ComplexTool* to help you find two points $z_1, z_2 \in \mathbb{D}$ such that $f(z_1) = f(z_2)$. To delete the shapes you have drawn, use the **Clear all** button and then regraph your image. [Hint: using *ComplexTool* find two distinct lines in the original domain that get mapped to the line on the real axis from -4 to 0 in the image domain; parametrize these two lines in such a way that for each t value, the image of these parametrized lines under f give the same image point.]

Try it out!

In Exploration 4.3, you may have noticed that the image of $\mathbb{D}$ under the function $g_2(z) = z - \frac{1}{2}z^2$ is similar to the image of $\mathbb{D}$ under the function $g_3(z) = 2z - z^2$. This is because $g_3 = 2g_2$. We want to avoid such repetitions. To do so, we will *normalize* all functions in the family of analytic, univalent functions. Suppose f_1 is univalent and analytic in $G \neq \mathbb{C}$. The Riemann Mapping Theorem can be stated in the following form:

THEOREM 4.5. **(Riemann Mapping Theorem)** Let $a \in G$. Then there exists a unique function $f_2 : G \to \mathbb{C}$ such that

(1) $f_2(a) = 0$ and $f_2'(a) > 0$;
(2) f_2 is univalent;
(3) $f_2(G) = \mathbb{D}$.

Thus $f_3 = f_1 \circ f_2^{-1} : \mathbb{D} \to f_1(G)$ with f_3 being univalent and analytic. So when studying mappings of simply-connected domains, we can simplify matters by letting $\mathbb{D}$ be our domain. Let $f_3 : \mathbb{D} \to \mathbb{C}$ be univalent and analytic. Since f_3 is analytic, it has a power series about the origin:

$$f_3(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$$

that is convergent in $\mathbb{D}$. Notice that adding a constant to f_3 merely translates the image domain and does not effect the univalency. Hence

$$f_4(z) = f_3(z) - \alpha_0 = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$$

is also univalent and analytic in $\mathbb{D}$. Next, note that $\alpha_1 \neq 0$ because f_4 being univalent implies $f_4'(z) \neq 0$ (for all $z \in \mathbb{D}$); but $f_4'(0) = \alpha_1$. So consider

$$f_5(z) = \frac{1}{\alpha_1}\, f_4(z) = z + \frac{\alpha_2}{\alpha_1}\, z^2 + \frac{\alpha_3}{\alpha_1}\, z^3 + \cdots .$$

Recall that multiplying f_4 by $\frac{1}{\alpha_1}$ merely rotates and/or stretches (or shrinks) the image domain. Hence f_5 is still univalent and analytic in $\mathbb{D}$. These steps have "normalized" our original function f_3.

DEFINITION 4.6. The family of analytic, normalized, univalent functions is denoted by S (from the German word "schlicht" which means "simple" or "plain"); that is,

$$S = \{f : \mathbb{D} \to \mathbb{C} \,|\, f \text{ is analytic and univalent with } f(0) = 0, f'(0) = 1\}.$$

Thus $f \in S$ implies $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.

EXERCISE 4.7. Show that $f(z) = z + a_2 z^2$ is univalent in $\mathbb{D} \Longleftrightarrow |a_2| \leq \frac{1}{2}$.
Try it out!

EXAMPLE 4.8. **(Polynomial map)**
Consider the function $g_2(z) = z - \frac{1}{2} z^2 \in S$. In Exploration 4.3, you graphed $g_2(\mathbb{D})$. While computer images are helpful, they can be misleading and sometimes inaccurate. So, it is important for us to be able to determine analytically such images. How can

we determine $g_2(\mathbb{D})$ analytically? Consider the image of the boundary of $\mathbb{D}$.

$$
\begin{aligned}
w = f(e^{i\theta}) =& e^{i\theta} - \frac{1}{2}e^{2i\theta} \\
=& (\cos\theta + i\sin\theta) - \frac{1}{2}(\cos 2\theta + i\sin 2\theta) \\
=& \Big(\cos\theta - \frac{1}{2}\cos 2\theta\Big) + i\Big(\sin\theta - \frac{1}{2}\sin 2\theta\Big) \\
=& u + iv
\end{aligned}
$$

Thus, $f(\partial\mathbb{D})$ is parametrized by

$$
\begin{aligned}
u(\theta) &= \cos\theta - \frac{1}{2}\cos 2\theta \\
v(\theta) &= \sin\theta - \frac{1}{2}\sin 2\theta.
\end{aligned}
$$

What is this image? It is a cardiod or an epicycloid with one cusp (see Figure 4.3).

Definition 4.9. An *epicycloid* is the path traced out by a point p on a circle of radius b rolling on the outside of a circle of radius a:

$$
\begin{aligned}
x(\theta) =& (a+b)\cos\theta - b\cos\left(\left(\frac{a}{b}+1\right)\theta\right) \\
y(\theta) =& (a+b)\sin\theta - b\sin\left(\left(\frac{a}{b}+1\right)\theta\right).
\end{aligned}
$$

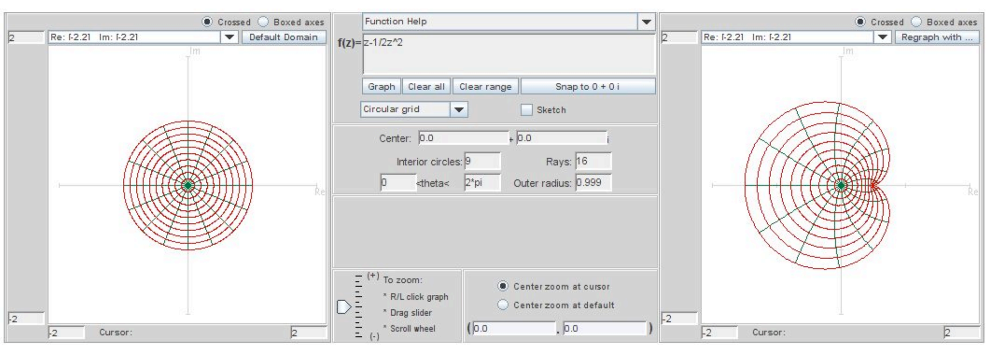


Figure 4.3. The image of the unit disk under the map $z - \frac{1}{2}z^2$.

Exploration 4.10. In Exercise 4.7, you showed that $f(z) = z + a_2z^2$ is univalent in $\mathbb{D} \iff |a_2| \leq \frac{1}{2}$. We want to make a conjecture about the generalization of this result. Use *ComplexTool* to graph $f(z) = z + a_3z^3$ for various values of a_3. What do you conjecture is the bound on a_3 for which f is univalent on $\mathbb{D}$? Do the same for $f(z) = z + a_4z^4$, $f(z) = z + a_5z^5$, etc. What do you conjecture is the bound on a_n

for which $f(z) = z + a_n z^n$ is univalent on $\mathbb{D}$? What do you conjecture $f(\mathbb{D})$ is when $a_n = -\frac{1}{n}$?

Try it out!

Let's look at determining $f(\mathbb{D})$ analytically for a few important examples that were included in Exploration 4.3.

Example 4.11. **(Right half-plane map)**
Consider the function

$$f_r(z) = \frac{z}{1-z} \in S.$$

Since $\dfrac{1}{1-z} = \sum\limits_{n=0}^{\infty} z^n$, we can multiply by z to get:

$$\frac{z}{1-z} = \sum_{n=1}^{\infty} z^n = z + z^2 + z^3 = \cdots .$$

Recall that this is the Möbius transformation that maps $\mathbb{D}$ onto the right half-plane whose boundary is the line $-\frac{1}{2} + ic$, where $c \in \mathbb{R}$ (see Figure 4.4).

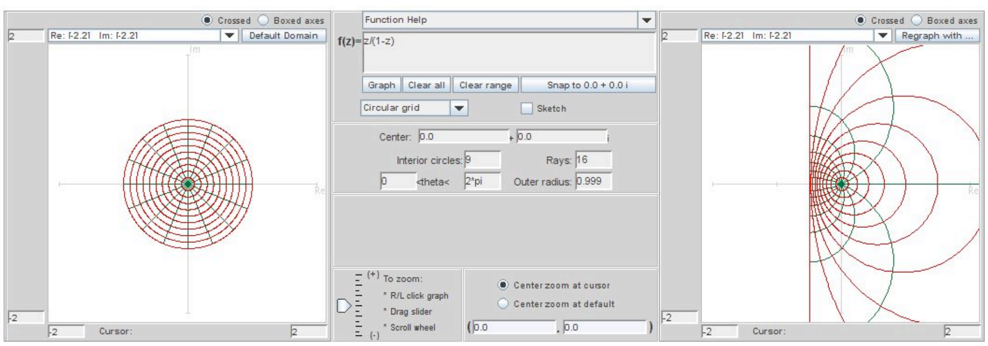


Figure 4.4. The image of the unit disk under the analytic right half-plane map in S.

Example 4.12. **(Koebe map)**
Next, consider the function

$$f_k(z) = \frac{z}{(1-z)^2} \in S.$$

We can compute the power series for f_k by differentiation the series for $\frac{1}{1-z}$ and then multiplying by z:

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots .$$

Notice, for this function, $a_n = n$ for all n. We will now show that the image of $\mathbb{D}$ under f_k is a slit domain; that is, a domain consisting of the entire complex plane except that a slit is cut out of it. To determine $f_k(\mathbb{D})$, consider the following sequence of maps:

$$u_1(z) = \frac{1+z}{1-z}, \quad u_2(z) = u_1^2(z), \quad u_3(z) = \frac{1}{4}\left[u_2(z) - 1\right].$$

Now,

$$u_3 \circ u_2 \circ u_1(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \frac{z}{(1-z)^2}.$$

Note that u_1 is the Möbius transformation that maps $\mathbb{D}$ onto the right half-plane whose boundary is the imaginary axis. Also, u_2 is the squaring function, while u_3 translates the image one space to the left and then multiplies it by a factor of $\frac{1}{4}$.

Thus the image $\mathbb{D}$ is the entire complex plane except for a slit along the negative real axis from $w = \infty$ to $w = -\frac{1}{4}$ (see Figure 4.5). The function $f_k(z) = \frac{z}{(1-z)^2}$ is known as the Koebe function.

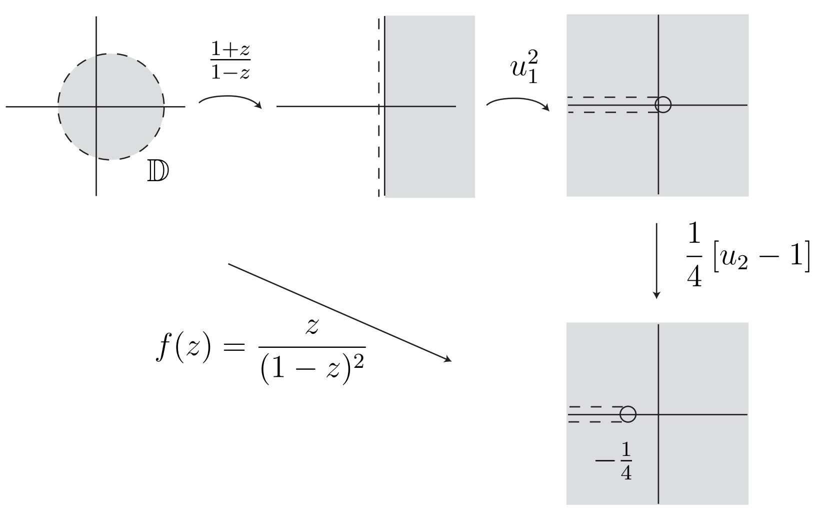


FIGURE 4.5. The image of the unit disk under the Koebe function.

EXPLORATION 4.13. It is difficult to interpret the image of $\mathbb{D}$ under the Koebe function using *ComplexTool*. One way to help understand the image is to use increasing values in the **Outer radius** box in the center panel of the applet. Graph $\mathbb{D}$ under the map $\frac{z}{(1-z)^2}$ several times using the values of 0.8, 0.85, 0.9, 0.95, and 0.999 in the **Outer radius** box.

Try it out!

Suppose we have an analytic function f with a Taylor series representation $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. One question to ask is for what values of a_n is f in the family of schlicht functions? Consider the case in which all $a_n = 0$ except possibly a_2. So, $f = z + a_2z^2$. By Exercise 4.7, we know that if $|a_2| \leq \frac{1}{2}$, then $f \in S$. The function $f(z) = z - \frac{1}{2}z^2$ from Example 4.8 is an extremal function. An extremal function is a function that is on the boundary between those that satisfy a condition and those that do not. Here, $f(z) = z - \frac{1}{2}z^2$ is extremal, because if we increase $|a_2| = \left| -\frac{1}{2}\right|$ then $f(z) = a + a_2z^2$ is no longer schlicht. In general, how large can $|a_n|$ be and f still be schlicht? Recall, for the Koebe function,

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n,$$

and so in this case we have that $a_n = n$. This lead Bieberbach to make his famous conjecture in 1916.

Bieberbach Conjecture For $f \in S$, $|a_n| \leq n$, for all n. In particular, $|a_2| \leq 2$.

Because the image of $\mathbb{D}$ under the Koebe function covers all of $\mathbb{C}$ except a slit along the real axis, it seems plausible that the the Bieberbach Conjecture is true with the Koebe function being extremal. It is true. However, it was not until 1984 that deBranges proved it.

We say an inequality is *sharp* if it is impossible to improve it (that is, we cannot decrease the upper bound or increase the lower bound). We can show that an inequality is sharp by finding a function with the desired properties and for which the inequality becomes equality. Such a function for which equality holds is known as an *extremal* function. Note that for deBranges Theorem (or Bieberbach Conjecture), the Koebe function is extremal.

There is another case in which the Koebe function is extremal. We know that if $f \in S$, then $f(\mathbb{D})$ is not the entire complex plane. That is, there is some point $a \in \mathbb{C}$ such that $a \notin f(\mathbb{D})$. This leads to the question of how small can $|a|$ be? For example, if $f(z) = z$, then $|a| = 1$; if $f(z) = \frac{z}{1-z}$, the right half-plane mapping in Example 4.11, then $|a| = \frac{1}{2}$. The answer to the question is that for all $f \in S$, $|a| \geq \frac{1}{4}$. This is known as the Koebe $\frac{1}{4}$ Theorem. The Koebe function is extremal in this case, because $|a| = \frac{1}{4}$ for the Koebe function. For these reasons and others, the Koebe function is very important in the study of schlicht functions.

Exploration 4.14. Consider the function $f(z) = \dfrac{z - tz^2}{(1-z)^2}$, where $0 \leq t \leq 1$. From what you have read so far, what is $f(\mathbb{D})$ when $t = 0$? What is $f(\mathbb{D})$ when $t = 1$? Using your answers to these two questions and not using *ComplexTool* yet, make a conjecture of what $f(\mathbb{D})$ is, when $0 < t < 1$. Now, use *ComplexTool* to modify or strengthen your conjecture. Using *ComplexTool*, what happens to $f(\mathbb{D})$ for $t > 1$? For $t = id, 0 \leq d \leq 1$?

Try it out!

The family S has been studied extensively for many years. Here are a few facts about normalized, analytic univalent functions that will be used later in our discussion:

(1) (uniqueness in the Riemann Mapping Theorem) Let $G \neq \mathbb{C}$ be a specific simply-connected domain with $a \in \mathbb{D}$. Because of the Riemann Mapping Theorem, the map $f \in S$ that maps $\mathbb{D}$ onto G with $f(0) = a$ and $f'(0) > 0$ is unique.

(2) (deBranges Theorem) For $f \in S$, $|a_n| \leq n$, for all n.

(3) (Koebe $\frac{1}{4}$ Theorem) The range of every function in class S contains the disk $G = \{w : |w| < \frac{1}{4}\}$

Remark: This is a consequence of the fact that $|a_2| \leq 2$ which was proved by Bieberbach in 1916.

(4) Let $f \in S$. Then $f(\mathbb{D})$ omits some value on each circle $\{w : |w| = R\}$ where $R \geq 1$. In other words, there is no function $f \in S$ for which $f(\mathbb{D})$ contains $\partial\mathbb{D}$, the unit circle.

4.3. The Family S_H of Normalized, Harmonic, Univalent Functions

About the same time that deBranges proved the Bierbach Conjecture, Clunie and Sheil-Small studied a family, S_H, of complex-valued harmonic functions that contained S as a proper subset and considered some of the same properties on S_H that had been investigated in S.

Recall that function $\phi(x, y)$ is harmonic if and only if $\phi_{xx} + \phi_{yy} = 0$.

Definition 4.15. A continuous function $f = u + iv$ defined in G is a *complex-valued, harmonic function* in G if u and v are real harmonic (but not necessarily harmonic conjugates) in G.

Example 4.16. The function

$$f(x, y) = u(x, y) + iv(x, y) = (x^2 - y^2) + i2xy$$

is complex-valued, harmonic because

$$\begin{aligned} u_{xx} + u_{yy} &= 2 - 2 = 0 \\ v_{xx} + v_{yy} &= 0 + 0 = 0. \end{aligned}$$

Exercise 4.17. Show that

$$f(x, y) = u(x, y) + iv(x, y) = \left(x + \frac{1}{2}x^2 - \frac{1}{2}y^2\right) + i(y - xy)$$

is complex-valued harmonic.

Try it out!

Although harmonic functions are more general than analytic functions, some familiar theorems about analytic functions have an equivalent form for harmonic functions. These include the mean-value theorem, the maximum-modulus theorem, Liouville's Theorem, and the Argument Principle. However, by considering all harmonic functions instead of just the subclass of analytic functions we can sometimes get more information as is in the case in the study of minimal surfaces.

One way of thinking of a function $f(x, y) = u(x, y) + iv(x, y)$ as being analytic is that f can be expressed solely in terms of $z = x + iy$ without using $\overline{z} = x - iy$. Hence, the function $f = z^2$ is analytic while $f = z\overline{z}$ is not. To explore this idea, let's say that $\zeta := z = x + iy$ and $\xi := \overline{z} = x - iy$. Then, we can "formally" write $x = \frac{1}{2}(\zeta + \xi)$ and $y = \frac{1}{2i}(\zeta - \xi)$. Using the chain rule with the function $f(x(\zeta, \xi), y(\zeta, \xi))$ and since $\zeta = z$ and $\xi = \overline{z}$, we can show that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{45}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{46}$$

Exercise 4.18.

(a) Derive eqs (45) and (46).

(b) Use these equations and the Cauchy-Riemann equations to prove that $f(x, y) = u(x, y) + iv(x, y)$ is analytic $\iff \dfrac{\partial f}{\partial \overline{z}} = 0$.

Try it out!

Exercise 4.19.

(a) Using $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$, rewrite

$$f(x, y) = u(x, y) + iv(x, y) = \left(x + \frac{1}{2}x^2 - \frac{1}{2}y^2\right) + i(y - xy)$$

in terms of z and $\overline{z}$.

(b) Use Exercise 4.18 to determine if f is analytic.

(c) Show that all analytic functions are complex-valued harmonic, but not all complex-valued harmonic functions are analytic.

Try it out!

The next theorem tells us that a complex-valued harmonic function defined on $\mathbb{D}$ is related to analytic functions.

Theorem 4.20. If $f = u + iv$ is harmonic in a simply-connected domain G, then $f = h + \overline{g}$, where h and g are analytic.

PROOF. Recall that if u and v are real harmonic on a simply-connected domain, then there exists analytic functions K and L such that $u = \operatorname{Re} K$ and $v = \operatorname{Im} L$. Hence,

$$f = u + iv = \operatorname{Re} K + i \operatorname{Im} L = \frac{K + \overline{K}}{2} + i\frac{L - \overline{L}}{2i} = \frac{K + L}{2} + \frac{\overline{K - L}}{2} = h + \overline{g}.$$

□

EXERCISE 4.21. Let

$$f(x, y) = u(x, y) + iv(x, y) = \left(x + \frac{1}{2}x^2 - \frac{1}{2}y^2\right) + i(y - xy)$$

be defined on $\mathbb{D}$. In Exercise 4.17 you showed that f is harmonic. Find analytic functions h and g such that $f = h + \overline{g}$.

Try it out!

We can use the applet *ComplexTool* to graph complex-valued harmonic functions. For example, to graph the image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{2}\overline{z}^2$, enter this function in *ComplexTool* in the form $z + 1/2$ `conj` $(z \wedge 2)$ (see Figure 4.6).

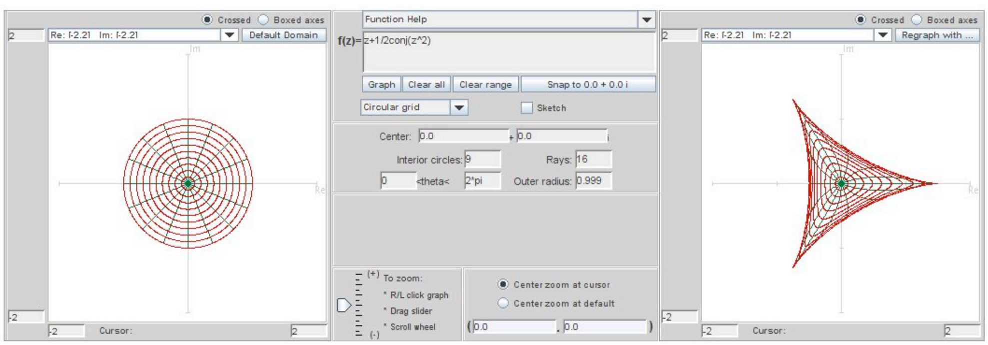


FIGURE 4.6. Image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{2}\overline{z}^2$

Note that the harmonic function $f(z) = h(z) + \overline{g}(z)$ can also be written in the form

$$f(z) = \operatorname{Re}\left\{h(z) + g(z)\right\} + i \operatorname{Im}\left\{h(z) - g(z)\right\}. \tag{47}$$

Hence, in the previous example, $f(z) = z + \frac{1}{2}\overline{z}^2$ can also be written as $f(z) = \operatorname{Re}\left\{z + \frac{1}{2}z^2\right\} + i \operatorname{Im}\left\{z - \frac{1}{2}z^2\right\}$. In *ComplexTool* you can also enter the harmonic function in this form. To do so, you would type in `re`$(z + 1/2z \wedge 2)$+ `i*im` $(z - 1/2z \wedge 2)$.

EXERCISE 4.22. Prove that the representations $f(z) = h(z) + \overline{g}(z)$ and $f(z) = \operatorname{Re}\left\{h(z) + g(z)\right\} + i \operatorname{Im}\left\{h(z) - g(z)\right\}$ are equivalent.

Try it out!

Exploration 4.23. Graph the image of $\mathbb{D}$ under the following harmonic maps. Describe characteristics that appear to be different for harmonic mappings as compare to analytic mappings.

(a) $f_1(z) = z + \frac{1}{3}\overline{z}^3$;

(b) $f_2(z) = \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right)$;

(c) $f_3(z) = \frac{z}{1-z} - \frac{1}{2}e^{\frac{\overline{z}+1}{\overline{z}-1}}$;

(d) $f_4(z) = \operatorname{Re}\left(\frac{i}{\sqrt{3}}\ln\left(\frac{1+e^{-i\frac{\pi}{3}}z}{1+e^{i\frac{\pi}{3}}z}\right)\right) + i\operatorname{Im}\left(\frac{1}{3}\ln\left(\frac{1+z+z^2}{1-2z+z^2}\right)\right)$;

(e) $f_5(z) = z + 2\ln(z+1) + (\overline{z}+1)e^{\frac{\overline{z}-1}{\overline{z}+1}}$.

Try it out!

Since $f = h + \overline{g}$, where h and g are analytic, f has the following series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n.$$

Hence, we normalize the harmonic univalent functions in a way similar to the normalized analytic univalent functions.

