CHAPTER 2

# Minimal Surfaces

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## 2.1. Introduction

Minimal surfaces are beautiful geometric objects with interesting properties that can be studied with the help of computers. Some standard examples of minimal surfaces in $\mathbb{R}^3$ are the plane, Enneper's surface, the catenoid, the helicoid, and Scherk's doubly periodic surface (see Figure 2.1; note that the images shown are just part of these surfaces and that each surface actually continues on forever).

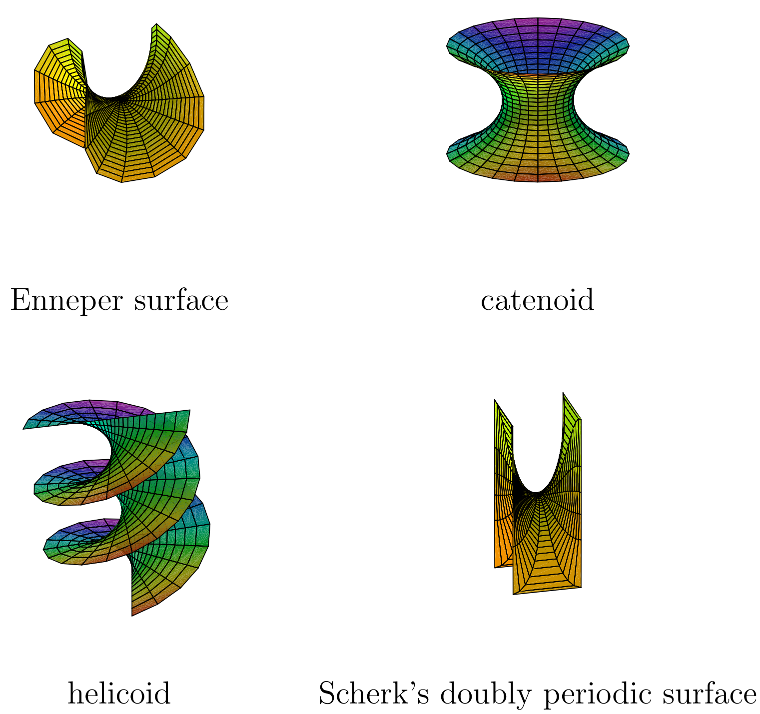



Figure 2.1. Examples of some minimal surfaces

Minimal surfaces are related to soap films that result when a wire frame is dipped in soap solution. To get a sense of this connection, consider the following problem.

**Steiner Problem:** Four houses are located so that they form the vertices of a square that has sides of length one mile. These neighbors want to connect their houses with a road of least length. What should the shape of the road be?

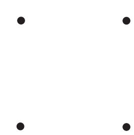
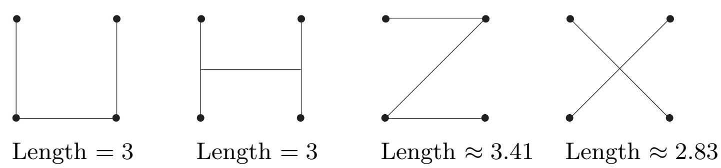


FIGURE 2.2. What is the shortest path connecting these 4 vertices?

Some possible solutions include the following:




However, none of these is the solution. The correct solution has a length of $1+\sqrt{3} \approx 2.7$ miles (see Figure 2.3). For more information about Steiner problems see [**5**] or [**13**].

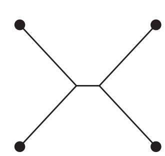
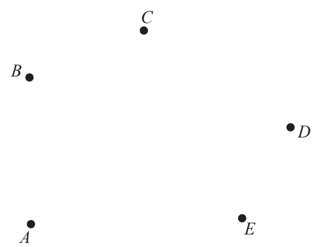
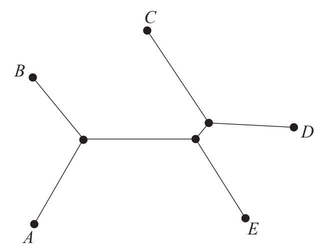


FIGURE 2.3. The shortest path connecting these 4 vertices.

How can we generalize this problem? One way is to have $n$-vertices. So the problem becomes, given $n$ cities find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system.




A $n$-vertices Steiner problem




The solution

Another way to generalize this idea is to move up a dimension. What is the analogue of the Steiner problem in one dimension higher? The Steiner Problem minimizes distance (1-dimensional object) in a plane (2-dimensional object). Soap film minimizes area (2-dimensional object) in space (3-dimensional object).

The answer to Steiner problems in the plane is related to the shape of a soap film. The soap film formed by dipping a cube frame into soap solution is shown in Figure 2.4. The projection of this soap film onto the plane suggests the solution to the Steiner problem above with 4 vertices.

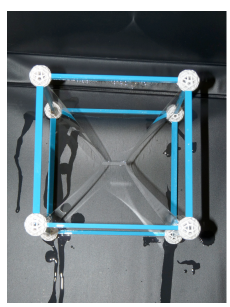


FIGURE 2.4. Soap film formed by a cube.

What is the connection between "minimization" problems such as a Steiner problem and soap films? Water molecules exert a force on each other. Near the surface of the water there is a greater force pulling the molecules toward the center of the water. This force creates surface tension which tends to minimize the surface area of the shape. Soap solution has a lower surface tension than water and this permits the formation of soap films which also tend to minimize geometric properties such as length and area. For more information along this line see [**21**].

Minimal surfaces can be created by dipping wire frame into soap solution. All of the minimal surfaces in Figure 2.1 can be formed by dipping a wire frame into soap solution.

Example 2.1. By dipping a wire frame of a "slinky" (or helix) with a straw in the middle connected to the ends of the "slinky" into soap solution we can create part of the minimal surface known as the helicoid.

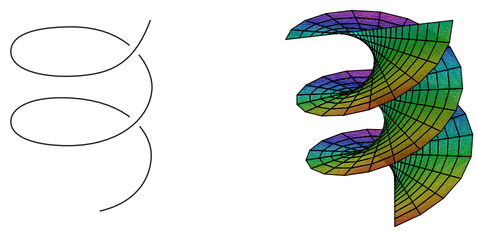


Figure 2.5. The wire frame of a slinky can be used to create part of the helicoid.

Example 2.2. By dipping a 3-dimensional version of the wire frame shown below (a box frame missing two parallel edges on the top and two parallel edges on the bottom) into soap solution we can create part of the minimal surface known as Scherk's doubly periodic surface.

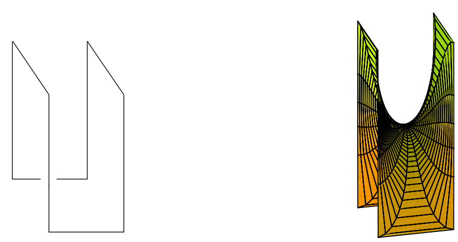


Figure 2.6. The wire frame of a box missing 4 edges can be used to create part of Scherk's doubly periodic surface.

Exploration 2.3. Each of the minimal surfaces shown in Figure 2.1 can be formed by dipping a wire frame in soap solution. Determine the shape of the wire frame that creates: (a) Enneper's surface; and (b) the catenoid.

***Try it out!***

Remark 2.4. To get a soap film of the part of Enneper's surface shown in Figure 2.1, we can dip a wire frame that matches the seams along a baseball. In fact, dipping such a wire frame in soap solution produces *two* minimal surfaces. The first is one half of the sphere of a baseball and the other is the complementary half of the "baseball." What is interesting is that if you start with one, you can deform it into the other by slightly and carefully blowing air into the soap film. There is a third, mysterious and unseen minimal surface one passes through while doing this, and this minimal surface is unstable. In other words, it cannot actually exist or remain in existence–disturbances cause it to pop or "wiggle" into another surface.

One area of minimal surface theory that has seen a lot of interest and results recently is the study of complete embedded minimal surfaces. Basically, these are minimal surfaces that are boundaryless (complete) and have no self-intersections (embedded). The plane, the catenoid, the helicoid, and Scherk's doubly periodic surface are examples of complete embedded minimal surfaces. However, the Enneper surface is not embedded, because it has self-intersections as its domain increases (see Exploration 2.11).

To begin to understand minimal surfaces, we need some tools from differentiable geometry and these are discussed in Section 2. Section 3 uses material from the previous section to define a minimal surface and discuss some examples and properties of minimal surfaces. Section 4 brings in complex analysis to study minimal surfaces and introduces the Weierstrass representation formula to efficiently describe and study properties of minimal surfaces. These three sections are fundamental and should be read first. In Sections 5 - 7, we begin to explore ideas that lead to beginning research problems for students. Section 5 and Section 6 are independent of each other. In Section 5 we present the Weierstrass representation in the form of the Gauss map and height differential which is the basis for much of the current research about minimal surfaces in $\mathbb{R}^3$. Section 6 connects ideas about minimal surfaces with planar harmonic mappings in geometric function theory (i.e., the study of complex analysis from a geometric viewpoint). Section 7 is a new area of investigation that combines the ideas of the previous two sections and has several problems that can be explored by beginning students. In this chapter, there are four applets used and they can be accessed online at http://www.jimrolf.com/explorationsInComplexVariables/chapter2.html:

- *DiffGeomTool* is used to visualize and explore basic differential geometry concepts in $\mathbb{R}^3$ such as the graph of a parametrization of a surface, curves on a surface, tangent planes on a surface, and unit normals on a surface.
- *MinSurfTool* is used to visualize and explore minimal surfaces in $\mathbb{R}^3$ by using various forms of the Weierstrass representation.
- *ComplexTool* is used to plot the image of domains in $\mathbb{C}$ under complex-valued functions.
- *LinComboTool* is used to plot and explore the convex combination of complex-valued harmonic polygonal maps.

Each section of this chapter contains examples, exercises, and explorations that involve using the applets. You should do all of the exercises and explorations many of which present surfaces and concepts that will be used later in the chapter (there are additional exercises at the end of the chapter). In addition, there are short projects and long projects that are suitable as research problems for undergraduates to explore. The goal of this chapter is not to give a comprehensive or step-by-step approach to this topic, but rather to get the reader engaged with the general notions, questions, and techniques of the area – but even more so, to encourage the reader to actively pose as well as pursue their own questions. To better understand the nature and purpose of this text, the reader should be sure to read the Introduction before proceeding.

## 2.2. Differential Geometry

Our goal is to develop the mathematics necessary to investigate minimal surfaces in $\mathbb{R}^3$. Such minimal surfaces minimize area locally and can be thought of as saddle surfaces. At each point, the bending upward in one direction is matched with the bending downward in the orthogonal direction. Such bending is known mathematically as curvature. So, to initially understand and investigate minimal surfaces we need to be able to understand the mathematics of curvature which comes from differential geometry, a field of mathematics in which the ideas and techniques of calculus are applied to geometric shapes.

We will begin our discussion of differential geometry by looking at a surface in $\mathbb{R}^3$. Every point on a surface $M \subset \mathbb{R}^3$ can be designated by a point, $(x, y, z) \in \mathbb{R}^3$, but it can also be represented by two parameters. Let $D$ be an open set in $\mathbb{R}^2$. Then the surface $M$ can be represented by a function $\mathbf{x} : D \to \mathbb{R}^3$, where $\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$ (that is, $M$ is the image of $\mathbf{x}(D)$). We will require that $\mathbf{x}$ be differentiable. That is, each coordinate function $x_k(u, v)$ has continuous partial derivatives of all orders in $D$. Such a function or mapping is called a *parametrization*.

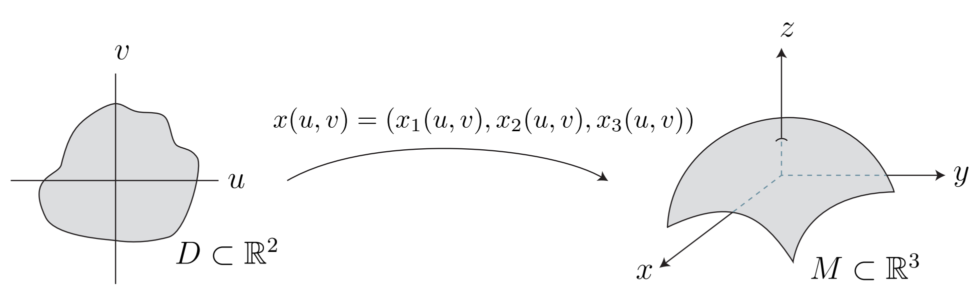



FIGURE 2.7. The parameterization of a surface

Let's consider two examples.

EXAMPLE 2.5. The Enneper surface is a minimal surface formed by bending a disk into a saddle surface. It can be parametrized by

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right),$$

where $u, v$ are in a disk of radius $r$. We can use the applet, *DiffGeomTool*, to graph this parametrization of the Enneper surface (see Figure 2.8). Open *DiffGeomTool* and enter the coordinate functions of the parametrization as

$$X(u, v) = u - 1/3 * u \wedge 3 + u * v \wedge 2$$
$$Y(u, v) = v - 1/3 * v \wedge 3 + u \wedge 2 * v$$
$$Z(u, v) = u \wedge 2 - v \wedge 2$$

into the appropriate boxes. In the gray part in the bottom right, click on `Circular grid` with radius min: 0.0, radius max: 1.0, theta min: 0.0, and theta max: 2* pi. This is because we want our $u, v$ values to be the unit disk. Then click the **Graph** button. To rotate the graph, place the cursor arrow on the image of the surface, and then click on and hold the left button on the mouse as you move the cursor. To increase the size of the image of the surface click on the left button on the mouse; to decrease the size, click on the right mouse button.

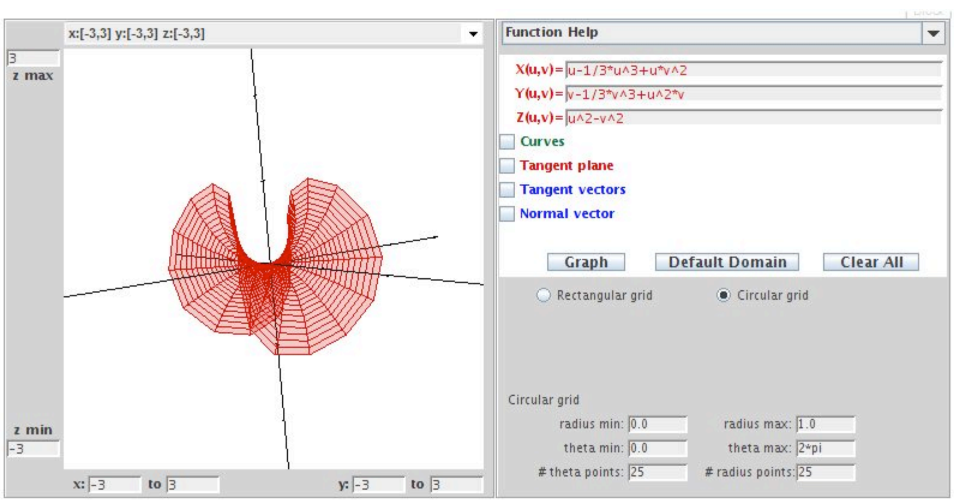



FIGURE 2.8. The Enneper surface.

EXAMPLE 2.6. If a heavy flexible cable is suspended between two points at the same height, then it takes the shape of a curve that can be described mathematically by the function $x_2 = a\cosh(x_1/a)$. Such a curve is called a catenary from the Latin word that means "chain". A *catenoid* is a surface that is generated by rotating a catenary on its side about the $x_3$-axis (see Figure 2.9). A catenoid is also a minimal surface. How do we parametrize this catenoid? If we let $x_1 = a\cosh v$ $(-\infty < v < \infty)$ and $x_3 = av$, then $r(v) = (a\cosh v, av)$ is a parametrization of the catenary curve on its side in the $x_1x_3$-plane. Rotating a line about an axis is a circular motion, and a circle can be parametrized by $(\cos u, \sin u)$. So, we can parametrized this rotation of the catenary curve about the $x_3$-axis by multiplying $a\cosh v$ by $\cos u$ for the $x_1$-coordinate function,

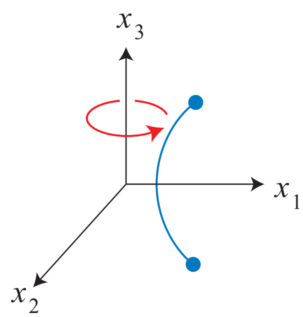



FIGURE 2.9. Creating a catenoid by rotating a catenary.

and multiplying $a \cosh v$ by $\sin u$ for the $x_2$-coordinate function. Hence, we get the following parametrization for this catenoid surface:

$$\mathbf{x}(u,v) = \Big(a \cosh v \cos u, a \cosh v \sin u, av\Big).$$

Using *DiffGeomTool*, we can graph this parametrization of a catenoid with $a = 1$, clicking on `Rectangular grid`, and setting the boxes to 0 <= u <= 2 pi and -2pi/3 <= v <= 2pi/3 (see Figure 2.10). Note that $\cosh v, \cos u$, and $\sin u$ should be entered as $\cosh(v), \cos(u)$, and $\sin(u)$, respectively.

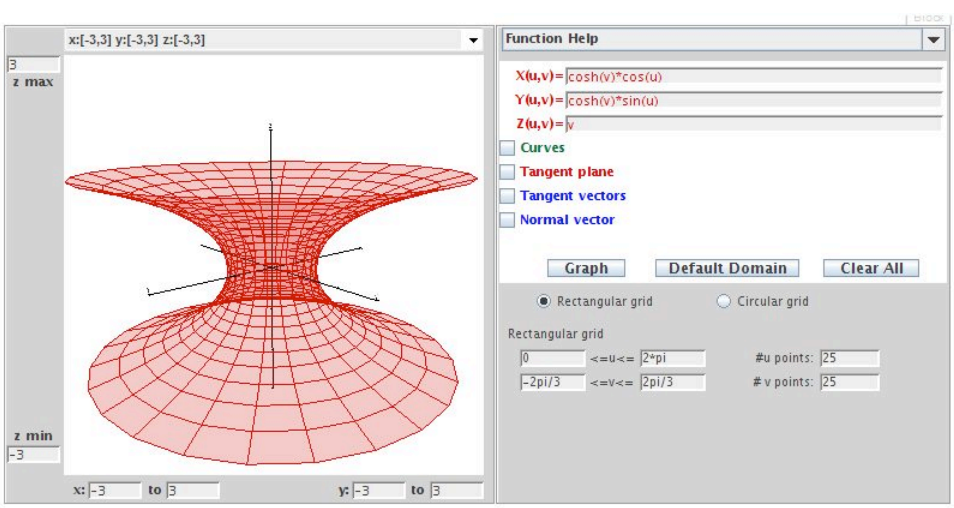



FIGURE 2.10. The catenoid.

Check out what happens if you change the $u, v$ values. For example, try:

(a) pi <= u <= 2pi, -2pi/3 <= v <= 2pi/3;
(b) 0 <= u <= 2pi, 0 <= v <= 2pi/3;
(c) 0 <= u <= 2pi, -pi/4 <= v <= pi/4;

(d) 0 <= u <= 2pi,  -pi <= v <= pi;

Exercise 2.7. A torus is a surface (but not a minimal surface) that can be formed by rotating a circle in the $x_1x_3$-plane about the $x_3$-axis. Let this be a circle of radius $b$ and whose center is a distance of $a$ from the origin.

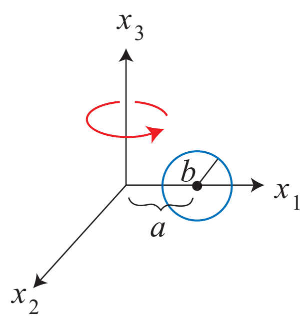



Figure 2.11. Creating a torus by rotating a circle.

Then the parametrization of this torus is

$$\mathbf{x}(u,v) = \Big((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v\Big),$$

where $a, b$ are fixed, $0 < u < 2\pi$, and $0 < v < 2\pi$.

(a) Show how to derive this parametrization for a torus.
(b) Use *DiffGeomTool* to sketch the graph of this torus when $a = 3$ and $b = 2$; use `Rectangular grid` with $0 < u < 2\pi$, and $0 < v < 2\pi$.

***Try it out!***

In pre-calculus, we talk about a function of one variable $y = F(x)$ as one that satisfies the vertical line test. The graph of $F(x)$ is a 1-dimensional object, living in $\mathbb{R}^2$, that can be parametrized by $(u, F(u))$. Analogously, we speak of a function of two variables $z = f(x, y)$, where the points $(x, y)$ lie in a two-dimensional domain and $f$ satisfies the vertical line test (here a line is vertical when it is parallel to the $z$-axis). The graph of $f(x, y)$ is a two-dimensional surface living in $\mathbb{R}^3$ with a height of $z = f(x, y)$ at a point $(x, y)$ in its domain. An example of such a graph is the minimal surface known as Scherk's doubly periodic surface. It can be parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right).$$

Exercise 2.8.

(a) In this parametrization of Scherk's doubly periodic surface, what are the restrictions on the $u$ and $v$ values in the domain?

(b) Use *DiffGeomTool* and your answer from part (a) to sketch a graph of Scherk's doubly periodic surface with -0.48 pi <= u,v <= 0.48 pi.

(c) Scherk's doubly periodic surface is a particular example of the graph of a function. Now, let $f(x, y)$ be any function. Find a parametrization of the graph of $f$ in general.

***Try it out!***

EXERCISE 2.9. Let $r$ be a differentiable curve whose derivative does not vanish (i.e., $r'(v) \neq 0$ for all values $v$ in the domain) and let $r$ lie in some plane in $\mathbb{R}^3$. A *surface of revolution* is a surface that forms by rotating $r$ about an axis in that plane such that the curve does not intersect the axis. The catenoid and torus are examples of this. For this exercise, let $r(v) = (f(v), 0, g(v))$ be such a curve in the $x_1x_3$-plane.

(a) Find a parametrization for the surface of revolution generated by rotating this curve about the $x_3$-axis.

(b) Check that your answer to part (a) matches the parametrizations of the catenoid and the torus given above.

***Try it out!***

EXPLORATION 2.10. Consider the torus $T_{a,b}$ whose parametrization is given in Exercise 2.7. Use *DiffGeomTool* to plot $T_{3,2}$ again. Describe what happens to the shape of the torus $T_{a,b}$ as as $a$ gets smaller and $b$ gets larger (HINT: in *DiffGeomTool*, plot each of the following tori: $T_(2.7, 2)$, $T_(2.4, 2)$, $T_(2, 2)$, $T_(3, 2.4)$, $T_(3, 2.7)$, and $T_(3, 3)$. What happens when $a < b$? Explain this in terms of how we derived the parametrization of the torus.

***Try it out!***

EXPLORATION 2.11. As mentioned earlier, the Enneper surface is not embedded; that is, it has self-intersections. Use *DiffGeomTool* and the parametrization given in Example 2.5 to graph the Enneper surface with the domain being a disk of radius 1.

(a) What happens to the Enneper surface as the radius $r$ of the disk increases?

(b) Estimate the largest value of $r$ for which the Enneper surface has no self-intersections.

(c) Assuming that the intersection occurs on the $x_3$-axis, prove your result from part (b).

***Try it out!***

So far we have discussed how a function (i.e., a parametrization) models a surface. Our goal is to determine the bending or curvature of curves on a surface. To do this, we next will need to use the parametrization of a surface to discuss the concepts of a tangent plane and a normal vector at a point on the surface. Suppose $\mathbf{x}(u, v)$ is a parametrization of a surface $M \subset \mathbb{R}^3$. If we fix $v = v_0$ and let $u$ vary, then $\mathbf{x}(u, v_0)$ depends on one parameter and is known as a *u-parameter curve*. Likewise, we can fix $u = u_0$ and let $v$ vary to get a *v-parameter curve* $\mathbf{x}(u_0, v)$ (see Figure 2.12). Tangent vectors for the $u$-parameter and $v$-parameter curves are computed by differentiating the component functions of $\mathbf{x}$ with respect to $u$ and $v$, respectively. That is, $\mathbf{x}_u$ and $\mathbf{x}_v$ are the tangent vectors defined by

$$\mathbf{x}_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right), \qquad \mathbf{x}_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right).$$

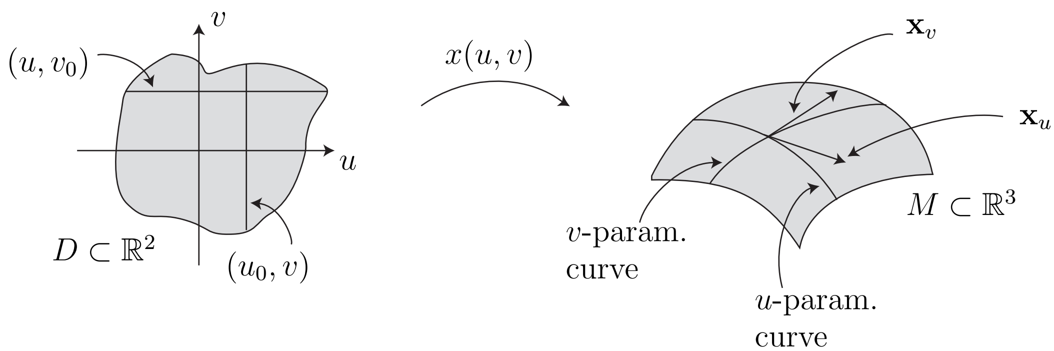



FIGURE 2.12. The $u$-parameter and $v$-parameter curves

Whenever we have a parametrization of a surface, we will require that $\mathbf{x}_u$ and $\mathbf{x}_v$ be linearly independent (i.e., not constant multiples of each other). Because of this, the span of $\mathbf{x}_u$ and $\mathbf{x}_v$ (i.e., the set of all vectors that can be written as a linear combination of $\mathbf{x}_u$, $\mathbf{x}_v$) forms a plane called the *tangent plane*.

DEFINITION 2.12. The *tangent plane* of a surface $M$ at a point $p$ is

$$T_pM = \{\mathbf{v} \big| \mathbf{v} \text{ is tangent to } M \text{ at } p\}.$$

DEFINITION 2.13. The *unit normal* to a surface $M$ at a point $p = \mathbf{x}(a, b)$ is

$$\mathbf{n}(a,b) = \left.\frac{\mathbf{x}_u \times \mathbf{x}_v}{\left|\mathbf{x}_u \times \mathbf{x}_v\right|}\right|_{(a,b)}.$$

Not every surface has a well-defined choice of a unit normal $\mathbf{n}$. Such surfaces are called *non-orientable*. An example of a non-orientable surface is given in Exercise 2.150. Note that the unit normal, $\mathbf{n}$, is orthogonal to the tangent plane at $p$ (see Figure 2.13). Also, if the surface $M$ is oriented, then geometrically there are two unit normals at each point $p \in M$ – an outward pointing normal and an inward pointing normal. However, the definition of $\mathbf{n}$ automatically chooses one of these normals.

EXAMPLE 2.14. Consider a torus parametrized by

$$\mathbf{x}(u,v) = \Big((3 + 2\cos v)\cos u, (3 + 2\cos v)\sin u, 2\sin v\Big),$$

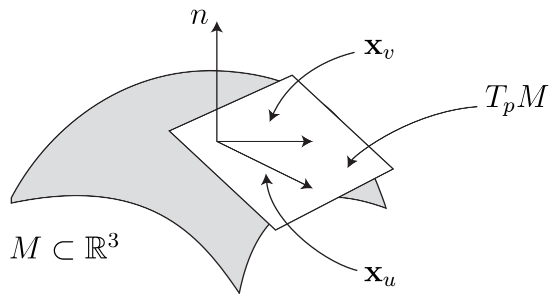



FIGURE 2.13. A tangent plane, $T_pM$, and unit normal vector, $\mathbf{n}$

where $0 < u, v < 2\pi$. For $v_0 = \frac{\pi}{3}$, the $u$-parameter curve is

$$\mathbf{x}\left(u, \frac{\pi}{3}\right) = (4\cos u, 4\sin u, \sqrt{3}).$$

For $u_0 = \frac{\pi}{2}$, the $v$-parameter curve is

$$\mathbf{x}\left(\frac{\pi}{2}, v\right) = (0, 3 + 2\cos v, 2\sin v).$$

Notice

$$\mathbf{x}_u(u,v) = (-(3+2\cos v)\sin u, (3+2\cos v)\cos u, 0)$$
$$\mathbf{x}_v(u,v) = (-2\sin v\cos u, -2\sin v\sin u, 2\cos v).$$

Now the $u$-parameter curve, $\mathbf{x}\big(u, \frac{\pi}{3}\big)$, and the $v$-parameter curve, $\mathbf{x}\big(\frac{\pi}{2}, v\big)$, intersect on the torus at $p = \mathbf{x}\big(\frac{\pi}{2}, \frac{\pi}{3}\big)$. Then the tangent vectors to the $u$- and $v$-parameter curves at the point $p$ are

$$\mathbf{x}_u\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = (-4, 0, 0)$$
$$\mathbf{x}_v\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = (0, -\sqrt{3}, 1).$$

These two vectors span the tangent plane, $T_pM$, at $p$. We compute that

$$\mathbf{x}_u\left(\frac{\pi}{2}, \frac{\pi}{3}\right) \times \mathbf{x}_v\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = (-4, 0, 0) \times (0, -\sqrt{3}, 1) = (0, 1, 4\sqrt{3}).$$

Hence,

$$\mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = \left(0, \frac{1}{7}, \frac{4\sqrt{3}}{7}\right).$$

We can use *DiffGeomTool* to display this $u$-parameter curve, $v$-parameter curve, $\mathbf{x}_u$, $\mathbf{x}_v$, and $\mathbf{n}$. Enter the parametrization in this example for the torus. Then click `Curves`. A `Point location` box along with a `fixed u` and a `fixed v` boxes will appear. In the `Point location` box, enter pi/2 into the first box (i.e., the fixed $u$ value) and pi/3 into the second box (i.e., the fixed $v$ value). Click the `fixed u` box and click on the **Graph** button. The $v$-parameter curve will appear. If you now click the `track fixed u curve` box, a slider will appear. Moving the slider with the cursor will move the point along the $v$-parameter curve on the torus. By clicking on the `track fixed v curve` box and clicking the `Graph` button again, the same can be done for the $u$-parameter curve. Next, click on each of the following boxes separately followed by the **Graph** button: `Tangent vectors` box, `Tangent plane` box, and `Normal vector` box. This will cause these geometric objects to appear. You should convince yourself that the images of the vectors at $(u, v) = (\frac{\pi}{2}, \frac{\pi}{3})$ match the computed values done earlier in Example 2.14.

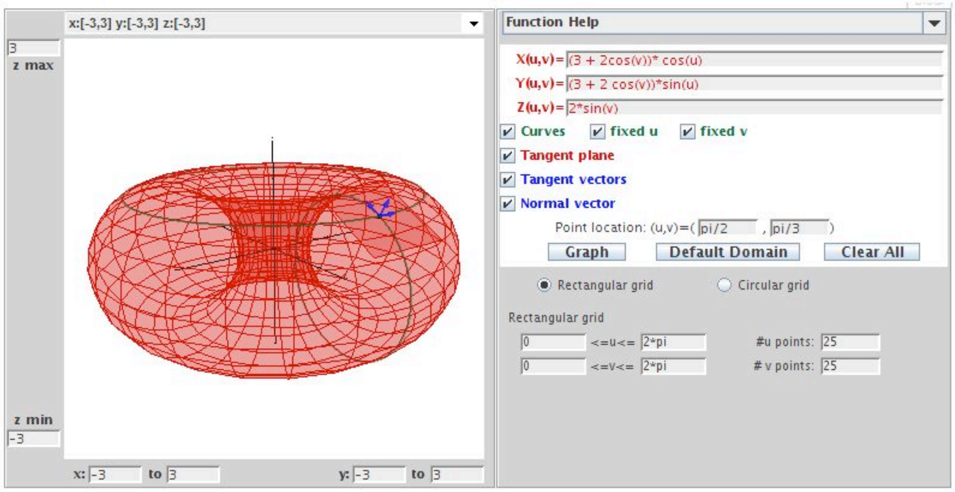



FIGURE 2.14. The torus with specific $u, v$-parameter curves, the tangent vectors, the tangent plane, and the normal vector.

EXERCISE 2.15. For a surface of revolution (see Exercise 2.9) parametrized by

$$\mathbf{x}(u, v) = \big(f(v)\cos u, f(v)\sin u, g(v)\big)$$

the $u$-parameter curves are called *parallels* and are the curves formed by horizontal slices, while the $v$-parameter curves are called *meridians* and are the curves formed by vertical slices. Describe the parallels and meridians for the catenoid in Example 2.6 and the torus in Exercise 2.7.

***Try it out!***

EXERCISE 2.16. Recall the parametrization of a catenoid

$$\mathbf{x}(u, v) = \Big(\cosh v \cos u, \cosh v \sin u, v\Big),$$

with $0 < u < 2\pi$ and $-\frac{2\pi}{3} < v < \frac{2\pi}{3}$.