Definition 4.24. Let S_H be the family of complex-valued harmonic, univalent mappings that are normalized on the unit disk; that is,

$$S_H = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is harmonic, univalent with } f(0) = a_0 = 0, f_z(0) = a_1 = 1\}.$$

$$S_H^O = \{f \in S_H \mid f_{\overline{z}}(0) = b_1 = 0\}.$$

Thus, $S \subset S_H^O \subset S_H$.

Let's look at some significant examples.

Example 4.25. **Harmonic polynomial map**
Consider

$$f(z) = h(z) + \overline{g(z)} = z + \frac{1}{2}\overline{z}^2.$$

In the next section we will prove that f is univalent and hence in S_H^O. But for now we will assume this and look at the image of $\mathbb{D}$ under f (see Figure 4.6). f maps $\mathbb{D}$ onto the interior of the region bounded by a hypocycloid with 3 cusps.

Definition 4.26. An *hypocycloid* is the curve produced by a fixed point p on a small circle of radius b rolling the inside of a larger circle of radius a:

$$x(\theta) = (a-b)\cos\theta + b\cos\left(\left(\frac{a}{b}-1\right)\theta\right)$$
$$y(\theta) = (a-b)\sin\theta + b\sin\left(\left(\frac{a}{b}-1\right)\theta\right).$$

In the next section, we will see that the function f is related to the function $F(z) = z - 1/2z^2$ which maps $\mathbb{D}$ to an epicycloid (see Example 4.8).

Exploration 4.27.

(a) Use *ComplexTool* to plot the image of $\mathbb{D}$ under the analytic polynomial map $F(z) = z - 1/2e^{it\frac{\pi}{6}}z^2$ for $t = 0, 1, 2, 3, 4, 5, 6$. Describe what happens to the image as t varies.
(b) Use *ComplexTool* to plot the image of $\mathbb{D}$ under the harmonic polynomial map $f(z) = z + 1/2e^{it\frac{\pi}{6}}\overline{z}^2$ for $t = 0, 1, 2, 3, 4, 5, 6$. Describe what happens to the image as t varies.
(c) What differences do you notice between the images in (a) and (b) as t increases? Explain why it is reasonable for this difference to occur.

Try it out!

Small Project 4.28.

(a) Use *ComplexTool* to plot the images of the following polynomials:

Harmonic Functions	Analytic Functions
(i) $f_1(z) = z + \frac{1}{3}\overline{z}^3$;	(ii) $F_1(z) = z + \frac{1}{3}z^3$;
(iii) $f_2(z) = z + \frac{1}{4}z^2 + \frac{1}{4}\overline{z}^2 + \frac{1}{3}\overline{z}^3$;	(iv) $F_2(z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3$;
(v) $f_3(z) = z + \frac{1}{6}\overline{z}^2 + \frac{1}{6}\overline{z}^4$;	(vi) $F_3(z) = z + \frac{1}{6}z^6$;
(vii) $f_4(z) = z + \frac{1}{6}z^2 + \frac{1}{6}\overline{z}^4$.	

(b) Write a list of similarities and differences between the images of $\mathbb{D}$ under harmonic and the analytic functions.
(c) Some questions to consider are: If the polynomial has three or more terms then how large can their coefficients be in modulus to guarantee univalency? If the polynomial has three terms, what difference does it make if the last two terms are $\overline{z}^2$ and $\overline{z}^3$ instead of z^2 and $\overline{z}^3$?
(d) Plot your own examples of harmonic and analytic polynomials and see if the properties in your list from (b) are still valid.

Optional

EXAMPLE 4.29. **Harmonic right half-plane map**
Consider

$$f(z) = h(z) + \overline{g(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \frac{\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2}$$
$$= \operatorname{Re}(h(z) - g(z)) + i\operatorname{Im}(h(z) - g(z)) = \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right).$$

We will prove that f is univalent in the next section. The image of $\mathbb{D}$ under f using *ComplexTool* is shown in Figure 4.7.

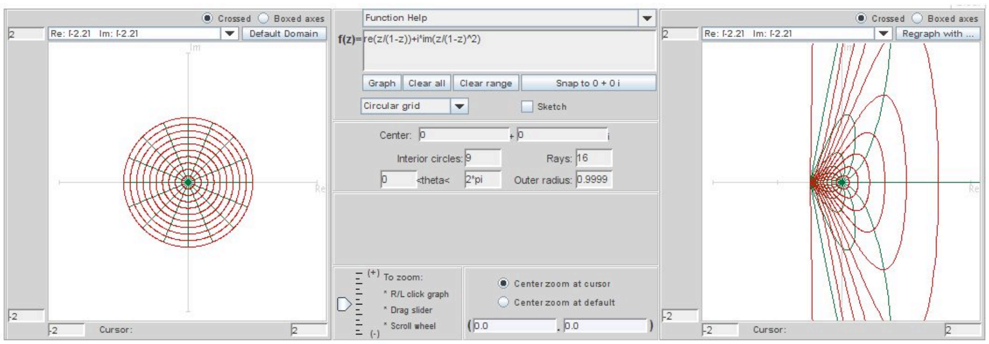


FIGURE 4.7. Image of $\mathbb{D}$ under $f(z) = \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right)$

It turns out that the image of $\mathbb{D}$ under this harmonic map is the right half-plane map $\{w \in \mathbb{C} \,|\, \operatorname{Re}\{w\} \geq -\frac{1}{2}\}$. This is the same domain as the image of $\mathbb{D}$ under the analytic map $\frac{z}{1-z}$ although the boundary behavior is different.

EXPLORATION 4.30.

(a) Use *ComplexTool* to plot $\mathbb{D}$ under the analytic right half-plane map $\frac{z}{1-z}$. Use the **Sketch** box to draw radial lines from the origin to the boundary of $\mathbb{D}$ in the original domain. What are the images of points on the unit circle under this analytic map?

(b) Use *ComplexTool* to plot $\mathbb{D}$ under the harmonic right half-plane map $\frac{z-\frac{1}{2}z^2}{(1-z)^2} - \frac{\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2}$. Use the **Sketch** box to draw radial lines from the origin to the boundary of $\mathbb{D}$ in the original domain. What is the image of points on the unit circle under this analytic map?

(c) Using (a) and (b), describe how the boundary behavior is different between the analytic right half-plane map and this harmonic right half-plane map?

Try it out!

We are interested in harmonic univalent functions. However, results about univalent functions are difficult to obtain. So it is sometimes useful to just consider locally univalent functions as opposed to globally univalent. Let's look at the idea of local univalence for a moment.

DEFINITION 4.31. A function f is *locally univalent* on G if $J_f \neq 0$ on G, where J_f is the Jacobian of $f = u + iv$:

$$\begin{aligned} J_f &= \det \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \det \begin{vmatrix} (\text{Re } h)_x + (\text{Re } g)_x & (\text{Re } h)_y + (\text{Re } g)_y \\ (\text{Im } h)_x - (\text{Im } g)_x & (\text{Im } h)_y - (\text{Im } g)_y \end{vmatrix}. \end{aligned}$$

For analytic functions F, the Cauchy-Riemann equations yield $(\text{Re } F)_y = -(\text{Im } F)_x$ and $(\text{Im } F)_y = (\text{Re } F)_x$. Hence we have

$$\begin{aligned} &= \det \begin{vmatrix} (\text{Re } h)_x + (\text{Re } g)_x & -(\text{Im } h)_x - (\text{Im } g)_x \\ (\text{Im } h)_x - \text{Im } g)_x & \text{Re } h_x - (\text{Re } g)_x \end{vmatrix} \\ &= (\text{Re } h)_x^2 - (\text{Re } g)_x^2 + (\text{Im } h)_x^2 - (\text{Im } h)_x^2 \\ &= |h'|^2 - |g'|^2. \end{aligned}$$

Thus, we want $|h'|^2 - |g'|^2 \neq 0$.

Besides local univalence, another important property of these functions is sense-preserving. What is sense-preserving? A continuous function f is *sense-preserving* or *orientation-preserving* if it preserves orientation. Consider the following example. Let f_1, f_2 be defined on the punctured disk $\mathbb{D} - \{0\}$ by

$$f_1(z) = \frac{1}{z} \qquad \text{and} \qquad f_2(z) = \overline{z}.$$

Both functions map the unit circle, $\partial\mathbb{D}$, onto itself. In particular, both functions map the points $A = 1$, $B = e^{i\frac{\pi}{4}}$, and $C = i$ to the points $A' = 1$, $B' = e^{-i\frac{\pi}{4}}$, and $C' = -i$, respectively. As we travel along $\partial\mathbb{D}$ in a counterclockwise direction, the left hand side domain (LHS) is in $\mathbb{D}$, while the right hand side domain (RHS) is in $\mathbb{C} - \mathbb{D}$. Where do these domains get mapped under these functions? Note that as we travel along $\partial\mathbb{D}$ in a counterclockwise direction, the image curve under both functions will be $\partial\mathbb{D}$ traversed in a clockwise direction. So, in the image domain, $\mathbb{D}$ is now RHS while $\mathbb{C} - \mathbb{D}$ is LHS. Now, f_1 maps the point $\frac{1}{2} \in \mathbb{D}$ to $2 \in \mathbb{C} - \mathbb{D}$ and so f_1 maps the LHS onto the LHS. Functions that map the LHS onto the LHS are sense-preserving. On the other hand, f_2 maps the point $\frac{1}{2} \in \mathbb{D}$ to $\frac{1}{2} \in \mathbb{D}$ and so f_1 maps the LHS onto the RHS. Functions that map the LHS onto the RHS are sense-reversing.

More exactly, as we travel counterclockwise along any simple closed contour $\gamma \in G$, there exists a left hand side domain (LHS) and a right hand side domain (RHS).

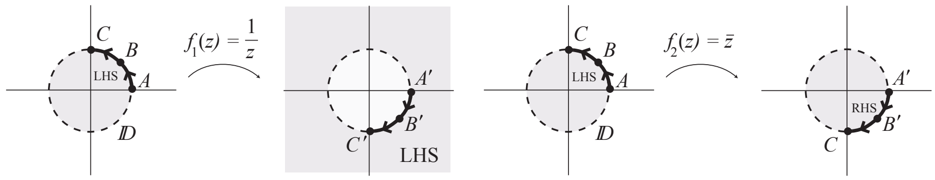


FIGURE 4.8. f_1 is sense-preserving, while f_2 is sense-reversing.

Consider the image curve $f(\gamma)$. f is *sense-preserving* if the original LHS domain with regard to γ is mapped to the LHS domain with regard to $f(\gamma)$. f is *sense-reversing* if the original LHS domain with regard to γ is mapped to the RHS domain with regard to $f(\gamma)$. Note that analytic functions are sense-preserving; if f is sense-preserving, then $\overline{f}$ is sense-reserving.

EXPLORATION 4.32. For each of the following harmonic functions, use *ComplexTool* to make a conjecture if the function is: (a) locally univalent; and (b) sense-preserving [Hint: for sense-preserving, you can use the Sketch feature to draw a counterclockwise curve on the unit circle and see its image under the function]

(a) $z + 2\overline{z}$;	(b) $z + \frac{1}{2}\overline{z}$
(c) $z + 2\overline{z}^2$;	(d) $z + \frac{1}{2}\overline{z}$
(e) $2z^2 + \overline{z}$;	(f) $\frac{1}{2}z^2 + \overline{z}$

Try it out!

Now, we will need the following important definition.

DEFINITION 4.33. $\omega(z) = g'(z)/h'(z)$ is known as the *dilatation* of f.

There is a connection between the dilatation of a harmonic function and its locally univalence and sense-preserving nature.

THEOREM 4.34 (Lewy). $f = h + \ \overline{g}$ is locally univalent and sense-preserving $\iff$ $|\omega(z)| < 1$, for all $z \in G$.

EXERCISE 4.35. Compute $\omega(z)$ for each of the functions in Exploration 4.32. Then use your results from Exploration 4.32 to verify that Lewy's Theorem for the functions in that Exploration.
Try it out!

EXERCISE 4.36. Show that $|\omega(z)| < 1$ for:

(a) $\omega_1(z) = e^{i\theta}z$, where $\theta \in \mathbb{R}$;

(b) $\omega_2(z) = z^n$, where $n = 1, 2, 3, \ldots$;

(c) $\omega_3(z) = \dfrac{z + a}{1 + \overline{a}z}$, where $|a| < 1$;

(d) $\omega_4(z)$ being the composition of any of the functions ω above.

Try it out!

Remark 4.37. Let $f = h + \overline{g}$ be a sense-preserving harmonic map that has $|\omega(z)| = 1$ for all $z \in$ arc of $\partial\mathbb{D}$. Then the image of the arc is either:

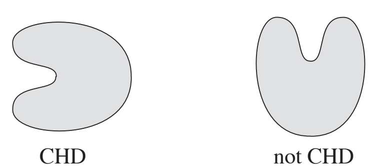
- a concave arc (see Example 4.25); or
- stationary (see Example 4.29).

We discuss the dilatation more in Section 4.5.

4.4. The Shearing Technique

Finding examples of univalent harmonic mappings that are not analytic is not easy. One very useful way to construct new examples of univalent harmonic mappings was provided by Clunie and Sheil-Small. It is known as the shearing technique. Before we proceed, we need to discuss certain types of domains.

Definition 4.38. A domain Ω is convex in the direction $e^{i\varphi}$ if for every $a \in \mathbb{C}$ the set $\Omega \cap \{a + te^{i\varphi} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain is convex in the direction of the real axis (CHD) if every line parallel to the real axis has a connected intersection with Ω.



Exercise 4.39. For which values of $n = 1, 2, 3, \ldots$ do the following functions map $\mathbb{D}$ onto a CHD domain?

(1) $f(z) = z^n$,
(2) $f(z) = z - \frac{1}{n}z^n$ (see Example 4.8 and Definition 4.9),
(3) $f(z) = \frac{z}{(1-z)^n}$ (see Examples 4.11 and 4.12 to get you started).

Try it out!

Theorem 4.40 (Clunie and Sheil-Small). A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain $\Longleftrightarrow$ $h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain.

Before we prove Theorem 4.40, let's look at an example.

Example 4.41. **Harmonic polynomial map**
In Example 4.25, we claimed that the harmonic polynomial $f(z) = z + \frac{1}{2}\overline{z}^2$ is univalent and is related to the analytic function $F(z) = z - \frac{1}{2}z^2$. We can use Theorem 4.40 to show this. The analytic univalent function

$$F(z) = h(z) - g(z) = z - \frac{1}{2}z^2$$

maps $\partial\mathbb{D}$ to an epicycloid with 1 cusp (see Example 4.8) that is convex in the direction of the real axis. Letting $\omega(z) = g'(z)/h'(z) = z$, we can apply the Shearing Method and solve for h and g:

$$\begin{aligned} h'(z) - g'(z) = 1 - z &\Rightarrow h'(z) - zh'(z) = 1 - z \\ &\Rightarrow h'(z) = 1 \\ &\Rightarrow h(z) = z. \end{aligned}$$

Since $g'(z) = zh'(z) = z$, we also have $g(z) = \frac{1}{2}z^2$. Notice that h and g are normalized; that is, $h(0) = 0$ and $g(0) = 0$. So, the corresponding harmonic univalent function is

$$f(z) = h(z) + \overline{g(z)} = z + \frac{1}{2}\overline{z}^2 \in S_H^O$$

which was derived from shearing the analytic univalent function

$$F(z) = z - \frac{1}{2}z^2 \in S$$

with the dilatation $\omega(z) = z$.

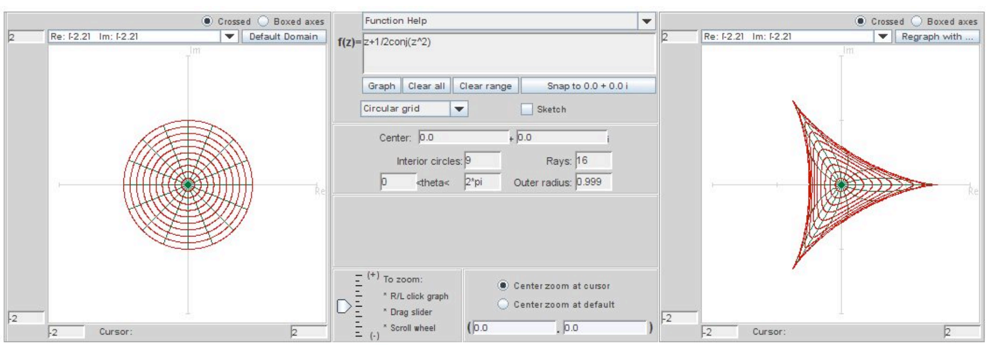


Figure 4.9. Image of $\mathbb{D}$ under the harmonic map $f(z) = z + \frac{1}{2}\overline{z}^2$

Remark 4.42. This technique is known as the "shear" method or "shearing" a function. The word *shear* means to cut (as in "shearing a sheep to get its wool"). In our situation, suppose $F = h - g$ is an analytic univalent function convex in the real direction. Then the corresponding harmonic shear is

$$f = h + \overline{g} = h - g + g + \overline{g} = h - g + 2\,\mathrm{Re}\{g\}.$$

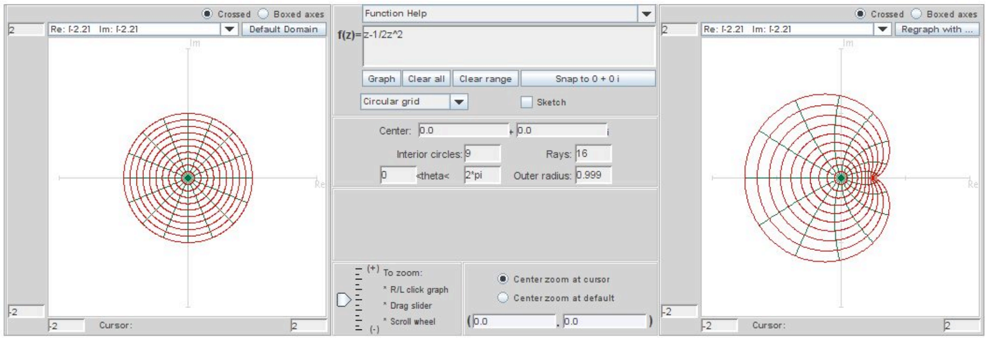


FIGURE 4.10. Image of $\mathbb{D}$ under the analytic map $F(z) = z - \frac{1}{2}z^2$.

So, the harmonic shear differs from the analytic function by adding a real function to it. Geometrically, you can think of this as taking F, the original analytic univalent function convex in the real direction, and cutting it up into thin horizontal slices which are then translated and/or scaled in a continuous way to form the corresponding harmonic function, f (see Figure 4.11). This is why the method is called "shearing."

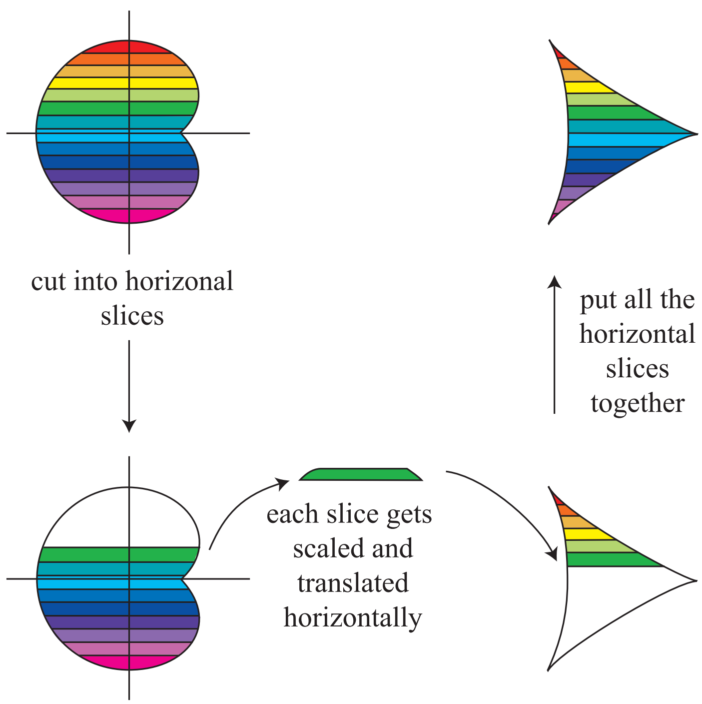


FIGURE 4.11. Shearing an analytic function to preserve univalency

Since F is univalent and convex in the real direction and we are only adding a continuous real function to it, the univalency is preserved.

EXERCISE 4.43. Let $f = h + \overline{g}$ with $h(z) - g(z) = z$ and $\omega(z) = z$. Compute h and g explicitly so that $f \in S_H^O$ and use *ComplexTool* to sketch $f(\mathbb{D})$.

EXERCISE 4.44. Let $f = h + \overline{g}$ with $h(z) - g(z) = z + \frac{1}{3}z^3$ and $\omega(z) = z^2$. Compute h and g explicitly so that $f \in S_H^O$ and use *ComplexTool* to sketch $f(\mathbb{D})$.

For exploring shears of functions, the applet *ShearTool* (see Figure 4.12) has an advantage over *ComplexTool*, because it easily allows you to see the image of $\mathbb{D}$ under a shear of $h(z) - g(z)$ without having to explicitly compute the resulting harmonic function $f = h + \overline{g}$. When using *ShearTool*, enter an analytic function that is convex in the direction of the real axis in **h-g=** box in the upper left section and enter the dilatation in the ω box below it. The click on the **Graph** button.

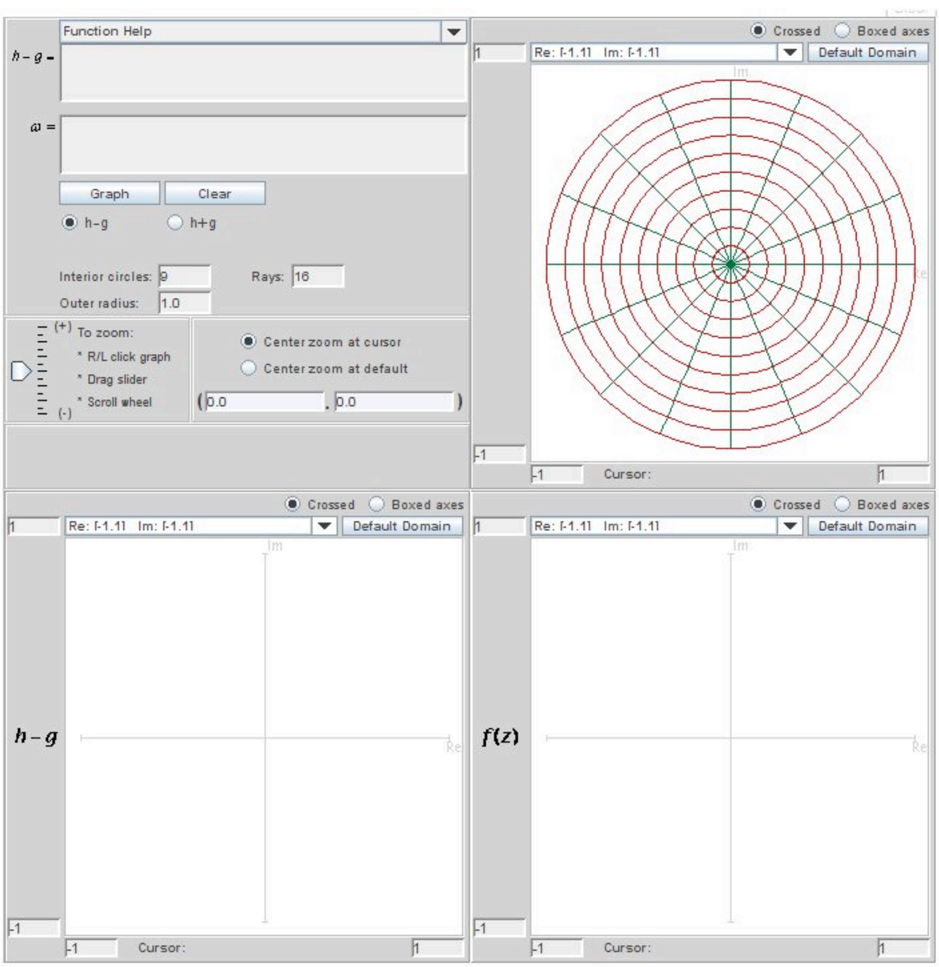


FIGURE 4.12. The applet *ShearTool*

In Example 4.41, we sheared $h(z) - g(z) = z - \frac{1}{2}z^2$ with $\omega(z) = z$. Entering these two functions into *ShearTool* we get the image of a hypocycloid with 3 cusps (see Figure 4.13).

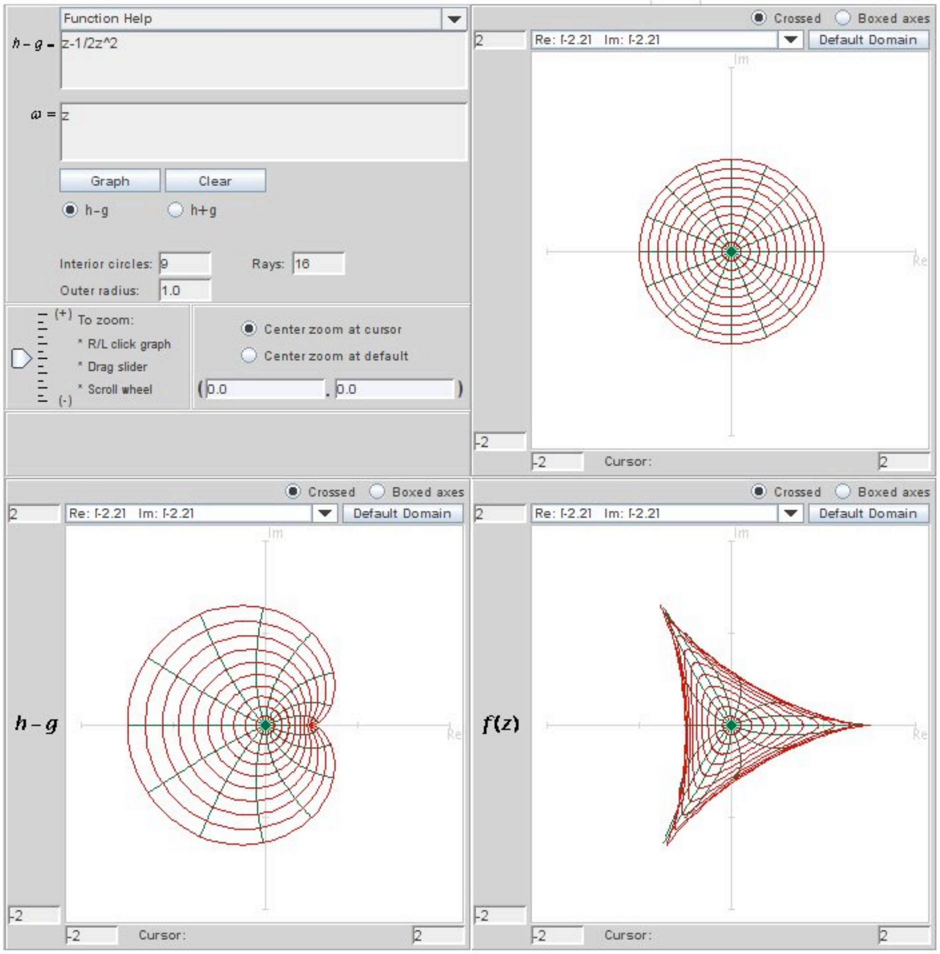


FIGURE 4.13. The image of $\mathbb{D}$ when shearing $h(z) - g(z) = z - \frac{1}{2}z^2$ with $\omega(z) = z$.

EXPLORATION 4.45.

(a.) Use *ShearTool* to graph the image of $\mathbb{D}$ under $f = h + \overline{g}$ where $h(z) - g(z) = z$ and $\omega(z) = -z$. Note the slight difference between this and Exercise 4.43.

(b.) Use *ShearTool* to graph the image of $\mathbb{D}$ under $f = h + \overline{g}$ where $h(z) - g(z) = z - \frac{1}{3}z^3$ and $\omega(z) = z^2$. Note the slight difference between this and Exercise 4.44.

Try it out!

To prove Theorem 4.40, we will use the following lemma

LEMMA 4.46. Let $\Omega \subset \mathbb{C}$ be a CHD domain and let ρ be a real-valued continuous function on Ω. Then the map $\omega \to \omega + \rho(\omega)$ is one-to-one in $\Omega \iff$ it is locally one-to-one. If it is one-to-one, then its range is a CHD domain.

PROOF OF LEMMA. ($\Rightarrow$) Trivial.

($\Leftarrow$) Suppose it is not $1{-}1$. That is, there are distinct points $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 + \rho(\omega_1) = \omega_2 + \rho(\omega_2)$. Let $\omega_k = u_k + iv_k, (k = 1, 2)$. Since ρ is real-valued

$$v_1 = \operatorname{Im}(\omega_1 + \rho(\omega_1)) = \operatorname{Im}(\omega_2 + \rho(\omega_2)) = v_2.$$

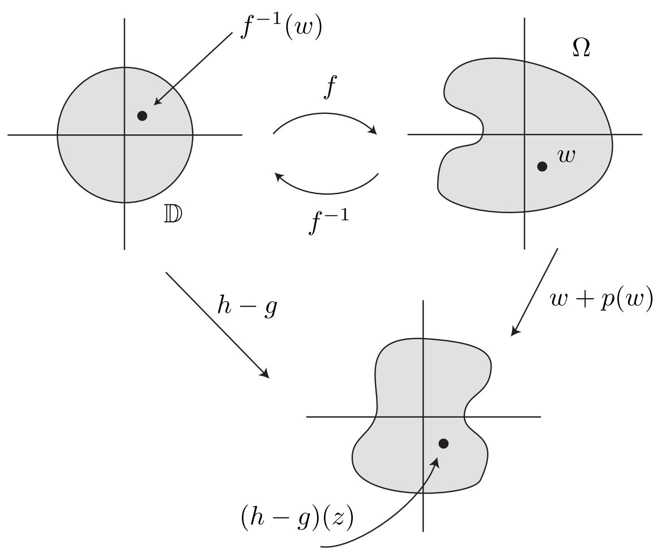
So $v_1 = v_2 = c$. Then the map $u \to u + \rho(u + ic)$ is not strictly monotonic, because there exists distinct points $u_1, u_2 \in \mathbb{R}$ such that their images are equal. Hence, this map is not locally $1{-}1$, and consequently, $\omega \to \omega + \rho(\omega)$ is not locally $1{-}1$.

Geometrically, the mapping acts as a "shear" in the horizontal direction and hence its range is CHD. $\square$

PROOF OF THEOREM 4.40. ($\Rightarrow$) Assume $f = h + \overline{g}$ is $1{-}1$ and $\Omega = f(\mathbb{D})$ is CHD. Note that $f = h - g + g + \overline{g} = h - g + 2\operatorname{Re}\{g\}$. Then the function

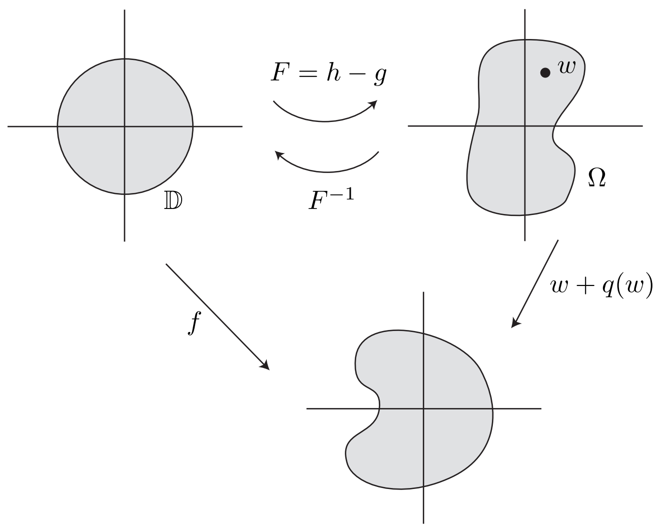
$$(h - g) \circ f^{-1}(w) = (f - 2\operatorname{Re}\{g\}) \circ f^{-1}(w) = w - 2\operatorname{Re}\{g(f^{-1}(w))\} = w + p(w)$$

may be defined in Ω, where p is real-valued and continuous. Since f is locally $1{-}1$, $|g'| < |h'| \iff g'(z) \neq h'(z)$, $\forall z \in \mathbb{D}$. Hence $h - g$ is locally $1{-}1$ in $\mathbb{D}$, and thus $w \to w + p(w)$ is also locally $1{-}1$ on Ω since it is the composition of locally $1{-}1$ functions. By the lemma, $w \to w + p(w)$ is in fact univalent and has range that is CHD. Hence, $(h - g)(z) = [w + p(w)] \circ f(z)$ is univalent being the composition of univalent functions, and its range is CHD.



(⇐) Now assume that $F = h - g$ is univalent on $\mathbb{D}$ and that $\Omega = F(\mathbb{D})$ is CHD. Then $f = F + 2\text{Re}\,\{\text{g}\,\}$

$$\begin{aligned} f(F^{-1}(w)) &= w + 2\text{Re}\{g(F^{-1}(w))\} \\ &= w + q(w) \end{aligned}$$



is locally 1-1 (being the composition of locally 1-1 functions) in Ω. By the Lemma, $f \circ F^{-1}$ is univalent in Ω and has a range that is CHD.

□

Example 4.47. **Harmonic Koebe map**

Since the Koebe function is an important function in S (see Example 4.12), let's see what happens when it is sheared with a standard dilatation. In particular, let

$$h(z) - g(z) = \frac{z}{(1-z)^2} \tag{48}$$

and let

$$\omega(z) = g'(z)/h'(z) = z.$$

Apply the Shearing Method:

$$\begin{aligned} h'(z) - g'(z) = \frac{1+z}{(1-z)^3} &\Rightarrow h'(z) - zh'(z) = \frac{1+z}{(1-z)^3} \\ &\Rightarrow h'(z) = \frac{1+z}{(1-z)^4}. \end{aligned}$$

Integrating $h'(z)$ and normalizing so that $h(0) = 0$, yields

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{49}$$

We can use this same method to solve for normalized $g(z)$, where $g(0) = 0$. However, this time we will find $g(z)$ by using eqs. (48) and (49).

$$g(z) = h(z) - \frac{z}{(1-z)^2} = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}.$$

So

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right) \in S_H^O.$$

What is the image of $\mathbb{D}$ under f? It turns out that $f(\mathbb{D})$ is similar to the image of $\mathbb{D}$ under the analytic Koebe function (see Figure 4.5) with the slit on the negative real axis except in this case the tip of the slit is at $-\frac{1}{6}$ instead of $-\frac{1}{4}$. To see this, let $\frac{1+z}{1-z} = w = u + iv$. Note that since $z \in \mathbb{D}$, $w = \frac{1+z}{1-z}$ is the right half-plane $\{w = u + iv \in \mathbb{C} \mid \operatorname{Re} w = u > 0, -\infty < v < \infty\}$. Then $z = \frac{w-1}{w+1}$. Substituting this into $h(z)$ and $g(z)$ and simplifying, we get:

$$h\left(\frac{w-1}{w+1}\right) = \frac{1}{8}\left[\frac{2}{3}w^3 + w^2 - \frac{5}{3}\right]$$
$$g\left(\frac{w-1}{w+1}\right) = \frac{1}{8}\left[\frac{2}{3}w^3 - w^2 + \frac{1}{3}\right].$$

Recall that $f = \operatorname{Re}(h+g) + i\operatorname{Im}(h-g)$. Thus,

$$f\left(\frac{w-1}{w+1}\right) = \frac{1}{6}\operatorname{Re}\left\{w^3 - 1\right\} + \frac{1}{4}i\operatorname{Im}\left\{w^2 - 1\right\}.$$

Using $w = u + iv$, and taking the real and imaginary parts, this becomes

$$\frac{1}{6}\left(u^3 - 3uv^2 - 1\right) + i\frac{1}{2}uv.$$

If we let $uv = 0$ (and so, $v = 0$), then the imaginary part vanishes and because $u > 0$, the real part varies from $-\frac{1}{6}$ to $+\infty$. Thus, for $uv = 0$, $f(\mathbb{D})$ contains the line segment on the real axis from $-\frac{1}{6}$ to $+\infty$. On the other hand, if we let $uv = c \neq 0$, then the imaginary part is constant and the real part is $\frac{u^3}{6} - \frac{c^2}{2u}$ which varies between $-\infty$ and $+\infty$. Thus, for any $c \neq 0$, $f(\mathbb{D})$ contains the entire line parallel to the real axis and through the point ic. Therefore, $f(\mathbb{D})$ is the entire complex plane except the slit on the negative real axis from $-\frac{1}{6}$ to $-\infty$.

Exercise 4.48. Let $f = h + \overline{g}$ with $h(z) - g(z) = \frac{z}{(1-z)^2}$ and $\omega(z) = z^2$. Compute h and g explicitly so that $f \in S_H^O$ and determine $f(\mathbb{D})$.

Try it out!

Exploration 4.49. If we shear $h(z) - g(z) = \frac{z}{1-z}$ with $\omega(z) = \frac{z^2+az}{1+az}$, where $-1 < a \leq 1$, then the image of $\mathbb{D}$ under $f = h + \overline{g}$ is a slit domain (like the image of $\mathbb{D}$ under the analytic and harmonic Koebe maps) with the tip of the slit varying as a varies. Use *ShearTool* to graph these domains while decreasing a from 1 to -1.

Try to place the cursor on the tip of the slit in the $f(z)-$domain box and look at the coordinates below to estimate the distance of the tip from the origin. How close to the origin can you get the tip of the slit? How far from the origin can you get the tip of the slit? Try finding other valid ω expressions that result in slit domains when shearing $h(z) - g(z) = \frac{z}{1-z}$ (This exploration is developed further in Small Project 4.59).

Try it out!

Because of the importance and extremal nature of the analytic Koebe function (see Example 4.12), it is conjectured that this "harmonic" Koebe function is extremal in similar ways. Note that the coefficients of this harmonic function

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \\
&= \operatorname{Re}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + i \operatorname{Im}\left(\frac{z}{(1-z)^2}\right)
\end{aligned} \tag{50}
$$

satisfy the properties $|a_n| = \frac{1}{6}(n+1)(2n+1)$, $|b_n| = \frac{1}{6}(n-1)(2n-1)$, and $\big||a_n| - |b_n|\big| = n$.

Conjecture 1 (Harmonic Bieberbach).

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \in S_H^O$. Then

$$
\begin{aligned}
|a_n| &\leq \frac{1}{6}(n+1)(2n+1), \\
|b_n| &\leq \frac{1}{6}(n-1)(2n-1), \\
\big||a_n| - |b_n|\big| &\leq n.
\end{aligned} \tag{51}
$$

In particular,

$$
|a_2| \leq \frac{5}{2} \tag{52}
$$

Exercise 4.50. Verify that the "harmonic" Koebe function given in (50) satisfies equality in eqs. (51) and (52).

Try it out!

Currently, the best established bound is that for all functions $f \in S_H^O$, $|a_2| < 49$ (see [**9**]). There is room for improvement here, if one can find a right approach.

Large Project 4.51. Read and understand the proof that $|a_2| \leq 2$ for analytic functions in S (for example, see [**1**], section 5.1). Read and understand two different proofs that give bounds on $|a_2|$ for harmonic functions in S_H^O (see [**5**], Theorem 4.1 and [**9**], p. 96). Investigate ways to modify any of these proofs or other proofs in order to establish that for $f \in S_H^O$, $|a_2| \leq K$ for some K, where $\frac{5}{2} \leq K < 49$.

Optional

Open Problem 4.52. Prove the Harmonic Bieberbach Conjecture.

In addition, because the tip of the the "harmonic" Koebe function is at $-\frac{1}{6}$, we have the following conjecture that is the analogue of the analytic $\frac{1}{4}$ Koebe Theorem.

Conjecture 2. The range of every function in class S_H^O contains the disk $G = \left\{w : |w| < \frac{1}{6}\right\}$.

Open Problem 4.53. Prove Conjecture 2. The best result so far is that the range of every $f \in S_H^O$ contains the disk $\left\{w : |w| < \frac{1}{16}\right\}$ (see [**5**]), so it would be interesting to increase the radius of this disk to some K, where $\frac{1}{16} < K \leq \frac{1}{6}$.

Recall Definition 4.38 of a domain convex in the general direction φ. The shearing theorem by Clunie and Sheil-Small can easily be generalized to apply to such domains.

Corollary 4.54. A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction $\varphi \Longleftrightarrow h - e^{2i\varphi}g$ is an analytic univalent mapping of $\mathbb{D}$ onto a domain convex in the direction φ.

Example 4.55. **Harmonic right half-plane map**
The analytic right half-plane function $\frac{z}{1-z}$ maps $\mathbb{D}$ onto a convex domain (that is, it is convex in all directions φ). So, in particular, it is convex in the direction of the imaginary axis ($\varphi = \frac{\pi}{2}$). Let's apply Corollary 4.54 with $\varphi = \frac{\pi}{2}$ to $\frac{z}{1-z}$ and use a dilatation that simplifies calculations. Consider

$$h(z) - e^{2i\varphi}g(z) = h(z) + g(z) = \frac{z}{1-z} \tag{53}$$

and let

$$\omega(z) = g'(z)/h'(z) = -z.$$

Computing h and g as in the previous two examples, yields

$$\begin{aligned} h(z) =& \frac{z - \frac{1}{2}z^2}{(1-z)^2} \\ g(z) =& -\frac{\frac{1}{2}z^2}{(1-z)^2}. \end{aligned} \tag{54}$$

Hence, the harmonic function is

$$f(z) = h(z) + \overline{g(z)} = \text{Re}\left(\frac{z}{1-z}\right) + i\,\text{Im}\left(\frac{z}{(1-z)^2}\right) \in S_H^O.$$

Exercise 4.56. Verify that shearing $h(z) + g(z) = \frac{z}{1-z}$ with $\omega(z) = -z$, yields eq. (54).

Try it out!

This is the harmonic map we discussed in Example 4.29.

EXPLORATION 4.57. Shear $h + g = \frac{z}{1-z}$ using $\omega = -z^n$, where $n = 1, 2, 3, \ldots$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ and n varies. Pay particular attention to the number of points the green lines go to as n increases.

Try it out!

One significance of $\mathbb{D}$ being mapped to the same domains under these two functions is that the uniqueness of the Riemann Mapping Theorem for analytic functions does not hold for harmonic functions. This leaves the open question:

OPEN PROBLEM 4.58. What is the analogue of the Riemann Mapping Theorem for harmonic functions?

SMALL PROJECT 4.59. Let $f = h + \overline{g}$ with $h(z) - g(z) = \dfrac{z}{(1-z)^2}$ and $\omega(z) = \dfrac{z^2 + az}{1 + az}$.

(1) Show that for $-1 \leq a \leq 1$, $|\omega(z)| < 1, \forall z \in \mathbb{D}$.
(2) Compute h and g explicitly so that $f \in S_H^O$.
(3) Show that for $a = -1$, $f(\mathbb{D})$ is a right half-plane.
(4) Show that for $-1 < a \leq 1$, $f(\mathbb{D})$ is a slit domain like the Koebe domain. For each value of a, determine where the tip of the slit is located.

Optional

EXAMPLE 4.60. **Harmonic square map**
Here is one more shearing example that we will use in some later sections and in our discussion of minimal sections. Let

$$h(z) - g(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right) \tag{55}$$

which is an analytic function that maps $\mathbb{D}$ onto a horizontal strip convex in the direction of the real axis and let

$$\omega(z) = g'(z)/h'(z) = -z^2.$$

Using the *ShearTool* we see that the shear of $h - g$ with $-z^2$ results in a univalent harmonic function that maps onto the interior of the region bounded by a square (see Figure 4.14).

Let's compute $h(z)$ and $g(z)$ explicitly and prove that the image is a square region. Applying the Shearing Method, we have

$$\begin{aligned} h'(z) - g'(z) = \frac{1}{1-z^2} &\Rightarrow h'(z) + z^2 h'(z) = \frac{1}{1-z^2} \\ &\Rightarrow h'(z) = \frac{1}{1-z^4} = \frac{1}{4}\left[\frac{1}{1+z} + \frac{1}{1-z} + \frac{1}{i+z} + \frac{1}{i-z}\right]. \end{aligned}$$

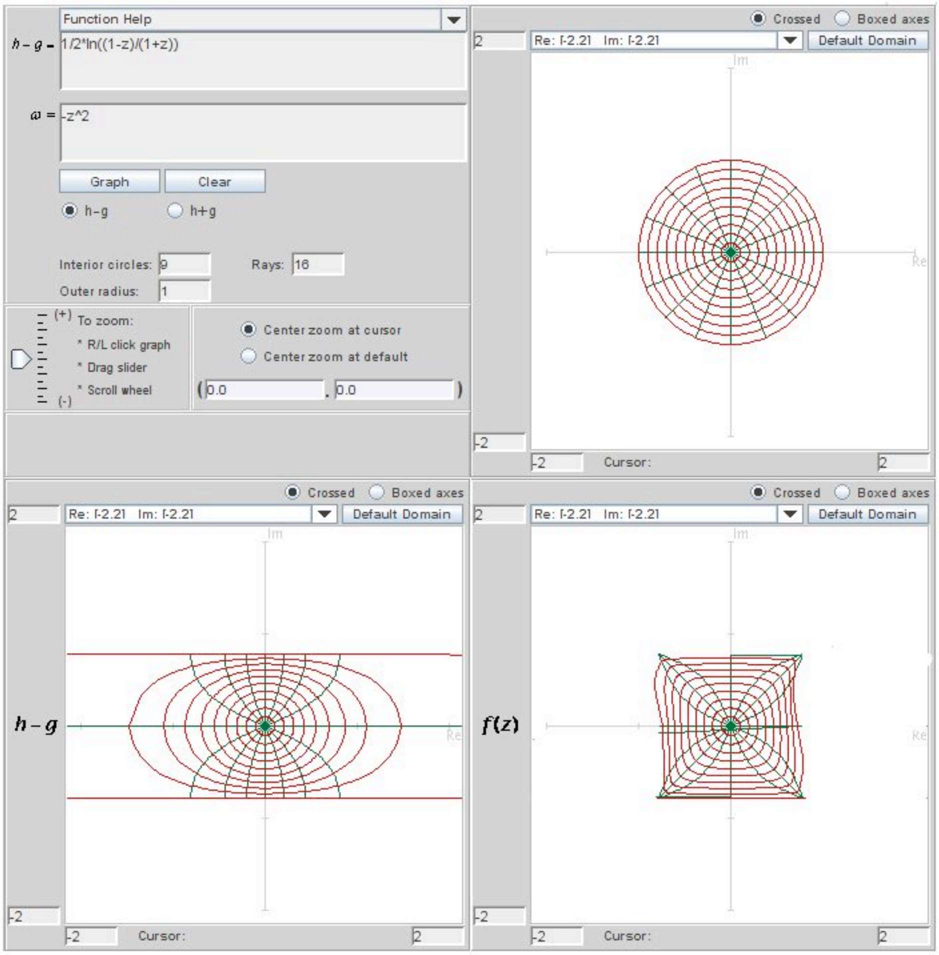


FIGURE 4.14. The image of $\mathbb{D}$ when shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = -z^2$.