(a) Use *DiffGeomTool* to sketch the $u$-parameter curve, $\mathbf{x}(u,0)$, and the $v$-parameter curve, $\mathbf{x}(0,v)$ on the catenoid. Also, sketch the vectors $\mathbf{x}_u(0,0)$, $\mathbf{x}_v(0,0)$, and $\mathbf{n}(0,0)$.
(b) Compute the vectors $\mathbf{x}_u(0,0)$, $\mathbf{x}_v(0,0)$, and $\mathbf{n}(0,0)$.

***Try it out!***

EXERCISE 2.17. Recall the parametrization of Scherk's doubly periodic surface

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right),$$

with $-0.48\pi < u, v < 0.48\pi$.

(a) Use *DiffGeomTool* to sketch the $u$-parameter curve, $\mathbf{x}\left(u, \frac{\pi}{4}\right)$, and the $v$-parameter curve, $\mathbf{x}\left(\frac{\pi}{4}, v\right)$ on Scherk's doubly periodic surface (make sure you use $-0.48\pi < u, v < 0.48\pi$). Also, sketch the vectors $\mathbf{x}_u\left(\frac{\pi}{4},\frac{\pi}{4}\right)$, $\mathbf{x}_v\left(\frac{\pi}{4},\frac{\pi}{4}\right)$, and $\mathbf{n}\left(\frac{\pi}{4},\frac{\pi}{4}\right)$. Note that you can slide these vectors by clicking the `track fixed u curve` box. This collection of vectors, $\mathbf{x}_u$, $\mathbf{x}_v$, $\mathbf{n}$), are known as a *moving frame* or *Frenet frame* of a curve. The way these vectors vary in $\mathbb{R}^3$ as the frame moves along the curve describes how the curve twists and turns in $\mathbb{R}^3$. For more details, see [**17**] or [**20**].
(b) Compute the vectors $\mathbf{x}_u\left(\frac{\pi}{4},\frac{\pi}{4}\right)$, $\mathbf{x}_v\left(\frac{\pi}{4},\frac{\pi}{4}\right)$, and $\mathbf{n}\left(\frac{\pi}{4},\frac{\pi}{4}\right)$.

***Try it out!***

Now that we have discussed the normal vector $\mathbf{n}$, we are ready to explore the idea of the curvature of a curve on a surface. This idea will help us later define minimal surfaces. Notice that any plane containing the normal $\mathbf{n}$ will intersect the surface $M$ in a curve, $\alpha$. For each curve $\alpha$, we can compute its curvature, which measures how fast the curve pulls away from the tangent line at $p$. So let's now consider some ideas about the curvature of a curve. Any curve in $\mathbb{R}^3$ can be parametrized by a function of one variable, say $\alpha(t)$, where $\alpha : [a,b] \to \mathbb{R}^3$. However, this parametrization is not unique.

EXERCISE 2.18. Find two different parametrizations of the unit circle in the $x_1x_2$-plane.

***Try it out!***

This lack of uniqueness can cause difficulties in exploring the concept of curvature. To eliminate these difficulties we will standardize our parametrization by requiring it to be a unit speed curve.

DEFINITION 2.19. A curve $\alpha$ is a *unit speed curve* if $\left|\alpha'(t)\right| = 1$.

If our parametrization of our regular curve $\alpha(t)$ is not of unit speed, we can always reparametrized it by arclength to form a unit speed curve $\alpha(s)$. Because of this, we will assume that the curves we will be discussing are unit speed curves $\alpha(s)$. This assumption means that we are only interested in the geometric shape of a regular curve since reparametrizing does not change its shape.

Given a curve $\alpha$, we want to discuss its curvature (or bending). The amount of bending of the curve is demonstrated by the measure of how rapidly the curve pulls away from the tangent line at $p$. In other words, it measures the rate of change of the angle $\theta$ that neighboring tangents make with the tangent at $p$. Thus, we are interested in the rate of change of the tangent vector (i.e., the value of the second derivative).

DEFINITION 2.20. The *curvature* of the unit speed curve $\alpha$ at $s$ is $\big|\alpha''(s)\big|$.

EXAMPLE 2.21. Consider a torus parametrized by

$$\mathbf{x}(u,v) = \Big((3+2\cos v)\cos u, (3+2\cos v)\sin u, 2\sin v\Big),$$

where $0 < u, v < 2\pi$. Let's compute the curvature for the $u$-parameter curves and the $v$-parameter curves. All the $v$-parameter curves (or meridians) are the curves formed by vertical slices of the torus, and hence are circles of radius $b = 2$. To compute the curvature of these $v$-parameter curves, we start with the parametrization of these curves

$$\mathbf{x}(v) = \mathbf{x}(u_0, v) = \Big((3+2\cos v)\cos u_0, (3+2\cos v)\sin u_0, 2\sin v\Big),$$

where $u_0$ is a fixed value. We next need to reparametrize $\mathbf{x}(v)$ so that it is a unit speed curve. Differentiating $\mathbf{x}(v)$ with respect to $v$ we get

$$\mathbf{x}'(v) = \big(-2\sin v\cos u_0, -2\sin v\sin u_0, 2\cos v\big).$$

Thus, $|\mathbf{x}'(v)| = 2$. To make $\mathbf{x}'(v)$ into a unit speed curve, we replace $v$ with $\frac{s}{2}$. So, our reparametrized curve is

$$\mathbf{x}(s) = \left(-2\sin\left(\frac{s}{2}\right)\cos u_0, -2\sin\left(\frac{s}{2}\right)\sin u_0, 2\cos\left(\frac{s}{2}\right)\right).$$

Then we compute

$$\begin{aligned}\mathbf{x}'(s) =& \left(-\sin\left(\frac{s}{2}\right)\cos u_0, -\sin\left(\frac{s}{2}\right)\sin u_0, \cos\left(\frac{s}{2}\right)\right)\\ \mathbf{x}''(s) =& \left(-\frac{1}{2}\cos\left(\frac{s}{2}\right)\cos u_0, -\frac{1}{2}\cos\left(\frac{s}{2}\right)\sin u_0, -\frac{1}{2}\sin\left(\frac{s}{2}\right)\right).\end{aligned}$$

Hence, the curvature of the $v$-parameter curves is

$$\big|\mathbf{x}''(s)\big| = \frac{1}{2}.$$

The $u$-parameter curves (or parallels) are the curves formed by horizontal slices of the torus, and so are circles of radius $3 + 2\cos v_0$, where $v_0 \in (0, 2\pi)$ is fixed; note that these radii vary between 1 and 5. These curves are parametrized by

$$\mathbf{x}(u) = \mathbf{x}(u, v_0) = \Big((3+2\cos v_0)\cos u, (3+2\cos v_0)\sin u, 2\sin v_0\Big),$$

which are reparametrized to the unit speed curve

$$\mathbf{x}(s) = \left((3+2\cos v_0)\cos\left(\frac{s}{3+2\cos v_0}\right), (3+2\cos v_0)\sin\left(\frac{s}{3+2\cos v_0}\right), 2\sin v_0\right).$$

Finally, computing the curvature of these $u$-parameter curves yields

$$\left|\mathbf{x}''(s)\right| = \frac{1}{3+2\cos v_0}.$$

So, the curvature of the these curves varies between $\frac{1}{5}$ and 1.

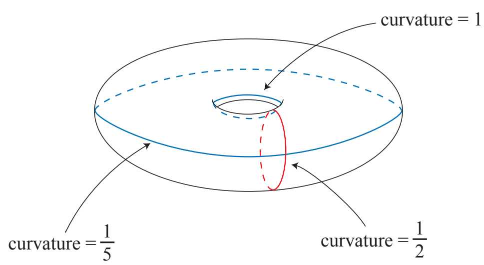



FIGURE 2.15. The curvature of the meridians and parallels on a torus.

EXERCISE 2.22. Compute the curvatures of the meridians and parallels of the catenoid parametrized by

$$\mathbf{x}(u,v) = \left(a\cosh v\cos u, a\cosh v\sin u, av\right).$$

***Try it out!***

EXERCISE 2.23. The curve parametrized by $\alpha(t) = (a\cos t, a\sin t, bt)$ is known as a *helix* which is a spiral that rises with a pitch of $2\pi b$ on the cylinder $x^2 + y^2 = a^2$.

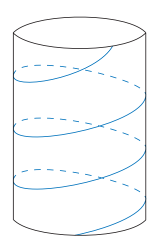


FIGURE 2.16. A helix in a cylinder.

We can create a surface by connecting a line from the axis $(0,0,bt)$ through the helix $(a\cos t, a\sin t, bt)$. This ruled surface is a minimal surface known as a *helicoid*. All minimal surfaces including the helicoid can be parametrized in several ways. For our purposes, we will use the following parametrization of the helicoid:

$$\mathbf{x}(u,v) = \Big(a\sinh v\cos u, a\sinh v\sin u, au\Big).$$

(a) Compute the curvatures of the $u$-parameter curves and $v$-parameter curves of this helicoid (note: making the $v$-parameter curve into a unit speed curve is not easy, so in doing this computation you may need to be creative).
(b) Use *DiffGeomTool* to graph this helicoid with $a = 1$.

***Try it out!***

EXERCISE 2.24. From the results of Example 2.21 you may have conjectured that the curvature of a circle of radius $r$ is $\frac{1}{r}$. This conjecture is correct. Prove this.

***Try it out!***

Now let's return to surfaces. Suppose we have a curve $\sigma(s)$ on a surface $M$. We can determine the unit tangent vector, $\mathbf{w}$ of $\sigma$ at $p \in M$ and the unit normal, $\mathbf{n}$ of $M$ at $p \in M$. Note that $\mathbf{w} \times \mathbf{n}$ forms a plane $\mathcal{P}$ that intersects $M$ creating a curve $\alpha(s)$.

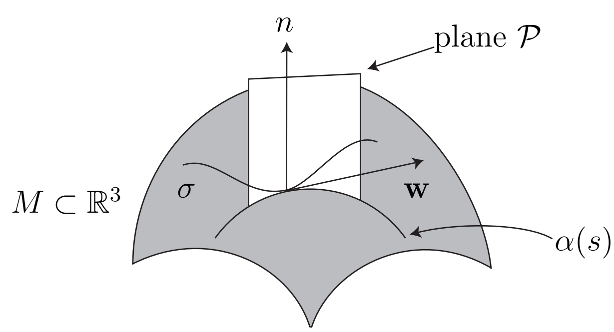



FIGURE 2.17. The normal curvature

DEFINITION 2.25. The *normal curvature* in the $\mathbf{w}$ direction is

$$k(\mathbf{w}) = \alpha'' \cdot \mathbf{n}.$$

Recall $\alpha'' \cdot \mathbf{n} = |\alpha''||\mathbf{n}|\cos\theta$, where $\theta$ is the angle between $\mathbf{n}$ and $\alpha''$. Hence $\alpha'' \cdot \mathbf{n}$ is the projection of $\alpha''$ onto the unit normal (hence, the name *normal* curvature). Intuitively, the normal curvature measures how much the surface bends towards $\mathbf{n}$ as you travel in the direction of the tangent vector $\mathbf{w}$ starting at point $p$. As we rotate the plane through the normal $\mathbf{n}$, we will get a set of curves on the surface each of which has a value for its curvature. Let $k_1$ and $k_2$ be the maximum and minimum curvature values at $p$, respectively. The directions in which the normal curvature attains its absolute maximum and absolute minimum values are known as the *principal directions*.

DEFINITION 2.26. The *mean curvature* (i.e., average curvature) of a surface $M$ at $p$ is

$$H = \frac{k_1 + k_2}{2}.$$

It turns out that $k_1$ and $k_2$ come from two orthogonal tangent vectors. The mean curvature depends upon the point $p \in M$. However, it can be shown that $H$ does not change if we choose any two orthogonal vectors and use their curvature values to compute $H$ at $p$. Also, we will use the convention that if $k_j$ is bending toward the unit normal $\mathbf{n}$, then $k_j > 0$ and if it is bending away from $\mathbf{n}$, then $k_j < 0$.

EXAMPLE 2.27. At any point on a sphere of radius $a$, all the curves $\alpha$ are circles of radius $a$ and hence have the same curvature value which can be computed to be $1/a$. Since these curves are bending away from $\mathbf{n}$, $k_1 = -1/a = k_2$. So the mean curvature is $-1/a$.

EXERCISE 2.28. Determine the mean curvature at all points on the cylinder parametrized by $\mathbf{x}(u, v) = (a \cos u, a \sin u, bv)$.

***Try it out!***

EXERCISE 2.29. Determine if there are points on the torus $\mathbf{x}(u, v) = \big((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v\big)$ where $H > 0$, $H = 0$, and $H < 0$.

***Try it out!***

In the next section of this chapter we will define a minimal surface in terms of mean curvature. Right now, we have mean curvature as given in Definition 2.26. However, this definition is not practical for determining the mean curvature of a surface since Definition 2.26 depends upon a specific point on the surface. Fortunately, there is a more useful formula for mean curvature using the coefficients of the first and second fundamental forms for a surface. Recall that $\alpha$ is a unit speed curve. Hence

$$\begin{aligned} 1 =& |\alpha'|^2 = \alpha' \cdot \alpha' \\ =& (\mathbf{x}_u\, du + \mathbf{x}_v\, dv) \cdot (\mathbf{x}_u\, du + \mathbf{x}_v\, dv) \\ =& \mathbf{x}_u \cdot \mathbf{x}_u\, du^2 + 2\mathbf{x}_u \cdot \mathbf{x}_v\, dudv + \mathbf{x}_v \cdot \mathbf{x}_v\, dv^2 \\ =& E\, du^2 + 2F\, dudv + G\, dv^2. \end{aligned} \tag{2}$$

The terms $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v$ are known as the *coefficients of the first fundamental form*. These describe how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$.

Next, recall $k(\mathbf{w}) = \alpha'' \cdot \mathbf{n}$. Note that $\alpha' \cdot \mathbf{n} = 0$, and so $(-\alpha' \cdot \mathbf{n})' = 0$, which implies $\alpha'' \cdot \mathbf{n} + \alpha' \cdot \mathbf{n}' = 0$, and thus $\alpha'' \cdot \mathbf{n} = -\alpha' \cdot \mathbf{n}'$. Similarly, $-\mathbf{x}_u \cdot \mathbf{n}_u = \mathbf{x}_{uu} \cdot \mathbf{n}$. So

$$\begin{aligned}
k(w) =& - \alpha' \cdot \mathbf{n}' \\
=& - (\mathbf{x}_u \, du + \mathbf{x}_v \, dv) \cdot (\mathbf{n}_u \, du + \mathbf{n}_v \, dv) \\
=& - \mathbf{x}_u \cdot \mathbf{n}_u \, du^2 - (\mathbf{x}_u \cdot \mathbf{n}_v + \mathbf{x}_v \cdot \mathbf{n}_u) \, dudv - \mathbf{x}_v \cdot \mathbf{n}_v \, dv^2 \\
=& \mathbf{x}_{uu} \cdot \mathbf{n} \, du^2 + 2\mathbf{x}_{uv} \cdot \mathbf{n} \, dudv + \mathbf{x}_{vv} \cdot \mathbf{n} \, dv^2 \\
=& e \, du^2 + 2f \, dudv + g \, dv^2.
\end{aligned}$$

The terms $e = \mathbf{x}_{uu} \cdot \mathbf{n}$, $f = \mathbf{x}_{uv} \cdot \mathbf{n}$, and $g = \mathbf{x}_{vv} \cdot \mathbf{n}$ are called *the coefficients of the second fundamental form*. These describe how much the surface bends away from the tangent plane.

EXAMPLE 2.30. Recall that a catenoid can be parametrized by

$$\mathbf{x}(u, v) = \Big(a \cosh v \cos u, a \cosh v \sin u, av\Big).$$

Using this parametrization, we compute that

$$\begin{aligned}
\mathbf{x_u} &= (-a \cosh v \sin u, a \cosh v \cos u, 0) \\
\mathbf{x_v} &= (a \sinh v \cos u, a \sinh v \sin u, a).
\end{aligned}$$

So, the coefficients of the first fundamental form are:

$$\begin{aligned}
E =& \mathbf{x}_u \cdot \mathbf{x}_u = a^2 \cosh^2 v; \\
F =& \mathbf{x}_u \cdot \mathbf{x}_v = 0; \\
G =& \mathbf{x}_v \cdot \mathbf{x}_v = a^2 \cosh^2 v.
\end{aligned}$$

What do these values for $E$, $F$, and $G$ tell us? Let $(u_0, v_0) \in D$ be a point in the domain and let's take a small square with a vertex at this point. Because $\mathbf{x}_u \cdot \mathbf{x}_v = F = 0$, we know that the orthogonal lines from the $u$-parameter curve and the $v$-parameter curve will remain orthogonal on the catenoid. That is, small squares will be mapped to small rectangles. Next, because $E = G$, adjacent sides of the image rectangle will have the same length. So, in fact, small squares in the domain $D$ will be mapped to small squares on the catenoid. Now suppose for simplicity sake that $a = 1$. Then $E = G = \cosh^2 v$. When $v = 0$, $E = G = 1$ and as $v$ gets farther away from 0, $E$ and $G$ get larger. This means that a small square containing the $u$-parameter curve $v = 0$ will get mapped to a small square of the small size on the catenoid. But as $v$ gets farther away from 0, the size of the side lengths of the image square will increase by a factor of $\cosh^2 v$. This can be seen in Figure 2.18 (note that when the $u$-parameter curve with $v = 0$ gets mapped to a parallel on the neck of the catenoid, and the $u$-parameter curve with $v = \frac{2\pi}{3}$ gets mapped to the edge of the catenoid as displayed in the figure).

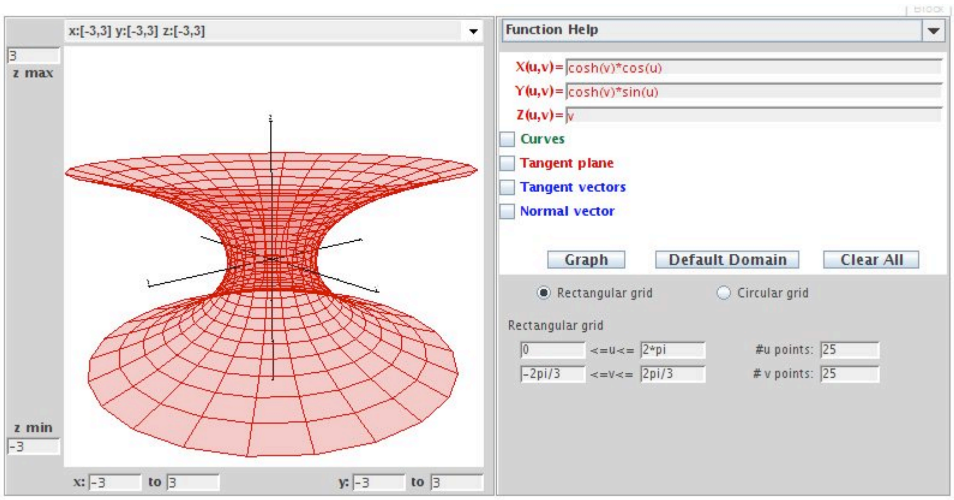



FIGURE 2.18. The catenoid.

In order to compute the coefficients of the second fundamental form, we need to compute $\mathbf{n} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$. Now,

$$\mathbf{x}_u \times \mathbf{x}_v = (a^2 \cosh v \cos u, a^2 \cosh v \sin u, -a^2 \cosh v \sinh v),$$

and so

$$|\mathbf{x}_u \times \mathbf{x}_v| = a^2 \cosh^2 v.$$

Hence

$$\mathbf{n} = \left( \frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\frac{\sinh v}{\cosh v} \right).$$

Also, we can compute that

$$\begin{aligned} \mathbf{x}_{uu} =& (-a \cosh v \cos u, -a \cosh v \sin u, 0) \\ \mathbf{x}_{uv} =& (-a \sinh v \sin u, -a \sinh v \cos u, 0) \\ \mathbf{x}_{vv} =& (a \cosh v \cos u, a \cosh v \sin u, 0) \end{aligned}$$

Therefore, the coefficients of the second fundamental form are:

$$\begin{aligned} e =& \mathbf{n} \cdot \mathbf{x}_{uu} = -a; \\ f =& \mathbf{n} \cdot \mathbf{x}_{uv} = 0; \\ g =& \mathbf{n} \cdot \mathbf{x}_{vv} = a. \end{aligned}$$

What do these values for $e$, $f$, and $g$ tell us? Again, let $(u_0, v_0) \in D$ be a point in the domain, and let $p \in M$ be the image of $(u_0, v_0)$ on the surface. Then at $p$, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ create the tangent plane $T_pM$ and the unit normal $\mathbf{n}$. For this catenoid, the $u$-parameter curve is bending away from $\mathbf{n}$ while the $v$-parameter curve is bending toward $\mathbf{n}$. However, both curves are bending the same amount away from the tangent plane. This is represented by the fact that $e = -g$ in this example.

EXERCISE 2.31. A torus has the parametrization

$$\mathbf{x}(u,v) = \Big((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v\Big).$$

(a) Compute the coefficients of the first and the second fundamental forms.
(b) Open *DiffGeomTool* and enter this parametrization for the torus with $a = 2$ and $b = 1$. Use the `Rectangular grid` with 0 <= u <= 2* pi and 0 <= v <= 2* pi. And set the # u points: to 20 and the # v points: to 20. Describe how the results from part (a) match with the image of the torus in *DiffGeomTool*.

EXERCISE 2.32. Compute the coefficients of the first and the second fundamental forms for Scherk's doubly periodic surface parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right).$$

***Try it out!***

Now we want to express the mean curvature $H$ in terms of these coefficients of the first and second fundamental forms. In particular, we will show that

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}.$$

PROOF. There is an elegant way to derive this formula. This approach requires some concepts that are interesting but beyond the scope of what we will need. So instead we will use a straightforward calculation that is in Oprea ([**21**], pp. 40-42). Although this calculation does not give much insight into the formula, it does provide a straightforward proof of this important formula. For a discussion involving the more advanced approach, see [**3**] or [**17**].

Let $\mathbf{w}_1, \mathbf{w}_2$ be any two perpendicular unit vectors. Let $k_1, k_2$ be their normal curvatures using the curves $\alpha_1$, $\alpha_2$ with parameters $u_1(s), v_1(s)$ and $u_2(s), v_2(s)$. Let's denote $p_1 = du_1 + i du_2$ and $p_2 = dv_1 + i dv_2$. Then

$$\begin{aligned} 2H = k_1 + k_2 &= e(du_1^2 + du_2^2) + 2f(du_1 dv_1 + du_2 dv_2) + g(dv_1^2 + dv_2^2) \\ &= e(p_1\ \overline{p_1}) + f(p_1\ \overline{p_2} +\ \overline{p_1} p_2) + g(p_2\ \overline{p_2}). \end{aligned}$$

We want to further simplify this so that it does not have $p_1$ and $p_2$. Recall eq (2):

$$1 = E\,du^2 + 2F\,dudv + G\,dv^2.$$

Now consider

$$\begin{aligned} Ep_1^2 + 2Fp_1p_2 + Gp_2^2 &= E\big[du_1^2 - du_2^2 + i2du_1du_2\big] + 2F\big[du_1dv_1 - du_2dv_2 + i(du_1dv_2 + du_2dv_1)\big] \\ &\qquad + G\big[dv_1^2 - dv_2^2 + i2dv_1dv_2\big] \\ &= 2i\big[Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2\big] \\ &\qquad + \big[Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2\big] - \big[Edu_2^2 + 2Fdu_1dv_2 + Gdv_2^2\big] \\ &= 0 + 1 - 1 \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} p_1 =& \frac{-2Fp_2 \pm \sqrt{4F^2p_2^2 - 4EGP_2^2}}{2E} = \left( -\frac{F}{E} \pm i\frac{\sqrt{EG - F^2}}{E} \right) p_2 \\ \overline{p_1} =& \left( -\frac{F}{E} \mp i\frac{\sqrt{EG - F^2}}{E} \right) \overline{p_2}. \end{aligned}$$

And so

$$p_1\,\overline{p_1} = \left( \frac{F^2}{E^2} + EG - F^2E \right) p_2\,\overline{p_2} = \frac{G}{E} p_2\,\overline{p_2} \tag{3}$$

$$p_1\,\overline{p_2} + \overline{p_1} p_2 = -\frac{2F}{E} p_2\,\overline{p_2}. \tag{4}$$

Now we have

$$\begin{aligned} 2H = k_1 + k_2 &= e(p_1\,\overline{p_1}) + f(p_1\,\overline{p_2} + \overline{p_1}p_2) + g(p_2\,\overline{p_2}) \\ &= \left[ e\frac{G}{E} + f\left( \frac{-2F}{E} \right) + g \right] p_2\,\overline{p_2}. \end{aligned}$$

We just need to get rid of $p_2\,\overline{p_2}$. Again using eq (2), we have

$$\begin{aligned} Ep_1\,\overline{p_1} &+ F(p_1\,\overline{p_2} + \overline{p_1}p_2) + Gp_2\,\overline{p_2} \\ &= E(du_1^2 + du_2^2) + 2F(du_1dv_1 + du_2dv_2) + G(dv_1^2 + dv_2^2) \\ &= 1 + 1 = 2. \end{aligned}$$

Using eqs (3) and (4),we derive

$$\begin{aligned} 2 &= E\left( \frac{G}{E} p_2\,\overline{p_2} \right) + 2F\left( \frac{-2F}{E} p_2\,\overline{p_2} \right) + Gp_2\,\overline{p_2} \\ \Rightarrow 2 &= \left[ 2G - \frac{2F^2}{E} \right] p_2\,\overline{p_2} \\ \Rightarrow p_2\,\overline{p_2} &= \frac{E}{EG - F^2} \end{aligned}$$

Therefore,

$$H = \frac{1}{2}\left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]p_2\,\overline{p_2} = \frac{Eg + eG - 2Ff}{2(EG - F^2)}.$$

□

## 2.3. Minimal Surfaces

Now that we have a foundation of some essential ideas from differential geometry, we can begin to explore minimal surfaces. Earlier we mentioned that minimal surfaces can be thought of as saddle surfaces. That is, at each point the bending upward in one direction is matched with the bending downward in the orthogonal direction. This picture can be described mathematically with the following definition.

DEFINITION 2.33. A *minimal surface* is a surface $M$ with the mean curvature $H = 0$ at all points $p \in M$.

Make sure that you understand how this definition fits with the picture of a surface bending upward in one direction while also bending downward in the orthogonal direction. At this point we can use the results from the previous section. First, we can use the formula

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \tag{5}$$

to show that a surface with a specific parametrization is minimal.

EXAMPLE 2.34. We will use eq (5) to show that the catenoid is a minimal surface. Recall that a catenoid can be parametrized by

$$\mathbf{x}(u, v) = \Big(a\cosh v \cos u, a\cosh v \sin u, av\Big).$$

From Example 2.30

$$\begin{aligned} E =& \mathbf{x}_u \cdot \mathbf{x}_u = a^2\cosh^2 v, \\ F =& \mathbf{x}_u \cdot \mathbf{x}_v = 0, \\ G =& \mathbf{x}_v \cdot \mathbf{x}_v = a^2\cosh^2 v, \end{aligned}$$

and

$$\begin{aligned} e =& \mathbf{n} \cdot \mathbf{x}_{uu} = -a, \\ f =& \mathbf{n} \cdot \mathbf{x}_{uv} = 0, \\ g =& \mathbf{n} \cdot \mathbf{x}_{vv} = a. \end{aligned}$$

Hence

$$H = \frac{eG - 2fF + Eg}{2(EG - F^2)} = 0.$$

And so the catenoid is a minimal surface.

EXERCISE 2.35. Using the parametrization for the helicoid

$$\mathbf{x}(u, v) = (a \sinh v \cos u, a \sinh v \sin u, au),$$

prove that the helicoid is a minimal surface. Using *DiffGeomTool* we display the graph of this helicoid when $a = 1$ (see Figure 2.19).

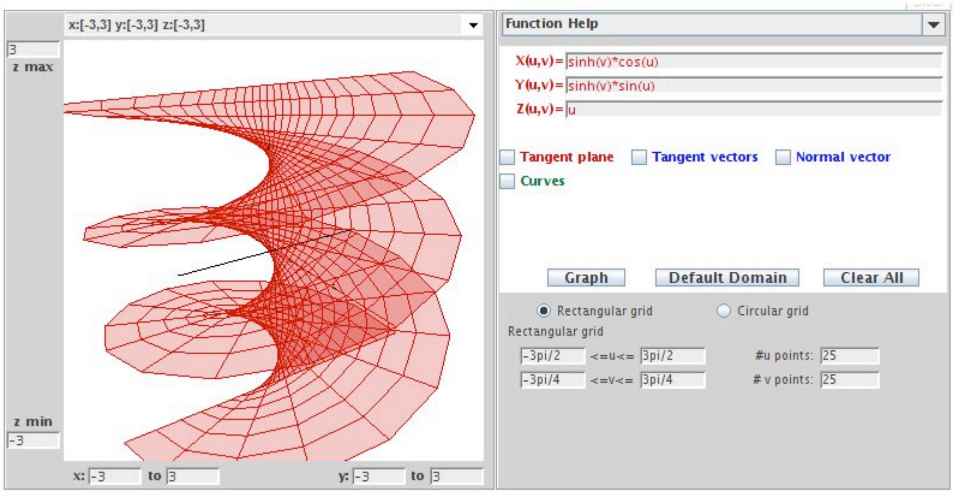



FIGURE 2.19. Helicoid.

***Try it out!***

EXERCISE 2.36. Using the parametrization for the torus

$$\mathbf{x}(u, v) = \big((a + b \cos v) \cos u, (a + b \cos v) \sin u, b \sin v\big),$$

prove that it is not a minimal surface.

***Try it out!***

EXERCISE 2.37. Suppose a surface $M$ is the graph of a function $f(x, y)$ of two variables (see the paragraph before Exercise 2.8). Then $M$ can be parametrized by

$$\mathbf{x}(x, y) = \big(x, y, f(x, y)\big),$$

where its domain is formed by the projection of $M$ onto the $xy$-plane.

(a) Compute the coefficients of the first and second fundamental forms for $M$.
(b) A *minimal graph* is a minimal surface that is a graph of a function. Prove

$$\text{(6)} \qquad M \text{ is a minimal graph } \iff f_{xx}\Big(1 + f_y^2\Big) - 2 f_x f_y f_{xy} + f_{yy}\Big(1 + f_x^2\Big) = 0.$$

***Try it out!***

In the paragraph before Exercise 2.8, we stated that Scherk's doubly periodic surface is a minimal graph. We will now use eq (6) to prove that. Applying this equation is usually not easy, because solving explicitly for $f$ can be complicated. However, one case in which we can do this is when $f$ can be separated into two functions each of which is dependent upon only one variable. In particular, suppose $f(x, y) = g(x) + h(y)$. Then the minimal surface equation becomes:

$$g''(x)[1 + (h'(y))^2] + h''(y)[1 + (g'(x))^2] = 0.$$

This is a separable differential equation and hence can be solved. To do so, separate all the terms with the $x$ variables from those with the $y$ variables by putting them on opposite sides. This yields:

$$-\frac{1 + (g'(x))^2}{g''(x)} = \frac{1 + (h'(y))^2}{h''(y)}. \tag{7}$$

What does this mean? If we fix $y$, the right side remains constant even if we change $x$ in the left side. The same is true if we fix $x$ and vary $y$. The only way such a situation can occur is if both sides are constant. So we have:

$$-\frac{1 + (g'(x))^2}{g''(x)} = k \qquad \Longrightarrow \qquad 1 + (g'(x))^2 = -kg''(x).$$

To solve this, let $\phi(x) = g'(x)$. Then $\frac{d\phi}{dx} = g''(x)$ and so

$$\begin{aligned} \int dx &= -k \int \frac{d\phi}{1 + \phi^2} \\ \Longrightarrow \quad x &= -k \arctan \phi + C \\ \Longrightarrow \quad \phi &= -\tan\left(\frac{x + C}{k}\right). \end{aligned}$$

For convenience, let $C = 0$ and $k = 1$. Since $\phi = g'$, we can integrate again to get:

$$g(x) = \ln[\cos x].$$

Completing the same calculations for the $y$-side of eq (7) yields:

$$h(y) = -\ln[\cos y].$$

Hence

$$f(x, y) = g(x) + h(y) = \ln\left[\frac{\cos x}{\cos y}\right]$$

which is an equation for Scherk's doubly periodic surface. Using *DiffGeomTool* we display the graph of Scherk's doubly periodic surface (see Figure 2.20).
Notice that $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$ and so this surface is defined over a square with side lengths $\pi$ centered at the origin. By a theorem known as the Schwarz Reflection Principle, we can fit pieces of Scherk's doubly periodic surface together horizontally and vertically to get a checkerboard domain (See Figure 2.21). Because one piece of this surface can

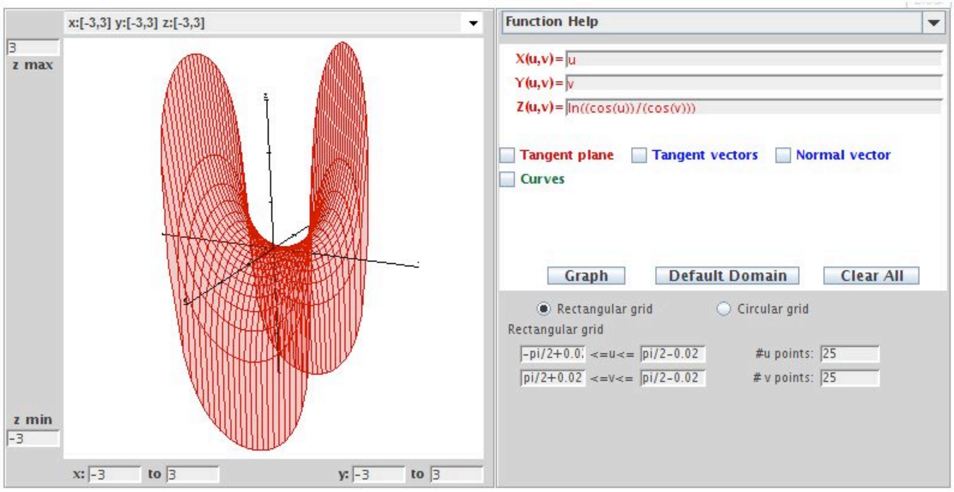



FIGURE 2.20. Scherk's doubly periodic surface.

be repeated or tiled in two directions, it is called a doubly periodic surface. This is really an exciting idea!

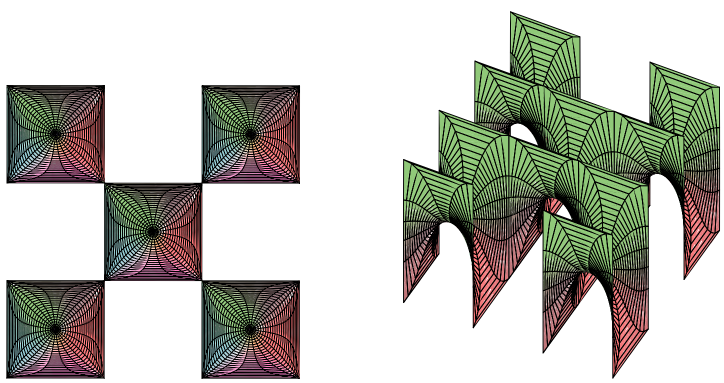


FIGURE 2.21. A tiling of Scherk's doubly periodic surface.

The following is a list of parametrizations for some minimal surfaces:

(1) The plane:
$$\mathbf{x}(u, v) = (u, v, 0).$$
(2) The Enneper surface:
$$\mathbf{x}(u, v) = \left(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + u^2v, u^2 - v^2\right).$$
(3) The catenoid:
$$\mathbf{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av).$$
(4) The helicoid:
$$\mathbf{x}(u, v) = (a \sinh v \cos u, a \sinh v \sin u, au).$$

(5) Scherk's doubly periodic surface:

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right).$$

(6) Scherk's singly periodic surface:

$$\mathbf{x}(u,v) = (\operatorname{arcsinh}(u), \operatorname{arcsinh}(v), \arcsin(uv)).$$

(7) Henneberg surface:

$$\mathbf{x}(u,v) = (-1 + \cosh(2u)\cos(2v), -\sinh(u)\sin(v) - \tfrac{1}{3}\sinh(3u)\sin(3v), -\sinh(u)\cos(v) + \tfrac{1}{3}\sinh(3u)\cos(3v)).$$

(8) Catalan surface:

$$\mathbf{x}(u,v) = (1 - \cos(u)\cosh(v), 4\sin(\tfrac{u}{2})\sinh(\tfrac{v}{2}), u - \sin(u)\cosh(v)).$$

In addition to the Enneper surface, the Henneberg surface and the Catalan surface are not embedded.

Before we proceed further, let us mention an interesting geometric result about minimal surfaces.

THEOREM 2.38. Any nonplanar minimal surface in $\mathbb{R}^3$ that is also a surface of revolution is contained in a catenoid.

As we have seen, determining if a surface is minimal basically involves solving second order differential equations. We can simplify these equations if we use a specific type of parametrization of a surface known as an isothermal parametrization.

DEFINITION 2.39. A parametrization $\mathbf{x}$ is *isothermal* if $E = \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v = G$ and $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$.

What does this mean? Recall that $E$, $F$, and $G$ describe how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$. So if $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$ then the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal and if $E = G$, then the amount of distortion is the same in these two orthogonal directions. Thus, we can think of an isothermal parametrization as mapping a small square in the domain to a small square on the surface. Sometimes, an isothermal parametrization is called a conformal parametrization, because the angle between a pair of curves in the domain is equal to the angle between the corresponding pair of curves on the surface.

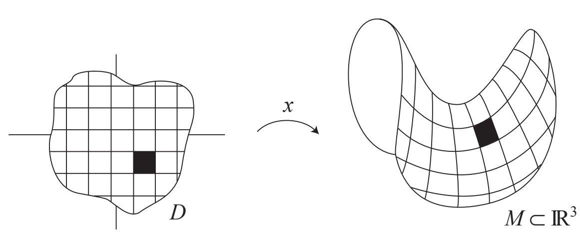



FIGURE 2.22. An isothermal parametrization maps small squares to small squares.

EXAMPLE 2.40. The parametrization

$$\mathbf{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$

for the catenoid is isothermal, because in Example 2.34 we derived that $E = a^2 \cosh^2 v = G$ and $F = 0$. We can get a geometric sense that this parametrization is isothermal by using *DiffGeomTool* to graph this parametrization of the catenoid. Open *DiffGeomTool* and enter this parametrization with $a = 1$. Using the `Rectangular grid`, set -pi <= u <= pi and -pi <= v <= pi. Also, set the # u points: to 20, and the # v points: to 20. This will make the domain a grid of squares that map onto the catenoid. Then click the **Graph** button. Notice that the grid of squares in the domain are pretty much mapped to a grid of squares as predicted above (see Figure 2.23).

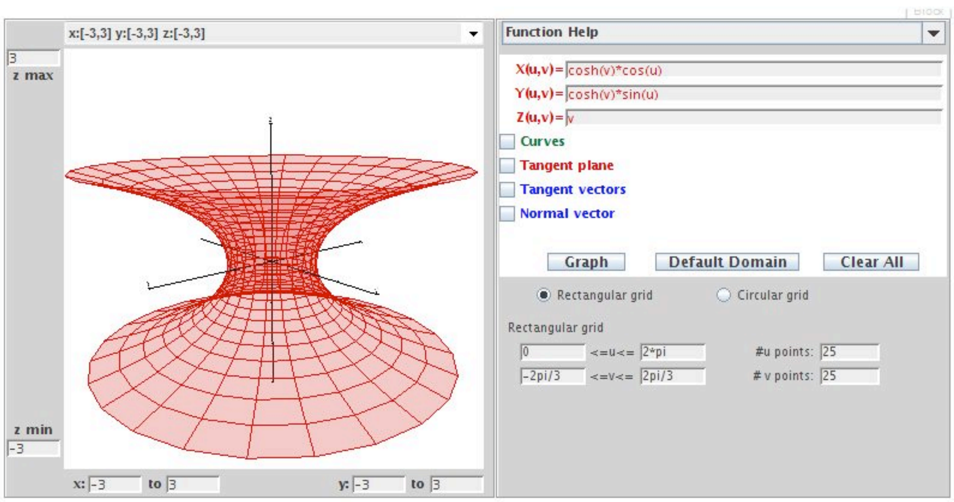



FIGURE 2.23. This parametrization of the catenoid is isothermal.

EXAMPLE 2.41. The parametrization

$$\mathbf{x}(u, v) = \big((a + b \cos v) \cos u, (a + b \cos v) \sin u, b \sin v\big)$$

for the torus is not isothermal. This is because in Exercise 2.31, you derived that

$$\begin{aligned} E =& (a + b \cos v)^2, \\ F =& 0, \\ G =& b^2. \end{aligned}$$

Because $F = 0$, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal on the torus. But $E \geq G$ with equality only when $v = \pi + 2\pi k$, $(k \in \mathbb{Z})$. Thus the image of squares in the domain will be nonsquare rectangles whenever $v \neq \pi + 2\pi k$. Again, open *DiffGeomTool* and enter this parametrization for the torus with $a = 2$ and $b = 1$. Set the `Rectangular grid` values to 0 <= u <= 2$*$ pi and 0 <= v <= 2$*$ pi. And set the # u points: to 20 and the # v points: to 20. Notice that the grid of squares in the domain are mapped to a grid of mostly nonsquare rectangles as mentioned above. The ratio, $\frac{length}{height}$, of the sides of the rectangles is largest for the part of the torus farthest away from the origin. This occurs when $v = 0$ (or $v = 2\pi$) resulting in $E = 4$ while $G = 1$. On the other hand, the rectangles are squares for the part of the torus closest to the origin. This occurs when $v = \pi$ resulting in $E = 1$ while $G = 1$. This helps us see why this parametrization of the torus is not isothermal (see Figure 2.24).

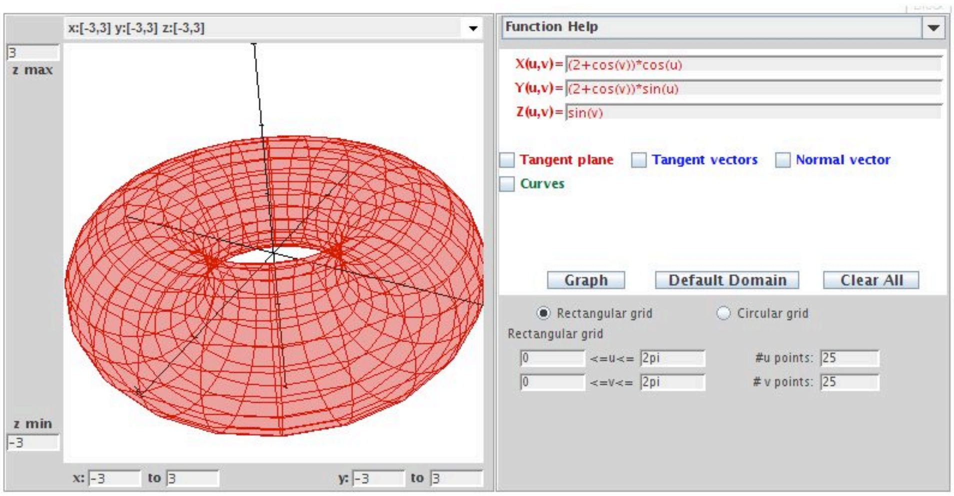



FIGURE 2.24. This parametrization of the torus is not isothermal.

EXERCISE 2.42. Using Definition 2.39 determine which of the following parametrizations of minimal surfaces is isothermal:

(a) The Enneper surface parametrized by

$$\mathbf{x}(u, v) = \left(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + u^2v, u^2 - v^2\right);$$

(b) Scherk's doubly periodic surface parametrized by

$$\mathbf{x}(u, v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right);$$

(c) The helicoid parametized by

$$\mathbf{x}(u, v) = (a \sinh v \cos u, a \sinh v \sin u, au).$$

***Try it out!***

EXPLORATION 2.43. Use *DiffGeomTool* to check the reasonableness of your answers in Exercise 2.42 by graphing each parametrization in that exercise as was done in Examples 2.40 and 2.41. Set the # u points: to 20 and the # v points: to 20, and use the following values for the `U/V domain` boxes:

(a) Enneper surface:

$-\frac{\pi}{3}$ <= u,v <= $\frac{\pi}{3}$;

(b) Scherk's doubly periodic surface:

$-\frac{\pi}{2} + 0.1$ <= u <= $\frac{\pi}{2} - 0.1$ and 0.1 <= v <= $\pi - 0.1$;

(c) Helicoid:

$-\pi$ <= u,v <= $\pi$.

***Try it out!***

From Exercise 2.42 and Exploration 2.43 you have seen there are parametrizations of minimal surfaces that are not isothermal. However, requiring minimal surfaces to have an isothermal parametrization is not a restriction because of the following theorem.

THEOREM 2.44. Every minimal surface in $\mathbb{R}^3$ has an isothermal parametrization.

REMARK 2.45. In fact, every differentiable surface has an isothermal parametrization. This is a very interesting result. Unfortunately, a proof of this is beyond the scope of this text, but if you are interested, a proof is given in [**2**], pp 15-35.

Recall that in Example 2.34, we derived that the isothermal parametrization for the catenoid

$$\mathbf{x}(u,v) = \Big(a\cosh v\cos u, a\cosh v\sin u, av\Big)$$

has

$$e = -g.$$

In general, we have the following result.

THEOREM 2.46. Let $M$ be a surface with isothermal parametrization. Then $M$ is minimal $\iff e = -g$.

EXERCISE 2.47. Prove Theorem 2.46.

EXPLORATION 2.48. Recall that $e = -g$ for the coefficients of the 2nd fundamental form represents that the $u$-parameter curve and the $v$-parameter curve are bending the same amount away from the normal $\mathbf{n}$ but in different directions. Use *DiffGeomTool* in connection with Theorem 2.46 to geometrically verify which of the following surfaces are minimal:

(a) Enneper surface:

$$\mathbf{x}(u,v) = \Big(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + u^2v, u^2 - v^2\Big);$$

(b) Cylinder:

$$\mathbf{x}(u,v) = \big(\cos u, \sin u, v\big);$$

(c) Helicoid:

$$\mathbf{x}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, av).$$

Now, here is an interesting and important result that brings in an idea from complex analysis. Recall from complex analysis, that if $f(z) = x(u,v) + iy(u,v)$ is an analytic function, then the Cauchy-Riemann equations hold for $f$. That is,

$$x_u = y_v, \qquad x_v = -y_u.$$

In such a case, $y$ is called the *harmonic conjugate* of $x$. Also, if $f$ is analytic, then

$$f'(z) = x_u + iy_u. \tag{8}$$

This concept allows us to relate a minimal surface to another minimal surface, known as its conjugate minimal surface.

DEFINITION 2.49. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugates. That is,

$$\mathbf{x}_u = \mathbf{y}_v \qquad \text{and} \qquad \mathbf{x}_v = -\mathbf{y}_u. \tag{9}$$

In such a case, $\mathbf{x}$ and $\mathbf{y}$ are called *conjugate minimal surfaces.*

EXAMPLE 2.50. Let's find the conjugate surface of the catenoid parametrized by

$$\mathbf{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av).$$

Let $\mathbf{y}(u, v)$ be the parametrization of this conjugate surface. By the first part of eq (9), we know

$$\mathbf{y}_v = \mathbf{x}_u = (-a \cosh v \sin u, a \cosh v \cos u, 0).$$

Integrating this with respect to $v$ yields

$$\mathbf{y} = (-a \sinh v \sin u + F_1(u), a \sinh v \cos u + F_2(u), F_3(u)),$$

where each $F_k(u)$ is a function independent of $v$. Similarly, by the second part of eq (9), we derive

$$\mathbf{y} = (-a \sinh v \sin u + G_1(v), a \sinh v \cos u + G_2(v), -au + G_3(v)).$$

Equating these two expressions for $\mathbf{y}$ we get that

$$\mathbf{y} = (-a \sinh v \sin u + K_1, a \sinh v \cos u + K_2, -au + K_3).$$

Using the substitution $u = \widetilde{u} - \frac{\pi}{2}$, $v = \widetilde{v}$, and letting $K_1 = 0$, $K_2 = 0$, and $K_3 = a\frac{\pi}{2}$, does not affect the geometry of this minimal surface, and yields the parametrization of a helicoid

$$\mathbf{y}(\widetilde{u}, \widetilde{v}) = (a \sinh \widetilde{v} \cos \widetilde{u}, a \sinh \widetilde{v} \sin \widetilde{u}, -a\widetilde{u})$$

given in Exercise 2.23 (Note that the negative sign in the third component function just has the effect of reflecting the surface through the $x_1x_2$-plane). Hence, the conjugate surface of this catenoid is a helicoid.

This idea of conjugate minimla surfaces gets really interesting. It turns out that any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces by the equation

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y},$$

where $t \in \mathbb{R}$. Note that when $t = 0$ we have the minimal surface parametrized by $\mathbf{x}$, and when $t = \frac{\pi}{2}$ we have the minimal surface parametrized by $\mathbf{y}$. So for $0 \leq t \leq \frac{\pi}{2}$, we have a continuous parameter of minimal surfaces known as *associated surfaces.* In other words, we can continuously "morph" one minimal surface into another minimal surface so that all the in-between surfaces are also minimal.

In Example 2.50, we saw that the helicoid and the catenoid are conjugate surfaces. Images of them and certain associated surfaces are shown in Figure 2.25.

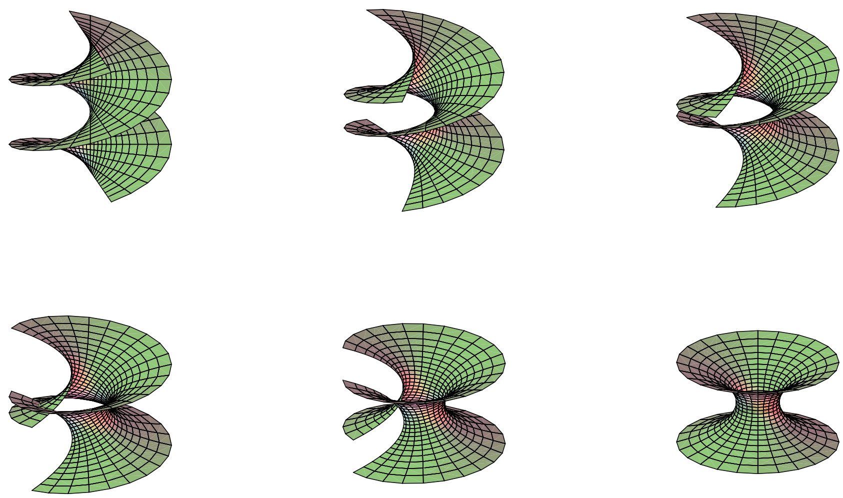


FIGURE 2.25. The helicoid, some associated surfaces, and the catenoid.

This is neat, but it is just the beginning. The rest of this section will explore properties of conjugate surfaces.

EXERCISE 2.51. Find the conjugate minimal surface for the Enneper surface

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right).$$

***Try it out!***

If we try to determine the conjugate minimal surface for Scherk's doubly periodic surface with the parametrization

$$\mathbf{x}(u,v) = \mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right),$$

this method will not work, because this parametrization is not isothermal. However, later we will see that Scherk's doubly periodic surface does have a conjugate surface. It is Scherk's singly periodic surface (see Figure 2.26).

EXPLORATION 2.52. You can see the associated surfaces that occur between Scherk's doubly periodic surface and Scherk's singly periodic surface by using another applet

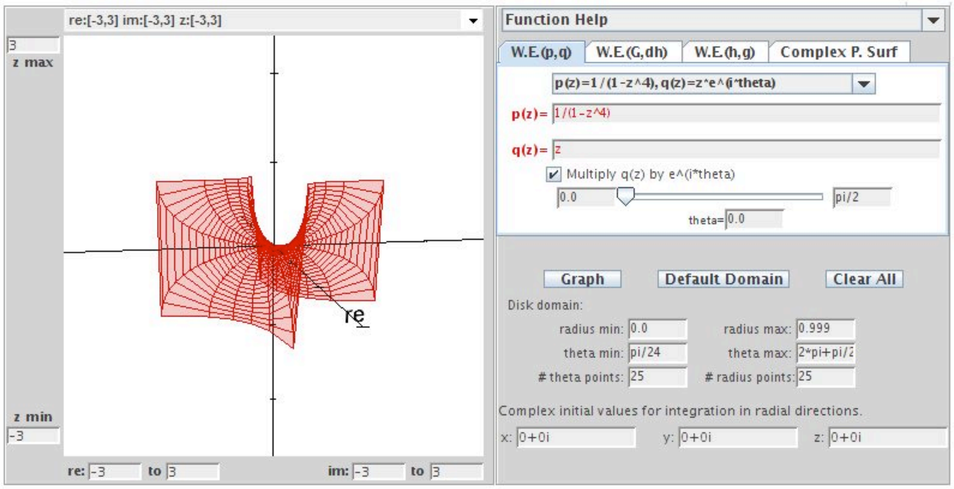



FIGURE 2.26. Scherk's singly periodic surface.

for this chapter. This applet is called *MinSurfTool*, and it can be used to visualize and explore minimal surfaces in $\mathbb{R}^3$. Now open *MinSurfTool*. On the right-hand side near the top, there is a set of tabs for different features of this applet. For this exploration, we want to use the **W.E.(p,q)** feature, so make sure that the **W.E.(p,q)** tab is on top (it should be a different color than the other tabs). In the **Pre-set functions** window, choose $p(z) = 1/(1-z^4)$, $q(z) = z * e \wedge (i * theta)$. Click on the **Graph** button, and one piece of Scherk's singly periodic surface will appear. Then move the slider arrow, that is above the **Graph** button, to the right to see how Scherk's singly periodic is transformed by way of the associated surfaces into Scherk's doubly periodic surface.

***Try it out!***

Recall that individual pieces of Scherk's doubly periodic surface can be put together in the $x_1x_2$-plane in a checkerboard fashion. So, these pieces repeat (or are periodic) in two directions, $x_1$ and $x_2$. Individual pieces of Scherk's singly periodic surface can fit together creating a tower in the $x_3$ direction. You can visualize adding two pieces together by taking one piece of Scherk's singly periodic surface and adding it to another piece that has been reflected across the $x_1x_2$-plane and shifted up in the $x_3$ direction. By continuing to do this, you can create a tower of several pieces (see Figure 2.27). Note that the helicoid is a singly periodic surface too.

Earlier in Example 2.30 we saw that the coefficients of the first fundamental form for the given parametrization of a catenoid are $E = a^2 \cosh^2 v$, $F = 0$, and $G = a^2 \cosh^2 v$. In Exercise 2.153 it can be shown that for the given parametrization of Enneper's surface these coefficients are $E = (1 + u^2 + v^2)^2$, $F = 0$, and $G = (1 + u^2 + v^2)^2$. Clearly, the $E$'s and $G$'s do not match up. However, for any two conjugate minimal surfaces and their associated minimal surfaces, the coefficients of the first fundamental

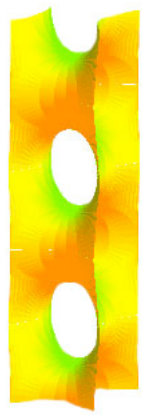


FIGURE 2.27. Scherk's singly periodic surface.

form are always the same. The following exercise will help you prove this surprising result.

EXERCISE 2.53.

(a) Prove that given two conjugate minimal surfaces, $\mathbf{x}$ and $\mathbf{y}$, all surfaces of the one-parameter family

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$$

have the same fundamental form: $E = \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{y}_u \cdot \mathbf{y}_u$, $F = 0$, $G = \mathbf{x}_v \cdot \mathbf{x}_v = \mathbf{y}_v \cdot \mathbf{y}_v$.

(b) Prove that all the surfaces in the one-parameter family $\mathbf{z}$ from part (a) are minimal for all $t \in \mathbb{R}$.

***Try it out!***

Finally, recall that the normal vector $\mathbf{n}$ at a point on a surface points orthogonally away from the surface. Since different minimal surfaces have different shapes, there is no reason to suspect that the normal vectors on one surface will be related to the normal vectors on another surface. However, for conjugate minimal surfaces and their associated minimal surfaces there is a strong connection. It turns out that for any point in the domain, the corresponding surface normal points in the same direction on all these minimal surfaces. The next theorem establishes this idea.

THEOREM 2.54. Let $\mathbf{x}, \mathbf{y} : D \to \mathbb{R}^3$ be isothermal parametrizations of conjugate minimal surfaces. Then for each $(u_0, v_0) \in D$, the corresponding surface unit normal is the same for all the associated surfaces.

PROOF. Let $(u_0, v_0) \in D$. Let $\mathbf{n}^{\mathbf{x}}$ and $\mathbf{n}^{\mathbf{y}}$ represent the surface normal for $\mathbf{x}$ and for $\mathbf{y}$, respectively. Then by the definition of conjugate surfaces, $\mathbf{x}$ and $\mathbf{y}$ have the same unit normal, because

$$\mathbf{n}^{\mathbf{x}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{\mathbf{y}_v \times -\mathbf{y}_u}{|\mathbf{y}_v \times -\mathbf{y}_u|} = \frac{\mathbf{y}_u \times \mathbf{y}_v}{|\mathbf{y}_u \times \mathbf{y}_v|} = \mathbf{n}^{\mathbf{y}}.$$

To show that this is true for the associated surfaces , let $\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$ be the parametrization of the associated surfaces. Then

$$\begin{aligned}
\mathbf{z}_u \times \mathbf{z}_v =& \big(\cos t\mathbf{x}_u + \sin t\mathbf{y}_u\big) \times \big(\cos t\mathbf{x}_v + \sin t\mathbf{y}_v\big) \\
=& \cos^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) + \cos t \sin t\big(\mathbf{x}_u \times \mathbf{y}_v\big) + \cos t \sin t\big(\mathbf{y}_u \times \mathbf{x}_v\big) + \sin^2 t\big(\mathbf{y}_u \times \mathbf{y}_v\big) \\
=& \cos^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) + \cos t \sin t\big(\mathbf{x}_u \times \mathbf{x}_u\big) + \cos t \sin t\big(-\mathbf{x}_v \times \mathbf{x}_v\big) + \sin^2 t\big(-\mathbf{x}_v \times \mathbf{x}_u\big) \\
=& \cos^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) + \cos t \sin t\big(\mathbf{0}\big) + \cos t \sin t\big(\mathbf{0}\big) + \sin^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) \\
=& \mathbf{x}_u \times \mathbf{x}_v.
\end{aligned}$$

□

The following example and exploration helps us visualize this idea.

EXAMPLE 2.55. Using *DiffGeomTool* we can graph the catenoid and its conjugate surface, the helicoid, whose parametrizations are given in Example 2.50. If we plot the normal $\mathbf{n}$ at the point $(\frac{2\pi}{3}, -\frac{\pi}{4})$ on these conjugate surfaces, we see that both normals point in the same direction as guaranteed by Theorem 2.54 (see Figure 2.28 and Figure 2.29).

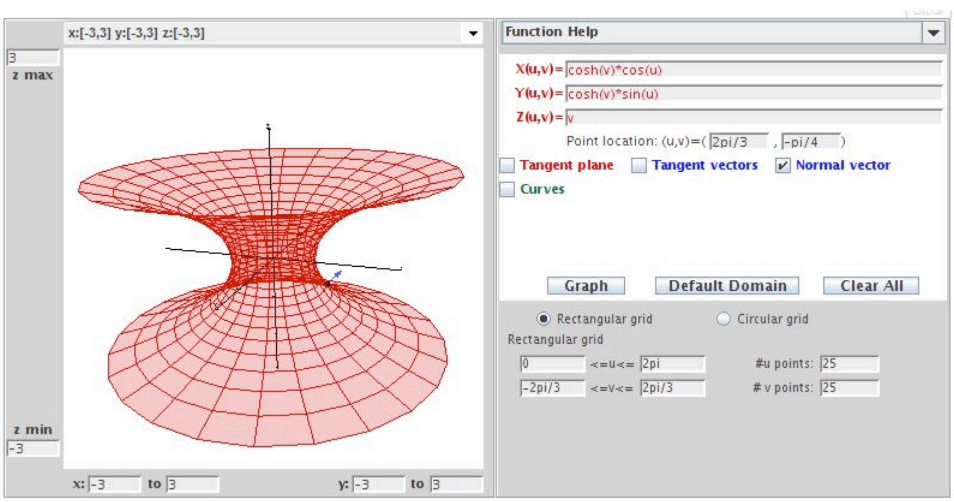



FIGURE 2.28. The catenoid with $\mathbf{n}$ at $(\frac{2\pi}{3}, -\frac{\pi}{4})$.

EXPLORATION 2.56. Open two separate windows of *DiffGeomTool*. In one plot the catenoid parametrized by

$$\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

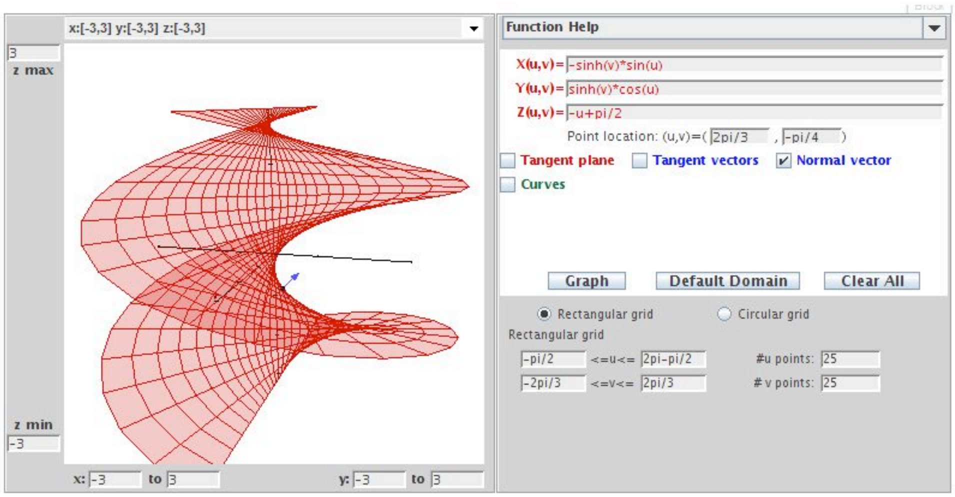



FIGURE 2.29. The conjugate helcoid with $\mathbf{n}$ at $(\frac{2\pi}{3}, -\frac{\pi}{4})$.

where $0 \leq u \leq 2\pi$ and $-\frac{2\pi}{3} \leq v \leq \frac{2\pi}{3}$. In the other plot its conjugate surface, the helicoid, with parametrization

$$\mathbf{y}(u,v) = \left( -\sinh v \cos u, \sinh v \sin u, -u + \frac{\pi}{2} \right),$$

where $-\frac{\pi}{2} \leq u \leq 2\pi - \frac{\pi}{2}$ and $-\frac{2\pi}{3} \leq v \leq \frac{2\pi}{3}$ (Note that the $u$ values for the helicoid are different than the values for the catenoid because we used the substitution $u = \widetilde{u} - \frac{\pi}{2}$ in Example 2.50). Then plot the following unit normals, $\mathbf{n}$, on each surface at the following points and observe that $\mathbf{n}$ points in the same direction as prescribed by Theorem 2.54:

(a) at $(\frac{\pi}{2}, 0)$, (b) at $(\frac{\pi}{4}, -\frac{\pi}{2})$, (c) at $(\frac{5\pi}{4}, \frac{\pi}{2})$.

## 2.4. Weierstrass Representation

At the end of the last section, we saw an application of complex analysis into minimal surface theory with the conjugate surfaces. In this section we are now ready to bring in more ideas from complex analysis. First, we will use the property of isothermal parametrization to give us a necessary and sufficient condition for a surface to be minimal. This condition is very important and useful. It will come as a corollary to the following theorem.

THEOREM 2.57. If the parametrization $\mathbf{x}$ is isothermal, then

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2EH\mathbf{n},$$

where $E$ is a coefficient of the first fundamental form and $H$ is the mean curvature.

Exercise 2.58 (Proof of Theorem 2.57). The set $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{n}\}$ forms a basis for $\mathbb{R}^3$. Assume $F = 0$. Then the vector $\mathbf{x}_{uu}$ can be expressed in terms of these bases vectors. That is,

$$\mathbf{x}_{uu} = \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + e\mathbf{n},$$

where the coefficients, $\Gamma^u_{uu}$ and $\Gamma^v_{uu}$, are known as Christoffel symbols and $e$ comes from the coefficient of the second fundamental form. That is, $e = \mathbf{n} \cdot \mathbf{x}_{uu}$.