Integrating $h'(z)$ and normalizing so that $h(0) = 0$, yields

$$h(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right). \tag{56}$$

We can use this same method to solve for normalized $g(z)$, where $g(0) = 0$. Note that we can also find $g(z)$ by using eqs. (55) and (56). Either way, we get

$$g(z) = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right).$$

So

$$f(z) = h(z) + \overline{g(z)} = \mathrm{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\,\mathrm{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] \in S_H^O.$$

What is $f(\mathbb{D})$? Notice that

$$\begin{aligned} f(z) =& \left[-\frac{1}{2} \arg \left(\frac{i+z}{i-z} \right) \right] + i \left[\frac{1}{2} \arg \left(\frac{1+z}{1-z} \right) \right] \\ =& u + iv. \end{aligned}$$

Let $z = e^{i\theta} \in \partial\mathbb{D}$. Then

$$\frac{i+z}{i-z} = \frac{i+e^{i\theta}}{i-e^{i\theta}} \frac{-i-e^{-i\theta}}{-i-e^{-i\theta}} = \frac{1-i(e^{i\theta}+e^{-i\theta})-1}{1+i(e^{i\theta}-e^{-i\theta})+1} = -i\frac{\cos\theta}{1-\sin\theta}.$$

Thus,

$$u = -\frac{1}{2} \arg \left(\frac{i+z}{i-z} \right) \Bigg|_{z=e^{i\theta}} = \begin{cases} \frac{\pi}{4} & \text{if } \cos\theta > 0, \\ -\frac{\pi}{4} & \text{if } \cos\theta < 0. \end{cases}$$

Likewise, we can show that

$$v = \begin{cases} \frac{\pi}{4} & \text{if } \sin\theta > 0, \\ -\frac{\pi}{4} & \text{if } \sin\theta < 0. \end{cases}$$

In summary, we have that $z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u + iv = \begin{cases} z_1 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{\pi}{4}} & = \frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (0, \frac{\pi}{2}), \\ z_3 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{3\pi}{4}} & = -\frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ z_5 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{5\pi}{4}} & = -\frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\pi, \frac{3\pi}{2}), \\ z_7 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{7\pi}{4}} & = \frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Thus, this harmonic function maps $\mathbb{D}$ onto the interior of the region bounded by a square with vertices at z_1, z_3, z_5 and z_7.

Note that this same function can be obtained by using the Poisson integral formula. In fact, using the Poisson formula we can find a harmonic function that maps $\mathbb{D}$ onto a regular n-gon for any $n \geq 3$. In particular, let

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} Re \left(\frac{1+ze^{-it}}{1-ze^{-it}} \right) e^{i\phi(t)} dt,$$

where $\phi(t) = \frac{\pi(2k+1)}{n} \quad (\frac{2\pi k}{n} \leq t < \frac{2\pi(k+1)}{n}, k = 0, ..., n-1)$. In this case, we can derive that

$$
\begin{aligned}
h(z) &= \sum_{m=0}^{\infty} \frac{1}{nm+1} z^{nm+1}, \\
g(z) &= \sum_{m=1}^{\infty} \frac{-1}{nm-1} z^{nm-1}, \\
h'(z) &= \frac{1}{1-z^n}, \\
g'(z) &= \frac{-z^{n-2}}{1-z^n}, \text{ and} \\
\omega(z) &= -z^{n-2}.
\end{aligned} \tag{57}
$$

Exercise 4.61. Let $f = h + \overline{g}$ with $h(z) - g(z) = \frac{1}{3}\log\left(\frac{1+z+z^2}{1-2z+z^2}\right)$ and $\omega(z) = -z$.

(1) Show that $h'(z) = \frac{1}{1-z^3}$ and $g'(z) = \frac{-z}{1-z^3}$.
(2) According to the previous paragraph, what should be the image of $\mathbb{D}$ under $f = h + \overline{g}$?
(3) Use *ShearTool* to sketch the image of f.
(4) Compute h and g explicitly so $f \in S_H^O$.

Try it out!

Exercise 4.62. Let $f = h + \overline{g}$ with $h(z) + g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and $\omega(z) = -z^2$ (note the difference between this exercise and Example 4.60). Compute h and g explicitly so that $f \in S_H^O$ and use *ComplexTool* to graph $f(\mathbb{D})$.

Try it out!

4.5. Properties of the dilatation

Because of the importance of the dilatation, we will examine some of its properties.

A result from complex analysis is that an analytic function will map infinitesimal circles to infinitesimal circles at any point where its derivative is nonzero. For harmonic functions, this result does not hold. This can be seen in the following exploration.

Exploration 4.63. In *ComplexTool*, enter values one at a time of 0.1, 0.4, 0.7, and 0.999 into the **Outer radius** box in the center panel to plot disks of various outer radii under the analytic right half-plane map

$$F(z) = \frac{z}{1-z}$$

and the harmonic right half-plane map

$$f(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \frac{\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2}.$$

[Note: You may need to resize of the image by clicking on the down arrow ▼ above the image and chose a different size; also,you can move the axes so that the image is centered by positioning the cursor over the image, clicking on the mouse button, and dragging the image to the left]. In all the cases look at the images of small circles under F and under f. What appears to be the image of circles under the harmonic function f?

Try it out!

Now, let's explore the ideas of the geometric dilatation, D_f, and the analytic dilatation, $\omega(z)$ which we discussed earlier.

EXERCISE 4.64.

(a.) Prove that the formulas

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \, \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

are equivalent to the formulas (45) and (46) given in section 4.3.

(b.) Prove that if $f = h + \overline{g}$ is harmonic, then

$$\frac{\partial f}{\partial z} = h'(z) \qquad \text{and} \qquad \frac{\partial f}{\partial \, \overline{z}} = \overline{g'(z)}.$$

Try it out!

Think of the differential of f as $df = f_z dz + f_{\overline{z}} d\,\overline{z}$. Thus we have

$$|df| = |f_z dz + f_{\overline{z}} d\,\overline{z}| = |h'(z)dz + \overline{g'(z)} d\,\overline{z}|.$$

Now bound this differential, and use the fact that if dz is very small, so also $d\,\overline{z}$ will be very small and approximately equal to dz. When we examine upper and lower bounds, we have

$$(|h'(z)| - |g'(z)|)|dz| \le |df| \le (|h'(z)| + |g'(z)|)|dz|.$$

Note that for sense-preserving harmonic functions, the left hand side is always positive. Now we take a ratio of the upper and lower bounds to get the *geometric dilatation* D_f defined by

$$D_f = \frac{|h'| + |g'|}{|h'| - |g'|}.$$

Since this is a ratio of the maximum to minimum $|df|$, if we evaluate D_f at a point z_0, that means that we will find a number that represents a ratio between the most and the least that an infinitesimal circle will be deformed by the function. Thus, if the function maps an infinitesimal circle to an infinitesimal ellipse, then the geometric dilatation gives the ratio of the major axis to the minor axis of the ellipse.

EXERCISE 4.65.

(a.) Prove that for analytic functions, D_f is always 1.
(b.) Prove that for sense-preserving harmonic functions, $D_f \geq 1$.
(c.) Find a formula for D_f for $z + \frac{\bar{z}^3}{3}$. What are the maximum and minimum of this function over the unit disk $\mathbb{D}$?

Try it out!

EXERCISE 4.66. Examine the geometric dilatation for $z + \frac{\bar{z}^3}{3}$ in greater detail. For the points $z = 0, 0.5, 0.9, 0.9e^{i\pi/4}$, and $0.9i$, find $D_f(z)$. Using *ComplexTool*, examine the images of circles of radius 0.05 centered at those points. Estimate the ratio of the major axis to the minor axis of the image ellipse. Does it match with your computation for D_f?
Try it out!

While the geometric dilatation provides some very useful information about the function, some information is lost when we take the modulus of $|h'(z)|$ and $|g'(z)|$. Instead, it is often useful to examine the analytic part versus the anti-analytic part of the function f. Thus we define what is sometimes called the *second complex dilatation* of f,

$$\omega_f(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} = \frac{g'(z)}{h'(z)},$$

where the representation in the last equality makes sense only for harmonic functions. When the function f is clear, we drop the subscript and refer only to the dilatation as ω. Because this function $\omega(z)$ is analytic if and only if $f(z)$ is harmonic, the second complex dilatation is also called the *analytic dilatation* of f.

EXERCISE 4.67.

(a.) Prove that $\omega(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)}$ is analytic if and only if $f(z)$ is harmonic.
(b.) Prove that $\omega(z)$ is identically 0 if and only if f is analytic.
(c.) Prove that for sense-preserving non-analytic harmonic functions f, $0 < |\omega(z)| < 1$.

Try it out!

Now one can naturally ask what the relationship is between the geometric dilatation and analytic dilatation.

EXERCISE 4.68. Prove that $D_f(z) \leq K$ if and only if $|\omega_f(z)| \leq \frac{K-1}{K+1}$.
Try it out!

Exploration 4.69. Re-examine the function $z + \frac{1}{3}\overline{z}^3$, now evaluating $\omega(z)$ at the points $z = 0, 0.5, 0.9, 0.9e^{i\pi/4}$, and $0.9i$. What do you observe about the relationship between ω at these points and images of a small circle centered at these points?

Try it out!

In Section 4.3, we remarked that if $f = h + \overline{g}$ is a sense-preserving harmonic map that has $|\omega(z)| = 1$ for all $z \in$ arc of $\partial\mathbb{D}$, then the image of the arc is either:

- a concave arc; or
- stationary.

To further explore this result, we will use *ShearTool* to graph the image of $\mathbb{D}$ under $f = h + \overline{g}$, when $h - g = z$ and ω has various values in order to see the effect of changing ω.

Exploration 4.70.

(1) Shear $h(z) - g(z) = z$ using $\omega(z) = e^{i\pi n/6}z$, where $n = 0, \ldots, 6$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ as n varies.

(2) Shear $h(z) - g(z) = z$ using $\omega(z) = z^n$, where $n = 1, 2, 3, 4$ and sketch $f(\mathbb{D})$ using *ShearTool*.

  (a) What patterns do you notice relating $f(\mathbb{D})$ and n?

  (b) Make a sketch on paper of what $f(\mathbb{D})$ looks like for $n = 5$. Then graph that shear using *ShearTool*.

  (c) Make a sketch on paper of what $f(\mathbb{D})$ looks like for $n = 6$. Then graph that shear using *ShearTool*.

(3) Shear $h(z) - g(z) = z$ using $\omega(z) = \frac{z+a}{1+\overline{a}z}$, for various values of $a \in \mathbb{D}$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ as a varies.

Try it out!

Small Project 4.71. Investigate the shearing of $h(z) - g(z) = z - \frac{1}{n^2}z^n$ $(n = 2, 3, 4, \ldots)$ with ω for various values of ω (note that the image of $\mathbb{D}$ under the analytic function $z - \frac{1}{n}z^n$ is not CHD for $n = 4, 5, 6, \ldots$; however, it is if we use $z - \frac{1}{n^2}z^n$). Use the approach of Exploration 4.70 as a starting point and then explore new approaches.

Optional

Up to this point, we have only used dilatations that are finite Blaschke products. A *finite Blaschke product* $B(z)$ can be expressed in the form

$$B(z) = e^{i\theta} \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right)^{m_j},$$

where $\theta \in \mathbb{R}$, $|a_j| < 1$, and m_j is the multiplicity of the zero a_j. The dilatations given in Exploration 4.70 and finite products of them are examples of finite Blaschke products. Harmonic univalent mappings whose dilatation is a finite Blaschke product have been studied (see [**13**]). However, little is known about mappings whose dilatation is not a finite Blaschke product. And so an interesting problem is to investigate the properties of harmonic univalent mappings whose dilatation is not a finite Blaschke product. One important type of mappings that are not a finite Blaschke product is a singular inner function. Hence, we will now investigate harmonic univalent mappings whose dilatation is a singular inner function.

First, we need to know what singular inner functions are. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function and denote its radial limit by

$$f^*(e^{i\theta}) := \lim_{r\to 1,\, r<1} f(re^{i\theta}).$$

DEFINITION 4.72. A bounded analytic function f is called an *inner function* if $|f^*(e^{i\theta})| = 1$ almost everywhere (with respect to Lebesgue measure on $\partial\mathbb{D}$). If f has no zeros on $\mathbb{D}$, then f is called a *singular inner function.*

Every inner function can be written in the form

$$f(z) = e^{i\alpha} B(z) e^{\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu\left(e^{i\theta}\right)\right)},$$

where $\alpha, \theta \in R$, μ is a positive measure on $\partial\mathbb{D}$, and $B(z)$ is a Blaschke product. The function $f(z) = e^{\frac{z+1}{z-1}}$ is an example of a singular inner function.

EXERCISE 4.73. Show that if $\omega(z) = e^{\frac{z+1}{z-1}}$, then $|\omega(z)| < 1, \forall z \in \mathbb{D}$.
Try it out!

It has been difficult to construct examples of harmonic mappings whose dilatation is a singular inner function. For awhile there were no known examples [**17**] until Weitsman [**27**] provided two examples. We present Weitsman's examples (see Example 4.75 and Example 4.80) giving a much shorter proof of his second example with this proof providing a method to find more examples.

One way to find an example of a harmonic map with a singular inner function as its dilatation is to use the shearing technique. However, being able to find a closed form for $f = h + \overline{g}$ is not often possible. For example, let $h(z) - g(z) = z$ and the dilatation be $\omega(z) = e^{\frac{z+1}{z-1}}$. Then by the shearing technique

$$h(z) = \int \frac{1}{1 - e^{\frac{z+1}{z-1}}}\, dz.$$

This integral does not have a closed form and so we cannot find an explicit representation for $f = h + \overline{g}$ in this case.

EXERCISE 4.74. Using the shearing technique with $h(z) - g(z) = z - \frac{1}{n}z^n$ and $\omega(z) = e^{\frac{z+1}{z-1}}$, express h as an integral. It is not possible to integrate h to get a closed-form solution.
Try it out!

The following is an example in which the shearing technique does allow us to find specific values for h and g.

Example 4.75. Consider shearing the analytic function

$$h(z) - g(z) = \frac{z}{1-z} + \frac{1}{2}e^{\frac{z+1}{z-1}}$$

with

$$\omega(z) = e^{\frac{z+1}{z-1}}.$$

To see that $h - g$ is convex in the direction of the real axis, we will use a remark by Pommerenke [**18**].

Theorem 4.76. Let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \neq 0$, and let

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})},$$

where $\theta \in \mathbb{R}$. If

$$\operatorname{Re}\left\{\frac{zf'(z)}{\varphi(z)}\right\} > 0, \text{ for all } z \in \mathbb{D},$$

then f is convex in the direction of the real axis.

Note that in this example

$$h'(z) - g'(z) = \frac{1}{(1-z)^2}\left[1 - e^{\frac{z+1}{z-1}}\right].$$

Using $\theta = \pi$ in Theorem 4.76, we have

$$\operatorname{Re}\left[1 - e^{\frac{z+1}{z-1}}\right] > 0$$

because $\left|e^{\frac{z+1}{z-1}}\right| < 1$. Hence $h - g$ is convex in the direction of the real axis.

Shearing $h - g$ with $\omega(z) = e^{\frac{z+1}{z-1}}$ and normalizing yields

$$h(z) = \int \frac{1}{(1-z)^2}\, dz = \frac{z}{1-z},$$

and solving for g we get

$$g(z) = -\frac{1}{2}e^{\frac{z+1}{z-1}}.$$

The image given by this map is similar to the image given by the right half-plane map $\frac{z}{1-z}$ except in this case there are an infinite number of cusps (see Figure 4.15).

Exercise 4.77. Let $f = h + \overline{g}$ with $h(z) - g(z) = \frac{z}{1-z}$ and $\omega(z) = e^{\frac{z+1}{z-1}}$. Use the shearing method to compute h and g explicitly so $f \in S_H^O$ and use *ComplexTool* to sketch $f(\mathbb{D})$ [Hint: In finding the specific function h, use a u-substitution to evaluate the integral].

Try it out!

Another technique to find harmonic mappings whose dilatations are singular inner functions involves using the following theorem by Clunie and Sheil-Small [**5**].

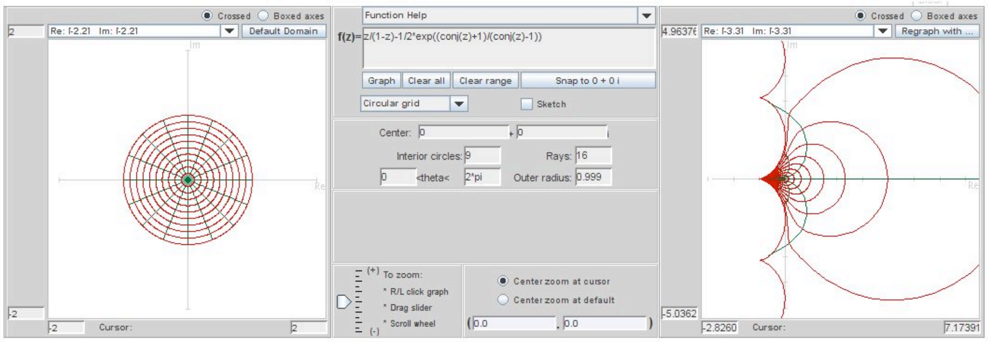


FIGURE 4.15. Image of $\mathbb{D}$ under $f(z) = \dfrac{z}{1-z} - \dfrac{1}{2} e^{\frac{\overline{z}+1}{\overline{z}-1}}$.

THEOREM 4.78. Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that $h + \epsilon g$ is convex for some $|\epsilon| \leq 1$. Then f is univalent.

Theorem 4.78 gives us a nice way to show that a harmonic function is univalent. To develop our present technique, we will let $\epsilon = 0$ in the theorem. This means that if h is analytic convex and if ω is analytic with $|\omega(z)| < 1$, then $f = h + \overline{g}$ is a harmonic univalent mapping.

Actually, the original theorem by Clunie and Sheil-Small gives us a bit more information about f. It states that f is close-to-convex. Geometrically, a close-to-convex function f is one whose image $f(re^{i\theta})$ has no "large hairpin" turns; that is, the tangent vector at $f(re^{i\theta})$ does not turn backward through an angle greater or equal to π anyway along the image of the curve $|z| = r$. However, at this point we are just interested in showing that f is univalent, so you do not need to be concerned about the close-to-convex property.

Also, to establish that a function f is convex, the following theorem is useful.

THEOREM 4.79. Let f be analytic and univalent in $\mathbb{D}$. Then f maps onto a convex domain if and only if

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geq 0, \text{ for all} z \in \mathbb{D}.$$

In the following example we will show how these ideas can be used to construct a harmonic univalent function whose dilatation is a singular inner function.

EXAMPLE 4.80. Let

$$h(z) = z - \frac{1}{4}z^2 \quad \text{with} \quad \omega(z) = g'(z)/h'(z) = e^{\frac{z+1}{z-1}}.$$

We will use Theorem 4.79 to show that h is convex. Let

$$T(z) = 1 + \frac{zh''(z)}{h'(z)} = 1 + \frac{-\frac{1}{2}z}{1 - \frac{1}{2}z} = \frac{1-z}{1-\frac{1}{2}z}.$$

Notice that $T(z)$ is a Möbius transformation. By the mapping properties of Möbius transformations we can show that T maps $\mathbb{D}$ onto $|z - \frac{2}{3}| = \frac{2}{3}$ which is the circle centered at $\frac{2}{3}$ of radius $\frac{2}{3}$. Hence, $\text{Re}\{T(z)\} > 0$ and h is convex.

Now solve for g.

$$g(z) = \int h'(z)\omega(z)\, dz = \int (1 - \frac{1}{2}z)e^{\frac{z+1}{z-1}}\, dz = -\frac{1}{4}(z-1)^2 e^{\frac{z+1}{z-1}}.$$

Hence,

$$f(z) = h(z) + \overline{g(z)} = z - \frac{1}{4}z^2 - \frac{1}{4}(\overline{z} - 1)^2 e^{\frac{\overline{z}+1}{\overline{z}-1}}.$$

Thus, by Theorem 4.78, $f = h + \overline{g}$ is univalent. The image of $\mathbb{D}$ under $f_1(z) = z - \frac{1}{4}z^2 - \frac{1}{4}(\overline{z}-1)^2 e^{\frac{\overline{z}+1}{\overline{z}-1}}$ is similar to the map of a harmonic polynomial but with an infinite number of cusps in the middle section on the right side (see Figure 4.16).

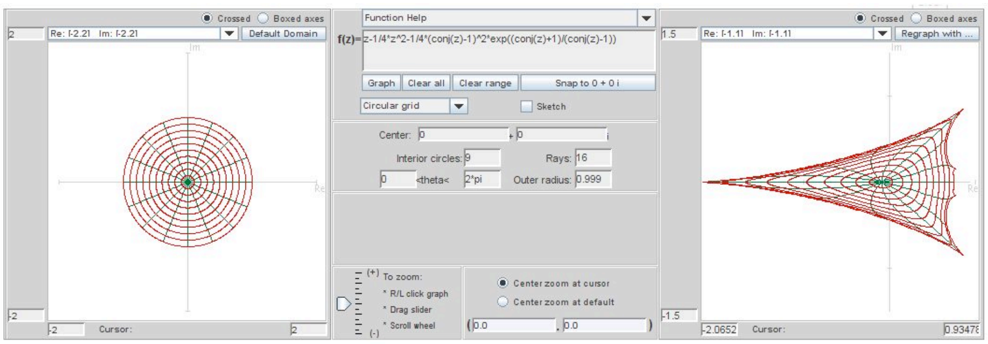


FIGURE 4.16. Image of $\mathbb{D}$ under $f_1(z) = z - \frac{1}{4}z^2 - \frac{1}{4}(\overline{z}-1)^2 e^{\frac{\overline{z}+1}{\overline{z}-1}}$.

EXERCISE 4.81. Let $h(z) = z + \frac{1}{11}z^3$ and $g(z) = -\frac{1}{11}(z-3)(z+1)^2 e^{\frac{z-1}{z+1}}$. Show that $f = h + \overline{g}$ is univalent and use *ComplexTool* to graph $f(\mathbb{D})$.

Try it out!

EXERCISE 4.82. Use the approach above to show that compute $f = h + \overline{g}$ is harmonic univalent, where $h(z) = z + 2\log(z+1)$ and $\omega(z) = e^{\frac{z-1}{z+1}}$. The graph of $\mathbb{D}$ under f is shown in Figure 4.17.

Try it out!

It was mentioned earlier that it is not often possible to find a closed form for $f = h + \overline{g}$ when the dilatation is a singular inner function. However, one can use *ShearTool* to explore images of $f(\mathbb{D})$ when the dilatation is a singular inner product.

EXPLORATION 4.83.

(1) If we shear $h(z) - g(z) = \frac{z}{(1-z)^2}$ with $\omega(z) = e^{\frac{z+1}{z-1}}$, then $f = h + \overline{g}$ will be univalent and convex in the direction of the real axis. Use *ShearTool* to sketch $f(\mathbb{D})$.

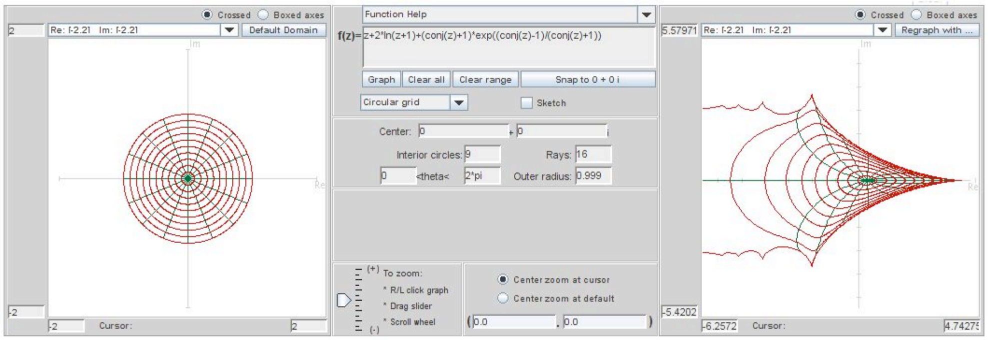


FIGURE 4.17. Image of $\mathbb{D}$ under $f(z) = h + \overline{g}$ in Exercise 4.82.