(a) Show that $\Gamma^u_{uu} = \frac{E_u}{2E}$ and $\Gamma^v_{uu} = -\frac{E_v}{2G}$ by taking the inner product of $\mathbf{x}_{uu}$ with each of the basis vectors. In a similar manner, it can be shown that

$$\mathbf{x}_{vv} = -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + g\mathbf{n}.$$

(b) Use the mean curvature equation (5) and the results from (a) to show that if the parametrization $\mathbf{x}$ is isothermal, then

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2EH\mathbf{n}.$$

***Try it out!***

Now, where do we go from here? Recall that for a minimal surface, $H \equiv 0$. So Theorem 2.57 tells us that $\mathbf{x}_{uu} + \mathbf{x}_{vv} \equiv 0$. But what does this last equation represent? It is Laplace's equation and relates to harmonic functions. Recall that $\varphi(u, v)$ is a real-valued *harmonic function* if $\varphi_{uu} + \varphi_{vv} = 0$ (for example, $\varphi(u, v) = u^2 - v^2$ is harmonic). This leads us to our next result which is very important and useful. At first, it may seem like this is a small result, because it is a brief corollary with a short proof. However, do not be mislead. We have spent a lot of time laying the foundation, and now we are fitting in the final pieces of the puzzle that will form the basis for describing minimal surfaces by using the Weierstrass representation.

Corollary 2.59. A surface $M$ with an isothermal parametrization $\mathbf{x}(u, v) = \big(x_1(u, v), x_2(u, v), x_3(u, v)\big)$ is minimal $\Longleftrightarrow$ $x_1, x_2, x_3$ are harmonic.

Make sure you understand the significance of this result. First, we need an isothermal parametrization for our surface, but this is not a difficulty because of Theorem 2.44. Then this result tells us we will have a minimal surface if and only if the coordinate functions of that parametrization are harmonic functions. This will provide us another way to create and to prove a surface is minimal.

Proof. ($\Rightarrow$) If $M$ is minimal, then $H = 0$ and so by Theorem 2.57 $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$, and hence the coordinate functions are harmonic. ($\Leftarrow$) Suppose $x_1, x_2, x_3$ are harmonic. Then $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$. So by Theorem 2.57 we have that $2(\mathbf{x}_u \cdot \mathbf{x}_u)H\mathbf{n} = 0$. But $\mathbf{n} \neq 0$ and $E = \mathbf{x}_u \cdot \mathbf{x}_u \neq 0$. Hence, $H = 0$ and $M$ is minimal. □

Exercise 2.60. Given the parametrization for the Enneper surface

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right),$$

use Corollary 2.59 to prove that the Enneper surface is a minimal surface.

***Try it out!***

The importance of Corollary 2.59 is not in proving specific surfaces are minimal. Instead, it lies in establishing a general formula that will guarantee any surface created by it will be minimal. This formula is known as the Weierstrass representation for minimal surfaces. This is neat, because it will provide us with a simple way to construct a lot of examples of minimal surfaces using functions from complex analysis. After stating the Weierstrass representation in Theorem 2.65 we will use the rest of this section to create more minimal surfaces. However, it turns out that not all minimal surfaces are of equal interest. So in section 2.5 of this chapter we leave behind the idea of creating arbitrary minimal surfaces and instead explore properties that make certain minimal surfaces more interesting.

Now we will derive this important formula, the Weierstrass Representation, and bring in the connection with complex analysis. Suppose $M$ is a minimal surface with an isothermal parametrization $\mathbf{x}(u,v)$. Let $z = u + iv$ be a point in the complex plane. Recall that $\overline{z} = u - iv$ is the conjugate of $z$. Using these representations for $z$ and $\overline{z}$ we can solve for $u, v$ in terms of $z, \overline{z}$ to get

$$u = \frac{z+\overline{z}}{2} \text{ and } v = \frac{z-\overline{z}}{2i}.$$

Then the parametrization of the minimal surface $M$ can be written in terms of the complex variables $z$ and $\overline{z}$ as:

$$\mathbf{x}(z,\overline{z}) = \big(x_1(z,\overline{z}), x_2(z,\overline{z}), x_3(z,\overline{z})\big).$$

EXERCISE 2.61. Let $f(u,v) = x(u,v) + iy(u,v)$ be a complex function. Using the notation $u = \frac{z+\overline{z}}{2}$ and $v = \frac{z-\overline{z}}{2i}$, we can express $f$ in terms of $z$ and $\overline{z}$ instead of $u$ and $v$. That is, we have the function $f(z,\overline{z})$. In this exercise you will prove the neat result that $f$ is analytic if and only if $f$ can be written in terms of $z = u + iv$ alone without using $\overline{z} = u - iv$.

(a) Using the chain rule, derive the following formulas:

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} - \frac{\partial x}{\partial v}\right),\\ \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial x}{\partial u} - \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} + \frac{\partial x}{\partial v}\right).\end{aligned}$$

(b) Show that $f$ is analytic $\Longleftrightarrow \frac{\partial f}{\partial \overline{z}} = 0$.

***Try it out!***

EXAMPLE 2.62. The function $f_1(z) = z^2$ is analytic, because $\frac{\partial f_1}{\partial \overline{z}}(z^2) = 0$. However, $f_2(z) = |z|^2 = z\overline{z}$ is not analytic, because $\frac{\partial f_1}{\partial \overline{z}} = z \neq 0$.

Exercise 2.63. Prove that

$$4\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right)\right) = f_{uu} + f_{vv}. \tag{10}$$

***Try it out!***

The next theorem expands upon Corollary 2.59 to establish the Weierstrass representation for minimal surfaces.

Theorem 2.64. Let $M$ be a surface with parametrization $\mathbf{x} = (x_1, x_2, x_3)$ and let $\phi = (\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_k = \frac{\partial x_k}{\partial z}$. Then $\mathbf{x}$ is isothermal $\iff \phi^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$. If $\mathbf{x}$ is isothermal, then $M$ is minimal $\iff$ each $\varphi_k$ is analytic.

Before we prove Theorem 2.64, let's look at applying it to a specific example to help us better understand what the theorem is saying. Suppose we have the parametrization $\mathbf{x} = (x_1, x_2, x_3) = (z - \frac{1}{3}z^3, -i(z + \frac{1}{3}z^3), z^2)$. Then $\varphi_1 = \frac{\partial x_1}{\partial z} = 1 - z^2$, $\varphi_2 = \frac{\partial x_2}{\partial z} = -i(1 + z^2)$, and $\varphi_3 = \frac{\partial x_3}{\partial z} = 2z$. Notice that $\phi^2 = [1 - z^2]^2 + [-i(1 + z^2)]^2 + [2z]^2 = 0$. Thus, by the theorem, the parametrization $\mathbf{x}$ is isothermal. Also, each $\varphi_k$ is a polynomial and hence analytic. So $\mathbf{x}$ is a parametrization of a minimal surface (in fact, it is Enneper's surface). Make sure you understand how this example relates to Theorem 2.64 before you read the following proof of the theorem.

Proof. Applying the complex differential operator $\frac{\partial f}{\partial z}$ from Exercise 2.61 to this situation and then squaring the terms, we have $(\varphi_k)^2 = \left(\frac{\partial x_k}{\partial z}\right)^2 = \left[\frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)\right]^2 = \frac{1}{4}\left[\left(\frac{\partial x_k}{\partial u}\right)^2 - \left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right]$. Also, recall that $\mathbf{x_u} \cdot \mathbf{x_u} = \left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial x_2}{\partial u}\right)^2 + \left(\frac{\partial x_3}{\partial u}\right)^2 = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2$ and similarly $\mathbf{x_v} \cdot \mathbf{x_v} = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2$. Hence,

$$\begin{aligned}
\phi^2 =& (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 \\
=& \frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 - \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\sum_{k=1}^{3}\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right] \\
=& \frac{1}{4}\big(\mathbf{x}_u \cdot \mathbf{x}_u - \mathbf{x}_v \cdot \mathbf{x}_v - 2i(\mathbf{x}_u \cdot \mathbf{x}_v)\big) \\
=& \frac{1}{4}(E - G - 2iF).
\end{aligned}$$

Thus, $\mathbf{x}$ is isothermal $\iff E = G, F = 0 \iff \phi^2 = 0$.

Now suppose that $\mathbf{x}$ is isothermal. By Corollary 2.59, it suffices to show that for each $k$, $x_k$ is harmonic $\iff \varphi_k$ is analytic. Using eq (10) and Exercise 2.61 this follows because

$$\frac{\partial^2 x_k}{\partial u \partial u} + \frac{\partial^2 x_k}{\partial v \partial v} = 4\left(\frac{\partial}{\partial \bar{z}}\left(\frac{\partial x_k}{\partial z}\right)\right) = 4\left(\frac{\partial}{\partial \bar{z}}\Big(\varphi_k\Big)\right) = 0.$$

∎

Note that if $\mathbf{x}$ is isothermal

$$
\begin{aligned}
|\phi|^2 &= \left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \left|\frac{\partial x_3}{\partial z}\right|^2 = \frac{1}{4}\left(\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 + \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2\right) \\
&= \frac{1}{4}\big(\mathbf{x}_u \cdot \mathbf{x}_u + \mathbf{x}_v \cdot \mathbf{x}_v\big) = \frac{1}{4}(E+G) = \frac{E}{2}.
\end{aligned}
$$

So if $|\phi|^2 = 0$, then all the coefficients of the first fundamental form are zero and $M$ degenerates to a point. Similarly, we want $|\phi|^2$ to be finite.

Finally, we need to solve $\varphi_k = \frac{\partial x_k}{\partial z}$ for $x_k$ since the parametrization of the surface is given as $\mathbf{x} = (x_1, x_2, x_3)$. The difficulty is that $x_k$ is a function of two variables, $z$ and $\overline{z}$, and we want to have a representation in which we only have to integrate with respect to one variable. To overcome this difficulty, we will use some ideas about differentials (see [**25**] for a nice introduction to differentials). First, since $x_k$ is also a function of the two variables $u$ and $v$, we can write

$$
dx_k = \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv. \tag{11}
$$

Also, $dz = du + idv$. Using Exercise 2.61 we have

$$
\begin{aligned}
\varphi_k dz &= \frac{\partial x_k}{\partial z}dz = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)(du + idv) \\
&= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv + i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right], \\
\overline{\varphi_k dz} = \overline{\varphi_k}\,\overline{dz} &= \overline{\frac{\partial x_k}{\partial z}}\,\overline{dz} = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} + i\frac{\partial x_k}{\partial v}\right)(du - idv) \\
&= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv - i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right].
\end{aligned}
$$

Adding these two equations yields

$$
\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv = \varphi_k dz + \overline{\varphi_k dz} = 2\,\mathrm{Re}\{\varphi_k dz\}. \tag{12}
$$

Combining eq (11) and eq (12), we have

$$
dx_k = 2\,\mathrm{Re}\{\varphi_k dz\}.
$$

Therefore, $x_k = 2\,\mathrm{Re}\int \varphi_k dz + c_k$. Since adding $c_k$ just translates the image by a constant amount and multiplying each coordinate function by 2 just scales the the surface, these constants do not affect the geometric shape of the surface. Hence, we do not need these constants and we will let our coordinate function be

$$
x_k = \mathrm{Re}\int \varphi_k dz.
$$

**Summary**: We can find a formula for constructing minimal surfaces by determining analytic functions $\varphi_k$ $(k = 1, 2, 3)$ such that

$$\phi^2 = 0 \qquad \text{and} \qquad |\phi|^2 \neq 0 \text{ and is finite,}$$

in which case, we have the parametrization

$$\mathbf{x} = \Big(\mathrm{Re}\int \varphi_1(z)dz, \mathrm{Re}\int \varphi_2(z)dz, \mathrm{Re}\int \varphi_3(z)dz\Big). \tag{13}$$

For example, consider

$$\begin{aligned}\varphi_1 =& p(1+q^2)\\ \varphi_2 =& -ip(1-q^2)\\ \varphi_3 =& -2ipq.\end{aligned}$$

Then

$$\begin{aligned}\phi^2 =&[p(1+q^2)]^2 + [-ip(1-q^2)]^2 + [-2ipq]^2\\ =&[p^2+2p^2q^2+p^2q^4] - [p^2-2p^2q^2+p^2q^4] - [4p^2q^2]\\ =&0,\end{aligned}$$

and

$$\begin{aligned}|\phi|^2 =&|p(1+q^2)|^2 + |-ip(1-q^2)|^2 + |-2ipq|^2\\ =&|p|^2[(1+q^2)(1+\bar{q}^2) + (1-q^2)(1-\bar{q}^2) + 4q\,\bar{q}]\\ =&|p|^2[2(1+2q\,\bar{q}+q^2\,\bar{q}^2)\\ =&4|p|^2(1+|q|^2)|^2 \neq 0 \qquad \text{(note: if } p=0\text{, then } \varphi_k = 0 \text{ for all } k).\end{aligned}$$

Notice that $p$, $pq^2$, and $pq$ have to be analytic in order for each $\varphi_k$ to be analytic. If $p$ is analytic with a zero of order $2m$ at $z_0$, then $q$ can have a pole of order no larger than $m$ at $z_0$. This leads to the following result.

Theorem 2.65 (Weierstrass Representation (p,q)). Every regular minimal surface has a local isothermal parametric representation of the form

$$\begin{aligned}\mathbf{x} =& (x_1(z), x_2(z), x_3(z))\\ =& \Big(\mathrm{Re}\Big\{\int_a^z p(1+q^2)dz\Big\},\\ & \mathrm{Re}\Big\{\int_a^z -ip(1-q^2)dz\Big\},\\ & \mathrm{Re}\Big\{\int_a^z -2ipqdz\Big\}\Big),\end{aligned}$$

where $p$ is an analytic function and $q$ is a meromorphic function in some domain $\Omega \subset \mathbb{C}$, having the property that at each point where $q$ has a pole of order $m$, $p$ has a zero of order at least $2m$, and $a \in \Omega$ is a constant.

EXAMPLE 2.66. For $p(z) = 1, q(z) = iz$, we get

$$\begin{aligned}\mathbf{x} &= \Big( \operatorname{Re}\Big\{ \int_0^z (1 - z^2)\, dz \Big\}, \operatorname{Re}\Big\{ \int_0^z -i(1+z^2)\, dz \Big\}, \operatorname{Re}\Big\{ \int_0^z 2z\, dz \Big\} \Big) \\ &= \Big( \operatorname{Re}\Big\{ z - \frac{1}{3}z^3 \Big\}, \operatorname{Re}\Big\{ -i\Big(z + \frac{1}{3}z^3\Big) \Big\}, \operatorname{Re}\Big\{ z^2 \Big\} \Big).\end{aligned}$$

Letting $z = u + iv$, this yields

$$\mathbf{x}(u,v) = \left( u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2 v, u^2 - v^2 \right)$$

which gives the Enneper surface.

You can use the applet, *MinSurfTool*, to graph an image of this surface using the functions $p$ and $q$. After opening *MinSurfTool* click on the `W.E.(p,q)` tab so it is on top. In the appropriate boxes, put $p(z) = 1$ and $q(z) = i * z$. Then click on the **Graph** button. Remember that you can increase the size of the image of the surface by clicking on the left button on the mouse, and you can decrease the size by clicking on the right mouse button. Also, you can rotate the surface by placing the cursor arrow on the image of the surface, then click on and hold the left button on the mouse as you move the cursor.

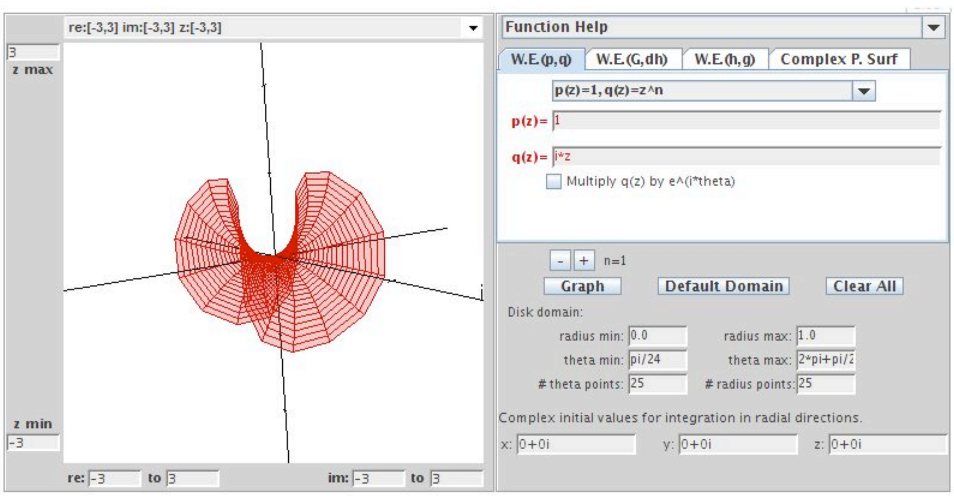



FIGURE 2.30. Enneper surface using $p(z) = 1$ and $q(z) = iz$.

EXAMPLE 2.67. Let $p(z) = 1$ and $q(z) = 1/z$ on the domain $\mathbb{C} - \{0\}$. Notice that $q$ is meromorphic with a pole of order 1 at $z_0 = 0$ while $p$ does not have a zero of order 2 at $z_0 = 0$. This does not violate the conditions of Theorem 2.65, because the domain is $\mathbb{C} - \{0\}$. We will show that this generates a helicoid. Using Weierstrass Representation (p,q) and letting $z = u + iv$, we get $\mathbf{x}(u,v) = (x_1, x_2, x_3)$, where

$$\begin{aligned}
x_1 &= \mathrm{Re} \int_1^z \left(1 + \frac{1}{z^2}\right) dz = \mathrm{Re}\left(z - \frac{1}{z}\right) = u - \frac{u}{u^2+v^2} \\
x_2 &= \mathrm{Re} \int_1^z -i\left(1 - \frac{1}{z^2}\right) dz = \mathrm{Im}\left(z + \frac{1}{z}\right) = v - \frac{v}{u^2+v^2} \\
x_3 &= \mathrm{Re} \int_1^z -2i\frac{1}{z}\, dz = 2\,\mathrm{Im}(\log z) = 2 \arg z = 2 \arctan\left(\frac{v}{u}\right).
\end{aligned}$$

Notice that this parametrization is different than the following parametrization we have been using for the helicoid:

$$\widetilde{\mathbf{x}}(\widetilde{u}, \widetilde{v}) = (\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}) = (a \sinh \widetilde{v} \cos \widetilde{u}, a \sinh \widetilde{v} \sin \widetilde{u}, a\widetilde{u}).$$

To show that $\mathbf{x}$ also gives an image of the helicoid, we will find a substitution that will change $\mathbf{x}$ into $\widetilde{\mathbf{x}}$. Note that

$$\begin{aligned}
x_1^2 + x_2^2 &= (u^2+v^2) - 2 + \frac{1}{u^2+v^2} \\
\widetilde{x_1}^2 + \widetilde{x_2}^2 &= a^2 \sinh^2 \widetilde{v} = a^2 \left(\frac{e^{\widetilde{v}} - e^{-\widetilde{v}}}{2}\right)^2.
\end{aligned}$$

Equating the right hand side of these equations and letting $a = 2$, we get that

$$u^2 + v^2 = e^{2\widetilde{v}}.$$

Also, with $x_3 = \widetilde{x_3}$, we see that

$$\frac{v}{u} = \tan \widetilde{u}.$$

Now, using these last two equations we can solve for $u$ and $v$ to get

$$u = e^{\widetilde{v}} \cos \widetilde{u} \quad \text{and} \quad v = e^{\widetilde{v}} \sin \widetilde{u}.$$

If we substitute these values for $u$ and $v$ into $\mathbf{x}(u,v)$ we get the parametrization $\widetilde{\mathbf{x}}(\widetilde{u}, \widetilde{v})$ for the helicoid.

Using the `W.E.(p,q)` tab in *MinSurfTool* with $p(z) = 1$ and $q(z) = 1/z$, we can get a graph of the helicoid. Since the domain is $\mathbb{C} \setminus \{0\}$, set the `Disk domain: radius min:` box to 0.1.

Exercise 2.68. Show that the minimal surfaces generated by using $p(z) = 1$ and $q(z) = 0$ on the domain $\mathbb{C}$ in the Weierstrass representation is the plane.

***Try it out!***

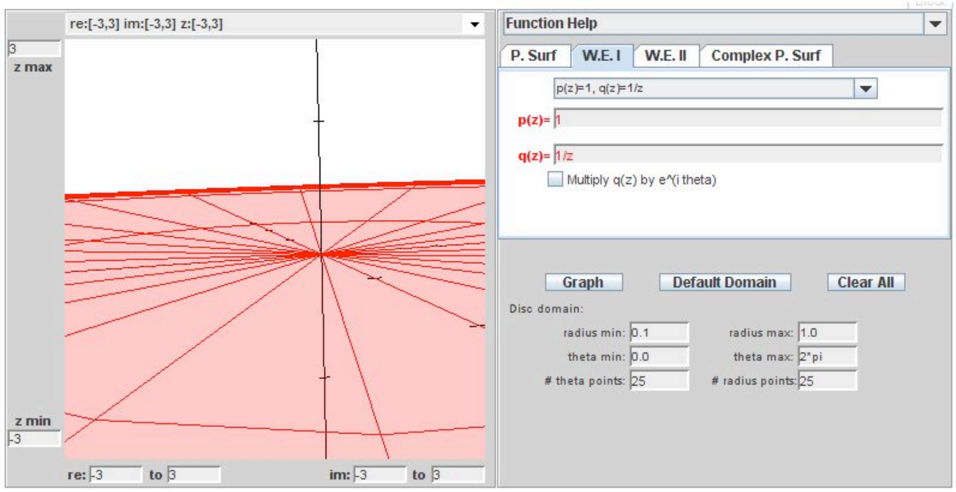



FIGURE 2.31. The helicoid using $p(z) = 1$ and $q(z) = \frac{1}{z}$.

EXERCISE 2.69. Show that the minimal surfaces generated by using $p(z) = 1$ and $q(z) = i/z$ on the domain $\mathbb{C} - \{0\}$ in the Weierstrass representation is the catenoid. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface.

***Try it out!***

EXPLORATION 2.70. Enneper's surface can be constructed also with $p(z) = 1$ and $q(z) = z$. Recall that it has four leaves (two pointing up and two pointing down). The number of leaves can be increased.

(a) Using $p(z) = 1$ and $q(z) = z^2$ on the domain $\mathbb{C}$ in the Weierstrass representation gives the Enneper surface with six leaves (see Figure 2.32). Compute the parametrization $\mathbf{x}(u, v)$ for this surface.

(b) Use *MinSurfTool* with the `W.E.(p,q)` tab to conjecture the values of $p$ and $q$ for the Enneper surface with $n$ leaves.

***Try it out!***

EXPLORATION 2.71. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of the surface generated by $p(z) = \frac{1}{1-z^4}$ and $q(z) = z$.

(a) What minimal surface is this?

(b) Click on the box "Multiply $q(z)$ by e∧(i theta)" and move the slider to generate a family of minimal surfaces. These surfaces are associated surfaces (see the paragraph after Definition 2.49). When $\theta = i$ you get the conjugate surface. In this case, what is the conjugate surface?

(c) Experiment with *MinSurfTool* to view the associated family and find the conjugate surface of various minimal surfaces discussed above.

***Try it out!***

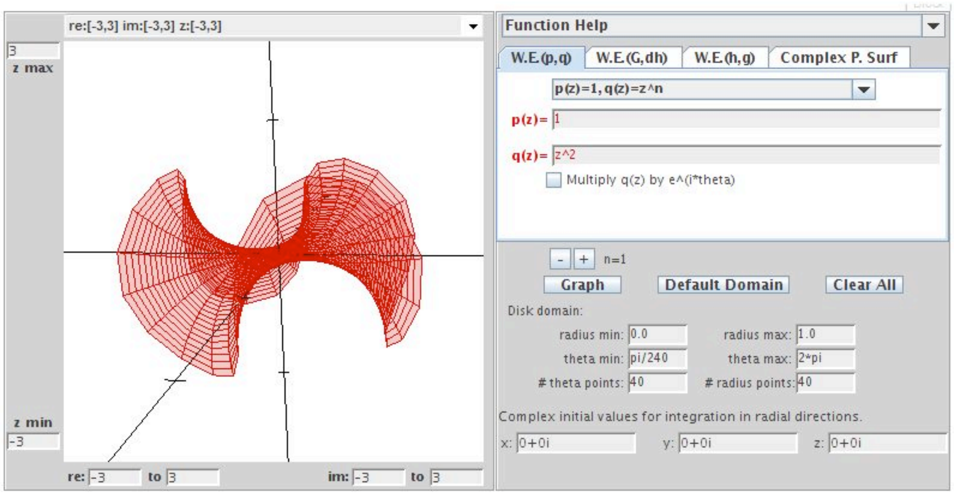



FIGURE 2.32. Enneper surface with 6 leaves using $p(z) = 1$ and $q(z) = z^2$.

EXPLORATION 2.72. Scherk's doubly periodic surface is generated with $p(z) = \frac{1}{1-z^4}$ and $q(z) = iz$. Using *MinSurfTool* with the `W.E.(p,q)` tab, graph an image of the surface generated by $p(z) = \frac{1}{1-z^{2n}}$ and $q(z) = iz^{n-1}$ for various values of $n = 2, 3, 4, \ldots$ on the domain the unit disk $\mathbb{D}$ in the Weierstrass representation.

(a) What happens to the surface as $n$ increases?
(b) Notice that the surface has leaves that alternate between going up and going down. How is $n$ related to the number of leaves?
(c) What is the image of the projection of the surface onto the $x_1x_2$-plane for each $n$?
(d) Using the previous parts conjecture how many leaves the surface would have if $p(z) = \frac{1}{1-z^5}$. Why could such a surface not exist?

***Try it out!***

EXPLORATION 2.73. Using *MinSurfTool* with the `W.E.(p,q)` tab, graph an image of the surface generated by $p(z) = \frac{1}{(1-z^4)^2}$ and $q(z) = iz^3$. This surface is known as the 4-noid (see Figures 2.33 and 2.34).

(a) Experiment with *MinSurfTool* to determine a $p$ and $q$ that will generate a 3-noid on the domain the unit disk $\mathbb{D}$ in the Weierstrass representation.
(b) Conjecture the values of $p$ and $q$ that will generate an $n$-noid.

***Try it out!***

While the Weierstrass Representation will generate a minimal surface, there is no guarantee that the minimal surface will be embedded. Recall that a surface is embedded if it has no self-intersections. The plane, the catenoid, the helicoid, and Scherk's doubly periodic surface are examples of embedded minimal surfaces. However,

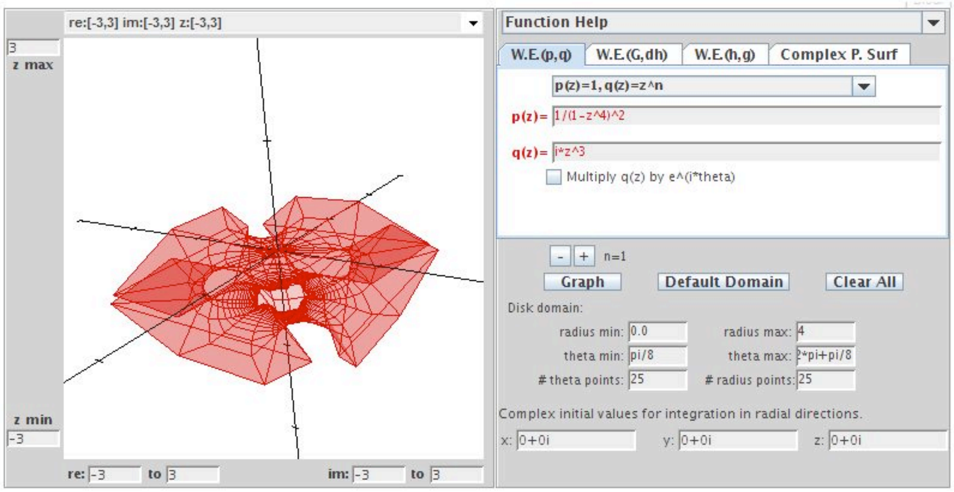



FIGURE 2.33. Image of the 4-noid minimal surface from the side.

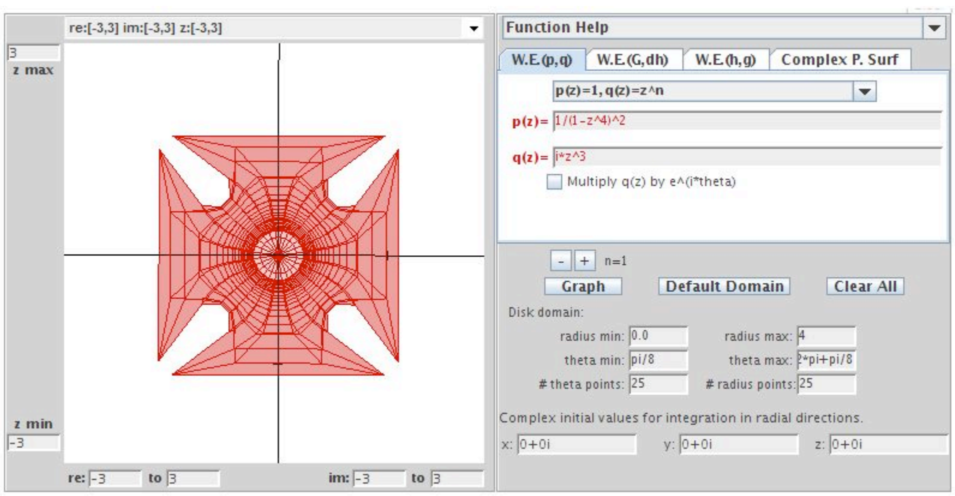



FIGURE 2.34. Image of the 4-noid minimal surface from above.

Enneper's surface is not embedded. In Exploration 2.11 you saw that Enneper's surface intersects itself when the domain contains a disk centered at the origin of radius $R \geq \sqrt{3}$.

EXPLORATION 2.74. Using *MinSurfTool* with the `W.E.(p,q)` tab, come up with three sets of functions $p$ and $q$ defined on the domain $\mathbb{D}$ that create other minimal surfaces that are not embedded.

***Try it out!***

An important area in minimal surface theory is the study of complete (boundaryless) embedded minimal surfaces. The following theorem tells us that any minimal surface without boundary cannot be closed and bounded.

THEOREM 2.75. If $M$ is a complete minimal surface in $\mathbb{R}^3$, then $M$ is not compact.

PROOF. By Theorem 2.44, we can assume that $M$ has an isothermal parametrization. Now, if $M$ were compact, then each coordinate function would attain a maximum. Since the real part of an analytic function is harmonic we see from Theorem 2.64, the coordinate functions in this parametrization are harmonic. But harmonic functions attain their maximum on the boundary of the set. So, $M$ must have a boundary which contradicts $M$ being complete. □

## 2.5. The Gauss map, $G$, and height differential, $dh$

We can use other representations for $\phi = (\varphi_1, \varphi_2, \varphi_3)$ to form different Weierstrass representations as long as $\phi^2 = 0$ and $|\phi|^2 \neq 0$ (see the Summary on page 141). An important representation employs the functions known as the Gauss map, $G$, and the height differential, $dh$. This representation is useful, because the functions $G$ and $dh$ describe the geometry of the minimal surface. To develop this, we first need some background about the Gauss map.

Recall that the curvature of a unit speed curve, $\alpha$, at a point $s$ is $\left|\alpha''(s)\right|$. That is, the curvature of a curve is described by the rate of change of the tangent vector. Similarly, the curvature of a surface is related to the change of the tangent plane. Since each tangent plane is essentially determined by its unit normal vector, $\mathbf{n}$, we can investigate the curvature of a surface by studying the variation of the unit normal vector. This is the idea behind the Gauss map.

DEFINITION 2.76. Let $M : \Omega \to \mathbb{R}^3$ be a surface with a chosen orientation (that is, a differentiable field of unit normal vectors $\mathbf{n}$). The *Gauss map*, $\mathbf{n}_p$, translates the unit normal on $M$ at a point $p$ to the unit vector at the origin pointing in the same direction as the unit normal and thus corresponds to a point on the unit sphere $S^2$.

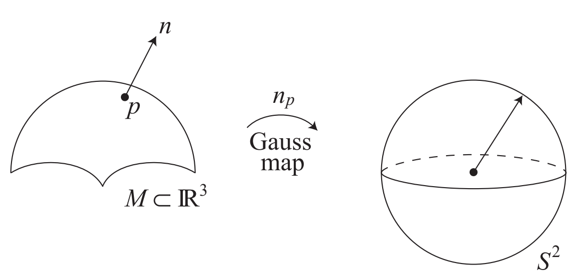



FIGURE 2.35. The Gauss map.

Example 2.77. Let's determine the image of the Gauss map for the catenoid. A meridian is a curve formed by a vertical slice on the surface (see Exercise 2.15). Consider a meridian on the entire catenoid (remember that the image in Figure 2.36 is just part of a catenoid and that it actually extends on forever). The Gauss map, $\mathbf{n_p}$, of this meridian will be a meridian on $S^2$ from the north pole, $(0,0,1)$, to the south pole, $(0,0,-1)$, that excludes these end points. Note that $(0,0,1)$ and $(0,0,-1)$ are excluded, because no matter how far the catenoid extends, the unit normal $\mathbf{n}$ never points exactly straight up or exactly straight down. Now, since the catenoid is a surface of revolution if we revolve this meridian on $S^2$, we get that the image of the Gauss map for the catenoid is $S^2 \setminus \{(0,0,1),(0,0,-1)\}$.

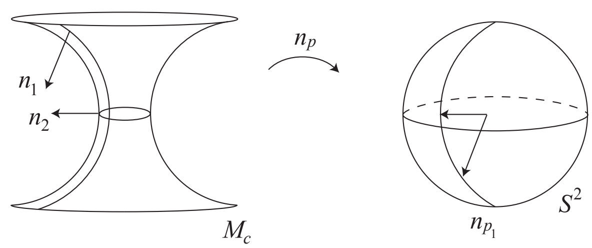



Figure 2.36. Image of a meridian on a catenoid under the Gauss map.

Exercise 2.78. Describe the image of the Gauss map for the following surfaces:
(a) a right circular cylinder;
(b) a torus;
(c) Ennepers surface defined just on $\mathbb{D}$;
(d) helicoid;
(e) Scherk's doubly periodic surface.

***Try it out!***

Theorem 2.79. Let $M$ be a minimal surface with an isothermal parametrization. Then the Gauss map of $M$ preserves angles.

While the Gauss map preserves angles, it reverses orientation. Such maps are known as *anticonformal*. To help visualize the fact that the Gauss maps reverses orientation, consider three points $A$, $B$, and $C$ on a curved path near the neck of the catenoid (see Figure 2.37). Since $A$ is above the neck of the catenoid, the outward pointing unit normal at $A$ will be pointing downward and hence the Gauss map will put it below the equator on $S^2$ at the point $A'$. The point $B$ is on the neck of the catenoid and so the outward pointing unit normal at $B$ will be horizontal. So, the Gauss map will put it on the equator of $S^2$ at the point $B'$. Similarly, the normal at $C$ will get mapped to $C'$. Thus, following the curve path from $A$ to $B$ to $C$ in the positive direction on the catenoid gets sent by the Gauss map to a curve from $A'$ to $B'$ to $C'$ in the negative direction on $S^2$. That is, we have an orientation-reversing map.

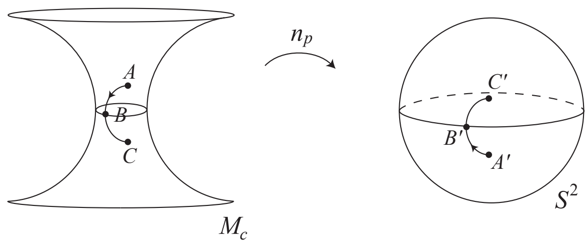



FIGURE 2.37. $\mathbf{n_p}$ is orientation reversing.

Since the Gauss map associates a point on $M$ with a point on $S^2$, we can also associate it with a point in the complex plane $\mathbb{C}$ by using stereographic projection. Recall that stereographic projection, $\sigma$, takes a point on $S^2$ to a point in the extended complex plane, $\mathbb{C} \cup \infty$. To do this, we place the complex plane through the equator of the sphere and take a line connecting the north pole, $(0, 0, 1) \in S^2$, with the given point $(x_1, x_2, x_3) \in S^2$. This line will intersect the extended complex plane at some point, $z = x + iy$. In such a setting the unit sphere is known as the *Riemann sphere*.

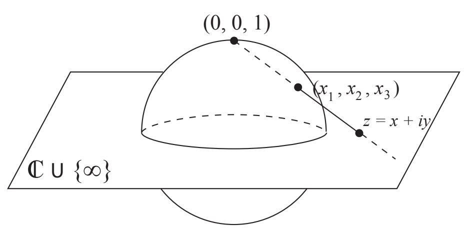



FIGURE 2.38. Stereographic projection.

EXERCISE 2.80. Describe the projections of the following sets on the Riemann sphere onto the extended complex plane:

(a) meridians;
(b) parallels;
(c) circles;
(d) circles that contain (i.e., touch) the point $(0, 0, 1)$;
(e) antipodal points (i.e., diametrically opposite points).

***Try it out!***

Finally, let $\overline{\sigma}$ be the projection of $(x_1, x_2, x_3) \in S^2$ to the point $x - iy \in \mathbb{C}$ given first by stereographic projection of $(x_1, x_2, x_3)$ to $z = x + iy$ followed by reflection of $z = x + iy$ across the real axis to $\overline{z} = x - iy$. Note that $\overline{\sigma}$ is anticonformal.

Now, let $G : D \subset \mathbb{C} \to \mathbb{C}$ be the map defined by $G = \overline{\sigma} \circ \mathbf{n} \circ \mathbf{x}$. Note that $G$ preserves angles since $\overline{\sigma}$, $\mathbf{n}$, and $\mathbf{x}$ preserves angles and so the composition preserves angles. Also,

$G$ is orientation preserving, because both $\overline{\sigma}$ and $\mathbf{n}$ are orientation reversing and so their composition is orientation preserving. Thus, $G$ is a meromorphic function. It is also called the Gauss map.

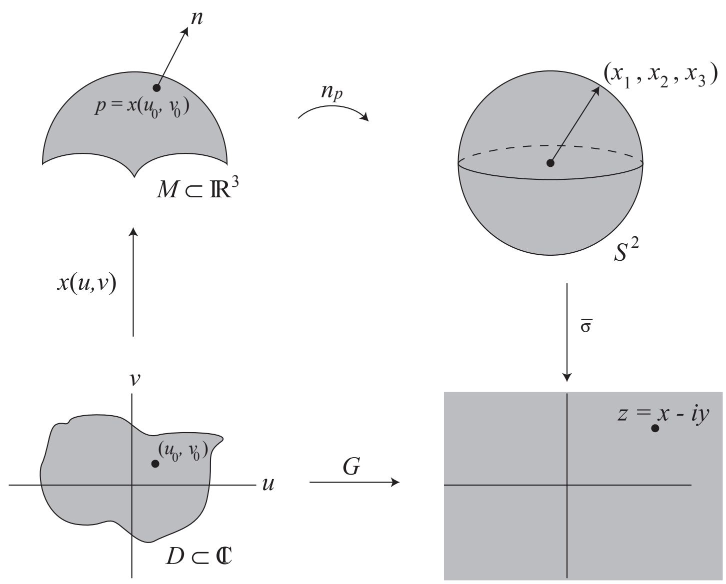



FIGURE 2.39. The map $G$.

EXAMPLE 2.81. Using the geometry of Enneper surface, $M_E$, we can determine specific values of a Gauss map, $G$, on $M_E$ even though we do not know what function $G$ is. What is $G(0)$? Enneper's surface is formed by bending a disk into a saddle surface. The point $0 \in \mathbb{C}$ should get mapped to the point in the center of the Enneper surface. For simplicity sake, we will take the downward pointing normal $\mathbf{n}$. Hence, the unit normal at the center of $M_E$ points straight down, and thus mapping it to $S^2$ under $\mathbf{n_p}$ gives the vector pointing at $(0, 0, -1)$. Taking the stereographic projection, $\sigma$, results in the point $z = 0 \in \mathbb{C}$ and reflecting this across the real axis does not change 0, so $\overline{\sigma}(0, 0, -1) = 0$. Hence, $G(0) = 0$.

Next, what is $G(r)$ when $r \in [0, 1]$? The points $r$ get mapped under $\mathbf{x}$ to a curve moving upward along one of the upward pointing leaves of the Enneper surface. The corresponding downward pointing unit normal, $n_r$, stays in the $x_1x_3$-half plane (where $x_1 \geq 0$) also moving upward (i.e., the $x_3$ value is increasing). As $r$ approaches 1, $n_r$ approaches being parallel to the $x_1$ axis. Thus, mapping these unit normals to $S^2$, the curve $\{r \in \mathbb{D} : 0 \leq r < 1\}$ traces a meridian on $S^2$ from $(0, 0, -1)$ to $(1, 0, 0)$. The stereographic projection of this onto the complex plane gives $\{r \in \mathbb{D} : 0 \leq r < 1\}$ and reflecting this across the real axis does not change the values. Hence, $G(r) = r$, where $0 \leq r \leq 1$.

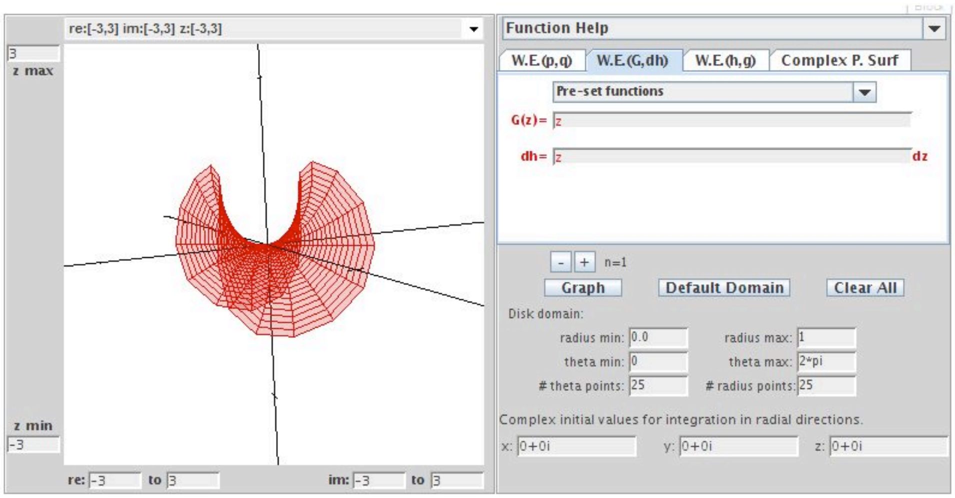



FIGURE 2.40. The Enneper surface.

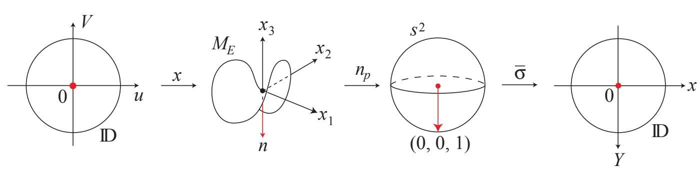



FIGURE 2.41. $G(0) = 0$ for the Enneper surface.

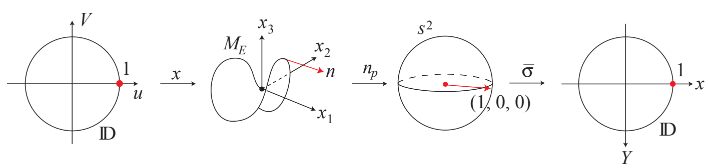



FIGURE 2.42. $G(1) = 1$ for the Enneper surface.

Finally, what is $G(e^{i\theta})$ for $0 \leq \theta \leq \frac{\pi}{2}$? If we restrict the domain of the Enneper surface to $\mathbb{D}$, these points get mapped to the edge of our image of the Enneper surface in a positive direction. At $\theta = 0$, the unit normal is pointing outward (i.e., away from the opposite leaf) and under the Gauss map, $n_p$, this corresponds to $(1, 0, 0) \in S^2$. As $\theta$ moves from 0 to $\frac{\pi}{2}$, the unit normal moves from pointing outward to pointing inward (i.e., toward the opposite leaf). So that at $\theta = \frac{\pi}{2}$, the unit normal is mapped under $n_p$ to $(0, -1, 0) \in S^2$ which projects under $\sigma$ to $-i \in \mathbb{C}$. Reflecting this across the real axis gives $\overline{\sigma}(0, -1, 0) = \overline{-i} = i$. A similar argument shows that the same thing happens for all $\theta \in [0, \frac{\pi}{2}]$. That is, $G(e^{i\theta}) = e^{i\theta}, (0 \leq \theta \leq \frac{\pi}{2})$.

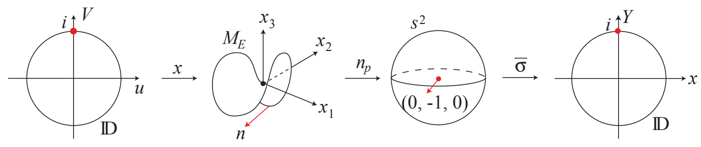



FIGURE 2.43. $G(i) = i$ for the Enneper surface.

EXAMPLE 2.82. Let's determine some specific values for the Gauss map $G$ for the singly periodic Scherk surface with six leaves, $M_S$. The domain used here is $\mathbb{D}$. Also, the leaves are centered at rays from the origin through each of the 6th roots of unity (i.e., $e^{i\pi k/3}$, $(k = 0, ..., 5)$) with the leaf centered at the positive real axis pointing upward and the subsequent leaves alternating between downward pointing and upward pointing (see Figure 2.44).

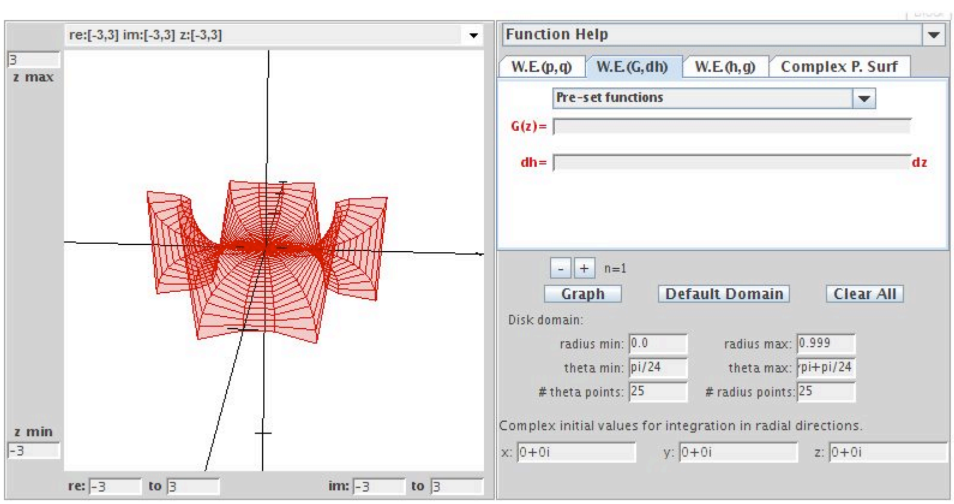



FIGURE 2.44. The singly periodic Scherk surface with six leaves.

As in the previous example we will use the downward pointing unit normal, and so $G(0) = 0$. Next, because the leaves are centered at the 6th roots of unity (i.e., $e^{i\pi k/3}, (k = 0, ..., 5)$), let's look at $G(e^{i\pi k/3})$, where $(k = 0, ..., 5)$. First, the point 1 gets mapped under $\mathbf{x}$ to the "edge" of $M_S$ above the positive real axis. By looking at the graph of $M_S$, we see that this is in the middle of an upward pointing leaf, and the corresponding unit normal $\mathbf{n}$ lies above the positive real axis and pointing away from the origin. Also, it lies in a plane parallel to the horizontal $x_1x_2$-plane. Mapping this normal under $\mathbf{n}_p$ and then $\overline{\sigma}$, results in the point 1. Hence, $G(1) = 1$. Now, consider $G(e^{i\pi/3})$. The point $e^{i\pi/3}$ gets mapped to the "edge" of $M_S$ above the line $re^{i\pi/3}, r > 0$. Since the leaves alternate between pointing upward and pointing downward, this is in the middle of a downward pointing leaf and points toward the origin. Mapping this normal under $\mathbf{n}_p$ and then $\overline{\sigma}$, results in the point $\overline{e^{i4\pi/3}} = e^{i2\pi/3}$. Hence, $G(e^{i\pi/3}) = e^{i2\pi/3}$.

In a similar way, we get the following values:

$$G(1) = 1, \qquad G\left(e^{i\frac{\pi}{3}}\right) = e^{i\frac{2\pi}{3}}, \qquad G\left(e^{i\frac{2\pi}{3}}\right) = e^{i\frac{4\pi}{3}},$$

$$G(-1) = 1, \qquad G\left(e^{i\frac{4\pi}{3}}\right) = e^{i\frac{2\pi}{3}} = e^{i\frac{8\pi}{3}}, \qquad G\left(e^{i\frac{5\pi}{3}}\right) = e^{i\frac{4\pi}{3}} = e^{i\frac{10\pi}{3}}.$$

EXERCISE 2.83. A picture of the half catenoid on its side defined on $\mathbb{D}$ is shown in Figure 2.45 with the positive real axis on the right.

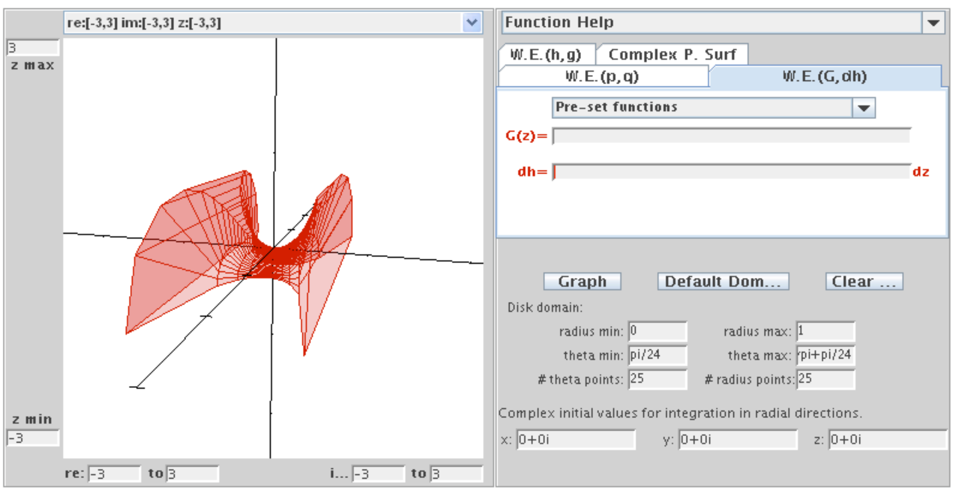



FIGURE 2.45. A view of the half catenoid on its side.

For this catenoid on its side determine:

(a) $G(0)$; (b) $G(1)$; (c) $G(-1)$; (d) $G(i)$; (e) $G(-i)$.

***Try it out!***

EXERCISE 2.84. For the 4-noid (see Figures 2.33 and 2.34), determine:

(a) $G(0)$; (b) $G(1)$; (c) $G(-1)$; (d) $G(i)$; (e) $G(-i)$.

***Try it out!***

We will use the Gauss map, $G$, to form another Weierstrass representation of a parametrized minimal surface. In doing so, we will also need the height differential,

$dh$, which is called such because it is locally (though not globally) the differential of the height coordinate. We will not get into the definition of differential forms; if you are interested in learning about differential forms, check out [**25**]. However, it is worth mentioning that at points where the Gauss map is vertical (i.e., $G = 0$ or $G = \infty$), the height function ought to have local minimums and maximums. Hence, $dh$ ought to have a zero at these points (for example, see Figure 2.41).

THEOREM 2.85 (Weierstrass Representation (G,dh)). Every regular minimal surface has a local isothermal parametric representation of the form

$$\mathbf{x} = \mathrm{Re} \int_a^z \left( \frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1 \right) dh, \tag{14}$$

where $G$ is the Gauss map, $dh$ is the height differential, and $a \in \Omega$ is a constant.

PROOF. From the Summary on page 141, we need

$$\phi^2 = 0 \quad \text{and} \quad |\phi|^2 \neq 0 \text{ and be finite.}$$

Comparing eq (14) and eq (13), we have that

$$\varphi_1 = \frac{1}{2}\left(\frac{1}{G} - G\right) dh, \quad \varphi_2 = \frac{i}{2}\left(\frac{1}{G} + G\right) dh, \quad \varphi_3 = dh.$$

In Exercise 2.86 you will show that

$$\phi^2 = 0 \qquad \text{and} \qquad |\phi|^2 \neq 0.$$

□

EXERCISE 2.86. Prove that $\phi^2 = 0$ and $|\phi|^2 \neq 0$ in the proof of Theorem 2.85.
***Try it out!***

Note that

$$G = \frac{\varphi_1 + i\varphi_2}{-\varphi_3} \qquad \text{and} \qquad dh = \varphi_3.$$

One advantage of using this Weierstrass representation with the Gauss map and height differential is that the complex analytic properties of $G$ and $dh$ are related to the geometry of a minimal surface. We will discuss this in a bit, but first we will look at some examples. The following is a list of the Weierstrass data for some common minimal surfaces.

| | | | |
|---|---|---|---|
| (a) The Enneper surface: | $G(z) = z$ | $dh = z\,dz$ | on $\mathbb{C}$. |
| (b) The catenoid: | $G(z) = z$ | $dh = \frac{1}{z}\,dz$ | on $\mathbb{C} \setminus \{0\}$. |
| (c) The helicoid: | $G(z) = z$ | $dh = \frac{i}{z}\,dz$ | on $\mathbb{C} \setminus \{0\}$. |
| (d) Scherk's doubly periodic surface: | $G(z) = z$ | $dh = \frac{z}{z^4-1}\,dz$ | on $\mathbb{D}$. |
| (e) Scherk's singly periodic surface: | $G(z) = z$ | $dh = \frac{iz}{z^4-1}\,dz$ | on $\mathbb{D}$. |
| (f) Polynomial Enneper: | $G(z) = p(z)$ | $dh = p(z)\,dz$ | on $\mathbb{C}$. |
| (g) Wavy plane: | $G(z) = z$ | $dh = dz$ | on $\mathbb{C} \setminus \{0\}$. |

EXAMPLE 2.87. For $G(z) = z^k$ and $dh = z^k\,dz$, where $k = 1, 2, \ldots$, we get

$$\begin{aligned}\mathbf{x} &= \mathrm{Re}\int_0^z \left(\frac{1}{2}\left(\frac{1}{z^k} - z^k\right), \frac{i}{2}\left(\frac{1}{z^k} + z^k\right), 1\right) z^k\,dh \\ &= \left(\mathrm{Re}\,\frac{1}{2}\left\{z - \frac{1}{2k+1}z^{2k+1}\right\}, \mathrm{Re}\,\frac{1}{2}\left\{-i\left(z + \frac{1}{2k+1}z^{2k+1}\right)\right\}, \mathrm{Re}\left\{\frac{z^{k+1}}{k+1}\right\}\right).\end{aligned}$$

This is the Enneper surface with $2k+2$ leaves (see Exploration 2.70).

EXERCISE 2.88. You may have noticed that the Weierstrass data for the catenoid and the helicoid, which are conjugate surfaces (see Definition 2.49), have the same Gauss map, $G$, while the height diffferentials $dh$ differ by a multiple of $i$. Prove that this is true for any conjugate surfaces.

***Try it out!***

EXERCISE 2.89. Let $G(z) = z^4$ and $dh = z^2\,dz$.

(a) Using eq (14), compute the parametrization.

(b) This minimal surface has a planar end (i.e., looks like a plane) and an Enneper end. Use *MinSurfTool* with the **W.E. (G,dh)** tab to graph the surface; use a disk domain with radius min: 0.3 and radius max: 1, theta min: pi/24 and theta max=2pi+pi/24 and initial values $x = \mathrm{Re}(-1/7 * z^7 - 1/z)$, $y = \mathrm{Re}(i/2 * (1/7 * z^7 - 1/z))$, and $z = \mathrm{Re}(1/3 * z^3)$.

***Try it out!***

The catenoid and the surface in Exercise 2.89 are examples of minimal surfaces with ends. Loosely, an *end* of a minimal surface is a piece that "goes on forever," or, more precisely, leaves all compact subsets of the minimal surface. Recall from Theorem 2.75 that all complete minimal surfaces in $\mathbb{R}^3$ are not compact, and hence they must possess at least one end.

EXERCISE 2.90. Determine the number of ends each of the following surfaces have: (a) the catenoid; (b) the plane; (c) the helicoid; (d) Enneper's surface.

***Try it out!***

Ends occur in a deleted neighborhood (i.e., a disk with the centered removed) centered at a singularity. Three common types of ends for minimal surfaces are: (1) Enneper ends; (2) catenoid ends; and (3) flat or planar ends. In discussing ends, we will need to represent $ds$, the metric (i.e., a way to measure distance) on a minimal surface, in terms of $G$ and $dh$. Using $ds^2 = |\phi|^2$ and eq (14), we derive

$$ds = \frac{1}{\sqrt{2}}\left(|G| + \frac{1}{|G|}\right)|dh|. \tag{15}$$

An Enneper end has $ds \sim |z^k| \cdot |dz|$, while a catenoidal end and a planar end has $ds \sim |dz|$ (i.e., the metric becomes Euclidean). A catenoidal end differs from a planar end in that the residue of $dh$ is logarithmic.

EXERCISE 2.91. Prove eq. (15).
***Try it out!***

EXAMPLE 2.92. We know that the catenoid has two catenoid ends, but let's show how we could prove this if we did not know what type of ends these are. Using the Weierstrass data, $G(z) = z$ and $dh = \frac{1}{z}\,dz$, for the catenoid, we have the parametrization

$$\mathbf{x}(z) = \left(\frac{1}{2}\operatorname{Re}\left(-\frac{1}{z} - z\right), \frac{i}{2}\operatorname{Re}\left(-\frac{1}{z} + z\right), \operatorname{Re}\left(\log z\right)\right).$$

So there is a singularity or pole of order 1 at 0. Also, there is a pole of order 1 at $\infty$. To see that there is a singularity at $\infty$, we replace $z$ with $\frac{1}{w}$ and look at the limit as $w$ goes to 0. Thus, the catenoid will have two ends (one at 0 and one at $\infty$). To determine what types of ends these are, we look at $ds$ at these points. Note that

$$ds = \frac{1}{\sqrt{2}}\left(|z| + \frac{1}{|z|}\right)\frac{1}{|z|}|dz|.$$

As $z \to \infty$, $ds \sim |dz|$ and because $x_3$ is logarithmic, we have a catenoid end.

At $z = 0$, plugging in 0 does not work, so instead we let $w = \frac{1}{z}$ and consider $w \to \infty$. Note that in this case,

$$dh = \frac{1}{z}\,dz = w\left(\frac{1}{w}\right)' = w\left(-\frac{1}{w^2}\,dw\right) = -\frac{1}{w}\,dw.$$

Therefore,

$$ds = \frac{1}{\sqrt{2}}\left(|w| + \frac{1}{|w|}\right)\frac{1}{|w|}|dw|.$$

As $w \to \infty$, $ds \sim |dw|$ and again because $x_3$ is logarithmic, we have a catenoid end.

EXAMPLE 2.93. The surface in Exercise 2.89 has $G(z) = z^4$ and $dh = z^2\,dz$, and the corresponding parametrization is

$$\mathbf{x}(z) = \left(\frac{1}{2}\operatorname{Re}\left(\frac{1}{7}z^7 - \frac{1}{z}\right), \frac{i}{2}\operatorname{Re}\left(\frac{1}{7}z^7 + \frac{1}{z}\right), \operatorname{Re}\left(\frac{1}{3}z^3\right)\right).$$

Note there are singularities at 0 and at $\infty$ and

$$ds = \frac{1}{\sqrt{2}}\left(|z|^4 + \frac{1}{|z|^4}\right)|z|^2|dz|.$$

As $z \to \infty$, $ds \sim |z|^6\,|dz|$ and so we have an Enneper end.

At $z = 0$, we again let $w = \frac{1}{z}$ and consider $w \to \infty$. Then $G(w) = w^4$ and $dh = -\frac{1}{w^4}\,dw$. Hence,

$$ds = \left(|w|^4 + \frac{1}{|w|^4}\right)\frac{1}{|w|^4}|dw|.$$

As $w \to \infty$, $ds \sim |dw|$, but because there is no logarithmic term, we have a planar end.

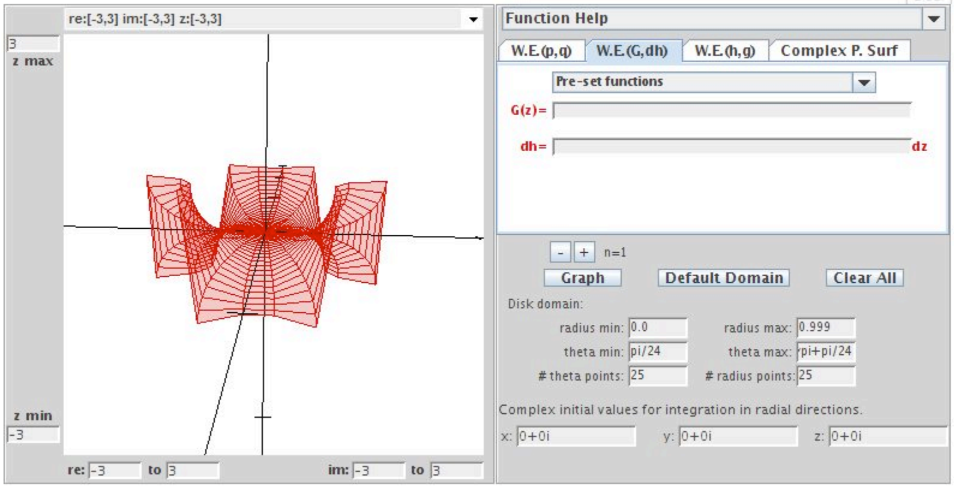



FIGURE 2.46. A minimal surface with Enneper and planar ends.

EXERCISE 2.94. Let $G(z) = \frac{z^2+3}{z^2-1}$ and $dh = \frac{z^2+3}{z^2-1}\,dz$.

(a) Using eq (14), compute the parametrization.
(b) Show that this minimal surface has one planar end and two catenoid ends.
(c) Use *MinSurfTool* with the **W.E. (G,dh)** tab to graph the surface; use a disk domain with radius min: 0.3 and radius max: 1.

***Try it out!***

The Gauss map and height differential also tell us about two important types of curves on a minimal surface. These are known as the asymptotic lines and the curvature lines. To understand what these lines are, we will review the terms normal curvature and principal directions both of which were discussed in Section 2. Let $p$ be a point on a curve on a minimal surface $M$. The tangent vector $\mathbf{w}$ and normal vector $\mathbf{n}$ at $p$ form a plane that intersects the surface in another curve, say $\alpha$ (see Figure 2.2). The normal curvature in the direction $\mathbf{w}$ is $\alpha'' \cdot \mathbf{n}$ and measures how much the surface bends toward $\mathbf{n}$ as you move in the direction of $\mathbf{w}$ at point $p$. An *asymptotic line* is a curve that is tangent to a direction in which the normal curvature is zero. As we rotate the plane through the normal $\mathbf{n}$, we will get a set of curves on the surface each of which has a value for its curvature. The directions in which the normal curvature attains its absolute maximum and absolute minimum values are known as the principal directions. *Curvature lines* are curves that are always tangent to a principal direction.

A nice relationship between these lines and the Weierstrass data is:

$$\text{A curve } z(t) \text{ is an asymptotic line} \iff \frac{dG}{G}(z) \cdot dh(z) \in i\mathbb{R}.$$

$$\text{A curve } z(t) \text{ is a curvature line} \iff \frac{dG}{G}(z) \cdot dh(z) \in \mathbb{R}.$$

Example 2.95. Let $G(z) = z$ and $dh(z) = dz$. This is a parametrization of the wavy plane. Computing the Weierstrass representation, we get the parametrization:

$$(x_1(z), x_2(z), x_3(z)) = \left( \text{Re}\left\{\frac{1}{2}\ln(z) - \frac{1}{4}z^2\right\}, \text{Re}\left\{\frac{i}{2}\ln(z) + \frac{i}{4}z^2\right\}, \text{Re}\left\{z\right\}\right).$$

Using *MinSurfTool* we plot an image of this surface. Note we let radius min: 0.001, radius max: 1, theta min: -pi+0.01, and theta max: pi-0.01 (see Figure 2.47).

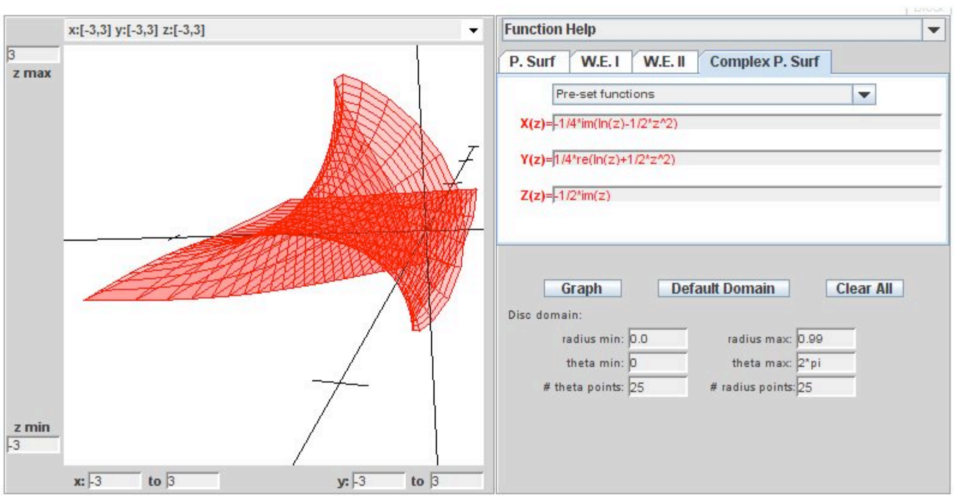



Figure 2.47. Side view of the wavy plane surface.