(2) Use *ShearTool* to sketch $f(\mathbb{D})$, where $h(z) - g(z) = z - \frac{1}{2}z^2$ with $\omega(z) = e^{\frac{z^2+1}{z^2-1}}$.

(3) Use *ShearTool* to sketch $f(\mathbb{D})$, where $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = e^{\frac{z+1}{z-1}}$.

Try it out!

OPEN PROBLEM 4.84. What are the properties of harmonic univalent mappings whose dilatation is a singular inner product?

4.6. Harmonic Linear Combinations

A common way to try to construct new functions with a given property is to take the linear combination of two functions with that property. This is done with derivatives and integrals in beginning calculus. And in Exploration 4.14, this done with the analytic Koebe mapping, f_k, and the right half-plane mapping, f_r, where

$$f_k(z) = \frac{z}{(1-z)^2} \qquad \text{and} \qquad f_r(z) = \frac{z}{1-z},$$

to derive the univalent analytic map

$$f_3(z) = tf_k(z) + (1-t)f_r(z) = \frac{z - tz^2}{(1-z)^2},$$

where $0 \le t \le 1$.

Is it true that the linear combination of two $1-1$ functions is also a $1-1$ function? Let's look at the case for real-valued functions. Suppose $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are $1-1$ functions. Will $f_3(x) = tf_1(x) + (1-t)f_2(x)$ also be $1-1$ when $0 \le t \le 1$? Not necessarily. Consider the example of $f_1(x) = x^3$, $f_2(x) = -x^3$, and $t = \frac{1}{2}$. Both f_1 and f_2 are $1-1$; they satisfy the horizontal line test. But $f_3(x) = tf_1(x) + (1-t)f_2(x) = 0$ which is not $1-1$.

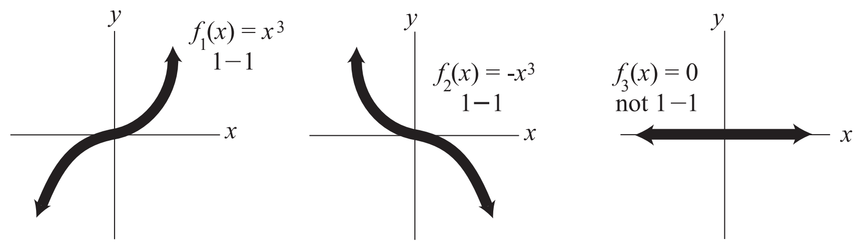


FIGURE 4.18. $f_1(x) = x^3$ and $f_2(x) = -x^3$ are $1-1$, but $f_3(x) = tf_1(x) + (1-t)f_2(x) = 0$ s not $1-1$.

In this case, the difficulty is that f_1 is an increasing $1-1$ function, f_2 is a decreasing $1-1$ function, and when $t = \frac{1}{2}$, the increase of f_1 cancels out the decrease of f_2. We can alleviate this difficulty by requiring that f_1, f_2 are either both increasing or both decreasing. This idea can be applied to complex-valued functions.

CONDITION A. Suppose f is complex-valued harmonic and non-constant in $\mathbb{D}$. There exists sequences z'_n, z''_n converging to $z = 1$, $z = -1$, respectively, such that

$$\begin{aligned} \lim_{n\to\infty} \mathrm{Re}\{f(z'_n)\} &= \sup_{|z|<1} \mathrm{Re}\{f(z)\} \\ \lim_{n\to\infty} \mathrm{Re}\{f(z''_n)\} &= \inf_{|z|<1} \mathrm{Re}\{f(z)\}. \end{aligned} \tag{58}$$

Note that the normalization in (58) can be thought of in some sense as if $f(1)$ and $f(-1)$ are the right and left extremes in the image domain in the extended complex plane.

EXAMPLE 4.85. We will show that Condition A is satisfied by $f(z) = z + \frac{1}{3}\overline{z}^3$ (see Figure 4.19).

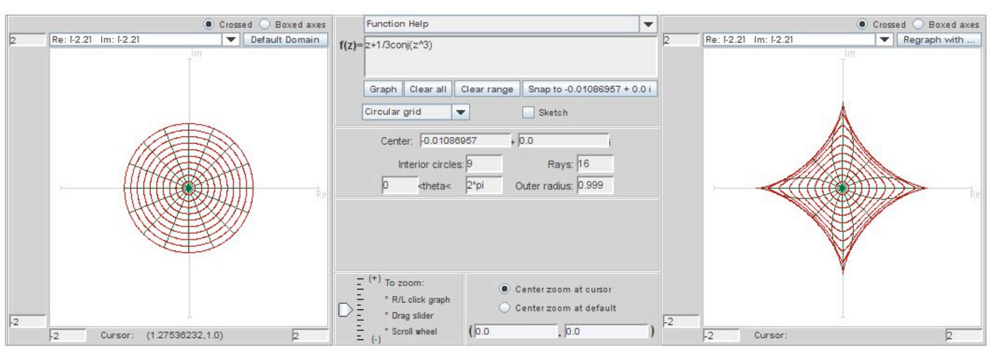


FIGURE 4.19. Image of $\mathbb{D}$ under $f(z) = z + \frac{1}{3}\overline{z}^3$

One can get a feel that f satisfies Condition A by using the **Sketch** option in *ComplexTool* to draw several paths $\{z'_n\} \in \mathbb{D}$ that approach 1 and see that the corresponding images of these paths approach the right-side cusp of the image of $\mathbb{D}$, a hypocycloid of 4 cusps (see Figure 4.19). Likewise, various paths $\{z''_n\} \in \mathbb{D}$ that approach -1 result in image paths that approach the left-side cusp of the hypocycloid.

To prove that f satisfies Condition A, note that

$$f(e^{i\theta}) = e^{i\theta} + \frac{1}{3}e^{-3i\theta} = (\cos\theta + \frac{1}{3}\cos 3\theta) + i(\sin\theta - \frac{1}{3}\sin 3\theta).$$

So, $\text{Re}\{f(e^{i\theta})\} = \cos\theta + \frac{1}{3}\cos 3\theta$. Hence, $-\frac{4}{3} \leq \text{Re}\{f(e^{i\theta})\} \leq \frac{4}{3}$ which means $\sup_{|z|<1} \text{Re}\{f(z)\} = \frac{4}{3}$. Letting $z'_n = 1 - \frac{1}{n} \to 1$, we have that

$$\lim_{n\to\infty} \text{Re}\{f(z'_n)\} = \frac{4}{3} = \sup_{|z|<1} \text{Re}\{f(z)\}.$$

Similarly, $\lim\limits_{n\to\infty} \text{Re}\{f(z''_n)\} = -\dfrac{4}{3} = \inf\limits_{|z|<1} \text{Re}\{f(z)\}$.

Exercise 4.86. Use the **Sketch** option in *ComplexTool* to determine which of the following harmonic functions satisfy Condition A:

(a) $f(z) = z + \frac{1}{2}\overline{z}^2$

(b) $\text{Re}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\,\text{Im}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right]$

(c) $\text{Re}\left(\frac{z}{1-z}\right) + i\,\text{Im}\left(\frac{z}{(1-z)^2}\right)$

(d) $\text{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\,\text{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$

(e) $\text{Re}\left(\frac{z+\frac{1}{3}z^3}{(1-z)^3}\right) + i\,\text{Im}\left(\frac{z}{(1-z)^2}\right)$

Try it out!

To prove a result about the linear combinations of harmonic functions, we will need the following result by Hengartner and Schober [**15**] that employs condition A. However, we won't use Theorem 4.87 afterwards.

Theorem 4.87 (Hengartner and Schober). Suppose f is holomorphic (i.e., analytic) and non-constant in $\mathbb{D}$. Then

$$\text{Re}\{(1-z^2)f'(z)\} \geq 0, z \in \mathbb{D}$$

if and only if

(1) f is univalent in $\mathbb{D}$,
(2) f is convex in the imaginary direction, and
(3) condition A holds.

We now seek to study conditions under which f_3 is globally univalent.

THEOREM 4.88. Let $f_1 = h_1 + \overline{g_1},\ f_2 = h_2 + \overline{g_2}$ be univalent harmonic mappings convex in the imaginary direction and $\omega_1 = \omega_2$. If $f_1,\ f_2$ satisfy condition A, then $f_3 = tf_1 + (1-t)f_2$ is convex in the imaginary direction (and univalent) $(0 \leq t \leq 1)$.

PROOF. To see that f_3 is locally univalent, use $g_1' = \omega_1 h_1'$ and $g_2' = \omega_2 h_2' = \omega_1 h_2'$. Then

$$\omega_3 = \frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'} = \frac{t\omega_1 h_1' + (1-t)\omega_1 h_2'}{th_1' + (1-t)h_2'} = \omega_1.$$

Next, by Clunie and Sheil-Small's shearing theorem (see Theorem 4.40), we know that each $h_j + g_j$ $(j = 1, 2)$ is univalent and convex in the imaginary direction. Also, $h_j + g_j$ satisfies Condition A since $\operatorname{Re}\{f_j\} = \operatorname{Re}\{h_j + g_j\}$. Applying Theorem 4.87 we have

$$\operatorname{Re}\{(1-z^2)(h_j'(z) + g_j'(z))\} \geq 0, (j = 1, 2).$$

Consider

$$\begin{aligned}\operatorname{Re}\{(1-&z^2)(h_3'(z) + g_3'(z))\} \\ &= \operatorname{Re}\{(1-z^2)[t(h_1'(z) + g_1'(z)) + (1-t)(h_2'(z) + g_2'(z))]\} \\ &= t\operatorname{Re}\{(1-z^2)(h_1'(z) + g_1'(z))\} + (1-t)\operatorname{Re}\{(1-z^2)(h_2'(z) + g_2'(z))\} \geq 0.\end{aligned}$$

By applying Theorem 4.87 in the other direction, we have that $h_3 + g_3$ is convex in the imaginary direction, and so by the shearing theorem, f_3 is convex in the imaginary direction. □

EXAMPLE 4.89. Consider the functions

$$\begin{aligned}f_1(z) &= \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\operatorname{Im}\left[-\frac{1}{2}\log\left(\frac{i+z}{i-z}\right)\right], \\ f_2(z) &= \operatorname{Re}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\operatorname{Im}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right].\end{aligned}$$

Now, f_1 maps $\mathbb{D}$ onto a square region (see Figure 4.20); the image is the same as for the harmonic square map in Example 4.60, but the function is different. In particular, f_1 has different arcs of the unit circle being mapped to the vertices, and the dilatation for f_1 is $\omega(z) = z^2$ which is different than the dilatation for the harmonic square map in Example 4.60. Condition A is satisfied for f_1 (see Example 4.92 for more details).

f_2 maps $\mathbb{D}$ onto a region similar to a hypocycloid with 4 cusps except instead of cusps the domain has ends that extend out to infinity (see Figure 4.21). The dilatation of f_2 is also $\omega(z) = z^2$ and Condition A is satisfied.

By Theorem 4.88, $f_3 = tf_1 + (1-t)f_2$ is univalent. The image of $\mathbb{D}$ when $t = \frac{1}{2}$ is shown in Figure 4.22.

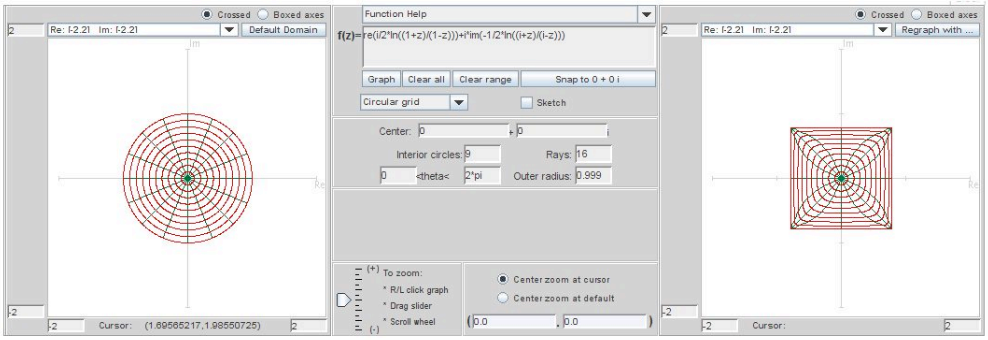


FIGURE 4.20. Image of $\mathbb{D}$ under $f_1(z) = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\operatorname{Im}\left[-\frac{1}{2}\log\left(\frac{i+z}{i-z}\right)\right]$.

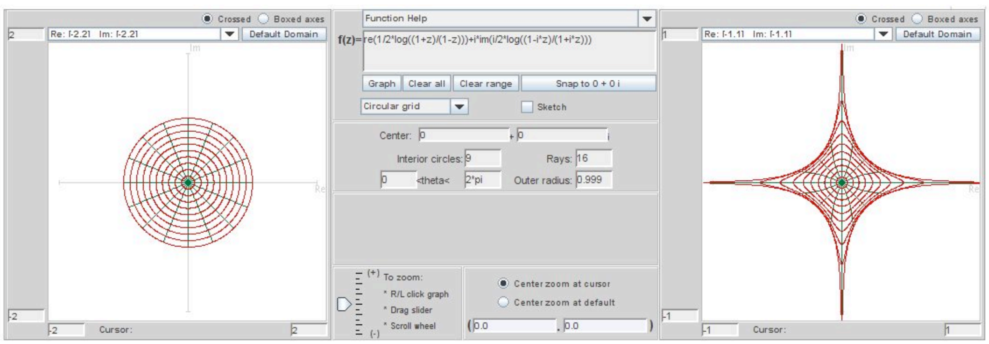


FIGURE 4.21. Image of $\mathbb{D}$ under $f_2(z) = \operatorname{Re}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\operatorname{Im}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right]$.

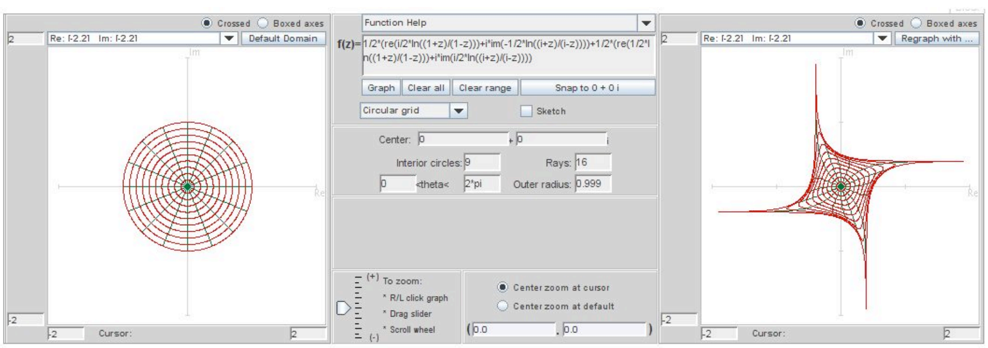


FIGURE 4.22. Image of $\mathbb{D}$ under $f_3(z) = \frac{1}{2}f_1(z) + \frac{1}{2}f_2(z)$

EXPLORATION 4.90. Let

$$f_1(z) = \operatorname{Re}\left[\frac{i}{\sqrt{3}}\ln\left(\frac{1+e^{-i\frac{\pi}{3}}z}{1+e^{i\frac{\pi}{3}}z}\right)\right] + i\operatorname{Im}\left[\frac{1}{3}\ln\left(\frac{1+z+z^2}{1-2z+z^2}\right)\right],$$

$$f_2(z) = \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right).$$

Show that f_1 and f_2 satisfies the conditions of Theorem 4.88 and then use *ComplexTool* to plot images of $f_3 = tf_1 + (1-t)f_2$ for various values of t.

Try it out!

LARGE PROJECT 4.91. Theorem 4.88 gives necessary but not sufficient conditions on f_1 and f_2 for the linear combination $f_3 = tf_1 + (1-t)f_2$ to be univalent. That f_3 can be univalent when f_1 does not satisfy Condition A is demonstrated by the functions

$$f_1(z) = \operatorname{Re}\left(\frac{z + 1/3z^3}{(1-z)^3}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right),$$
$$f_2(z) = \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right).$$

Construct f_3 for various values of t and use *ComplexTool* to see the images of $\mathbb{D}$ under f_3.

In fact, the following functions suggest that several of the hypotheses of Theorem 4.88 can fail and still f_3 be univalent:

$$f_1(z) = z - \frac{1}{m}\overline{z}^m \qquad \text{and} \qquad f_2(z) = z - \frac{1}{n}\overline{z}^n,$$

where $m, n \geq 2$. For various values of m, n, and t construct f_3 and use *ComplexTool* to see the images of $\mathbb{D}$ under f_3.

Investigate the examples above, and then construct other examples in which f_3 is univalent but the hypotheses of Theorem 4.88 do not hold. Using these examples make a conjecture for hypotheses of a new theorem that guarantees f_3 will be univalent. Prove this new theorem.

Optional

Now, let us look at an example that initially is surprising and is related to the nonconvex polygons described by Duren, McDougall, and Schaubroeck [**11**].

EXAMPLE 4.92. Let $f_1 = h_1 + \overline{g_1}$ be the harmonic square map in Example 4.89, where

$$h_1(z) = \frac{i}{4}\log\left(\frac{1+z}{1-z}\right) - \frac{1}{4}\log\left(\frac{i+z}{i-z}\right)$$
$$g_1(z) = \frac{i}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{4}\log\left(\frac{i+z}{i-z}\right).$$

We can write this as

$$f_1(z) = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\operatorname{Im}\left[-\frac{1}{2}\log\left(\frac{i+z}{i-z}\right)\right].$$

Using the same approach as in Example 4.60, we see that $z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u_1 + iv_1 = \begin{cases} z_1 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{\pi}{4}} & \text{if } \theta \in (\frac{-\pi}{2}, 0), \\ z_3 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{3\pi}{4}} & \text{if } \theta \in (0, \frac{\pi}{2}), \\ z_5 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{5\pi}{4}} & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ z_7 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{7\pi}{4}} & \text{if } \theta \in (\pi, \frac{3\pi}{2}). \end{cases}$$

So, f_1 maps $\mathbb{D}$ onto a square region with vertices at z_1, z_3, z_5 and z_7 (see Figure 4.23). The dilatation for f_1 is $\omega = z^2$ and Condition A is satisfied. For example, for any sequence of points, z_n', in the fourth quadrant approaching 1,

$$\lim_{n\to\infty} \text{Re}\{f_1(z_n')\} = \frac{\pi}{2\sqrt{2}} = \sup_{|z|<1} \text{Re}\{f_1(z)\}$$

and for any sequence of points, z_n'', in the second quadrant approaching -1,

$$\lim_{n\to\infty} \text{Re}\{f_1(z_n'')\} = -\frac{\pi}{2\sqrt{2}} = \inf_{|z|<1} \text{Re}\{f_1(z)\}.$$

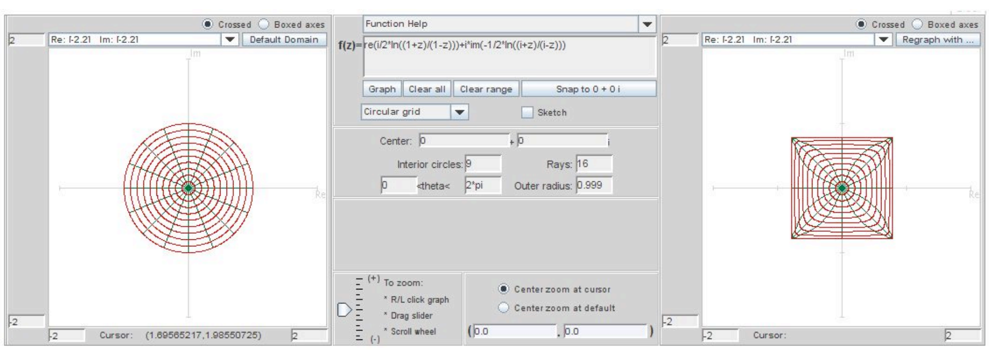


FIGURE 4.23. Image of $\mathbb{D}$ under f_1

Next, let $f_2 = h_2 + \overline{g_2}$, where

$$\begin{aligned} h_2(z) &= -\frac{1}{4}e^{-i\frac{3\pi}{4}} \log\left(\frac{e^{i\frac{\pi}{4}} + z}{e^{i\frac{\pi}{4}} - z}\right) - \frac{1}{4}e^{-i\frac{\pi}{4}} \log\left(\frac{e^{i\frac{3\pi}{4}} + z}{e^{i\frac{3\pi}{4}} - z}\right) \\ g_2(z) &= \frac{1}{4}e^{i\frac{3\pi}{4}} \log\left(\frac{e^{i\frac{\pi}{4}} + z}{e^{i\frac{\pi}{4}} - z}\right) + \frac{1}{4}e^{i\frac{\pi}{4}} \log\left(\frac{e^{i\frac{3\pi}{4}} + z}{e^{i\frac{3\pi}{4}} - z}\right). \end{aligned}$$

Similar to above

$$f_2(z) = \mathrm{Re}\left\{\frac{i}{2\sqrt{2}}\left[\log\left(\frac{e^{i\frac{\pi}{4}}+z}{e^{i\frac{\pi}{4}}-z}\right)+\log\left(\frac{e^{i\frac{3\pi}{4}}+z}{e^{i\frac{3\pi}{4}}-z}\right)\right]\right\}$$
$$+\, i\,\mathrm{Im}\left\{\frac{1}{2\sqrt{2}}\left[\log\left(\frac{e^{i\frac{\pi}{4}}+z}{e^{i\frac{\pi}{4}}-z}\right)-\log\left(\frac{e^{i\frac{3\pi}{4}}+z}{e^{i\frac{3\pi}{4}}-z}\right)\right]\right\},$$

$z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u_2 + iv_2 = \begin{cases} z_0 = \frac{\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{-\pi}{4}, \frac{\pi}{4}), \\ z_2 = \frac{i\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{\pi}{4}, \frac{3\pi}{4}), \\ z_4 = -\frac{\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{3\pi}{4}, \frac{5\pi}{4}), \\ z_6 = -\frac{i\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{5\pi}{4}, \frac{7\pi}{4}). \end{cases}$$

That is, f_2 maps $\mathbb{D}$ onto a rotated square region with vertices at z_0, z_2, z_4 and z_6 (see Figure 4.24) with $\omega = z^2$, and it also satisfies Condition A.

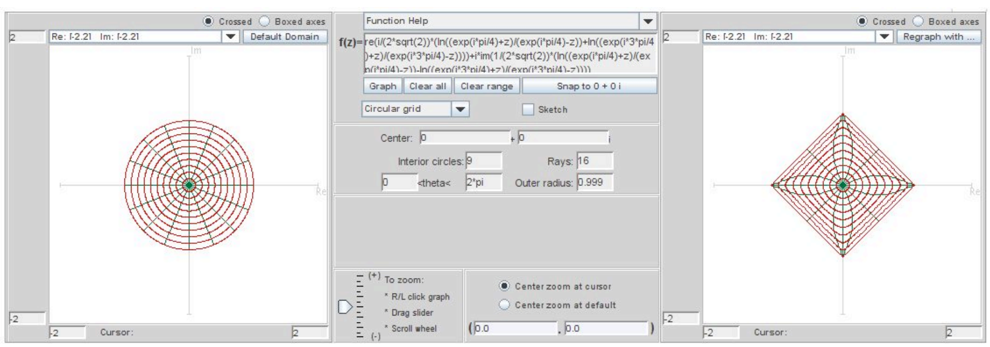


FIGURE 4.24. Image of $\mathbb{D}$ under f_2

By Theorem 4.88, $f_3 = tf_1 + (1-t)f_2$ is univalent. What is the image of $\mathbb{D}$ under f_3? Let's look at the specific case when $t = \dfrac{1}{2}$. You might think that $f_3(\mathbb{D})$ would be just an overlaying of $f_1(\mathbb{D})$ and $f_2(\mathbb{D})$ (see Figure 4.25(a)). However, it is not. Instead, it is the nonconvex star shown in Figure 4.25(b).

Why is the correct image the nonconvex star in Figure 4.25(b)? Let's look where arcs of the unit circle are mapped under f_3. Notice that $f_1(e^{i\theta})$ and $f_2(e^{i\theta})$ depend upon which of eight arcs θ is in. For example, if $\theta \in (-\frac{\pi}{4}, 0)$, then $f_1(e^{i\theta}) = z_1$ and $f_2(e^{i\theta}) = z_0$, and so in this interval $f_3(e^{i\theta}) = \frac{z_1+z_0}{2}$ (that is, it is the midpoint between z_1 and z_0). However, if $\theta \in (0, \frac{\pi}{4})$, then $f_1(e^{i\theta}) = z_3$ and $f_2(e^{i\theta}) = z_0$, and $f_3(e^{i\theta}) = \frac{z_3+z_0}{2}$.

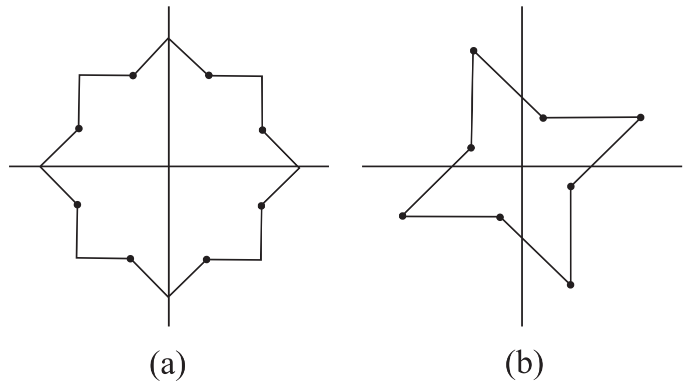


FIGURE 4.25. Which is the image of $f_3(\mathbb{D})$?

Specifically,

$$f_3(e^{i\theta}) = \begin{cases} w_1 = \frac{z_1+z_0}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{\pi}{8}e^{i\frac{\pi}{8}} & \text{if } \theta \in (\frac{-\pi}{4}, 0), \\ w_2 = \frac{z_3+z_0}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{3\pi}{8}e^{i\frac{3\pi}{8}} & \text{if } \theta \in (0, \frac{\pi}{4}), \\ w_3 = \frac{z_3+z_2}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{\pi}{8}e^{i\frac{5\pi}{8}} & \text{if } \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \\ w_4 = \frac{z_5+z_2}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{3\pi}{8}e^{i\frac{7\pi}{8}} & \text{if } \theta \in (\frac{\pi}{2}, \frac{3\pi}{4}), \\ w_5 = \frac{z_5+z_4}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{\pi}{8}e^{i\frac{9\pi}{8}} & \text{if } \theta \in (\frac{3\pi}{4}, \pi), \\ w_6 = \frac{z_7+z_4}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{3\pi}{8}e^{i\frac{11\pi}{8}} & \text{if } \theta \in (\pi, \frac{5\pi}{4}), \\ w_7 = \frac{z_7+z_6}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{\pi}{8}e^{i\frac{13\pi}{8}} & \text{if } \theta \in (\frac{5\pi}{4}, \frac{3\pi}{2}), \\ w_8 = \frac{z_1+z_6}{2} = \frac{\pi}{2\sqrt{2}}\cos\frac{3\pi}{8}e^{i\frac{15\pi}{8}} & \text{if } \theta \in (\frac{3\pi}{2}, \frac{7\pi}{4}). \end{cases}$$

Note that the vertices w_1, w_3, w_5 and w_7 lie equally spaced on a circle of radius $r_{outer} = \frac{\pi}{2\sqrt{2}}\cos\frac{\pi}{8} \approx 1.026$, while the vertices w_2, w_4, w_6 and w_8 lie equally spaced on a circle of radius $r_{inner} = \frac{\pi}{2\sqrt{2}}\cos\frac{3\pi}{8} \approx 0.425$.

We can visualize the boundary of $f_3(\mathbb{D})$ by plotting the eight vertices $z_0, z_1, \ldots z_7$ and drawing the midpoints $w_1, \ldots, w_8$ (see Figure 4.26).

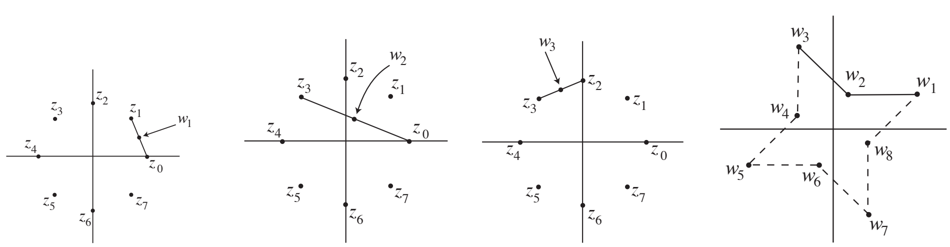


FIGURE 4.26. Visualizing the image of the boundary of $f_3(\mathbb{D})$

We can also explore the linear combination of these two functions that mapped onto rotated square regions by using the applet, *LinComboTool* (see Figure 4.27).

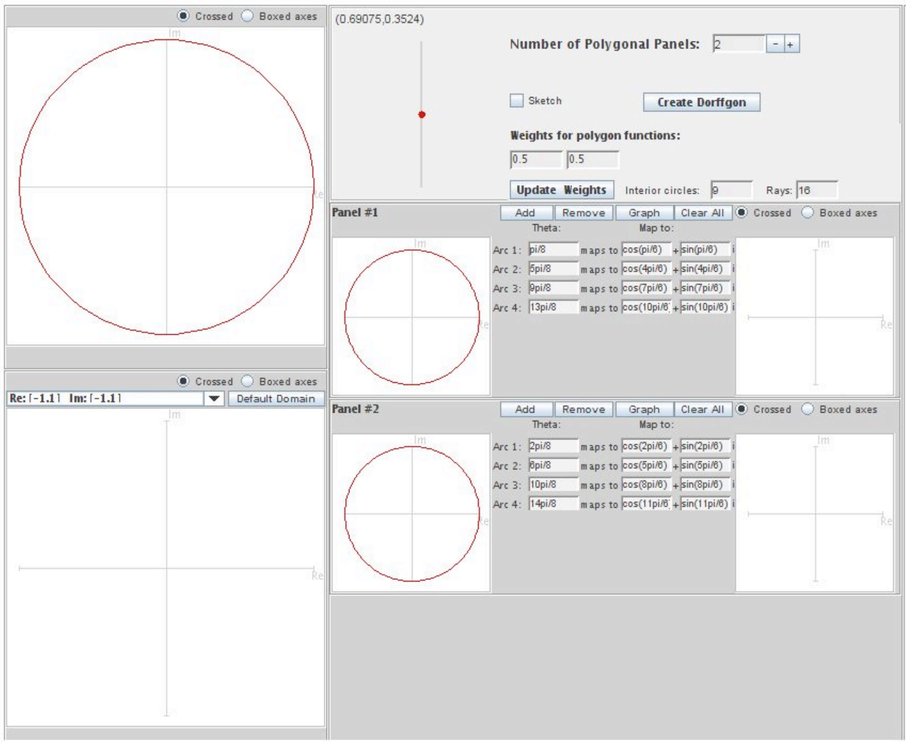


FIGURE 4.27. The applet *LinComboTool*

Open up *LinComboTool*. Make sure that at the top of the page, the **Number of Polygonal Pa...** is 2. In **Panel #1** enter the left end points of the intervals for the arcs of the unit circle used in function f_1 (these endpoints need to be positive numbers). Then enter the real and imaginary values of the image of this arc under the function f_1. For example, if we take the interval $(0, \frac{\pi}{2})$ for Arc 1 (note that we are starting with this interval because we need to use nonnegative values), then enter `0` for Arc 1, `pi/(2*sqrt(2))*cos(pi/4)` for the real value of its image, and `pi/(2*sqrt(2))*sin(pi/4)` for the imaginary value of its image. Remember that for Arc 4, we will use `3pi/2`. If there are not enough boxes for the arcs, click on the **Add** button to add an arc. Similarly, click on **Remove**, if there are too many boxes for the arcs. When you are done entering the points, click on **Graph** to produce the image $f_1(\mathbb{D})$. Then go to **Panel #2** and enter the points for f_2 and graph $f_2(\mathbb{D})$. After these are both graphed, click on **Create LinCombogon** and the corresponding image will appear in the lower lefthand box (see Figure 4.28).

EXERCISE 4.93. Using *LinComboTool*, start with the same arc values and corresponding point values as in Example 4.92. Note that you can change the value of t

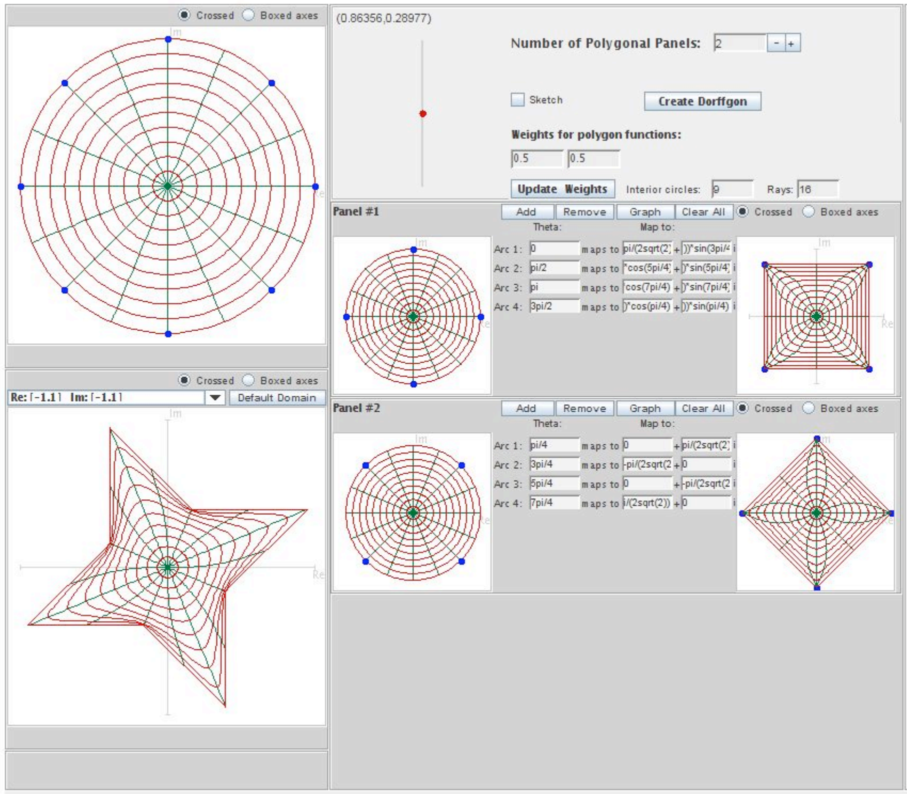


FIGURE 4.28. Image of $\mathbb{D}$ under f_3

by sliding up and down the red dot near the top of the page. Describe what happens as t varies from 0 to 1. In the example above we showed that when $t = \frac{1}{2}$, $r_{outer} = \frac{\pi}{2\sqrt{2}} \cos \frac{\pi}{8} \approx 1.026$ and $r_{inner} = \frac{\pi}{2\sqrt{2}} \cos \frac{3\pi}{8} \approx 0.425$. Compute r_{outer} and r_{inner} for any t $(0 \leq t \leq 1)$.

Try it out!

REMARK 4.94. In Theorem 4.88, we do not need that $\omega_1 = \omega_2$. Looking over the proof of this theorem, what is really needed is just that f_3 is locally univalent. This can be achieved if we have that

$$|\omega_3| = \left| \frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'} \right| < 1. \tag{59}$$

EXERCISE 4.95. We can have one pair of functions f_1, f_2 mapping onto image domains G_1, G_2, respectively, and another pair of functions $\tilde{f}_1$, $\tilde{f}_2$ that also map onto these same image domains G_1, G_2, but the linear combinations f_3 and $\tilde{f}_3$ map onto a different image domains.

Repeat the steps in Example 4.92 using the same function for f_2 but replacing f_1 with the harmonic square map in Example 4.60, where

$$\begin{aligned} h_1(z) =& \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right) \\ g_1(z) =& -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right). \end{aligned}$$

Note that

$$\begin{aligned} h_1'(z) &= \frac{1}{1-z^4}, \qquad & g_1'(z) &= \frac{-z^2}{1-z^4} \\ h_2'(z) &= \frac{1}{1+z^4}, \qquad & g_2'(z) &= \frac{z^2}{1+z^4}. \end{aligned}$$

(a.) In this case, $\omega_1(z) = -z^2$ while $\omega_2(z) = z^2$. Using eq. (59) in Remark 4.94 above, show that f_3 is locally univalent.
(b.) Use *LinComboTool* find the image of $f_3(\mathbb{D})$ using this f_1 and f_2.
(c.) Explain why this happens by using the approach in Example 4.92 to compute the new values of $w_1, \ldots, w_8$ and then use the visualization technique in the example to plot the eight vertices $z_0, \ldots, z_7$ and draw the midpoints $w_1, \ldots, w_8$.

Try it out!

EXERCISE 4.96. Repeat the steps in Exercise 4.95 using the same function for f_1 but replacing f_2 with the harmonic hexagon map that can be derived from eq (57) for h' and g' at the end of Example 4.60 , where

$$\begin{aligned} h_2'(z) =& \frac{1}{1-z^6} \Rightarrow \\ h_2(z) =& \frac{1}{6}\log\left(\frac{1+z}{1-z}\right) + \frac{e^{\frac{-i\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i\pi}{3}}z}{1-e^{\frac{i\pi}{3}}z}\right) + \frac{e^{\frac{-i2\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i2\pi}{3}}z}{1-e^{\frac{i2\pi}{3}}z}\right) \\ g_2'(z) =& \frac{-z^4}{1-z^6} \Rightarrow \\ g_2(z) =& -\frac{1}{6}\log\left(\frac{1+z}{1-z}\right) - \frac{e^{\frac{i\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i\pi}{3}}z}{1-e^{\frac{i\pi}{3}}z}\right) - \frac{e^{\frac{i2\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i2\pi}{3}}z}{1-e^{\frac{i2\pi}{3}}z}\right). \end{aligned}$$

(a.) In this case, $\omega_1(z) = -z^2$ while $\omega_2(z) = -z^4$. Using eq. (59) in the remark above, show that f_3 is locally univalent.
(b.) Use *LinComboTool* find the image of $f_3(\mathbb{D})$ using this f_1 and f_2.
(c.) Explain why this happens by using the approach in Example 4.92 to compute the new values of the vertices of $f_3(\mathbb{D})$.

Try it out!

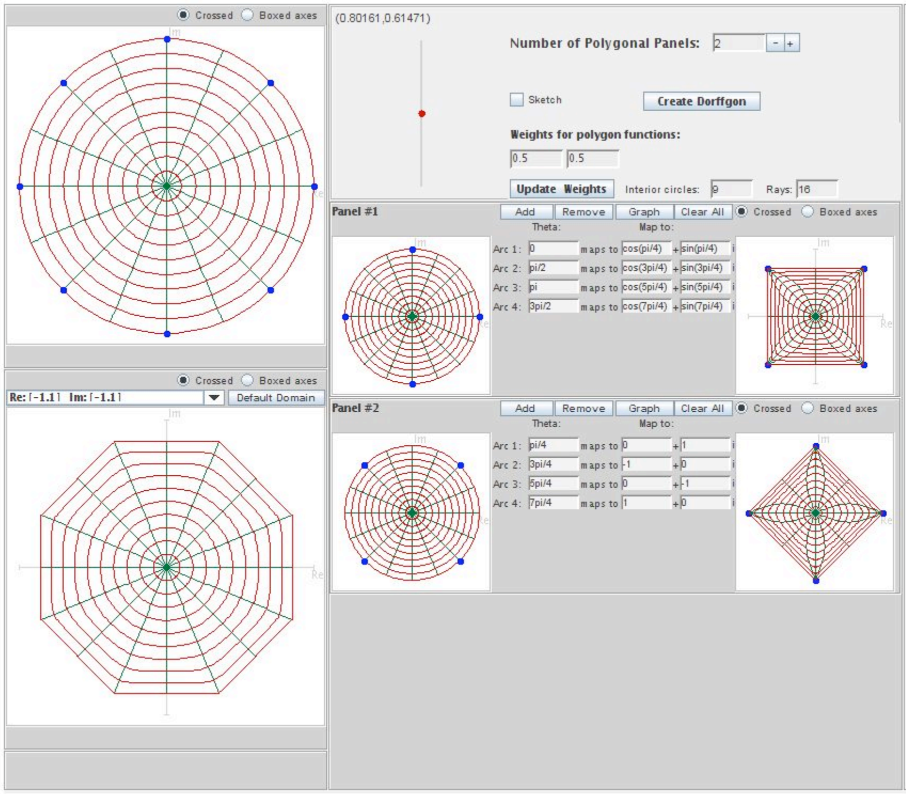


FIGURE 4.29. Image of $\mathbb{D}$ under f_3 in Exercise 4.95

We can generalize Theorem 4.88 to include the linear combination of n functions $f_1, \dots, f_n$.

THEOREM 4.97. Let $f_1 = h_1 + \overline{g_1}$, …, $f_n = h_n + \overline{g_n}$ be n univalent harmonic mappings convex in the imaginary direction and $\omega_1 = \cdots = \omega_n$. If f_1, …, f_n satisfy condition A, then $F = t_1 f_1 + \cdots + t_n f_n$ is convex in the imaginary direction, where $0 \leq t_n \leq 1$ and $t_1 + \cdots + t_n = 1$.

EXERCISE 4.98. Prove Theorem 4.97.
Try it out!

EXPLORATION 4.99. Using the Theorem 4.97, create three maps in three different panels of *LinComboTool*, where each map takes 4 arcs on the unit circle to 4 vertices of a square. Make sure that these maps satisfy the conditions of the theorem. Then click on the **Create LinCombogon** button to see the resulting image domain. Explore this idea by using different maps in the panels. For an example, see Figure 4.30
Try it out!

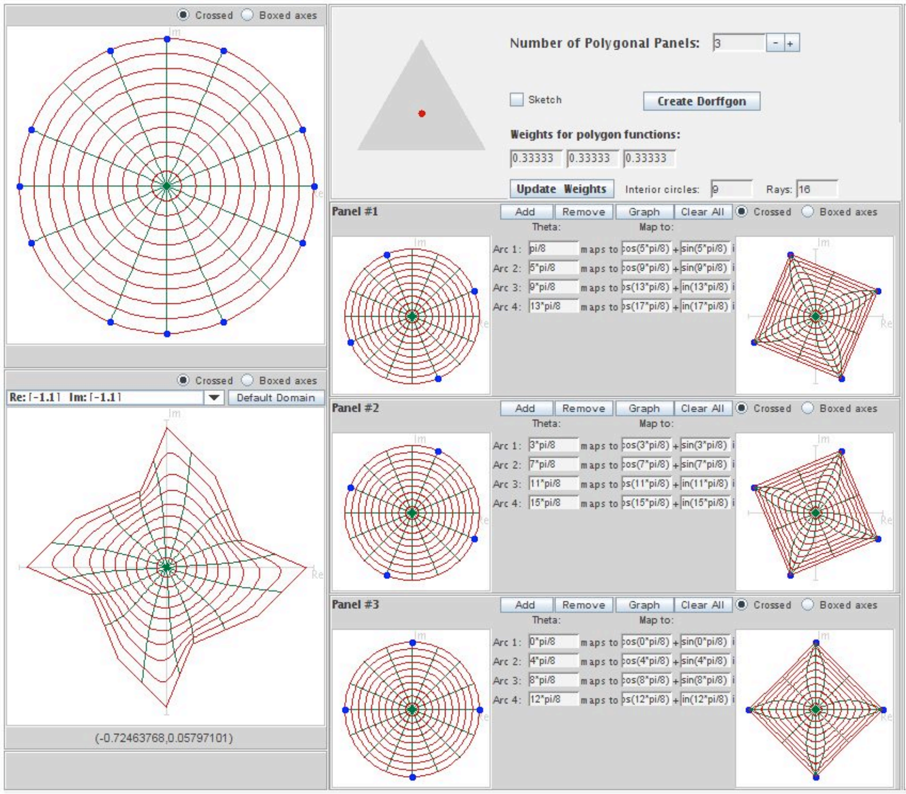


FIGURE 4.30. Example of image of $\mathbb{D}$ under the linear combination of three squares

LARGE PROJECT 4.100. In Example 4.92 and in Exercise 4.95 we took the linear combinations of two harmonic square mappings and ended up with fundamentally different images. Explore this with other n-gons. In particular, use *LinComboTool* to determine what and how many fundamentally different (i.e., not rotations or not scalings) images can be constructed when taking the linear combination with $t = \frac{1}{2}$ of two harmonic 5-gon maps? 6-gon maps? n-gon maps. Make sure that Condition A holds in every case and that $|\omega_3| < 1$.

Optional

LARGE PROJECT 4.101. In Exercise 4.96 we took the linear combinations of a harmonic 4-gon mapping and a harmonic 6-gon mapping with dilatations $-z^2$ and $-z^4$, respectively. Use *LinComboTool* to determine what combinations are possible and what images can be constructed when taking the linear combination with $t = \frac{1}{2}$ of a harmonic m-gon and n-gon, where $m < n$. Make sure that Condition A holds in every case and that $|\omega_3(z)| < 1$.

Optional

4.7. Convolutions

Another way of combining two univalent functions is the Hadamard product or convolution. For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad F(z) = \sum_{n=0}^{\infty} A_n z^n,$$

their convolution is defined as

$$f(z) * F(z) = \sum_{n=0}^{\infty} a_n A_n z^n.$$

EXAMPLE 4.102. Consider the convolution of the right half-plane function (see Example 4.11)

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

and the Koebe function (see Example 4.12)

$$F(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n.$$

Then

$$\begin{aligned} f(z) * F(z) =& \frac{z}{1-z} * \frac{z}{(1-z)^2} \\ =& \sum_{n=1}^{\infty} z^n * \sum_{n=1}^{\infty} n z^n \\ =& (z + z^2 + z^3 + z^4 + \cdots) * (z + 2z^2 + 3z^3 + 4z^4 + \cdots) \\ =& (z + 2z^2 + 3z^3 + 4z^4 + \cdots) \\ =& \frac{z}{(1-z)^2}. \end{aligned}$$

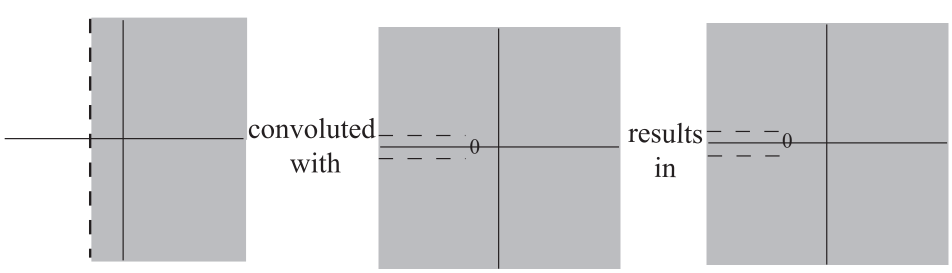


FIGURE 4.31. Right half-plane map convoluted with the Koebe function yields the Koebe function.