Now, for the wavy plane

$$\frac{dG}{G}(z) \cdot dh(z) = \frac{dz}{z} \cdot dz.$$

If we let $z = e^{i\theta}$ (since we let radius max=1), then $dz = ie^{i\theta}d\theta$ and

$$\left.\frac{dG}{G}(z) \cdot dh(z)\right|_{z=e^{i\theta}} = -e^{i\theta}(d\theta)^2.$$

So, from the equations above, we get that for $k \in \mathbb{R}$: (1) the asymptotic lines occur when $\theta = \frac{\pi}{2} + k\pi$; and (2) the curvature lines occur when $\theta = k\pi$.

If we use *MinSurfTool* to plot the wavy plane with theta min $= -\frac{\pi}{2}$ and theta max $= \frac{\pi}{2}$, we see that these asymptotic lines lie in the $x_1x_2$-plane. Similarly, if we plot theta min $= 0$ and theta max $= \pi$, we see that these curvature lines are reflection lines through which the wavy plane can be reflected as if through a mirror to get a smooth continuation of the minimal surface.

Exercise 2.96. Using $G(z) = z$ and $dh = zdz$ for Ennepers surface, show that for $z = e^{i\theta}$ the asymptotic lines occur when $\theta = \frac{\pi}{4} + \frac{k\pi}{2}$, where $k \in \mathbb{R}$, and the curvature lines occur when $\theta = \frac{k\pi}{2}$, where $k \in \mathbb{R}$. Use *MinSurfTool* to plot these lines on Ennepers surface.

***Try it out!***

Exercise 2.97. Prove that for conjugate surfaces, the asymptotic lines (and curvature lines) of one surface are the curvature lines (and asymptotic lines) of the other surface.

***Try it out!***

As mentioned earlier, an advantage of using the Gauss map and height differential is that the complex analytic properties of $G$ and $dh$ are related to the geometry of a minimal surface. So, let's look at the complex analytic properties of $G$ and $dh$. First, recall from the Summary on page 141, we need $|\phi|^2$ to be finite and nonzero. Because of eq (15), this leads to the following condition.

Proposition 2.98. At a nonsingular point, $G$ has a zero or pole of order $n \iff dh$ has a zero of order $n$.

Second, note that the integrals in the Weierstrass representation in eq (14) might depend upon the path of integration if the domain of $G$ and $dh$ is not simply connected. This means that for all closed paths $\gamma$ in the domain

$$\begin{aligned} &\mathrm{Re}\left(\frac{1}{2}\int_\gamma \left(\frac{1}{G} - G\right)\, dh\right) = 0;\\ &\mathrm{Re}\left(\frac{i}{2}\int_\gamma \left(\frac{1}{G} + G\right)\, dh\right) = 0;\\ &\mathrm{Re}\int_\gamma dh = 0. \end{aligned}$$

These three equations can be reduced to the following two period conditions:

$$(16) \qquad \begin{aligned} &(i)\ \int_\gamma G\, dh = \overline{\int_\gamma \frac{1}{G}\, dh} \quad \text{(horizontal period condition)},\\ &(ii)\ \mathrm{Re}\int_\gamma dh = 0 \quad \text{(vertical period condition)}. \end{aligned}$$

for all closed paths $\gamma$ in the domain.

Exercise 2.99. Show that the conditions

$$\mathrm{Re}\left(\frac{1}{2}\int_\gamma \left(\frac{1}{G} - G\right)\, dh\right) = 0, \text{ and } \mathrm{Re}\left(\frac{i}{2}\int_\gamma \left(\frac{1}{G} + G\right)\, dh\right) = 0$$

are equivalent to the condition

$$\int_\gamma G\, dh = \overline{\int_\gamma \frac{1}{G}\, dh}.$$

***Try it out!***

These period conditions are useful in determining horizontal periods (e.g., Scherk's doubly periodic surface), vertical periods (e.g., Scherk's singly periodic surface) and in determining possible constant values in $G$ and $dh$.

EXAMPLE 2.100. Consider Scherk's doubly periodic surface with the Weierstrass data $G(z) = z$ and $dh(z) = \frac{z}{z^4-1}dz$. The horizontal period condition is $\int_\gamma G\ dh = \overline{\int_\gamma \frac{1}{G}\ dh}$, for all closed paths $\gamma$ in the domain. Note that both integrands are meromorphic with poles of order 1 at $\pm 1, \pm i$. So, the only paths that concern us are ones that include one, two, three, or all of these poles. A nice way to calculate these integrals along such paths is to use the Residue Theorem that states if $\gamma$ is a simple closed positively-oriented contour and $f$ is analytic inside and on $\gamma$ except at the points $z_1, ldots, z_n$ inside $\gamma$, then

$$\int_\gamma f(z)\ dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j).$$

Recall that for poles of order 1,

$$\mathrm{Res}(f, z_j) = \lim_{z\to z_j} (z - z_j) f(z).$$

Thus, for $\displaystyle\int_\gamma G\,dh$, we have

$$\mathrm{Res}(G\,dh, z_j) = \lim_{z\to z_j} \frac{z^3 - z_j z^2}{z^4 - 1} = \lim_{z\to z_j} \frac{3z^2 - 2z_j z}{4z^3} = \lim_{z\to z_j} \frac{3z^3 - 2z_j z^2}{4z^4} = \frac{z_j^3}{4}.$$

In particular,

$$\mathrm{Res}(G\,dh, 1) = \frac{1}{4} \qquad \mathrm{Res}(G\,dh, i) = \frac{-i}{4}$$

$$\mathrm{Res}(G\,dh, -1) = \frac{-1}{4} \qquad \mathrm{Res}(G\,dh, -i) = \frac{i}{4}.$$

Similarly, we can compute that

$$\mathrm{Res}\left(\frac{1}{G}\,dh, 1\right) = \frac{1}{4} \qquad \mathrm{Res}\left(\frac{1}{G}\,dh, i\right) = \frac{i}{4}$$

$$\mathrm{Res}\left(\frac{1}{G}\,dh, -1\right) = \frac{-1}{4} \qquad \mathrm{Res}\left(\frac{1}{G}\,dh, -i\right) = \frac{-i}{4}.$$

Now, if the path $\gamma_1$ just contains the pole at $z_1 = 1$, then the horizontal period conditions result in

$$\int_{\gamma_1} G\,dh = 2\pi i\,\mathrm{Res}(G\,dh, 1) = \frac{i\pi}{2}, \qquad \overline{\int_\gamma \frac{1}{G}\,dh} = \overline{2\pi i\,\mathrm{Res}\left(\frac{1}{G}\,dh, 1\right)} = \frac{-i\pi}{2}.$$

These integrals should be equal, which occurs if the minimal surface is periodic in the imaginary direction with period of $\frac{\pi}{2}$. Likewise, if we take a path $\gamma_2$ that just contains the pole $z_2 = i$, then we get

$$\int_{\gamma_2} G\, dh = \frac{\pi}{2}, \qquad \overline{\int_{\gamma} \frac{1}{G}\, dh} = \frac{-\pi}{2},$$

and the minimal surface is periodic in the real direction with period $\frac{\pi}{2}$. All other paths $\gamma$ are covered by these two cases. Finally, if we look at the vertical period condition, we get that the condition is automatically true for all paths $\gamma$ and so the minimal surface is not periodic in the vertical direction. This matches up with what is true for Scherk's doubly periodic surface.

EXERCISE 2.101. Show that the period conditions given in eq. (16) result in Scherk's singly periodic surface being periodic in the vertical direction.

***Try it out!***

Let's look at example of how all of this can help us use the geometry of a minimal surface to determine $G$ and $dh$.

EXAMPLE 2.102. From the list of Weierstrass data on page 154, we know that $G(z) = z$ and $dh = z\ dz$ for the Enneper surface. However, we want to show how this Weierstrass data can be determined by using the geometric shape of the surface. First, let's determine a plausible candidate for $G$. To do this, we will make a guess based on the value of $G$ at a few specific points. From Example 2.81, we know that $G(0) = 0$, $G(r) = r$ for $0 \leq r \leq 1$, and $G(e^{i\theta}) = e^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$. Therefore, it seems plausible to let $G(z) = z$. Second, given this $G$, let's determine $dh$. Because eq (15) must be finite and the Enneper surface has no ends in $\mathbb{C}$, $dh$ cannot have any poles in $\mathbb{C}$. However, from the sentence before Exercise 2.90, we know that Enneper's surface must have at least one end. This end corresponds to the point at infinity, $z = \infty$, and so $dh$ has a pole at $\infty$. Thus, $dh = \rho z^n$, for some $n \in \mathbb{N}$ and $\rho \in \mathbb{C}$. Since $G(z) = z$ has a zero of order 1 at 0, by Proposition 2.98 $dh$ must also have a zero of order 1 at 0 and no other zeros. Thus, $dh = \rho z\ dz$. For simplicity sake, we let $\rho = 1$. Finally, notice that the period conditions in eq (16) hold, because there are no poles in $\mathbb{C}$, and so every integral along any closed path $\gamma$ will equal 0. Hence, the Weierstrass data

$$G(z) = z, \quad dh = z\ dz$$

generates a minimal surface.

EXAMPLE 2.103. Consider the singly periodic Scherk surface with six leaves, $M_S$ (see Figure 2.44). Note that these leaves go off to infinity. Hence, we will have 6 poles. Because of symmetry, we will choose these poles to be at the 6th roots of unity (i.e., $e^{i\pi k/3}, (k = 0, ..., 5)$). This means that $dh$ will have the term $z^6 - 1$ in its denominator. However, we will need to determine $G$ first in order to know what should be in the numerator of $dh$. From the results in Example 2.82, it seems reasonable that $G(z) = z^2$.

Since $G(z) = z^2$ has a zero of order 2 at 0, by Proposition 2.98 $dh$ must also have a zero of order 2 at 0 and no other zeros. Thus, we so far have

$$G(z) = z^2, \qquad dh = \rho \frac{z^2}{z^6 - 1}\, dz,$$

where $\rho \in \mathbb{C}$. To determine possible values of $\rho$, consider the period conditions in eq (16). There are poles of order 1 at $e^{ik\pi/3}, k = 0, \ldots, 5$. We compute that

$$\operatorname{Res}(G\, dh, z_j) = \frac{\rho z_j^5}{6}, \qquad \operatorname{Res}\left(\frac{1}{G}\, dh, z_j\right) = \frac{\rho z_j}{6}.$$

Hence, if $\gamma$ contains the pole $z_j$, then the horizontal period condition requires

$$\int_\gamma G\, dh = 2\pi i \operatorname{Res}(G\, dh, z_j) = \frac{\rho \pi i z_j^5}{3}, \text{ and}$$

$$\overline{\int_\gamma \frac{1}{G}\, dh} = \overline{2\pi i \operatorname{Res}\left(\frac{1}{G}\, dh, z_j\right)} = \frac{-\overline{\rho}\pi i \overline{z}_j}{3}$$

to be equal (since there is no periodicity of $M_S$ in the horizontal direction).These integrals will be equal for these poles $z_j^5 = \overline{z}_j$ if $\rho = -\overline{\rho}$. That is, $\rho$ is purely imaginary. Without loss of generality, we let $\rho = i$ and we check that the vertical period condition holds. Hence, we have that the Weierstrass data for singly periodic Scherk surface with six leaves is

$$G(z) = z^2, \qquad dh = \frac{iz^2}{z^6 - 1}\, dz.$$

Exercise 2.104. Let $M$ be the Enneper surface with 8 leaves. Using the approach of Example 2.102 determine $G$ and $dh$ for this surface.

***Try it out!***

Exercise 2.105. Let $M$ be the 3-noid with ends symmetrically placed so that if the surface is rotated by $\frac{2\pi}{3}$ you will get the same image. Determine $G$ and $dh$ for this surface.

***Try it out!***

Small Project 2.106. Let $M$ be a minimal surface that has 6 symmetrically-placed ends with 4 ends along the side (like a 4-noid), 1 end on the top, and 1 end on the bottom. So, $M$ will look the same if it is rotated horizontally by $\frac{\pi}{2}$ and if it is rotated vertically by $\frac{\pi}{2}$. Determine $G$ and $dh$ for this surface.

**Optional**

Small Project 2.107. For Scherk's singly periodic the four ends are symmetrically placed so that if the surface is rotated by $\frac{\pi}{2}$ you will get the same image. This is because the denominator of $dh$ is $z^4 - 1$ which has zeros that are equally spaced on the unit circle. It is possible to create a variation of Scherk's singly periodic surface that has four ends with rotational symmetry of $\pi$. That is, if the ends are labelled

$E_1, \ldots, E_4$, then $E_2$ will be closer to $E_1$ than to $E_3$ (and likewise, $E_4$ will be closer to $E_3$ than to $E_1$) and if the surface is rotated by $\pi$ you will get the same image. Determine $G$ and $dh$ for this surface.

**Optional**

Large Project 2.108. Describe and classify the possible minimal surfaces where one of the coordinates of the parametrization is fixed while the other two coordinates vary. For example, if $x_3$ is fixed to a specific function, what are the possible coordinate functions for $x_1$ and $x_2$. Try to generalize this approach as much as possible.

**Optional**

Large Project 2.109. Describe and classify the possible minimal surfaces with $G(z) = z^m$ and $dh = z^n\ dz$, for all $n, m \in \mathbb{N}$ (see Example 2.87 and Example 2.93). There are several distinct cases to consider. Determine how to separate $m, n$ into these distinct cases remembering to discuss types of surface, types of ends, lines of symmetry, etc.

**Optional**

## 2.6. Minimal Surfaces and Harmonic Univalent Mappings

In the Summary on page 141 before the first Weierstrass representation, we learned that each coordinate function of the parametrization $\mathbf{x}$ of a minimal surface had the form $x_k = \operatorname{Re} \int \varphi_k dz$ with $\varphi_k$ being analytic. Since the real part of an analytic function is a harmonic function, we see that each $x_k$ is harmonic. Also from this Summary, we have that $(\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$. This means that if we know the functions $\varphi_1$ and $\varphi_2$, we can determine the function $\varphi_3$. So, another way to get a Weierstrass representation for minimal surfaces is to use two harmonic functions $x_1$ and $x_2$. In other words, we can investigate minimal surfaces by studying harmonic mappings in the complex plane. Such mappings are known as planar harmonic mappings and have been studied independently of minimal surfaces.

In this section we will develop another Weierstrass representation. In this case we will use planar harmonic mappings instead of $p$ and $q$ as in Section 2.4 or $G$ and $dh$ as in Section 2.5. What benefit do we obtain from this new approach? The benefit is in establishing the embeddedness (i.e., no self-intersections) of minimal surfaces, and as we have mentioned earlier embeddedness is an important property of minimal surfaces. Getting embedded minimal surfaces from certain planar harmonic mappings will be a result of requiring the harmonic mappings to be $1-1$ functions. In the complex plane, $1-1$ functions is the same as in $\mathbb{R}$. That is, $f$ will be $1-1$ in $G$ means that if $f(z_1) = f(z_2)$, then $z_1 = z_2$. Geometrically, this means that the image, $f(G)$, will not overlap or intersect itself. When we use this new Weierstrass representation with $1-1$ planar harmonic mappings, the corresponding minimal surface will be a minimal graph and hence will be embedded (for a refresher on minimal graphs read the paragraph before Exercise 2.8). For example, the $1-1$ planar harmonic function given by

$$f(z) = h(z) + \overline{g(z)} = \text{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\,\text{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$$

maps the unit disk onto a square region. This square region is the projection (i.e., shadow) of Scherk's doubly periodic surface onto the plane. In other words, we can

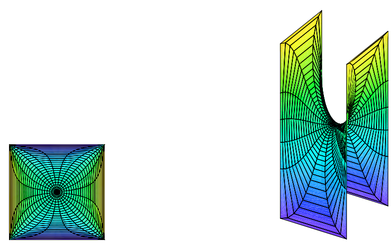


FIGURE 2.48. The image of $f(\mathbb{D})$ and Scherk's doubly periodic surface

lift $f$ from $\mathbb{C}$ into $\mathbb{R}^3$ and to get Scherk's doubly periodic surface. Planar harmonic mappings that are $1-1$ are also known as harmonic univalent mappings. Harmonic univalent mappings can be studied on their own without bringing in minimal surfaces and such a study is the topic of chapter 4 of this book.

EXERCISE 2.110.

(a). Although all minimal graphs are embedded, the converse is not true. Give an example of an embedded minimal surface that is not a minimal graph.

(b). Suppose you have a nonunivalent harmonic mapping. Why could it not be the projection of a minimal graph?

***Try it out!***

Now that we have given an overview of this section, let's briefly discuss harmonic univalent mappings. A planar harmonic mapping is a function $f = u(x,y) + iv(x,y)$ where $u$ and $v$ are real harmonic functions. This concept is more general than that of an analytic function, because we do not require $u$ and $v$ to be harmonic conjugates. However, the following theorem allows us to relate a planar harmonic mapping to analytic functions. For our purposes, we will assume that the domain of $f$ is the unit disk, $\mathbb{D}$.

THEOREM 2.111. Define a function $f = u + iv$, where u and v are real harmonic functions. If $D$ is a simply-connected domain and $f : D \to \mathbb{C}$, then there exist analytic functions $h$ and $g$ such that $f = h + \overline{g}$.

EXERCISE 2.112.

(a) Show that $f(x,y) = u(x,y)+iv(x,y) = (x^3-3xy^2)+i(-3x^2y+y^3)$ is complex-valued harmonic by showing that $u$ and $v$ are real harmonic functions.

(b) Using $x = \frac{1}{2}(z+\overline{z})$ and $y = \frac{1}{2i}(z-\overline{z})$, rewrite $f(x,y) = (x^3-3xy^2)+i(-3x^2y+y^3)$ in terms of $z$ and $\overline{z}$.

(c) Determine the analytic functions $h$ and $g$ such that $f = h + \overline{g}$.

***Try it out!***

Example 2.113. In the previous exercise, we saw that the planar harmonic map $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(x,y) = u(x,y) + iv(x,y) = (x^3 - 3xy^2) + i(-3x^2y + y^3)$$

can be written as

$$f(z) = h(z) + \overline{g}(z) = z + \frac{1}{3}\overline{z}^3.$$

What is the image of $\mathbb{D}$ under $f$? It is a hypocycloid with 4 cusps. This fact can be computed by considering $f(e^{i\theta}) = u(\theta)+iv(\theta)$ and comparing the component functions, $u(\theta)$ and $v(\theta)$, to the parametrized equation for a hypocycloid with 4 cusps. To help us visualize the image, we can use the applet *ComplexTool*. To graph the image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{3}\overline{z}^3$, enter this function in *ComplexTool* in the form $z + 1/3$ `conj` $(z \wedge 3)$ (see Figure 2.49). Remember this example; we will show that this function is related to a minimal graph.

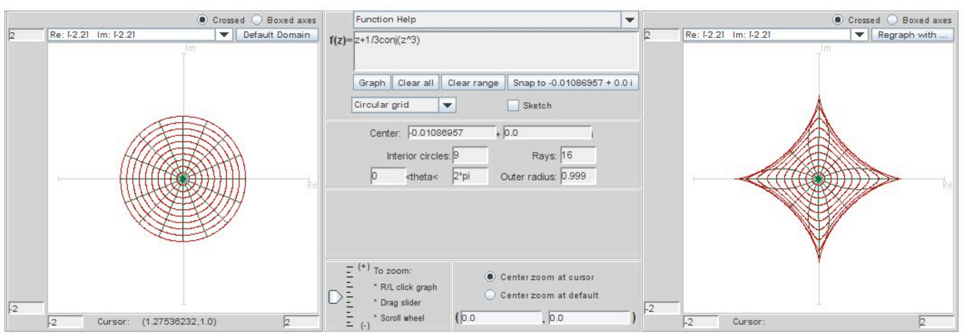



Figure 2.49. Image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{3}\overline{z}^3$

Note that the harmonic function $f(z) = h(z) + \overline{g}(z)$ can also be written in the form

$$f(z) = \operatorname{Re}\left\{h(z) + g(z)\right\} + i\operatorname{Im}\left\{h(z) - g(z)\right\}. \tag{17}$$

This is because,

$$\operatorname{Re}\{h+g\} = \frac{1}{2}[(h+g) + \overline{(h+g)}] \quad \text{and} \quad \operatorname{Im}\{h-g\} = \frac{1}{2i}[(h-g) - \overline{(h-g)}].$$

Hence, in the previous example, $f(z) = z + \frac{1}{3}\overline{z}^3$ can also be written as $f(z) = \operatorname{Re}\left\{z + \frac{1}{3}z^3\right\} + i\operatorname{Im}\left\{z - \frac{1}{3}z^3\right\}$.

We are interested in harmonic functions that are $1-1$ or univalent, because this is one necessary condition in order to lift the harmonic mapping to a minimal graph. One theorem that establishes univalency requires the following background material.

DEFINITION 2.114. The *dilatation* of $f = h + \overline{g}$ is $\omega(z) = g'(z)/h'(z)$.

THEOREM 2.115. $f = h + \overline{g}$ is locally univalent and orientation-preserving $\iff$ $|g'(z)/h'(z)| < 1$, for all $z \in \mathbb{D}$.

EXERCISE 2.116. Show that if $z \in \mathbb{D}$, then $|\omega(z)| < 1$ for:

(a) $\omega_1(z) = e^{i\theta}z$, where $\theta \in \mathbb{R}$;

(b) $\omega_2(z) = z^n$, where $n = 1, 2, 3, \ldots$;

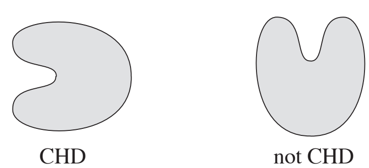
(c) $\omega_3(z) = \dfrac{z+a}{1+\overline{a}z}$, where $|a| < 1$;

(d) $\omega_4(z)$ being the composition of any of the functions $\omega$ above.

***Try it out!***

Creating nontrivial examples of harmonic univalent mappings that lift to minimal graphs is not easy. However, one way to do this is to use the shearing technique of Clunie and Sheil-Small. Before we proceed, we need to discuss a certain type of domain.

DEFINITION 2.117. A domain $\Omega$ is *convex in the direction of the real axis* (or convex in the horizontal direction, CHD) if every line parallel to the real axis has a connected intersection with $\Omega$.




THEOREM 2.118 (Clunie and Sheil-Small). A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain $\iff$ $h-g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain.

REMARK 2.119. This technique is known as the "shear" method or "shearing" a function. In our situation, suppose $F = h - g$ is an analytic univalent function convex in the real direction. Then the corresponding harmonic shear is

$$f = h + \overline{g} = h - g + g + \overline{g} = h - g + 2\,\mathrm{Re}\{g\}.$$

So, the harmonic shear differs from the analytic function by adding a real function to it. Geometrically, you can think of this as taking $F$, the original analytic univalent function convex in the real direction, and cutting it up into thin horizontal slices which are then translated and/or scaled in a continuous way to form the corresponding harmonic function, $f$. This is why the method is called "shearing." Since $F$ is univalent and convex in the real direction and we are only adding a continuous real function to it, the univalency is preserved.

Example 2.120. Let

$$h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \tag{18}$$

which is an analytic function that maps $\mathbb{D}$ onto a horizontal strip convex in the direction of the real axis (see Figure 2.50).

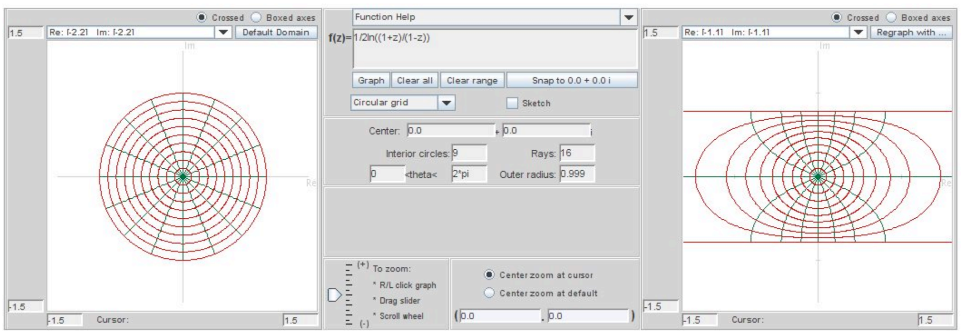



Figure 2.50. Image of $\mathbb{D}$ under the analytic function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

Let

$$\omega(z) = g'(z)/h'(z) = -z^2.$$

Applying the shearing method from Theorem 2.118 with the substitution $g'(z) = -z^2h'(z)$, we have

$$\begin{aligned} h'(z) - g'(z) = \frac{1}{1-z^2} &\Rightarrow h'(z) + z^2h'(z) = \frac{1}{1-z^2} \\ &\Rightarrow h'(z) = \frac{1}{1-z^4} = \frac{1}{4}\left[\frac{1}{1+z} + \frac{1}{1-z} + \frac{1}{i+z} + \frac{1}{i-z}\right]. \end{aligned}$$

Integrating $h'(z)$ and normalizing so that $h(0) = 0$, yields

$$h(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right). \tag{19}$$

We can use this same method to solve for normalized $g(z)$, where $g(0) = 0$. Note that we can also find $g(z)$ by using eqs. (55) and (56). Either way, we get

$$g(z) = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right).$$

So

$$f(z) = h(z) + \overline{g(z)} = \text{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\,\text{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

What is $f(\mathbb{D})$? Notice that

$$f(z) = \left[-\frac{1}{2}\arg\left(\frac{i+z}{i-z}\right)\right] + i\left[\frac{1}{2}\arg\left(\frac{1+z}{1-z}\right)\right] = u + iv.$$

Let $z = e^{i\theta} \in \partial\mathbb{D}$. Then

$$\frac{i+z}{i-z} = \frac{i+e^{i\theta}}{i-e^{i\theta}}\frac{-i-e^{-i\theta}}{-i-e^{-i\theta}} = \frac{1-i(e^{i\theta}+e^{-i\theta})-1}{1+i(e^{i\theta}-e^{-i\theta})+1} = -i\frac{\cos\theta}{1-\sin\theta}.$$

Thus,

$$u = -\frac{1}{2}\arg\left(\frac{i+z}{i-z}\right)\bigg|_{z=e^{i\theta}} = \begin{cases} \frac{\pi}{4} & \text{if } \cos\theta > 0, \\ -\frac{\pi}{4} & \text{if } \cos\theta < 0. \end{cases}$$

Likewise, we can show that

$$v = \begin{cases} \frac{\pi}{4} & \text{if } \sin\theta > 0, \\ -\frac{\pi}{4} & \text{if } \sin\theta < 0. \end{cases}$$

Therefore, we have that $z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u + iv = \begin{cases} z_1 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{\pi}{4}} & = \frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (0, \frac{\pi}{2}), \\ z_2 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{3\pi}{4}} & = -\frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ z_3 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{5\pi}{4}} & = -\frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\pi, \frac{3\pi}{2}), \\ z_4 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{7\pi}{4}} & = \frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Thus, this harmonic function maps $\mathbb{D}$ onto the interior of the region bounded by a square with vertices at $z_1, z_2, z_3$ and $z_4$.

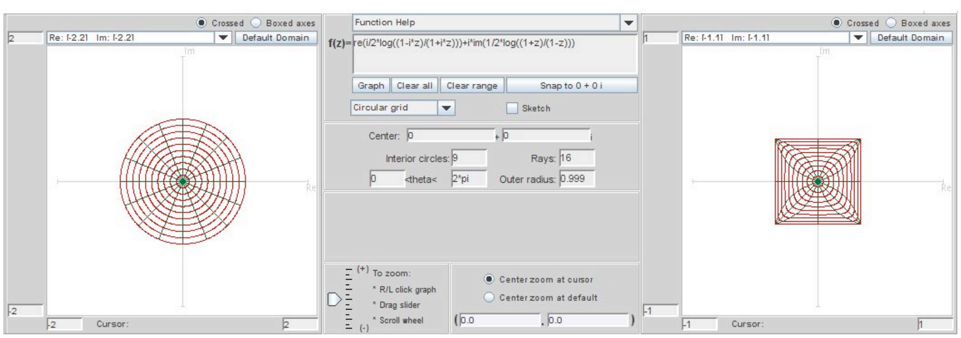



FIGURE 2.51. Image of $\mathbb{D}$ under $f(z) = \text{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\,\text{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$.

EXERCISE 2.121. Verify that shearing $h(z) - g(z) = z - \frac{1}{3}z^3$ with $\omega(z) = z^2$ yields $f(z) = z + \frac{1}{3}\overline{z}^3$ from Example 2.113.

***Try it out!***

To actually find the minimal graph that is associated with specific types of harmonic univalent mappings, we need to develop the appropriate Weierstrass representation as outlined in eq (13). Recall that it must satisfy the properties $\phi^2 = 0$ and $|\phi|^2 \neq 0$, and we want it to use planar harmonic mappings. A natural choice is to consider

$$\begin{aligned} x_1 &= \operatorname{Re}(h+g) = \operatorname{Re}\int (h' + g')\,dz = \operatorname{Re}\int \varphi_1\,dz \\ x_2 &= \operatorname{Im}(h-g) = \operatorname{Re}\int -i(h' - g')\,dz = \operatorname{Re}\int \varphi_2\,dz \\ x_3 &= \operatorname{Re}\int \varphi_3\,dz \end{aligned}$$

and then solve for $\varphi_3$.

EXERCISE 2.122. Derive that $\varphi_3 = 2ih'\sqrt{g'/h'} = 2i\sqrt{g'h'}$.

***Try it out!***

We need $\varphi_3$ to be analytic and so we require the dilatation $\omega = g'/h'$ to be a perfect square.

THEOREM 2.123 (Weierstrass Representation (h,g)). If $f = h + \overline{g}$ is a sense-preserving harmonic univalent mapping of $\mathbb{D}$ onto some domain $\Omega \in \mathbb{C}$ with dilatation $\omega = q^2$ for some function $q$ analytic in $\mathbb{D}$, then the isothermal parametrization

$$\begin{aligned} \mathbf{x}(u,v) &= \big(x_1, x_2, x_3\big) \\ &= \left(\operatorname{Re}\{h(z)+g(z)\}, \operatorname{Im}\{h(z)-g(z)\}, 2\operatorname{Im}\left\{\int_0^z \sqrt{g'(\zeta)h'(\zeta)}d\zeta\right\}\right) \end{aligned}$$

defines a minimal graph whose projection onto the complex plane is $f$. Conversely, if a minimal graph $\mathbf{x}(u,v) = \big\{(u,v,F(u,v)) : u + iv \in \Omega\big\}$ is parametrized by sense-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, then the projection onto its base plane defines a harmonic univalent mapping $f(z) = u + iv = \operatorname{Re}\{h(z) + g(z)\} + i\operatorname{Im}\{h(z) - g(z)\}$ of $\mathbb{D}$ onto $\Omega$ whose dilatation is the square of an analytic function.

**Summary:** Let $f = h + \overline{g}$ defined on $\mathbb{D}$ be a harmonic univalent mapping such that the dilatation $\omega = g'/h'$ is the square of an analytic function and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Then $f$ lifts to a minimal graph using the Weierstrass formula given in Theorem 2.123.

EXAMPLE 2.124. Recall from Example 2.113 the harmonic univalent mapping

$$f(z) = z + \frac{1}{3}\overline{z}^3 = \operatorname{Re}\left(z + \frac{1}{3}z^3\right) + i\operatorname{Im}\left(z - \frac{1}{3}z^3\right).$$

Note that $h(z) = z$ and $g(z) = \frac{1}{3}z^3$. Also, $\omega(z) = z^2$ which is the square of an analytic function. Hence this harmonic mapping lifts to a minimal graph. We compute that

$$x_3 = 2\,\mathrm{Im}\int_0^z \sqrt{h'(\zeta)g'(\zeta)}\,d\zeta = \mathrm{Im}\left(z^2\right).$$

This yields a parametrization of a surface that is the conjugate of the Enneper's surface given in Example 2.66:

$$\mathbf{x} = \left(\mathrm{Re}\left\{z + \frac{1}{3}z^3\right\}, \mathrm{Im}\left\{z - \frac{1}{3}z^3\right\}, \mathrm{Im}\left\{z^2\right\}\right)$$

and hence yields Enneper's surface. Note that the projection of the Enneper surface onto the $x_1x_2$-plane is the image of $\mathbb{D}$ under the harmonic mapping $f$. Also, while Enneper's surface is not a graph over $\mathbb{C}$, it is a graph over $\mathbb{D}$ as this result proves. You can see this by using *MinSurfTool* with the **W.E. (h,g)** tab. Enter in the functions $h(z) = z$ and $g(z) = \frac{1}{3}z^3$. Make sure to use the Disk domain for the unit disk (i.e., radius min: 0; radius max: 1; theta min: 0; theta max: 2 pi). The minimal surface is colored red while the $f(\mathbb{D})$ is colored green. As you move the image so that it is viewed from the top, the projection of the minimal surface matches the image of $f(\mathbb{D})$.