EXAMPLE 4.103. Now, consider the convolution of the Koebe function, $f(z) = \frac{z}{(1-z)^2}$, and the horizontal strip map, $F(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. What is the Hadamard product, $f(z) * F(z)$? We need to compute the Taylor series for F. To do so, notice that

$$\log(1-z) = \int \frac{-1}{1-z}\,dz = -\int \sum_{n=0}^{\infty} z^n\,dz = \sum_{n=0}^{\infty} \frac{-1}{n+1} z^{n+1}.$$

Likewise,

$$\log(1+z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1} z^{n+1}.$$

Hence,

$$\begin{aligned}\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1} z^{n+1} - \sum_{n=0}^{\infty} \frac{-1}{n+1} z^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}.\end{aligned}$$

Thus,

$$\begin{aligned}f(z) * F(z) =& \frac{z}{(1-z)^2} * \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \\ =& \sum_{n=1}^{\infty} nz^n * \sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1} \\ =& (z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + \cdots) * (z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots) \\ =& z + z^3 + z^5 + \cdots.\end{aligned}$$

Since $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$, we have that $\frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \cdots$ and $\frac{z}{1-z^2} = z + z^3 + z^5 + \cdots$. That is,

$$f(z) * F(z) = \frac{z}{(1-z)^2} * \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \frac{z}{1-z^2}.$$

EXERCISE 4.104. Let $f(z) = -\log(1-z)$ and $F(z) = \frac{z}{(1-z)^2}$. Determine $f(z) * F(z)$.

Try it out!

PROPOSITION 4.105.

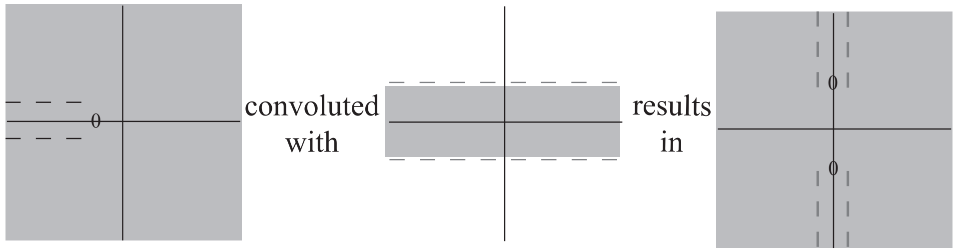


FIGURE 4.32. The Koebe function convoluted with a horizontal strip map yields a double-slit map.

(a) The right half-plane mapping, $f(z) = \dfrac{z}{1-z}$, acts as the convolution identity; that is, if F is an analytic function, then $\dfrac{z}{1-z} * F(z) = F(z)$.

(b) The Koebe function, $f(z) = \dfrac{z}{(1-z)^2}$, acts as a differential operator; that is, if $F(z)$ is an analytic function, then $\dfrac{z}{(1-z)^2} * F(z) = zF'(z)$.

(c) Convolution is commutative; that is, if f_1, f_2 are analytic functions, then $f_1 * f_2 = f_2 * f_1$.

(d) If f_1, f_2 are analytic functions, then $(f_1(z) * f_2(z))' = zf_1'(z) * f_2(z)$.

EXERCISE 4.106. Prove Proposition 4.105 (a)-(d).
Try it out!

Note that if $f_1, f_2 \in S$, then $f_1 * f_2$ may not be in S. For example,

$$\frac{z}{(1-z)^2} * \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n * \sum_{n=1}^{\infty} nz^n$$
$$= \sum_{n=1}^{\infty} n^2 z^n \notin S.$$

Why do we know that $\sum_{n=1}^{\infty} n^2 z^n \notin S$?

However, we do have the following results. Note that if the analytic function, $f \in S$, maps onto a domain that is convex, then we will denote that by writing $f \in K$. Similarly, if the harmonic function, $f \in S_H$, maps onto a domain that is convex, then we will write $f \in K_H$.

THEOREM 4.107 (Ruscheweyh and Sheil-Small, [**21**]). Let $f, f_1 \in K$. Then $f * f_1 \in K$. In addition, if $f_2, f_3 \in S$ map onto a close-to-convex, and a starlike domain, respectively. Then $f * f_2, f * f_3$ are in S and map onto a close-to-convex, and a starlike domain, respectively.

Now let's consider the case of harmonic convolutions.

Definition 4.108. For harmonic univalent functions

$$f(z) = h(z) + \overline{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \, \overline{z}^n \text{ and}$$

$$F(z) = H(z) + \overline{G}(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \, \overline{z}^n,$$

define the *harmonic convolution* as

$$f(z) * F(z) = h(z) * H(z) + \overline{g(z) * G(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \, \overline{z}^n. \tag{60}$$

As mentioned above, for the convolution of analytic functions it is known that if $f_1, f_2 \in K$, then $f_1 * f_2 \in K$. Is such a similar result true for harmonic univalent convex mappings?

There are a few known results about harmonic convolutions of functions on $\mathbb{D}$.

Theorem 4.109 (Clunie and Sheil-Small, [**5**]). If $f \in K_H$ and $\varphi \in S$, then the functions

$$f \, * \, (\alpha\overline{\varphi} + \varphi) \in S_H$$

map $\mathbb{D}$ onto a close-to-convex domain, where ($|\alpha| \leq 1$).

Clunie and Sheil-Small posed the following open problem (see [**5**]).

Open Problem 4.110. Let $f \in K_H$, then what are the collection of harmonic functions F such $f * F \in K_H$?

As partial answers to this open question, there are the following results.

Theorem 4.111 (Ruscheweyh and Salinas, [**20**]). Let g be analytic in $\mathbb{D}$. Then

$$f \,\widetilde{*}\, g = \mathrm{Re}\{f\} * g + \overline{\mathrm{Im}\{f\} * g} \in K_H$$

for all $f \in K_H \iff$ for each $\gamma \in \mathbb{R}, g + i\gamma z g'$ is convex in the imaginary direction.

Theorem 4.112 (Goodloe, [**12**]). Let $f_m, f_n \in K_H$ be the canonical harmonic functions that map $\mathbb{D}$ onto the regular m-gon and n-gon, respectively. Then $f_m * f_n \in K_H$ and the image of $\mathbb{D}$ is a p-gon, where $p = lcm(m,n)$.

EXERCISE 4.113. Compute $f_k = f_4 * f_6$, where $f_4 = h_4 + \overline{g_4}$ is the canonical square map (see Example 4.89) given by

$$h_4(z) = \frac{i}{4}\log\left(\frac{1+z}{1-z}\right) - \frac{1}{4}\log\left(\frac{i+z}{i-z}\right) = \int \frac{1}{1-z^4}\,dz$$
$$g_4(z) = \frac{i}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{4}\log\left(\frac{i+z}{i-z}\right) = \int \frac{-z^2}{1-z^4}\,dz.$$

and $f_6 = h_6 + \overline{g_6}$ is the canonical regular hexagon map (see Exercise 4.96) given by

$$h_6(z) = \frac{1}{6}\log\left(\frac{1+z}{1-z}\right) + \frac{e^{\frac{-i\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i\pi}{3}}z}{1-e^{\frac{i\pi}{3}}z}\right) + \frac{e^{\frac{-i2\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i2\pi}{3}}z}{1-e^{\frac{i2\pi}{3}}z}\right) = \int \frac{1}{1-z^6}\,dz$$
$$g_6(z) = -\frac{1}{6}\log\left(\frac{1+z}{1-z}\right) - \frac{e^{\frac{i\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i\pi}{3}}z}{1-e^{\frac{i\pi}{3}}z}\right) - \frac{e^{\frac{i2\pi}{3}}}{6}\log\left(\frac{1+e^{\frac{i2\pi}{3}}z}{1-e^{\frac{i2\pi}{3}}z}\right) = \int \frac{-z^4}{1-z^6}\,dz.$$

Sketch $f_k(\mathbb{D})$ using *ComplexTool*.

Try it out!

In considering Open Problem 4.110, let's look at a simple problem: if $f_1, f_2 \in K_H$, then when is $f_1 * f_2 \in S_H$?

Recall Lewy's Theorem that $f = h + \overline{g}$ with $h'(z) \neq 0$ in $\mathbb{D}$ is locally univalent and sense-preserving if and only if $\omega(z) = g'(z)/h'(z)| < 1, \forall z \in \mathbb{D}$.

THEOREM 4.114 (Dorff, [**6**]). Let $f_1 = h_1 + \overline{g}_1, f_2 = h_2 + \overline{g}_2 \in K_H$ with $h_k(z) + g_k(z) = \frac{z}{1-z}$ for $k = 1, 2$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H$ and is convex in the direction of the real axis.

PROOF. Since $h(z) + g(z) = \dfrac{z}{1-z}$ and $F(z) * \dfrac{z}{1-z} = F(z)$ for any analytic function F, we have that

$$\begin{aligned} h_2 - g_2 &= (h_1 + g_1) * (h_2 - g_2) \\ &= h_1 * h_2 - h_1 * g_2 + h_2 * g_1 - g_1 * g_2 \\ h_1 - g_1 &= (h_1 - g_1) * (h_2 + g_2) = \\ &\quad h_1 * h_2 + h_1 * g_2 - h_2 * g_1 - g_1 * g_2. \end{aligned}$$

Thus,

$$h_1 * h_2 - g_1 * g_2 = \tfrac{1}{2}[(h_1 - g_1) + (h_2 - g_2)]. \tag{61}$$

We will now show that $(h_1 - g_1) + (h_2 - g_2)$ is convex in the direction of the real axis. Note that

$$h'(z) - g'(z) = \big(h'(z) + g'(z)\big)\left(\frac{h'(z) - g'(z)}{h'(z) + g'(z)}\right) = \big(h'(z) + g'(z)\big)\left(\frac{1-\omega(z)}{1+\omega(z)}\right) = \frac{p(z)}{(1-z)^2},$$

where $\text{Re}\{p(z)\} > 0, \forall z \in \mathbb{D}$.
So, letting $\varphi(z) = z/(1-z)^2$, we have

$$\begin{aligned}\text{Re}\left\{\frac{z[(h_1'(z) - g_1'(z)) + (h_2'(z) - g_2'(z))]}{\varphi(Z)}\right\} &= \text{Re}\left\{\frac{\frac{z}{(1-z)^2}[p_1(z) + p_2(z)]}{\frac{z}{(1-z)^2}}\right\} \\ &= \text{Re}\Big\{p_1(z) + p_2(z)\Big\} > 0.\end{aligned}$$

Therefore, by Theorem 4.76 in Section 4.5 and eq.(61), $h_1 * h_2 - g_1 * g_2$ is convex in the direction of the real axis.

Finally, since we assumed that $f_1 \widetilde{*} f_2$ is locally univalent, we apply Clunie and Sheil-Small's Shearing Theorem (see Theorem 4.40) to get that $f_1 \widetilde{*} f_2 = h_1 * h_2 - g_1 * g_2$ is convex in the direction of the real axis. □

It is known (see [**7**]), that for any right half-plane mapping $f = h + \overline{g} \in K_H$,

$$h(z) + g(z) = \frac{z}{1-z}.$$

Hence, Theorem **??** applies to harmonic right half-plane mappings.

Example 4.115. Let $f_0 = h_0 + \overline{g}_0$ be the canonical right half-plane mapping given in Example 4.11 with $h_0(z) + g_0(z) = \frac{z}{1-z}$ with $\omega(z) = -z$. Then

$$\begin{aligned}h_0(z) =& \frac{z - \frac{1}{2}z^2}{(1-z)^2} \\ g_0(z) =& \frac{\frac{1}{2}z^2}{(1-z)^2}.\end{aligned}$$

Next, let $f_1 = h_1 + \overline{g}_1$, where $h_1(z) + g_1(z) = \frac{z}{1-z}$ with $\omega(z) = z$. Then

$$\begin{aligned}h_1(z) &= \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} \\ g_1(z) &= -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z}.\end{aligned}$$

Note that f_1 is a right half-strip mapping (see Figure 4.33).

Consider $F_1 = f_0 * f_1 = H_1 + \overline{G_1}$. Note that

$$\begin{aligned}H_1(Z) &= h_0(z) * h_1(z) = \frac{1}{2}\big[h_1(z) + zh_1'(z)\big] = \frac{1}{8}\log\left(\frac{1+z}{1-z}\right) + \frac{\frac{3}{4}z - \frac{1}{4}z^3}{(1-z)^2(1+z)} \\ G_1(z) &= g_0(z) * g_1(z) = \frac{1}{2}\big[g_1(z) - zg_1'(z)\big] = -\frac{1}{8}\log\left(\frac{1+z}{1-z}\right) + \frac{\frac{1}{4}z - \frac{1}{2}z^2 - \frac{1}{4}z^3}{(1-z)^2(1+z)},\end{aligned}$$

with

$$\widetilde{\omega}(z) = -z\left(\frac{2z^2 + z + 1}{z^2 + z + 2}\right).$$

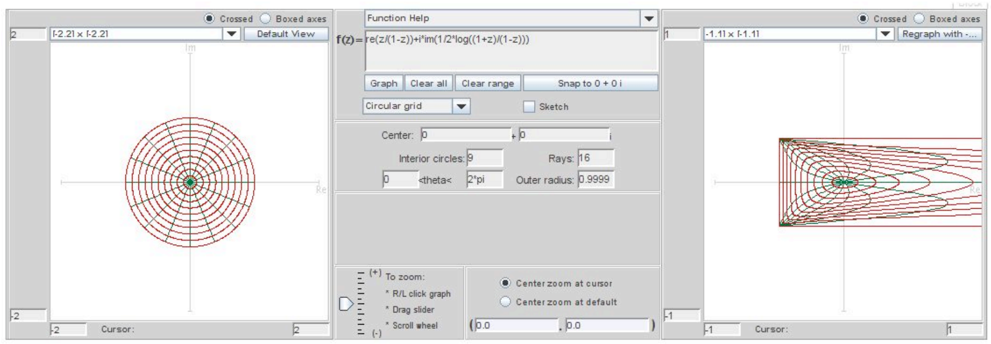


FIGURE 4.33. Image of $\mathbb{D}$ under f_1.

The image of $\mathbb{D}$ under $F_1 = f_0 * f_1$ is shown in Figure 4.34.

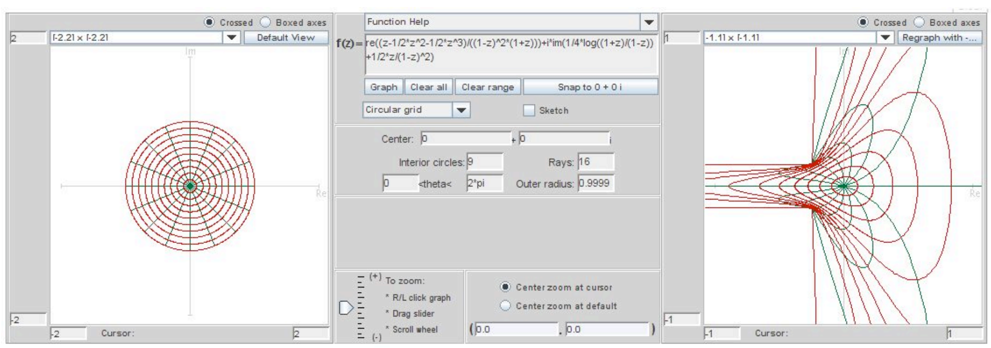


FIGURE 4.34. Image of $\mathbb{D}$ under $F_1 = f_0 * f_1$.

EXERCISE 4.116. Compute $F = H + \overline{G}$, where $F = f_0 * f_0$. Sketch $F(\mathbb{D})$ using *ComplexTool*.

Try it out!

Throughout the rest of this section we will consider the question "For which dilatation functions, $\omega = g'/h'$, is the function $f = h + \overline{g}$ locally univealent. In doing so, let

$$f_0(z) = h_0(z) + \overline{g_0(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}}$$

be the canonical right half-plane mapping given in Example 4.29.

Also, as mentioned in the proof above, the collection of functions $f = h + \overline{g} \in S_H^O$ that map $\mathbb{D}$ onto the right half-plane, $R = \{w : \text{Re}(w) > -1/2\}$, have the form

$$h(z) + g(z) = \frac{z}{1-z}.$$

We will use the following method to prove that local univalency holds:

Method 1. (Cohn's Rule, see [**19**], p 375) Given a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

of degree n, let

$$f^*(z) = z^n\, \overline{f(1/\,\overline{z})} = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_0} z^n.$$

Denote by p and s the number of zeros of f inside the unit circle and on it, respectively. If $|a_0| < |a_n|$, then

$$f_1(z) = \frac{\overline{a_n} f(z) - a_0 f^*(z)}{z}$$

is of degree $n-1$ with $p_1 = p-1$ and $s_1 = s$ the number of zeros of f_1 inside the unit circle and on it, respectively.

Theorem 4.117. Let $f = h + \overline{g} \in K_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega(z) = e^{i\theta} z^n$ ($n \in \mathbb{N}$ and $\theta \in \mathbb{R}$). If $n = 1, 2$, then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Proof. Let the dilatation of $f_0 * f$ be given by $\widetilde{\omega} = (g_0 * g)'/(h_0 * h)'$. By Theorem 4.114 and by Lewy's Theorem, we just need to show that $|\widetilde{\omega}(z)| < 1, \forall z \in \mathbb{D}$.

First, note that if F is analytic in $\mathbb{D}$ and $F(0) = 0$, then

$$\begin{aligned} h_0(z) * F(z) &= \frac{1}{2}\big[F(z) + zF'(z)\big] \\ g_0(z) * F(z) &= \frac{1}{2}\big[F(z) - zF'(z)\big]. \end{aligned}$$

Also, since $g'(z) = \omega(z) h'(z)$, we know $g''(z) = \omega(z) h''(z) + \omega'(z) h'(z)$.

Hence

$$\widetilde{\omega}(z) = -\frac{z g''(z)}{2h'(z) + z h''(z)} = \frac{-z\omega'(z) h'(z) - z\omega(z) h''(z)}{2h'(z) + z h''(z)}. \tag{62}$$

Using $h(z) + g(z) = \frac{z}{1-z}$ and $g'(z) = \omega(z) h'(z)$, we can solve for $h'(z)$ and $h''(z)$ in terms of z and $\omega(z)$:

$$h'(z) = \frac{1}{(1+\omega(z))(1-z)^2}$$

$$h''(z) = \frac{2(1+\omega(z)) - \omega'(z)(1-z)}{(1+\omega(z))^2 (1-z)^3}.$$

Substituting these formulas for h' and h'' into the equation for $\widetilde{\omega}$, we derive:

$$\begin{aligned}\widetilde{\omega}(z) =& \frac{-z\omega'(z)h'(z) - z\omega(z)h''(z)}{2h'(z) + zh''(z)} \\ =& - z\frac{\omega^2(z) + [\omega(z) - \frac{1}{2}\omega'(z)z] + \frac{1}{2}\omega'(z)}{1 + [\omega(z) - \frac{1}{2}\omega'(z)z] + \frac{1}{2}\omega'(z)z^2}.\end{aligned} \tag{63}$$

Now, consider the case in which $\omega(z) = e^{i\theta}z$. Then eq (63) yields

$$\widetilde{\omega}(z) = - ze^{2i\theta}\frac{\left(z^2 + \frac{1}{2}e^{-i\theta}z + \frac{1}{2}e^{-i\theta}\right)}{\left(1 + \frac{1}{2}e^{i\theta}z + \frac{1}{2}e^{i\theta}z^2\right)} = -ze^{2i\theta}\frac{p(z)}{q(z)}.$$

Note that $q(z) = z^2\ \overline{p(1/\overline{z})}$. In such a situation, if z_0 is a zero of p, then $\frac{1}{\overline{z_0}}$ is a zero of q. Hence,

$$\widetilde{\omega}(z) = - ze^{2i\theta}\frac{(z + A)(z + B)}{(1 + \ \overline{A}z)(1 + \ \overline{B}z)}.$$

Using Method 1, we have

$$p_1(z) = \frac{\overline{a_2}p(z) - a_0p^*(z)}{z} = \frac{3}{4}z + \left(\frac{1}{2}e^{-i\theta} - \frac{1}{4}\right).$$

So, p_1 has one zero at $z_0 = \frac{1}{3} - \frac{2}{3}e^{-i\theta} \in \mathbb{D}$. By Cohn's Rule, p has two zeros, namely A and B, with $|A|, |B| < 1$.

Next, consider the case in which $\omega(z) = e^{i\theta}z^2$. In this case,

$$|\widetilde{\omega}(z)| = \left|z^2\right|\left|\frac{z^3 + e^{-i\theta}}{1 + e^{i\theta}z^3}\right| = |z|^2 < 1.$$

□

Example 4.118. Let $f_2 = h_2 + \ \overline{g}_2$ be the harmonic mapping in $\mathbb{D}$ such that $h_1(z) + g_2(z) = \frac{z}{1-z}$ and $\omega_2(z) = -z^2$. Then we can compute

$$\begin{aligned}h_2(z) =& \frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} + \frac{1}{4}\frac{z}{(1-z)^2} \\ g_2(z) =& -\frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} - \frac{1}{4}\frac{z}{(1-z)^2}\end{aligned}$$

and the image of $\mathbb{D}$ under f_2 is the right half-plane, $R = \{w \in \mathbb{C} \big| \operatorname{Re}\{w\} \geq -\frac{1}{2}\}$. Note that $f_2(e^{it}) = \frac{1}{2} + i\frac{\pi}{16}$, if $0 < t < \pi$, and $f_2(e^{it}) = \frac{1}{2} - i\frac{\pi}{16}$, if $\pi < t < 2\pi$ (see Figure 4.35).

Let

$$F_2 = h_0 * h_2 + \ \overline{g_0 * g_2} = H_2 + \ \overline{G_2}.$$

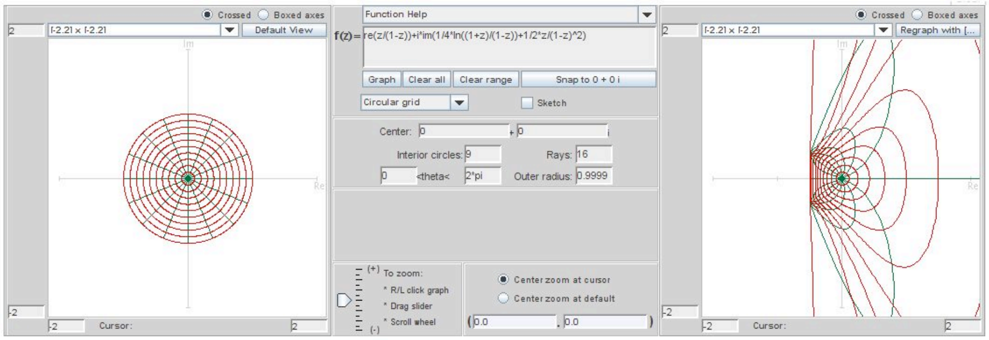


FIGURE 4.35. Image of $\mathbb{D}$ under f_2.

Then we can compute that

$$
(64)\quad
\begin{aligned}
H_2(z) =& \frac{1}{16}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{4}\frac{z}{1-z} + \frac{1}{8}\frac{z}{(1-z)^2} + \frac{1}{2}\frac{z}{(1-z)^3(1+z)} \\
G_2(z) =& -\frac{1}{16}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{4}\frac{z}{1-z} - \frac{1}{8}\frac{z}{(1-z)^2} + \frac{1}{2}\frac{z^3}{(1-z)^3(1+z)}.
\end{aligned}
$$

It can be shown analytically that $F_2(\mathbb{D})$ is the entire complex plane minus two half-lines given by $x \pm \frac{\pi}{16}i$, $x \leq -\frac{1}{4}$. This is not clear if we use *ComplexTool* with the standard settings to view this image (see Figure 4.36). However, using both this image and the image of just the unit circle (see Figure 4.37), this result seems reasonable [Note: to graph the image of $\partial\mathbb{D}$ in *ComplexTool*, change the settings in the middle box of *ComplexTool* to **Interior circles: 0** and **Rays: 0**].

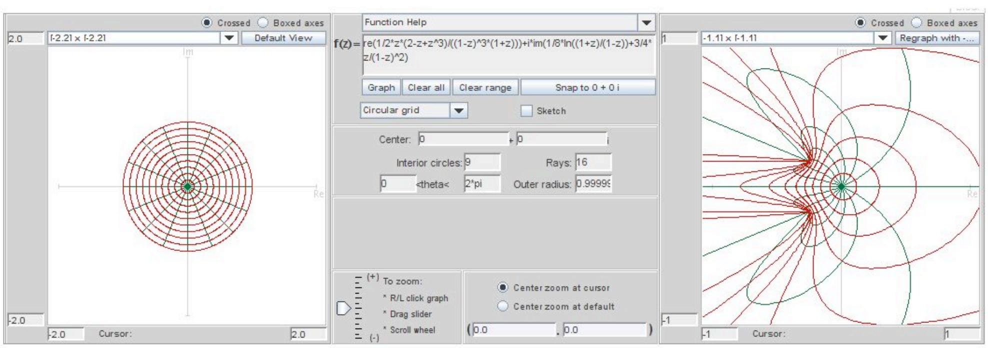


FIGURE 4.36. Image of $\mathbb{D}$ under $F_2 = f_0 * f_2$.

REMARK 4.119. If we assume the hypotheses of the previous theorem with the exception of making $n \geq 3$, then for each n we can find a specific $\omega(z) = e^{i\theta}z^n$ such

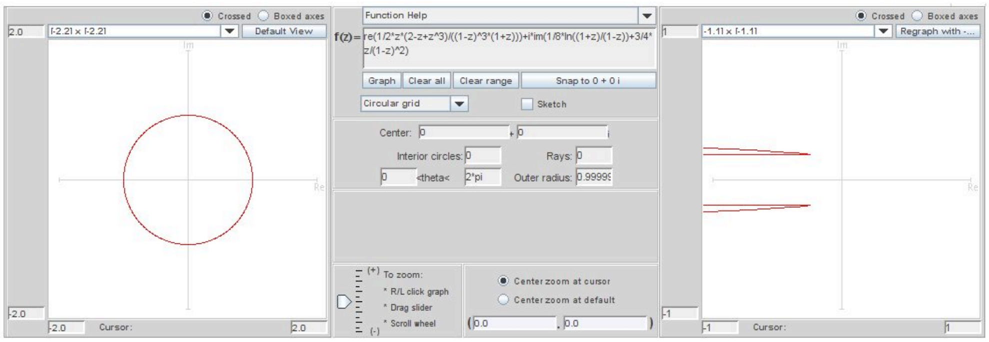


FIGURE 4.37. Image of $\partial\mathbb{D}$ under $F_2 = f_0 * f_2$.

that $f_0 * f \notin S_H^O$. For example, if n is odd, let $\omega(z) = -z^n$ and then eq (63) yields

$$\widetilde{\omega}(z) = -z^n \frac{z^{n+1} + \left(\frac{n}{2} - 1\right)z - \frac{n}{2}}{1 + \left(\frac{n}{2} - 1\right)z^n - \frac{n}{2}z^{n+1}}.$$

It suffices to show that for some point $z_0 \in \mathbb{D}$, $|\widetilde{\omega}(z_0)| > 1$. Let $z_0 = -\frac{n}{n+1} \in \mathbb{D}$. Then

$$\widetilde{\omega}(z_0) = \left(\frac{n}{n+1}\right)^n \frac{\left(\frac{n}{n+1}\right)^{n+1} - \left(\frac{n}{2} - 1\right)\left(\frac{n}{n+1}\right) - \frac{n}{2}}{1 - \left(\frac{n}{2} - 1\right)\left(\frac{n}{n+1}\right)^n - \left(\frac{n}{2}\right)\left(\frac{n}{n+1}\right)^{n+1}}$$

(65)

$$= 1 + \frac{\left[\left(\frac{n+1}{n}\right)^n - \left(\frac{n}{n+1}\right)^{n+1}\right] + \left[1 - \frac{n}{n+1}\right]}{\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{n+1}\right) + \left(\frac{n+1}{n}\right)^n}.$$

Note that $\left[\left(\frac{n+1}{n}\right)^n - \left(\frac{n}{n+1}\right)^{n+1}\right] + \left[1 - \frac{n}{n+1}\right] > 0$. Also, $\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{n+1}\right) + \left(\frac{n+1}{n}\right)^n > 0$ since $\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{n+1}\right) > n - \frac{3}{2} > e$ and $\left(\frac{n+1}{n}\right)^n$ is an increasing series converging to e. Thus, if $n \geq 5$ is odd, $|\widetilde{\omega}(z_0)| > 1$. If $n = 3$, it is easy to compute that $|\widetilde{\omega}(z_0)| = \left(\frac{3}{4}\right)^3 \left|\frac{3^4 - \frac{1}{2}\cdot 3\cdot 4^3 - \frac{3}{2}\cdot 4^4}{4^4 - \frac{1}{2}\cdot 3^3\cdot 4 - \frac{3}{2}\cdot 3^4}\right| > 2$. Now, if n is even, let $\omega(z) = z^n$ and $z_0 = -\frac{n}{n+1}$. This simplifies to the same $\widetilde{\omega}(z_0)$ given eq (65) and the argument above also holds for $n \geq 6$. If $n = 4$, $|\widetilde{\omega}(z_0)| = \left(\frac{4}{5}\right)^4 \left|\frac{4^5 - 4\cdot 5^4 - 2\cdot 5^5}{5^5 - 4^4\cdot 5 - 2\cdot 4^5}\right| > 15$.

EXPLORATION 4.120. Using *ComplexPlot*, graph $\widetilde{\omega}(\mathbb{D})$ given in eq (63) for $\omega(z) = -z^n$, where $n = 1, 2, 3, 4$. Explain how these images support Theorem 4.117 and Remark 4.119.

Try it out!

THEOREM 4.121. Let $f = h + \overline{g} \in K_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega(z) = \frac{z+a}{1+az}$ with $a \in (-1, 1)$. Then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Proof. Using $\omega(z) = \frac{z+a}{1+az}$, where $-1 < a < 1$, we have

$$\begin{aligned}\widetilde{\omega}(z) &= -\,z\frac{\left(z^2 + \frac{1+3a}{2}z + \frac{1+a}{2}\right)}{\left(1 + \frac{1+3a}{2}z + \frac{1+a}{2}z^2\right)}\\ &= -\,z\frac{f(z)}{f^*(z)}\\ &= -\,z\frac{(z+A)(z+B)}{(1+\ \overline{A}z)(1+\ \overline{B}z)}\\ &= -\,z\frac{p(z)}{q(z)}.\end{aligned}$$

Again using Method 1,

$$p_1(z) = \frac{\overline{a_2}p(z) - a_0p^*(z)}{z} = \frac{(a+3)(1-a)}{4}z + \frac{(1+3a)(1-a)}{4}.$$

So p_1 has one zero at $z_0 = -\frac{1+3a}{a+3}$ which is in the unit circle since $-1 < a < 1$. Thus, $|A|$, $|B| < 1$. □

Large Project 4.122. In Theorem 4.114, we require that the resulting convolution function satisfy the dilatation condition

$$|\omega(z)| = \left|\frac{g'(z)}{h'(z)}\right| < 1, \forall z \in \mathbb{D}.$$

Determine various ω functions for which the dilatation condition holds and ones for which it does not hold. See Theorem 4.117, Theorem 4.121, and Remark 4.119 for examples.

Optional

Large Project 4.123. Similar to the right half-plane map, $f = h +\ \overline{g}$ is an asymmetric vertical strip map if $h(z) + g(z) = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$, where $0 < \alpha < \pi$. Theorem 4.114 can be stated in terms of asymmetric vertical strip mappings instead of right half-plane mappings.

> Theorem: Let $f = h + \overline{g} \in K_H^O$ with $\omega = g'/h'$ be such that $h+g = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$, where $0 < \alpha < \pi$. Then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Determine various ω functions for which the dilatation condition holds for this theorem and ones for which it does not hold.

Optional

4.8. Conclusion

We have presented an introduction to harmonic univalent mappings and described a few topics to entice a beginner. Our emphasis is on the geometric aspects of harmonic univalent mappings that students can explore using the exercises, the exploratory problems, and the projects along with the applets. There are more interesting and deeper topics in harmonic univalent mappings. Here is a short list along with some resources: (a) coefficient estimates and conjectures ([**5**], [**9**], [**23**]); (b) a generalized Riemann Mapping Theorem ([**9**], [**14**], [**26**]); (c) properties of special classes of functions such as convex, close-to-convex, starlike, and typically real ([**5**], [**9**], [**23**]); (d) harmonic polynomials ([**4**], [**24**], [**28**]); (e) extremal problems ([**9**]); (f) harmonic meromorphic functions ([**16**], [**25**]); (g) inner mapping radius ([**2**], [**8**]); and (h) multiply connected domains ([**9**], [**10**]). Another topic is the connection between harmonic mappings and minimal surfaces. This topic is discussed in the chapter on minimal surfaces in this book. In addition, there are several nice general resources to learn more about harmonic univalent functions. These include Peter Duren's book [**9**], Clunie and Sheil-Small's original article [**5**], Bshouty and Hengartner's article [**3**], and Schober's article [**22**]. Finally, Bshouty and Hengartner complied a list of open problems and conjectures [**2**].

4.9. Additional Exercises

The Family S of Analytic, Normalized, Univalent Functions

EXPLORATION 4.124.

(a) Using *ComplexTool* guess the smallest $k > 0$ such that $(z+k)^2$ is univalent in $\mathbb{D}$.
(b) Prove your guess from (a).
(c) Using *ComplexTool* guess the smallest $k > 0$ such that $(z+k)^3$ is univalent in $\mathbb{D}$.
(d) Prove your guess from (c).

EXERCISE 4.125. Show that $f(z) = z + a_3 z^3$ is univalent in $\mathbb{D} \iff |a_3| \leq \frac{1}{3}$. Determine $f(\mathbb{D})$ analytically when $a_3 = -\frac{1}{3}$.

EXERCISE 4.126. Work out the details to show that $\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots$

EXERCISE 4.127. Determine $f(\mathbb{D})$ analytically $f(z) = \dfrac{z - cz^2}{(1-z)^2}$, where $0 < c < 1$.

EXERCISE 4.128. Prove that $f(z) = \dfrac{1}{2} \log \left(\dfrac{1+z}{1-z} \right)$ is univalent. Determine $f(\mathbb{D})$ analytically.

EXPLORATION 4.129. Consider the function

$$f_c(z) = \frac{1}{2c} \left[\left(\frac{1+z}{1-z} \right)^c - 1 \right].$$

(a) Show that if $c = 2$, then $f_c(z)$ is the Koebe function.
(b) Show that if $c = 1$, then $f_c(z)$ is the right half-plane mapping.
(c) Use *ComplexTool* to view the image of $\mathbb{D}$ under f_c for various values $0 < c < 2$. For what values of c does f_c appear to be univalent.

EXERCISE 4.130. Find the image of $\mathbb{D}$ analytically under the univalent function $f(z) = \dfrac{z}{1 - z^2}$.

The Family S_H of Normalized, Harmonic, Univalent Functions

EXERCISE 4.131. Determine if $f(x,y) = u(x,y) + iv(x,y) = (x^3 + xy^2) + i(x^2 y + y^3)$ is complex-valued harmonic.

EXERCISE 4.132. Prove that $f(x,y) = u(x,y) + iv(x,y)$ is harmonic $\iff \dfrac{\partial^2 f}{\partial z \partial \overline{z}} = 0$.

EXERCISE 4.133. Rewrite $f(x,y) = u(x,y) + iv(x,y) = (x - \frac{1}{2}x^2 + \frac{1}{2}y^2) + i(y - xy)$ in terms of z and $\overline{z}$ and then determine if f is analytic.

EXERCISE 4.134. Prove that for all functions $f \in S_H^O$, the sharp inequality $|b_2| \leq \frac{1}{2}$ holds.

EXERCISE 4.135. Verify that the image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{2}\bar{z}^2$ is a hypocycloid with 3 cusps.

EXERCISE 4.136. If a domain is convex in the direction $e^{i\varphi}$ for every value of $\varphi \in [0, \pi)$, then the domain is called a *convex* domain. For example, a disk is a convex domain. For which values of $n = 1, 2, 3, \ldots$ are the following functions that map $\mathbb{D}$ onto a convex domain?

(a) $f(z) = z^n$,
(b) $f(z) = z - \frac{1}{n}z^n$ (see Example 4.8 and Definition 4.9),
(c) $f(z) = \frac{z}{(1-z)^n}$ (see Examples 4.11 and 4.12 to get you started).

The Shearing Technique

EXPLORATION 4.137. Let $f = h + \overline{g}$ with $h(z) - g(z) = z - \frac{1}{n}z^n$ and $\omega(z) = z^{n-1}$. Use *ShearTool* to sketch the graph of $f(\mathbb{D})$ for different values of n and then compute h and g explicitly so that $f \in S_H^O$.

EXERCISE 4.138. Let $f = h + \overline{g}$ with $h(z) - g(z) = \frac{z}{(1-z)^2}$ and $\omega(z) = -z$. Compute h and g explicitly so that $f \in S_H^O$ and determine $f(\mathbb{D})$.

EXERCISE 4.139. Let $f = h + \overline{g}$ with $h(z) - g(z) = \dfrac{z}{(1-z)^2}$ and $\omega(z) = z\,\dfrac{z + \frac{1}{2}z}{1 + \frac{1}{2}z}$.

(a) Show that $|\omega(z)| < 1, \forall z \in \mathbb{D}$.
(b) Compute h and g explicitly so that $f \in S_H^O$.
(c) Show that $f(\mathbb{D})$ is a slit domain like the Koebe domain. Determine where the tip of the slit is located.
(d) What is the significance of this example in relationship to the Riemann Mapping Theorem?

EXERCISE 4.140. Let $f = h + \overline{g}$ with $h(z) + g(z) = \dfrac{z}{1-z}$ and $\omega(z) = e^{i\theta}z$, where $\theta \in [0, 2\pi)$. Use *ShearTool* to sketch the graph of $f(\mathbb{D})$ for different values of n and then compute h and g explicitly so that $f \in S_H^O$.

EXPLORATION 4.141. We can find harmonic functions, $f_n = h_n + \overline{g}_n$, that map onto regular n-gons by generalizing the ideas from Example 4.60. Use *ShearTool* to explore the images of $\mathbb{D}$ under $f = h + \overline{g}$, where f comes from shearing

$$h_n(z) - g_n(z) = \sum_{k=0}^{n-1} \frac{-2\cos\left(\frac{2\pi k}{n}\right)}{n} \log\left(1 - ze^{i\frac{2\pi k}{n}}\right)$$

with $\omega(z) = -z^{n-2}$.

EXPLORATION 4.142. Let $f = h + \overline{g}$ with $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and $\omega(z) = m^2z^2$, where $m = e^{i\theta}(0 \le \theta \le \frac{\pi}{2})$.Use *ShearTool* to sketch the graph of $f(\mathbb{D})$ for different values of n and then compute h and g explicitly so that $f \in S_H^O$ [Preview: This Exploration fits nicely with minimal surfaces, because when $m = 1$ f lifts to a canonical minimal surface, Scherk's doubly-periodic, and when $m = i$ f lifts to a different canonical minimal surface, the helicoid].

Dilatations

EXPLORATION 4.143. Shear $h(z) - g(z) = \frac{z}{1-z}$ using $\omega(z) = az$, where $-1 \le a \le 1$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ as a varies.

EXPLORATION 4.144. Shear $h(z) - g(z) = \frac{z}{1-z}$ using $\omega(z) = z^n$, where $n = 1, 2, 3, 4, 5$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ as a varies.

EXPLORATION 4.145. Shear $h(z) - g(z) = \log\left(\frac{1-z}{1+z}\right)$ using $\omega(z) = e^{i\pi n/6}z$, where $n = 0, \ldots, 6$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ as n varies.

EXPLORATION 4.146. Shear $h(z) - g(z) = \frac{z}{1-z} + ae^{\frac{z+1}{z-1}}$ using $\omega(z) = e^{\frac{z+1}{z-1}}$, where $-0.5 \le a \le 0.5$ and sketch $f(\mathbb{D})$ using *ShearTool*. Describe what happens to $f(\mathbb{D})$ as a varies.

EXERCISE 4.147. Let $h_\alpha(z) = \frac{z}{1+ze^{-i\alpha}}$, where $0 < \alpha < \pi$, and $\omega_\alpha(z) = e^{-i\left(\frac{z+e^{-i\alpha}}{1+ze^{-i\alpha}}\right)}$. Compute $f_\alpha = h_\alpha + \overline{g}_\alpha$ and show that $f_\alpha \in S_H^O$. Use *ComplexTool* to sketch $f_\alpha(\mathbb{D})$ for various values of α [Note: as α approaches 0, you should get the image shown in Figure 4.15].

EXERCISE 4.148. Let $h_\gamma(z) = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right)$, where $\frac{\pi}{2} \le \gamma < \pi$, and $\omega_\gamma(z) = e^{-\left(\frac{2\sin(\pi-\gamma)}{(\pi-\gamma)}h(z)-1\right)}$. Compute $f_\gamma = h_\gamma + \overline{g}_\gamma$ and show that $f_\gamma \in S_H^O$. Use *ComplexTool* to sketch $f_\gamma(\mathbb{D})$ for various values of γ [Note: as γ approaches π, you should get the image shown in Figure 4.15, but each $f_\gamma(\mathbb{D})$ is different than any $f_\alpha(\mathbb{D})$ in the Exercise 4.147].

Harmonic Linear Combinations

EXERCISE 4.149. Let

$$f_1(z) = \mathrm{Re}\left\{-z - 2\log(1-z)\right\} + i\,\mathrm{Im}\left\{z\right\},$$
$$f_2(z) = \mathrm{Re}\left\{\frac{z + 1/3z^3}{(1-z)^3}\right\} + i\,\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\}.$$

(a) Show that f_1 can be derived by shearing $h(z) - g(z) = z$ with $g'(z) = zh'(z)$.

(b) Use *ComplexTool* to plot the image of $\mathbb{D}$ under f_1. Recall that f_2 is the "harmonic Koebe" function. What is the image of $\mathbb{D}$ under f_2?

(c) Use *ComplexTool* to see that $f_3 = tf_1 + (1-t)f_2$ is not univalent for at least one value of $t, (0 \leq t \leq 1)$. Why does this not contradict Theorem 4.88?

EXPLORATION 4.150. Let

$$f_1(z) = \operatorname{Re}\left\{\frac{z}{(1-z)^2}\right\} + i\operatorname{Im}\left\{\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right\},$$
$$f_2(z) = \operatorname{Re}\left\{\frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right)\right\} + i\operatorname{Im}\left\{\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right\}.$$

Show that f_1 and f_2 satisfies the conditions of Theorem 4.88 and then use *ComplexTool* to plot images of $f_3 = tf_1 + (1-t)f_2$ for various values of t.

EXPLORATION 4.151. Let

$$f_1(z) = \operatorname{Re}\left\{\frac{z}{1-z} - \frac{1}{2}e^{\frac{z+1}{z-1}}\right\} + i\operatorname{Im}\left\{\frac{z}{1-z} + \frac{1}{2}e^{\frac{z+1}{z-1}}\right\},$$
$$f_2(z) = \operatorname{Re}\left\{z - \frac{1}{4}z^2 - \frac{1}{4}(z-1)^2e^{\frac{z+1}{z-1}}\right\} + i\operatorname{Im}\left\{z - \frac{1}{4}z^2 + \frac{1}{4}(z-1)^2e^{\frac{z+1}{z-1}}\right\}.$$

Show that f_1 and f_2 satisfies the conditions of Theorem 4.88 and then use *ComplexTool* to plot images of $f_3 = tf_1 + (1-t)f_2$ for various values of t.

EXERCISE 4.152. Repeat the steps in Example 4.92 using the same function for f_1 but replacing f_2 with the harmonic square map in Example 4.60, where

$$h_2(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right)$$
$$g_2(z) = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right).$$

(a.) In this case, $\omega_1(z) = z^2$ while $\omega_2(z) = -z^2$. Using eq. (59) in the remark above, show that f_3 is locally univalent.

(b.) Use *LinComboTool* find the image of $f_3(\mathbb{D})$ using this f_1 and f_2.

(c.) Explain why this happens by using the approach in Example 4.92 to compute the new values of $w_1, \ldots, w_8$ and then use the visualization technique in the example to plot the eight vertices $z_0, \ldots, z_7$ and draw the midpoints $w_1, \ldots, w_8$.

EXPLORATION 4.153. Using *LinComboTool*, start with the same arc values and corresponding point values as in Example 4.92. In **Panel #1** increase the arc values by increments of $\frac{\pi}{16}$ while not changing the point values, and decrease the arc values in **Panel #2** by the same amount. Note that you can do this either by changing the specific value in the **Arc** n box or by just using the cursor to move the four blue dots the same amount in the same direction in **Panel #1** and the same amount in the opposite direction in **Panel #2** on the unit circle of the domain in each panel.

Describe how the image domain changes as the arc values in **Panel #1** increase by a total of $\frac{\pi}{4}$ and decrease in **Panel #2** by the same amount.

EXPLORATION 4.154. Using *LinComboTool*, create a map in **Panel #1** that maps 3 arcs on the unit circle to 3 vertices of an equilateral triangle. Then create a second map in **Panel #2** that maps 3 different arcs on the unit circle to 3 vertices of a rotated equilateral triangle. Make sure that these maps satisfy the conditions of Theorem 4.88. Click on the **Create LinCombogon** button to see the resulting image domain. Explore this idea by using different maps in the panels to get at least three different resulting image domains.

EXPLORATION 4.155. Using *LinComboTool*, create a map in **Panel #1** that maps 6 arcs on the unit circle to 6 vertices of a regular hexagon. Then create a second map in **Panel #2** that maps 6 different arcs on the unit circle to 6 vertices of a rotated regular hexagon. Make sure that these maps satisfy the conditions of Theorem 4.88. Click on the **Create LinCombogon** button to see the resulting image domain. Explore this idea by using different maps in the panels to get at least three different resulting image domains.

Convolutions

EXERCISE 4.156. Let

$$f(z) = \int \frac{1}{1-z^2}\, dz = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and

$$F(z) = \int \frac{1}{1-z^3}\, dz = \frac{1}{3}e^{\frac{i5\pi}{3}}\log\left(1+e^{\frac{i\pi}{3}}z\right) + \frac{1}{3}e^{\frac{i5\pi}{3}}\log\left(1+e^{\frac{i5\pi}{3}}z\right) - \frac{1}{3}\log\left(1-z\right).$$

Using eq (57) at the end of Example 4.60, determine $f * F$ and the image of $\mathbb{D}$ under this convolution. In general, what is $f * F$ when $f'(z) = \frac{1}{1-z^m}$ and $F(z) = \frac{1}{1-z^n}$?

EXERCISE 4.157. In Theorem 4.109, let f be the canonical right half-plane mapping $f_0 \in K_H$ and let $\varphi(z) = \frac{z}{(1-z)^2} \in S$. Compute $F = f_0 * (\overline{\varphi} + \varphi)$ and use *ComplexTool* to sketch $F(\mathbb{D})$.

EXERCISE 4.158. Derive the expressions for H_2 and G_2 given in eq (64).

SMALL PROJECT 4.159. Compute $f_a = h_a + \overline{g}_a$, where $h_a(z) + g_a(z) = \frac{z}{1-z}$ and $\omega(z) = \frac{z+a}{1+az}$. From Theorem 4.121, we know that $F_a = f_0 * f_a \in S_H$ for $-1 < a < 1$. Compute F_a and use *ComplexTool* to sketch $F_a(\mathbb{D})$ for various values of $a(-1 < a < 1)$. Describe what happens as a varies between 1 and -1.

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