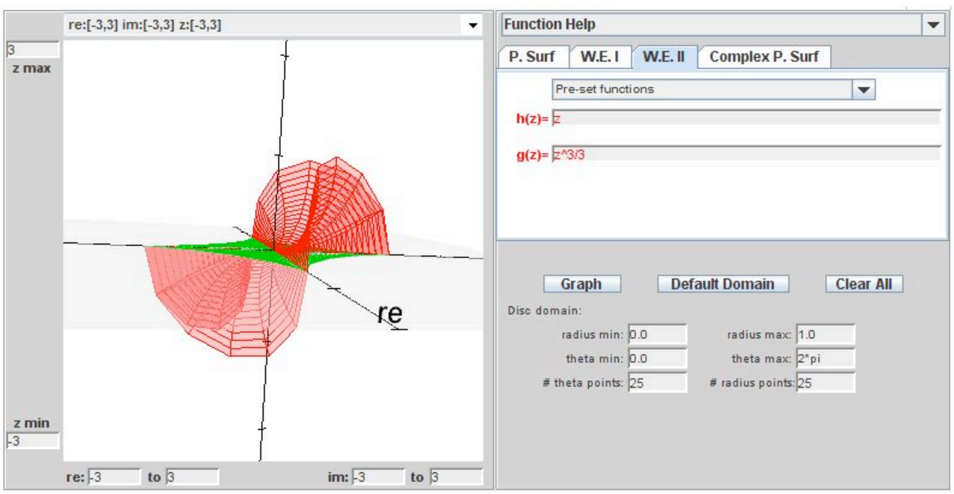



FIGURE 2.52. Side view of the Enneper surface and the image of the unit disk under the harmonic map.

EXPLORATION 2.125. In the Weierstrass representation (h,g), we require that $\omega = g'/h'$ be the square of an analytic function. This is necessary because $\varphi_3 = ih'\sqrt{g'/h'}$, and if $g'/h'$ were not the square of an analytic function, then there would be two branches of the square root. Geometrically, we can see that this is necessary. Use *MinSurfTool* with the **W.E. (h,g)** tab to graph the following images and describe why the geometry of those functions $f = h + \overline{g}$ in the left column do lift to a minimal

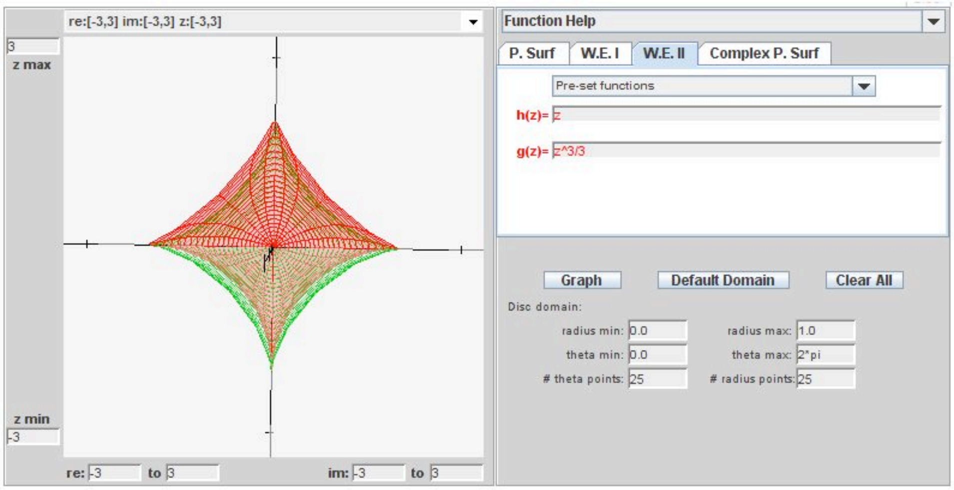



FIGURE 2.53. The projection of the Enneper surface is the image of the unit disk under the harmonic map.

graph while those in the right column do not.

(a) $z + \frac{1}{3}\overline{z}^3$ (note: $\omega = z^2$); (b) $z + \frac{1}{2}\overline{z}^2$ (note: $\omega = z$);

(c) $z - \frac{1}{5}\overline{z}^5$ (note: $\omega = -z^4$); (d) $z - \frac{1}{4}\overline{z}^4$ (note: $\omega = -z^5$).

EXAMPLE 2.126. Consider the harmonic univalent mapping from Example 2.120 given by

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

Because $\omega(z) = -z^2$ is the square of an analytic function, we can lift this harmonic mapping to a minimal graph. We compute that

$$x_3 = 2\operatorname{Im}\int_0^z \frac{iz}{1-z^4}\,d\zeta = \frac{1}{2}\operatorname{Im}\left\{i\log\left(\frac{1+z^2}{1-z^2}\right)\right\}.$$

This yields a parametrization of Scherk's doubly periodic minimal surface:

$$\mathbf{x} = \left(\operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right], \operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right], \frac{1}{2}\operatorname{Im}\left[i\log\left(\frac{1+z^2}{1-z^2}\right)\right]\right).$$

Again we can use *MinSurfTool* with the **W.E. (h,g)** tab to plot the minimal graph and the image of the unit disk under the planar harmonic mapping. Because of the singularities at $\pm 1, \pm i$, set radius max to 0.999; also, to get a better display set theta min: pi/8 and theta max: 2 pi + pi/8. Notice that the projection of Scherk's doubly periodic surface onto the $x_1x_2$-plane is a square which is the image of $\mathbb{D}$ under the harmonic mapping $f$.

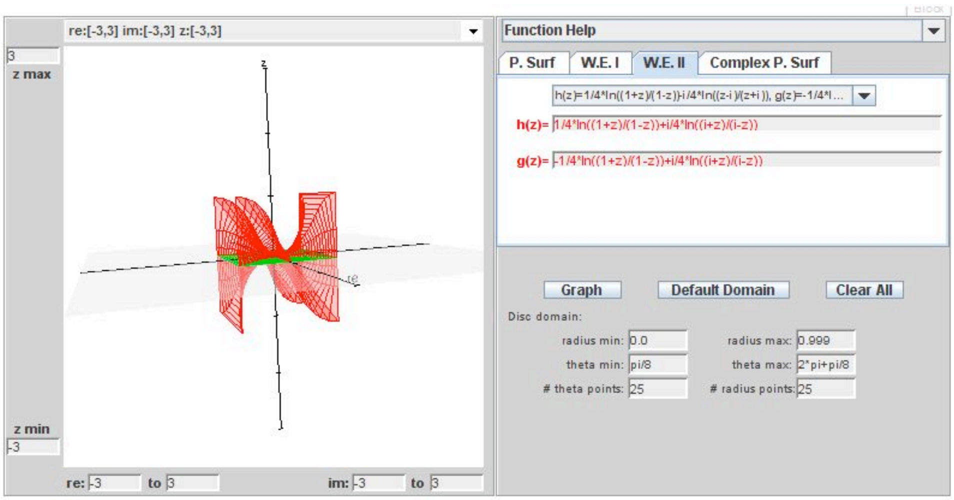



FIGURE 2.54. Side view of Scherk's doubly periodic surface and the image of the unit disk under the harmonic map.

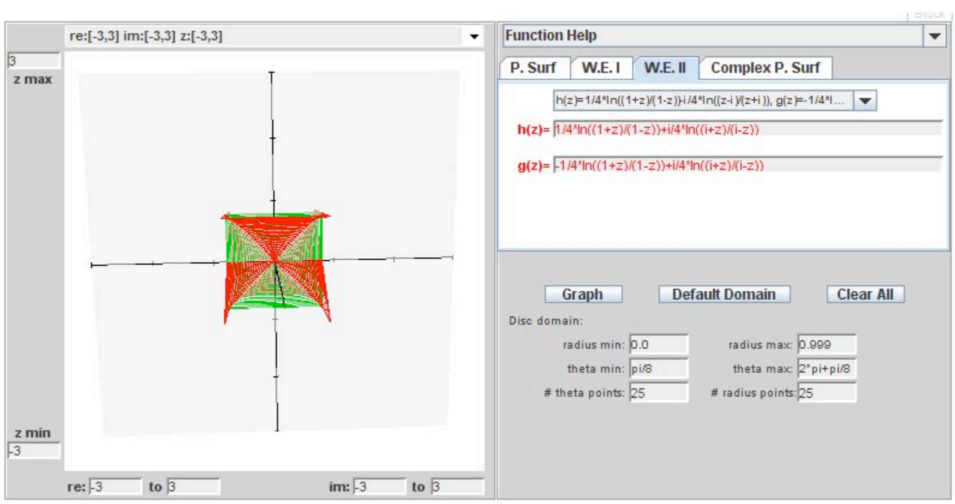



FIGURE 2.55. The projection of Scherk's doubly periodic surface is the image of the unit disk under the harmonic map.

EXPLORATION 2.127. Use the **W.E. (h,g)** tab in *MinSurfTool* to plot the minimal graphs associated with the given functions $h$ and $g$ for the planar harmonic mappings. Determine which minimal surfaces these are.

(a) $h(z) = z$, $g(z) = \frac{1}{2n+1} z^{2n+1} (n = 1, 2, 3, \ldots)$, with domain $= \mathbb{D}$;

(b) $h(z) = \frac{1}{4} \log\left(\frac{i+z}{i-z}\right) - \frac{i}{4} \log\left(\frac{1+z}{1-z}\right)$, $g(z) = \frac{1}{4} \log\left(\frac{i+z}{i-z}\right) + \frac{i}{4} \log\left(\frac{1+z}{1-z}\right)$, and domain $= \mathbb{D}$;

(c) $h(z) = \frac{1}{z}$, $g(z) = z$, and domain $= \{z \in \mathbb{C} : 0.1 \leq |z| \leq 1\}$;

(d) $h(z) = \frac{1}{z}$, $g(z) = iz$, and domain $= \{z \in \mathbb{C} : 0.1 \leq |z| \leq 1\}$;

(e) $h(z) = \frac{1}{5}z^5$, $g(z) = -\frac{1}{z}$, and domain $= \{z \in \mathbb{C} : 0.1 \leq |z| \leq 1\}$.

***Try it out!***

Starting with a minimal graph and finding the corresponding harmonic univalent mapping is fairly straightforward. This is because the harmonic univalent mapping is the projection of the minimal graph onto the $x_1x_2$-plane and so the harmonic mapping can be represented by the first two coordinate functions in the parametrization of the minimal graph. However, going in the other direction is not so easy. If we start with a harmonic univalent mapping we can use Theorem 2.123 to find the parametrization of a minimal graph, but we do not necessarily know which minimal graph this is. There have been several research papers in the field of harmonic univalent mappings that have used this approach to create minimal graphs from harmonic univalent mappings (e.g., [**8**], [**9**], [**10**], [**11**], [**16**]). However, many of them have not identified the specific minimal graph created.

**Question:** Given a harmonic univalent mapping we can use Theorem 2.123 to find the parametrization of a minimal graph. Can we determine which minimal graph this is?

EXAMPLE 2.128. By shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = g'(z)/h'(z) = m^2z^2$, where $|m| \leq 1$, it was shown in [**9**] that the harmonic function $f = h + \overline{g}$ is univalent, where

$$\begin{aligned} h(z) &= \frac{1}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right) \\ g(z) &= \frac{m^2}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right). \end{aligned}$$

When $m = e^{i\frac{\pi}{2}}$, the function $f$ is the same as in Example 2.120 and the image of $\mathbb{D}$ under $f = h + \overline{g}$ is a square. In fact, for every $m$ such that $|m| = 1$, the image of $\mathbb{D}$ under $f = h + \overline{g}$ is a parallelogram.

Since $\omega(z) = g'(z)/h'(z) = m^2z^2$, we can lift $f$ to a minimal graph. We can compute

$$x_3 = \operatorname{Im}\left\{\frac{m}{1-m^2}\log\left(\frac{1-m^2z^2}{1-z^2}\right)\right\}.$$

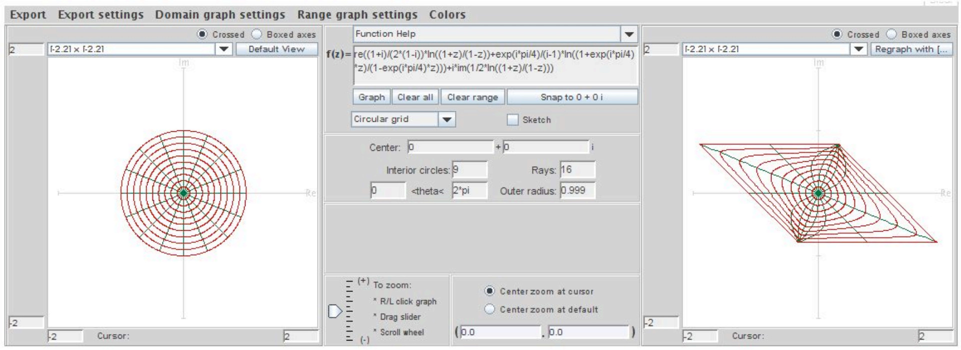



FIGURE 2.56. Image of $\mathbb{D}$ under $f = h + \bar{g}$ when $m = e^{i\frac{\pi}{4}}$.

Hence, the corresponding parametrization of the minimal graph is

$$\mathbf{x} = \left( \operatorname{Re}\left\{ \frac{1+m^2}{2(1-m^2)} \log\left(\frac{1+z}{1-z}\right) + \frac{m}{(m^2-1)} \log\left(\frac{1+mz}{1-mz}\right) \right\}, \right.$$

$$\left. \operatorname{Im}\left\{ \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) \right\}, \operatorname{Im}\left\{ \frac{m}{1-m^2} \log\left(\frac{1-m^2z^2}{1-z^2}\right) \right\} \right).$$

When $m = e^{i\frac{\pi}{2}}$, the minimal graph is Scherk's doubly periodic surface. For $m = e^{i\theta}, (0 < \theta < \frac{\pi}{2})$, the minimal graphs are slanted Scherk's surfaces.

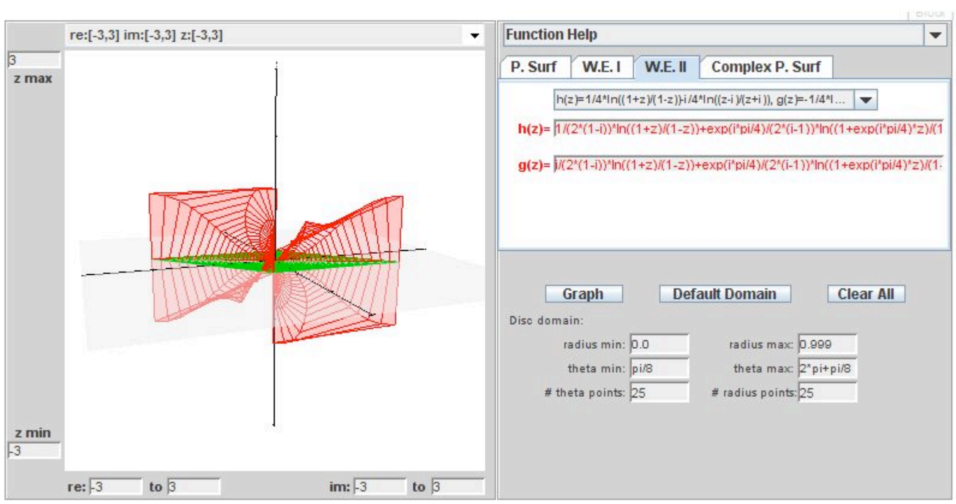



FIGURE 2.57. Side view of slanted Scherk's surface and the image of the unit disk under the harmonic map.

What is the minimal graph when $m = 1$? In the limit (i.e., $\theta = 0$) we have the equation

$$\mathbf{x} = \left( \text{Re}\left\{ \frac{z}{1 - z^2} \right\}, \text{Re}\left\{ -\frac{i}{2} \log\left( \frac{1+z}{1-z} \right) \right\}, \text{Re}\left\{ \frac{-iz^2}{1 - z^2} \right\} \right).$$

Using the substitution $z \longmapsto \frac{e^z - 1}{e^z + 1}$ and the fact that $\text{Re}\left\{ \frac{-iz^2}{1-z^2} \right\} = \text{Re}\left\{ \frac{1}{2i} \frac{1+z^2}{1-z^2} \right\}$, this equation is equivalent to

$$X = \left( \frac{1}{2} \sinh u \cos v, \frac{1}{2} v, \frac{1}{2} \sinh u \sin v \right),$$

which is an equation of a helicoid.

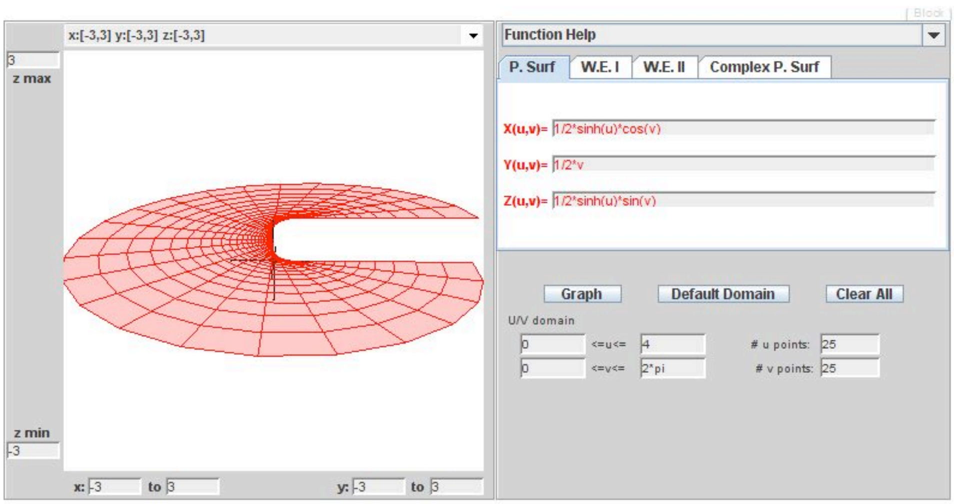



FIGURE 2.58. Side view of helicoid that is the limit function of the slanted Scherk's surfaces.

EXERCISE 2.129. Show that $\text{Re}\left\{ \frac{-iz^2}{1-z^2} \right\} = \text{Re}\left\{ \frac{1}{2i} \frac{1+z^2}{1-z^2} \right\}$.

***Try it out!***

EXAMPLE 2.130. If we shear $h(z) - g(z) = \frac{z}{1-z}$ with $z^2$, then we get the harmonic univalent mapping $f = h + \overline{g}$, where

$$h = \frac{1}{8} \ln\left( \frac{z+1}{z-1} \right) + \frac{3z - 2z^2}{4(1-z)^2} \quad \text{and} \quad g = \frac{1}{8} \ln\left( \frac{z+1}{z-1} \right) - \frac{z - 2z^2}{4(1-z)^2}.$$

This function is interesting in complex analysis, because

$$h(z) - g(z) = \frac{z}{1-z}$$

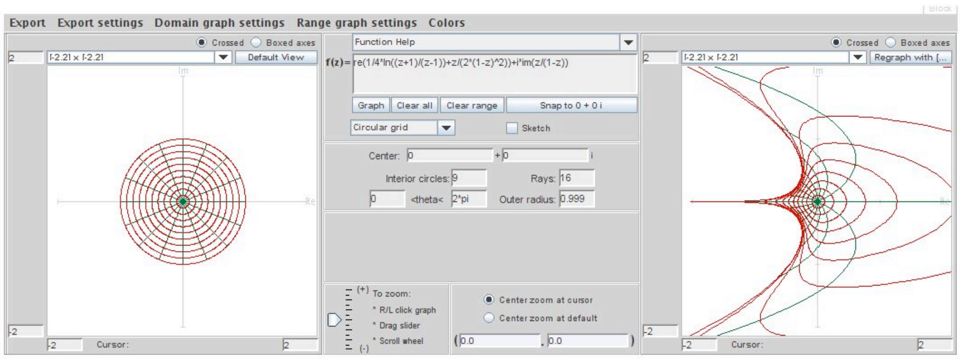



FIGURE 2.59. Image of $\mathbb{D}$ under $f = \text{Re}\left(\frac{1}{4}\log\left(\frac{z+1}{z-1}\right) + \frac{z}{2(1-z)^2}\right) + \text{Im}\left(\frac{z}{1-z}\right)$.

is a right half-plane mapping that has several interesting properties.
Note that $\omega(z) = z^2$, and so we can use Theorem 2.123 to find the parametrization of the corresponding minimal graph:

$$\mathbf{x} = \left(\text{Re}\left\{2\log\left(\frac{z+1}{z-1}\right)+\frac{z}{2(1-z)^2}\right\}, \text{Im}\left\{\frac{z}{1-z}\right\}, \text{Im}\left\{\frac{1}{4}\log\left(\frac{z+1}{z-1}\right)-\frac{z}{2(1-z)^2}\right\}\right).$$

In this form, the coordinate functions do not look familiar. However, we can use a Möbius transformation which will not affect the geometry of the minimal graph to rewrite these coordinate functions. In particular, letting $z \mapsto \frac{\hat{z}+1}{\hat{z}-1}$, we get:

$$\hat{\mathbf{x}} = \left(\frac{1}{4}\,\text{Re}\left\{\log(\hat{z}) + \frac{1}{2}\hat{z}^2 - \frac{1}{2}\right\}, -\frac{1}{2}\,\text{Im}\left\{\hat{z}\right\}, -\frac{1}{4}\,\text{Im}\left\{\log(\hat{z}) - \frac{1}{2}\hat{z}^2 + \frac{1}{2}\right\}\right).$$

This transformation is useful, because it simplifies the log terms in $x_1$ and $x_3$. Next, we notice that by switching the coordinate functions and factoring out $\frac{1}{2}$ we have something that looks more like the wavy plane.

$$\widetilde{\mathbf{x}} = \left(-\frac{1}{2}\left[\frac{1}{2}\,\text{Im}\left\{\log(\widetilde{z}) - \frac{1}{2}\widetilde{z}^2\right\}\right], \frac{1}{2}\left[\frac{1}{2}\,\text{Re}\left\{\log(\widetilde{z}) + \frac{1}{2}\widetilde{z}^2\right\}\right], -\frac{1}{2}\left[\text{Im}\left\{\widetilde{z}\right\}\right]\right).$$

The coordinates above correspond to the conjugate surface of the wavy plane scaled by $\frac{1}{2}$. This is clear given the actual coordinates of the wavy plane below:

$$\mathbf{W} = \left(\frac{1}{2}\,\text{Re}\left\{\log(z) + \frac{1}{2}z^2 + c\right\}, -\frac{1}{2}\,\text{Im}\left\{\log(z) - \frac{1}{2}z^2 + c_2\right\}, -\,\text{Re}\left\{z + c_3\right\}\right).$$

Since the wavy plane is its own conjugate surface, this means that it is accurate to describe our surface as the wavy plane.

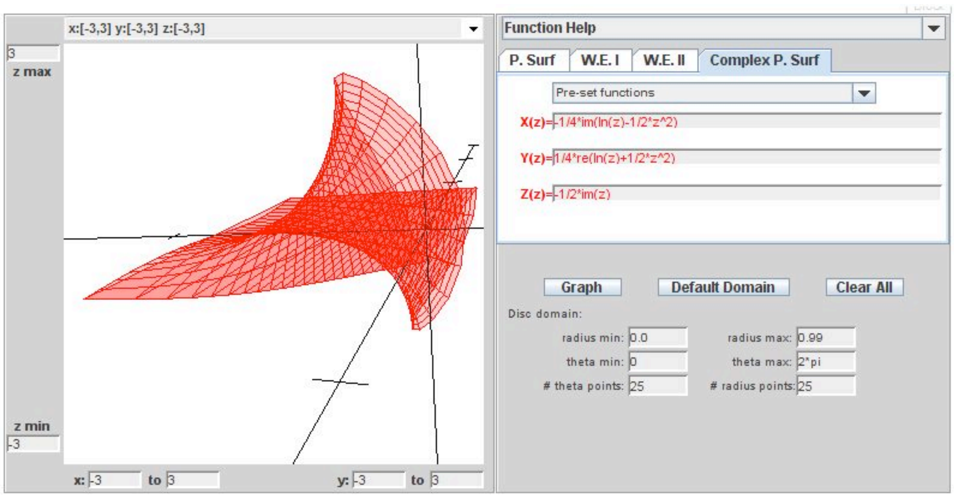



FIGURE 2.60. Side view of the wavy plane surface.

EXERCISE 2.131. Consider the harmonic univalent map $f(z) = h(z) + \overline{g}(z)$, where

$$h = \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) + \frac{\frac{1}{2}z}{1-z^2} \quad \text{and} \quad g = \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{\frac{1}{2}z}{1-z^2}.$$

(a) Use Theorem 2.123 to find the parametrization of the minimal graph that $f$ lifts to.
(b) Use *ComplexTool* to graph the image of $\mathbb{D}$ under $f$ and *MinSurfTool* with tab **W.E.(h,g)** to sketch the corresponding minimal graph.
(c) Use the approach of Example 2.128 to show analytically that this minimal graph is the catenoid.

***Try it out!***

EXERCISE 2.132. An important function is complex analysis is the Koebe function given by $\frac{z}{(1-z)^2}$. By shearing

$$h(z) - g(z) = \frac{z}{(1-z)^2} \quad \text{with} \quad \omega(z) = z^2,$$

we derive the harmonic univalent mapping $f = h + \overline{g}$, where

$$h = \frac{z - z^2 + \frac{1}{3}z^3}{(1-z)^3} \quad \text{and} \quad g = \frac{\frac{1}{3}z^3}{(1-z)^3}.$$

(a) Use Theorem 2.123 to find the parametrization of the minimal graph that $f$ lifts to.
(b) Use *ComplexTool* to graph the image of $\mathbb{D}$ under $f$ and *MinSurfTool* with tab **W.E.(h,g)** to sketch the corresponding minimal graph.

(c) Use the approach of Example 2.130 to show analytically that this minimal graph is the Enneper surface.

***Try it out!***

Large Project 2.133. The analytic function, $F(z) = z$, maps the unit disk, $\mathbb{D}$, onto itself. Shear $h(z) - g(z) = z$ with various dilatations, $\omega$, that satisfy the condition $|\omega| < 1$ for all $z \in \mathbb{D}$ (e.g., $\omega = z^{2n} (n \in \mathbb{N}), \omega = e^{i\theta} z^2 (\theta \in \mathbb{R}), \omega = \left(\frac{z-a}{1-\bar{a}z}\right)^2 (|a| < 1)$). Determine the corresponding minimal graphs.

**Optional**

Large Project 2.134. The analytic function, $F(z) = \frac{z}{1-z}$, maps the unit disk, $\mathbb{D}$, onto a right half-plane and is an important function. Shear $h(z) + g(z) = \frac{z}{1-z}$ with various dilatations, $\omega$, that satisfy the condition $|\omega| < 1$ for all $z \in \mathbb{D}$ (e.g., $\omega = z^{2n} (n \in \mathbb{N}), \omega = e^{i\theta} z^2 (\theta \in \mathbb{R}), \omega = \left(\frac{z-a}{1-\bar{a}z}\right)^2 (|a| < 1)$). Determine the corresponding minimal graphs.

**Optional**

## 2.7. Convex Combinations of Minimal Graphs

We are interested in finding ways to construct embedded minimal surfaces. In this section we will explore the idea of taking a convex combination of minimal graphs. The background from the previous two sections lay the foundation for this section.

Definition 2.135. A convex combination

$$x = t_1 x_1 + \cdots + t_n x_n$$

is a linear combination of a finite number of "points" $x_1, \ldots, x_n$, where each scalar $t_k$ is non-negative and $\displaystyle\sum_{k=1}^{n} t_k = 1$.

Example 2.136. The set of all convex combinations, $t_1 P_1 + t_2 P_2$, of two points, $P_1$ and $P_2$, is the line segment between these points. In the definition, "points" can be more general. For example, the expression $\frac{1}{2} e^{i\theta} + \frac{1}{2} e^{-i\theta} = \cos\theta$ is a convex combination of the functions $f_1(\theta) = e^{i\theta}$ and $f_2(\theta) = e^{-i\theta}$.

Let $M_1$, $M_2$ be minimal graphs in $\mathbb{R}^3$ with Weierstrass data $(G_1, dh_1)$, $(G_2, dh_2)$, respectively. It is not true that the convex combination $M = t_1 M_1 + t_2 M_2$ must be a minimal graph. Consider the following example.

Example 2.137.

However, we can guarantee that the convex combination will be a minimal graph if we include a few conditions. Before we state this result, we need to present a few background ideas.

DEFINITION 2.138. A domain $\Omega \subset \mathbb{C}$ is convex in the direction $e^{i\varphi}$ if for every $a \in \mathbb{C}$ the set $\Omega \cap \{a + te^{i\varphi} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain is convex in the imaginary direction if every line parallel to the imaginary axis has a connected intersection with $\Omega$.

**Condition A**: Let $F$ be a non-constant analytic function in $\mathbb{D}$, and there exists sequences $z_n'$, $z_n''$ converging to $z = 1$, $z = -1$, respectively, such that

$$\begin{aligned} \lim_{n\to\infty} \mathrm{Re}\{F(z_n')\} &= \sup_{|z|<1} \mathrm{Re}\{F(z)\} \\ \lim_{n\to\infty} \mathrm{Re}\{F(z_n'')\} &= \inf_{|z|<1} \mathrm{Re}\{F(z)\}. \end{aligned} \tag{20}$$

Note that the normalization in (20) can be thought of in some sense as if $F(1)$ and $F(-1)$ are the right and left extremes in the image domain in the extended complex plane.

Now, we can state our main result.

THEOREM 2.139. Let $M_1, \dots, M_n : \mathbb{D} \to \mathbb{R}^3$ be minimal graphs with isothermal parametrizations $\phi_k = \mathrm{Re}(\phi_k^1, \phi_k^2, \phi_k^3)$ in terms of the Gauss map $G_k$ and height differential $dh_k$ ($k = 1, \dots, n$) as given in (14). Let $G_k = G_1$, for each $k$. Also, let each $D_k$, the projection of $M_k$ onto the $x_1x_2$-plane be convex in the imaginary direction and let condition A hold for each $\phi_k^1$, for $k = 1, \dots, n$. Then the convex combination $M = t_1M_1 + \cdots + t_nM_n$ is a minimal graph, for all $0 \le t_k \le 1$, where $t_1 + \cdots + t_n = 1$ with $G = G_1$ and $dh = t_1dh_1 + \cdots + t_ndh_n$.

In order to prove Theorem 2.139, we need some background material. First, we will need some results about univalent harmonic mappings, $f = h + \overline{g}$, where $h$, $g$ are analytic in $\mathbb{D}$ that were discussed in Section . Recall from Theorem 2.123 that harmonic univalent mappings are connected with minimal graphs in $\mathbb{R}^3$ through a Weierstrass representation. Note that the Gauss map $G(z)$ and height differential $dh(z)$ discussed in Section relate to the the univalent harmonic mapping $f = h + \overline{g}$ by:

$$G(z) = \sqrt{\frac{g'(z)}{h'(z)}}, dh(z) = -2i\sqrt{g'(z)h'(z)}\, dz. \tag{21}$$

EXERCISE 2.140. Using Weierstrass Representation (G,dh) and Weierstrass Representation (h,g) derive the formulas in (21).

***Try it out!***

The next theorem from [**5**] can be used to show that the harmonic function $f = h+\overline{g}$ is univalent.

THEOREM 2.141. A harmonic function $f = h+\overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the imaginary direction if and only if $h + g$ is a univalent analytic mapping of $\mathbb{D}$ onto a domain convex in the imaginary direction.

There is a result in [**15**] about univalent analytic mappings that map onto domains convex in the imaginary direction.

THEOREM 2.142. Suppose $f$ is holomorphic and non-constant in $\mathbb{D}$. Then

$$\operatorname{Re}\{(1-z^2)f'(z)\} \geq 0, z \in \mathbb{D},$$

if and only if $f$ is univalent in $\mathbb{D}$, $f(\mathbb{D})$ is convex in the imaginary direction, and Condition A holds.

Now, we will prove our main result, Theorem 2.139.

PROOF. By Theorem 2.123, the projection of each minimal graph $M_k$ onto the $x^1x^2$-plane defines a univalent harmonic mapping $f_k = h_k + \overline{g_k}$ with dilatation $\omega_k = g_k'/h_k'$. Let

$$f = h + \overline{g} = (t_1h_1 + \cdots + t_nh_n) + \overline{(t_1g_1 + \cdots + t_ng_n)}.$$

We will show that $f$ is a univalent harmonic mapping of $\mathbb{D}$ onto a domain convex in the imaginary direction.

Since $G_1 = G_k$, we see from (21) that $\omega_1 = \omega_k$ for all $k = 2, \ldots, n$. Also, $\omega = g'/h'$ equals $\omega_1$, because

$$\omega = \frac{t_1g_1' + \cdots + t_ng_n'}{t_1h_1' + \cdots + t_nh_n'} = \frac{t_1h_1'\omega_1 + \cdots + t_nh_n'\omega_n}{t_1h_1' + \cdots + t_nh_n'} = \omega_1.$$

Hence, $f$ is locally univalent since $|\omega(z)| = |\omega_1(z)| < 1, \forall z \in \mathbb{D}$.

We now will show that $h + g$ is a univalent analytic mapping of $\mathbb{D}$ onto a domain convex in the imaginary direction, so we can apply Theorem 2.141. By Theorem 2.141, we know that each $h_k + g_k$ is univalent and convex in the imaginary direction. Also, $h_k + g_k$ satisfies condition A since $\operatorname{Re}\{h_k + g_k\} = \operatorname{Re}\{\phi_k^1\}$. Applying Theorem 2.142 we have

$$\operatorname{Re}\{(1-z^2)(h_k'(z) + g_k'(z))\} \geq 0.$$

Then

$$\begin{aligned}
\operatorname{Re}\{(1-&z^2)(h'(z) + g'(z))\} \\
&= \operatorname{Re}\{(1-z^2)[t_1(h_1'(z) + g_1'(z)) + \cdots + t_n(h_n'(z) + g_n'(z))]\} \\
&= t_1 \operatorname{Re}\{(1-z^2)(h_1'(z) + g_1'(z))\} + \cdots + t_n \operatorname{Re}\{(1-z^2)(h_n'(z) + g_n'(z))\} \geq 0.
\end{aligned}$$

By applying Theorem 2.142 in the other direction, we have that $h + g$ is convex in the imaginary direction, and so by Theorem 2.141, $f$ is univalent mapping with $f(\mathbb{D})$ being convex in the imaginary direction.

We can now apply the Weierstrass representation from Theorem 2.123, to lift $f = h + \overline{g}$ to a minimal graph $\widetilde{M} = (u, v, f(u, v))$. Notice that

$$\begin{aligned}
u =& \operatorname{Re}\{h + g\} \\
=& \operatorname{Re}\{(t_1h_1 + t_1g_1) + \cdots + (t_nh_n + t_ng_n)\} \\
=& t_1 \operatorname{Re}\{\phi_1^1\} + \cdots + t_n \operatorname{Re}\{\phi_n^1\}.
\end{aligned}$$

Similarly, $v = \operatorname{Im}\{h-g\} = t_1 \operatorname{Re}\{\phi_1^2\} + \cdots + t_n \operatorname{Re}\{\phi_n^2\}$.
Finally,

$$\begin{aligned}
F(u,v) =& 2\operatorname{Im}\left\{\int_0^z \sqrt{(t_1 g_1'(\zeta)+\cdots+t_n g_n'(\zeta))(t_1 h_1'(\zeta)+\cdots+t_n h_n'(\zeta))}\, d\zeta\right\}\\
=& 2\operatorname{Im}\left\{\int_0^z \sqrt{\big(t_1\omega_1(\zeta)h_1'(\zeta)+\cdots+t_n\omega_n(\zeta)h_n'(\zeta)\big)\big(t_1 h_1'(\zeta)+\cdots+t_n h_n'(\zeta)\big)}\, d\zeta\right\}\\
=& 2\operatorname{Im}\left\{\int_0^z \sqrt{\omega_1(\zeta)}\Big(t_1 h_1'(\zeta)+\cdots+t_n h_n'(\zeta)\Big)\, d\zeta\right\}\\
=& 2\operatorname{Im}\left\{\int_0^z \Big(t_1\sqrt{g_1'(\zeta)h_1'(\zeta)}+\cdots+t_n\sqrt{g_n'(\zeta)h_n'(\zeta)}\Big)\, d\zeta\right\}\\
=& t_1 \operatorname{Re}\left\{\phi_1^3\right\} + \cdots + t_n \operatorname{Re}\{\phi_n^3\}.
\end{aligned}$$

Thus, $\widetilde{M} = t_1 M_1 + \cdots + t_n M_n = M$. □

REMARK 2.143. The hypothesis of Theorem 2.139 that $G_k = G_1$ for all $k$ is not necessary.

EXAMPLE 2.144. Consider Scherk's doubly periodic surface $M_1$ with Weierstrass data

$$G_1(z) = z \quad \text{and} \quad dh_1(z) = \frac{z\,dz}{1-z^4},$$

and Scherk's singly periodic surface $M_2$ with Weierstrass data

$$G_2(z) = z \quad \text{and} \quad dh_2(z) = \frac{-iz\,dz}{1-z^4}.$$

These satisfy the conditions of Theorem 2.139. Hence, the Weierstrass data

$$G(z) = z \quad \text{and} \quad dh_t(z) = \frac{(t-i(1-t))z\,dz}{1-z^4}$$

is a minimal graph for all $t \in [0,1]$. Note $M_1$ and $M_2$ are conjugate surfaces. The corresponding associate surfaces are determined by $G(z) = z$ and $dh_\theta(z) = \frac{e^{i\theta}z\,dz}{1-z^4}$ which is different than the surface created by the convex combination of $M_1$ and $M_2$.

EXERCISE 2.145. Add conjugate surfaces that satisfy condition A, such as a catenoid on its side and a helicoid on its side.

EXAMPLE 2.146. Consider the minimal surface $M_1$ with Weierstrass data

$$G_1(z) = z \quad \text{and} \quad dh_1(z) = \frac{z\,dz}{1-z^4}.$$

The domain is $\mathbb{D}$ and by the Weierstrass representation given in (14), $M_1$ is parametrized by

$$\begin{aligned}
\mathbf{x}(\mathbf{z}) &= (u_1, v_1, w_1) \\
&= \left( \operatorname{Re}\left\{ \frac{i}{2} \log\left( \frac{1+z}{1-z} \right) \right\}, \operatorname{Re}\left\{ \frac{i}{2} \log\left( \frac{i+z}{i-z} \right) \right\}, \operatorname{Re}\left\{ \frac{1}{2} \log\left( \frac{1+z^2}{1-z^2} \right) \right\} \right).
\end{aligned}$$

What is the projection of this surface onto the $x_1x_2$-plane? In other words, what is the image of $\mathbb{D}$ under complex function $u_1 + iv_1$? Note that

$$u_1 = -\frac{1}{2} \arg\left( \frac{1+z}{1-z} \right), \quad v_1 = -\frac{1}{2} \arg\left( \frac{i+z}{i-z} \right).$$

Let $z = e^{i\theta} \in \partial\mathbb{D}$. Then

$$\frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} \frac{1-e^{-i\theta}}{1-e^{-i\theta}} = \frac{1+e^{i\theta}-e^{-i\theta}-1}{1-e^{i\theta}-e^{-i\theta}+1} = i\frac{\sin\theta}{1-\cos\theta}.$$

Thus,

$$u_1 = -\frac{1}{2} \arg\left( \frac{1+z}{1-z} \right)\bigg|_{z=e^{i\theta}} = \begin{cases} \frac{\pi}{4} & \text{if } \sin\theta < 0, \\ -\frac{\pi}{4} & \text{if } \sin\theta > 0. \end{cases}$$

Likewise, we can show that

$$v_1 = \begin{cases} \frac{\pi}{4} & \text{if } \cos\theta > 0, \\ -\frac{\pi}{4} & \text{if } \cos\theta < 0. \end{cases}$$

In summary, we have that $z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u_1 + iv_1 = \begin{cases} z_1 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{3\pi}{4}} &= -\frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (0, \frac{\pi}{2}), \\ z_3 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{5\pi}{4}} &= -\frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ z_5 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{7\pi}{4}} &= \frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\pi, \frac{3\pi}{2}), \\ z_7 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{9\pi}{4}} &= \frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Thus, the projection of this minimal surface onto the $x_1x_2$-plane is the interior of the region bounded by a square with vertices at $z_1, z_3, z_5$ and $z_7$.

Notice that $w_1 = \frac{1}{2}\ln\left(\frac{1+z^2}{1-z^2}\right)$ has singularities at $z = \pm 1, \pm i$. As $z \to \pm 1$, $w_1 \to +\infty$ while as $z \to \pm i$, $w_3 \to -\infty$. Our surface $M_1$ is a parametrization of Scherk's doubly periodic surface.

Next, consider the minimal surface $M_2$ with the Weierstrass data

$$G_2(z) = z \qquad \text{and} \qquad dh_2(z) = \frac{-iz\, dz}{1+z^4}.$$

This is another parametrization of Scherks doubly-periodic surface given by

$$\begin{aligned}
\mathbf{x_2(z)} &= (u_2, v_2, w_2) \\
&= \Bigg( \mathrm{Re}\left\{ \frac{i}{2\sqrt{2}} \left[ \log\left( \frac{e^{i\frac{\pi}{4}} + z}{e^{i\frac{\pi}{4}} - z} \right) + \log\left( \frac{e^{i\frac{3\pi}{4}} + z}{e^{i\frac{3\pi}{4}} - z} \right) \right] \right\}, \\
&\qquad \mathrm{Re}\left\{ \frac{-i}{2\sqrt{2}} \left[ \log\left( \frac{e^{i\frac{\pi}{4}} + z}{e^{i\frac{\pi}{4}} - z} \right) - \log\left( \frac{e^{i\frac{3\pi}{4}} + z}{e^{i\frac{3\pi}{4}} - z} \right) \right] \right\}, \\
&\qquad \mathrm{Re}\left\{ \frac{1}{2} \log\left( \frac{i + z^2}{i - z^2} \right) \right\} \Bigg).
\end{aligned}$$

Similar to above, the projection of $M_2$ onto the $x_1x_2$-plane is a rotated square region with vertices at $z_0, z_2, z_4$ and $z_6$ since $z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u_2 + iv_2 = \begin{cases} z_0 = \frac{\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{-\pi}{4}, \frac{\pi}{4}), \\ z_2 = \frac{i\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{\pi}{4}, \frac{3\pi}{4}), \\ z_4 = -\frac{\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{3\pi}{4}, \frac{5\pi}{4}), \\ z_6 = -\frac{i\pi}{2\sqrt{2}} & \text{if } \theta \in (\frac{5\pi}{4}, \frac{7\pi}{4}). \end{cases}$$

Also, $w_2 = \frac{1}{2}\ln\left(\frac{i+z^2}{i-z^2}\right)$ has singularities at $z = \pm e^{i\pi/4}, \pm e^{-i\pi/4}$. As $z \to \pm e^{i\pi/4}$, $w_2 \to +\infty$ while as $z \to \pm e^{-i\pi/4}$, $w_2 \to -\infty$.

The hypotheses for Theorem 2.139 are satisfied. Therefore,

$$M = tM_1 + (1-t)M_2, \qquad (0 \le t \le 1)$$

is a minimal graph in $\mathbb{R}^3$ with

$$G(z) = z \qquad \text{and} \qquad dh_t(z) = t\frac{z\,dz}{1-z^4} + (1-t)\frac{-iz\,dz}{1+z^4}.$$

To determine what surfaces these are, consider the specific case when $t = \frac{1}{2}$. The projection of the surface $M$ onto the complex plane is the nonconvex star shown in the bottom left column of Figure 4.28.

To see why this is, we can look at where arcs of the unit circle are mapped under $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$. Notice that $f_1(e^{i\theta})$ and $f_2(e^{i\theta})$ depend upon which of eight arcs $\theta$ is in. For example, if $\theta \in (-\frac{\pi}{4}, 0)$, then $f_1(e^{i\theta}) = z_1$ and $f_2(e^{i\theta}) = z_0$, and so in this interval $f(e^{i\theta}) = \frac{z_1+z_0}{2}$ (that is, it is the midpoint between $z_1$ and $z_0$). However, if $\theta \in (0, \frac{\pi}{4})$,

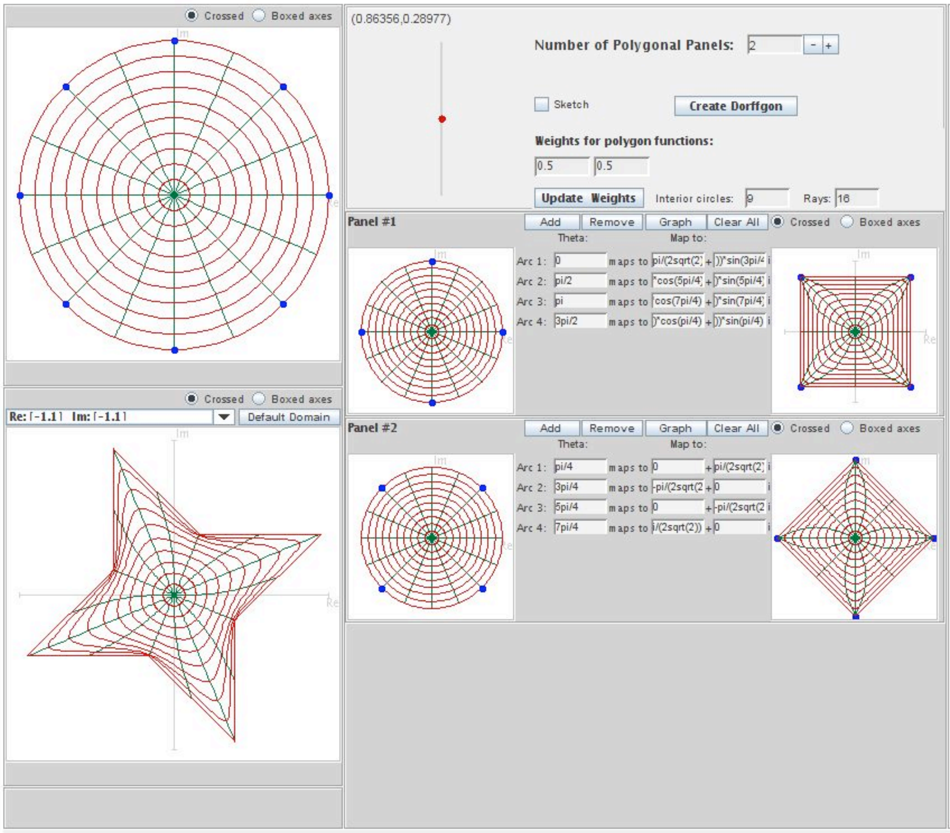



FIGURE 2.61. Image $\mathbb{D}$ under $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$

then $f_1(e^{i\theta}) = z_3$ and $f_2(e^{i\theta}) = z_0$, and $f(e^{i\theta}) = \frac{z_3+z_0}{2}$. Specifically,

$$f(e^{i\theta}) = \begin{cases} \zeta_1 = \frac{z_1+z_0}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{\pi}{8} e^{i\frac{\pi}{8}} & \text{if } \theta \in (\frac{-\pi}{4}, 0), \\ \zeta_2 = \frac{z_3+z_0}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{3\pi}{8} e^{i\frac{3\pi}{8}} & \text{if } \theta \in (0, \frac{\pi}{4}), \\ \zeta_3 = \frac{z_3+z_2}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{\pi}{8} e^{i\frac{5\pi}{8}} & \text{if } \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \\ \zeta_4 = \frac{z_5+z_2}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{3\pi}{8} e^{i\frac{7\pi}{8}} & \text{if } \theta \in (\frac{\pi}{2}, \frac{3\pi}{4}), \\ \zeta_5 = \frac{z_5+z_4}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{\pi}{8} e^{i\frac{9\pi}{8}} & \text{if } \theta \in (\frac{3\pi}{4}, \pi), \\ \zeta_6 = \frac{z_7+z_4}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{3\pi}{8} e^{i\frac{11\pi}{8}} & \text{if } \theta \in (\pi, \frac{5\pi}{4}), \\ \zeta_7 = \frac{z_7+z_6}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{\pi}{8} e^{i\frac{13\pi}{8}} & \text{if } \theta \in (\frac{5\pi}{4}, \frac{3\pi}{2}), \\ \zeta_8 = \frac{z_1+z_6}{2} = \frac{\pi}{2\sqrt{2}} \cos\frac{3\pi}{8} e^{i\frac{15\pi}{8}} & \text{if } \theta \in (\frac{3\pi}{2}, \frac{7\pi}{4}). \end{cases}$$

Note that the vertices $\zeta_1, \zeta_3, \zeta_5$ and $\zeta_7$ lie equally spaced on a circle of radius $r_{outer} = \frac{\pi}{2\sqrt{2}} \cos\frac{\pi}{8} \approx 1.026$, while the vertices $\zeta_2, \zeta_4, \zeta_6$ and $\zeta_8$ lie equally spaced on a circle of radius $r_{inner} = \frac{\pi}{2\sqrt{2}} \cos\frac{3\pi}{8} \approx 0.425$.

We can visualize the boundary of $f(\mathbb{D})$ by plotting the eight vertices $z_0, z_1, \ldots z_7$ and drawing the midpoints $\zeta_1, \ldots, \zeta_8$ (see Figure 4.26).

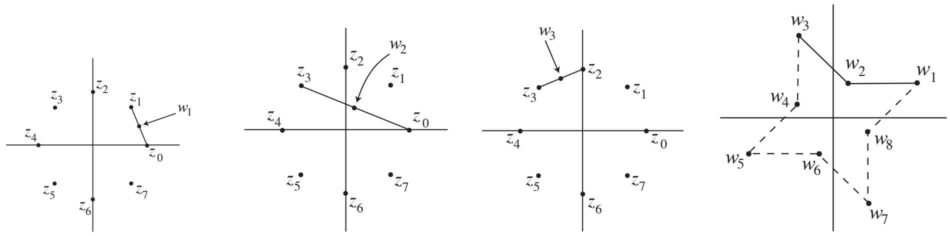



FIGURE 2.62. Visualizing the image of the boundary of $f(\mathbb{D})$

Now, let's look at the behavior of the third coordinate function, $w$, for this convex combination map, $f$. Note that there are singularities at $\pm 1$, $\pm e^{i\pi/4}$, $\pm i$, and $\pm e^{-i\pi/4}$. As $z \to \pm 1$, $w_1 \to +\infty$ while $w_2$ remains finite, and so $w \to +\infty$. On the other hand, as $z \to \pm i$, $w_1 \to -\infty$ while $w_2$ remains finite, and so $w \to -+\infty$. Similarly, $w \to +\infty$ as $z \to \pm e^{i\pi/4}$, while $w \to -\infty$ as $z \to \pm e^{-i\pi/4}$.

By changing the value of $t$ we can get different asymmetric nonconvex polygonal regions with the same behavior for $w$, and the corresponding surfaces $M$ are known as Jenkins-Serrin surfaces.

EXERCISE 2.147. We can have two different parametrizations, $\mathbf{x_1}$ and $\mathbf{x_2}$, of the same surface with the same image projected onto the $x_1x_2$-plane. Yet, when we take the convex combination of each of these with a parametrization of another independent surface $\mathbf{x_3}$, the resulting surfaces can be different.

Repeat the steps in Example 2.146 using the same $G_2$ and $dh_2$ for $M_2$ but replacing the Weierstrass data for $M_1$ with

$$G_1(z) = iz \qquad \text{and} \qquad dh_1(z) = \frac{z\,dz}{1 - z^4}.$$

(a.) Show that $G_1 = iz$ and $G_2 = z$ satisfies eq. (**??**).
(b.) Determine the image of the projection onto the $x_1x_2$-plane of the convex combination of $M_1$ and $M_2$. Do this by using the approach in Example 2.146 to compute the new values of $w_1, \ldots, w_8$ and then use the visualization technique in the example to plot the eight vertices $z_0, \ldots, z_7$ and draw the midpoints $w_1, \ldots, w_8$.
(c.) Use *LinComboTool* to verify your result in part (b.).
(d.) Determine the behavior of the third coordinate function, $w$, for this convex combination map using the approach in Example 2.146.

***Try it out!***

EXERCISE 2.148. Repeat the steps in Exercise 2.147 using the same Weierstrass data,

$$G_1 = iz \quad \text{and} \quad dh_1 = \frac{z\,dz}{1-z^4}$$

for $M_1$ but replacing $M_2$ with the Weierstrass data

$$G_2 = iz^2 \quad \text{and} \quad dh_1 = \frac{z^2\,dz}{1-z^6}.$$

(a.) Show that $G_1 = iz$ and $G_2 = iz^2$ satisfies eq. (**??**).
(b.) Determine the image of the projection onto the $x_1x_2$-plane of the convex combination of $M_1$ and $M_2$ by using the approach in Example 2.146 to compute the new values of the vertices.
(c.) Use *LinComboTool* to verify your result in part (b.).
(d.) Determine the behavior of the third coordinate function, $w$, for this convex combination map using the approach in Example 2.146.

***Try it out!***

## 2.8. Conclusion

We have presented an introduction to minimal surfaces and described a few topics that students can explore using the exercises, the exploratory problems, and the projects along with the applets. For a deeper and thorough explanation of differential geometry consult [**7**], [**17**], or [**20**] for beginners, and [**3**] for intermediates. Also, you should consider Spivak's five volume work [**23**]. For more background on minimal surfaces we recommend [**24**], [**14**], [**15**], [**6**], [**22**], and [**19**].

## 2.9. Additional Exercises

### Differential Geometry

EXPLORATION 2.149. An oblique cylinder can be parametrized by

$$\mathbf{x}(u,v) = (\cos u, \sin u + v\cos\theta, v\sin\theta),$$

where $\theta \in \left(0, \frac{\pi}{2}\right)$ is a fixed value. Use *DiffGeomTool* to explore what happens to the oblique cylinder as $\theta$ varies between 0 and $\frac{\pi}{2}$.

EXERCISE 2.150. Use *DiffGeomTool* to graph the surface parametrized by

$$\mathbf{x}(u,v) = \left(\cos u\left(1 + v\sin\left(\frac{1}{2}u\right)\right), \sin u\left(1 + v\sin\left(\frac{1}{2}u\right)\right), v\cos\left(\frac{1}{2}u\right)\right),$$

where $-\pi < u < \pi$, $-\frac{1}{2} < v < \frac{1}{2}$. This surface is known as the Möbius strip and is nonorientable; that is, the normal vector can change from pointing outward to pointing inward as it travels along a closed path on the surface. You can see this in *DiffGeomTool* by clicking on the **Normal vector** box and setting the `Point locator: (u,v)=` to $(\pi - 0.1, 0)$. Next, change this $u$ coordinate to each of the following values: $u = \pi - 0.1 - 1$, $u = \pi - 0.1 - 2$, $u = \pi - 0.1 - 3$, $u = \pi - 0.1 - 4$, $u = \pi - 0.1 - 5$, and $u = \pi - 0.1 - 6$. As you do so, observe that $\mathbf{n}$ will make nearly a complete path along a closed curve but it will change the direction it is pointing from where it started to where it ended.

EXERCISE 2.151. Describe the $u$-parameter and $v$-parameter curves on the Enneper surface.

EXERCISE 2.152. In Exploration 2.11(c), you proved the largest value of $r$ for which the Enneper surface has no self-intersections assuming that the intersection occurs on the $x_3$-axis. In this exercise, prove the same result without assuming the intersection occurs on the $x_3$-axis.

EXERCISE 2.153. Compute the coefficients of the first and the second fundamental forms for the Enneper surface whose parametrization is

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right).$$

EXERCISE 2.154. A *CMC (Constant Mean Curvature) surface* is a surface that has the same mean curvature everywhere on the surface. Minimal surfaces are a subset of CMC surfaces. Using *DiffGeomTool* sketch the following surfaces and determine which are CMC surfaces:

(a). $\mathbf{x}(u,v) = (u - v, u + v, 2(u^2 + v^2))$, where $-1 < u < 1$, $-1 < v < 1$;
(b). $\mathbf{x}(u,v) = (\cos u, \sin u, v)$, where $-\pi < u < \pi$, $-2 < v < 2$;
(c). $\mathbf{x}(u,v) = \Big((2 + \cos v)\cos u, (2 + \cos v)\sin u, \sin v\Big)$, where $0 < u, v < 2\pi$;
(d). $\mathbf{x}(u,v) = (\sqrt{1 - u^2}\cos v, \sqrt{1 - u^2}\sin v, u)$, where $-1 < u < 1$, $-\pi < v < \pi$;

## Minimal Surfaces

EXERCISE 2.155. Use eq (5) to show that the Enneper surface parametrized by

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right)$$

is a minimal surface.

EXERCISE 2.156. Prove Theorem 2.38 in the special case that the surface of revolution has the parametrization

$$\mathbf{x}(u,v) = \big(f(v)\cos u, f(v)\sin u, v\big).$$

EXERCISE 2.157. An oblique cylinder is a cylinder whose side forms an angle $\theta$ with the $x_1x_2$-plane, where $0 < \theta \le \frac{\pi}{2}$. For a fixed $\theta$ it can be parametrized by

$$\mathbf{x}(u,v) = (\cos u, \sin u + v\cos\theta, v\sin\theta).$$

Determine the values of $\theta$ for which $\mathbf{x}$ is isothermal.

EXERCISE 2.158. Show that the parametrization

$$\mathbf{x}(u,v) = \Bigg(\arctan\left(\frac{2u}{1-(u^2+v^2)}\right), \arctan\left(\frac{-2v}{1-(u^2+v^2)}\right), \frac{1}{2}\ln\left(\frac{(u^2-v^2+1)^2+4u^2v^2}{(u^2-v^2-1)^2+4u^2v^2}\right)\Bigg)$$

is an isothermal parametrization of Scherk's doubly periodic surface (that is, show that it is isothermal and that there is transformation that maps this parametrization to the parametrization given in Exercise 2.42(b) for Scherk's doubly periodic surface).

## Weierstrass Representation

EXERCISE 2.159. In Example 2.66, show the details in going from the Enneper surface parametrization

$$\mathbf{x} = \left(\mathrm{Re}\left\{z - \frac{1}{3}z^3\right\}, \mathrm{Re}\left\{-i\left(z + \frac{1}{3}z^3\right)\right\}, \mathrm{Re}\left\{z^2\right\}\right)$$

to the parametrization

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2\right)$$

that is also for the Enneper surface.

EXERCISE 2.160. Compute the parametrization for the minimal surfaces generated by using $p(z) = \frac{1}{2z}$ and $q(z) = iz$ on the domain $\mathbb{C} - \{0\}$ in the Weierstrass representation. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface which is known as the *wavy plane*. [Use radius min=0.001, radius max=1.3, theta min=-pi, theta max=pi with initial values $x = \text{Re}(1/2 * \ln(z) - 1/4 * z^2)$, $y = \text{Im}(1/2 * \ln(z) + 1/4 * z^2)$, and $z = \text{Re}(z)$.]

EXERCISE 2.161. Compute the parametrization for the minimal surfaces generated by using $p(z) = z^2$ and $q(z) = \frac{i}{z^2}$ on the domain $\mathbb{C} - \{0\}$ in the Weierstrass representation. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface which is known as *Richmond's surface*. [Use radius min=0.1, radius max=1, theta min=pi/24, theta max=2pi+pi/24 with initial values $x = \text{Re}(1/3 * z^3 + 1/z)$, $y = \text{Im}(1/3 * z^3 - 1/z)$, and $z = \text{Re}(2 * z)$.]

EXERCISE 2.162. Compute the parametrization for the minimal surfaces generated by using $p(z) = \frac{(z+1)^2}{z^4}$ and $q(z) = \frac{z^2(z-1)}{z+1}$ on the domain $\mathbb{D} - \{0\}$ in the Weierstrass representation. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface which is known as the *wavy plane*. [Use radius min=0.1, radius max=0.9, theta min=pi/24, theta max=2pi+pi/24 with initial values $x =$, $y =$, and $z =$.]

### The Gauss map, $G$, and height differential, $dh$

EXERCISE 2.163. Show that if $z = x + iy$ is the projection of the point $(x_1, x_2, x_3)$ on the Riemann sphere onto to complex plane, then

$$x = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{1 - x_3}.$$

EXERCISE 2.164. For Scherk's doubly periodic surface find:

(a) $G(0)$; (b) $G(1)$; (c) $G(-1)$; (d) $G(i)$; (e) $G(-i)$.

EXERCISE 2.165. The Weierstrass data for a 4-noid are

$$G(z) = z^3 \quad \text{and} \quad dh = \frac{z^3}{(z^4 - 1)^2}\, dz.$$

Show that the ends of the 4-noid are catenoid ends.

EXERCISE 2.166. Determine the asymptotic and curvature lines for Scherk's doubly periodic surface with $G(z) = z$ and $dh(z) = \frac{iz}{z^4-1} dz$.

EXERCISE 2.167. Determine the period conditions for the wavy plane with $G(z) = z$ and $dh(z) = dz$.

EXERCISE 2.168. Let $M$ be the Scherk doubly periodic surface with 6 ends. Using the approach of Example 2.103 determine $G$ and $dh$ for this surface.

### Minimal Surfaces and Harmonic Univalent Mappings

EXERCISE 2.169. Prove that if $f = u + iv$ is harmonic in a simply-connected domain $G$, then $f = h + \overline{g}$, where $h$ and $g$ are analytic.

EXERCISE 2.170. Prove that the representations $f(z) = h(z) + \overline{g}(z)$ and $f(z) = \operatorname{Re}\{h(z)+g(z)\} + i\operatorname{Im}\{h(z)-g(z)\}$ are equivalent.

EXERCISE 2.171. Shear $h(z) - g(z) = \frac{z}{1-z}$ with $\omega(z) = z^2$ to get the harmonic univalent function $f = h + \overline{g}$ given in Example 2.130, where

$$h = \frac{1}{8}\ln\left(\frac{z+1}{z-1}\right) + \frac{3z-2z^2}{4(1-z)^2} \quad \text{and} \quad g = \frac{1}{8}\ln\left(\frac{z+1}{z-1}\right) - \frac{z-2z^2}{4(1-z)^2}.$$

EXERCISE 2.172. Shear $h(z) - g(z) = \frac{z}{(1-z)^2}$ with $\omega(z) = z^2$ to get the harmonic univalent function $f = h + \overline{g}$ given in Exercise 2.132, where

$$h = \frac{1}{8}\ln\left(\frac{z+1}{z-1}\right) + \frac{3z-2z^2}{4(1-z)^2} \quad \text{and} \quad g = \frac{1}{8}\ln\left(\frac{z+1}{z-1}\right) - \frac{z-2z^2}{4(1-z)^2}.$$

EXERCISE 2.173. Show that the parametrization:

$$\mathbf{x} = \left(\operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right], \operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right], \frac{1}{2}\operatorname{Im}\left[i\log\left(\frac{1+z^2}{1-z^2}\right)\right]\right)$$

is equivalent to the parametrization in Exercise 2.158 that gives Scherk's doubly periodic minimal surface.

LARGE PROJECT 2.174. The analytic function, $F(z) = \frac{z}{(1-z)^2}$, maps the unit disk, $\mathbb{D}$, onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4})$ and is an important function. Shear $h(z) - g(z) = \frac{z}{(1-z)^2}$ with various dilatations, $\omega$, that satisfy the condition $|\omega| < 1$ for all $z \in \mathbb{D}$ (e.g., $\omega = z^{2n} (n = \mathbb{N}) \omega = e^{\imath\theta} z^2, (\theta \in \mathbb{R}), \omega = \left(\frac{z-a}{1-\overline{a}z}\right)(|a| < 1)$). Determine the corresponding minimal graphs.

## Convex Combinations of Minimal Surfaces

EXERCISE 2.175. Repeat the steps in Exercise 2.147 using the Weierstrass data,

$$G_1 = z \quad \text{and} \quad dh_1 = \frac{z\,dz}{1-z^4}$$

for $M_1$ and

$$G_2 = iz \quad \text{and} \quad dh_1 = \frac{z\,dz}{1-z^4}$$

for $M_2$.

(a.) Show that eq. (**??**) is satisfied.
(b.) Determine the image of the projection onto the $x_1x_2$-plane of the convex combination of $M_1$ and $M_2$ by using the approach in Example 2.146 to compute the new values of the vertices.
(c.) Use *LinComboTool* to verify your result in part (b.).
(d.) Determine the behavior of the third coordinate function, $w$, for this convex combination map using the approach in Example 2.146.

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